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Analyticity of the Stokes semigroup in spaces of bounded functions

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1. Introduction

We consider the initial-boundary problem for the Stokes equations

$$v_t-\Delta v+\nabla q=0 \qquad \text{in } \Omega\times(0,T), \tag{1.1}$$

$$\operatorname{div} v=0 \qquad \text{in } \Omega\times(0,T), \tag{1.2}$$

$$v=0 \qquad \text{on } \partial\Omega\times(0,T), \tag{1.3}$$

$$v(x,0)=v_0 \qquad \text{on } \Omega\times\{0\}, \tag{1.4}$$

in a domain Ω in $\mathbf{R}^n$ with $n\geqslant 2$. It is well known that the solution operator

$$S(t)\colon v_0\longmapsto v(t)=v(\cdot\,,t)$$

forms an analytic semigroup in the solenoidal L^r space $L^r_\sigma(\Omega)$ for $r\in(1,\infty)$ for various kind of domains Ω including smoothly bounded domains [27], [55]. However, it has been a long-standing open problem whether or not the Stokes semigroup $\{S(t)\}_{t\geqslant 0}$ is analytic in L^∞-type space even if Ω is bounded. When Ω is a half space it is known that the Stokes semigroup $\{S(t)\}_{t\geqslant 0}$ is analytic in L^∞-type space since explicit solution formulas are available [14], [45], [58].

The goal of this paper is to give an affirmative answer to this open problem at least when Ω is bounded as a typical case. For a precise statement let $C_{0,\sigma}(\Omega)$ denote the L^∞-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all smooth solenoidal vector fields with compact support in Ω. When Ω is bounded, $C_{0,\sigma}(\Omega)$ agrees with the space of all solenoidal vector fields continuous in $\overline{\Omega}$ and vanishing on the boundary $\partial\Omega$ [43]. The following is one of our main results.

THEOREM 1.1. (Analyticity in $C_{0,\sigma}$) *Let Ω be a bounded domain in $\mathbf{R}^n$ with C^3 boundary. Then the solution operator* (*the Stokes semigroup*) *$S(t)\colon v_0 \mapsto v(t)$ $(t \geqslant 0)$ is a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$.*

A key observation is a gradient estimate of harmonic pressure of the form

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla q(x,t)| \leqslant C\|\nabla v\|_{L^\infty(\partial\Omega)}(t), \tag{1.5}$$

with C depending only on Ω, where d_Ω is the distance function from $\partial\Omega$. The estimate (1.5) follows from a property of the Helmholtz decomposition or the inhomogeneous Neumann problem for the Laplace equation. Such a property is not limited to a bounded domain, so we call such a domain *admissible*; for a precise definition see Definition 2.3.

Based on (1.5), we are able to derive a necessary bound for

$$N(v,q)(x,t) = |v(x,t)| + t^{1/2}|\nabla v(x,t)| + t|\nabla^2 v(x,t)| + t|\partial_t v(x,t)| + t|\nabla q(x,t)|,$$

which is a key to prove analyticity results. We state such a-priori estimates in a general domain (not necessarily bounded) for $\tilde{L}^r$-solutions, introduced by R. Farwig, H. Kozono and H. Sohr [16], [17], [18], with $\tilde{L}^r_\sigma = L^r_\sigma \cap L^2_\sigma$ $(r \geqslant 2)$, where L^r_σ is the closure of $C^\infty_{c,\sigma}(\Omega)$ in L^r. (The $\tilde{L}^r$ theory is useful to handle the Navier–Stokes equations in a general domain [21].) It is by now well known [24] that if a uniformly C^3-domain admits the Helmholtz decomposition in L^r, then there exists an L^r-solution and the Stokes semigroup $S(t)$ is analytic in L^r_σ. However, in general, the Helmholtz decomposition in L^r space may not hold (see [11] and [46]). Fortunately, $\tilde{L}^r$ theory is available for a general domain so we establish a-priori estimates for an $\tilde{L}^r$-solution.

THEOREM 1.2. (A priori L^∞-estimates) *Let Ω be an admissible, uniformly C^3-domain in $\mathbf{R}^n$ and let $r > n$. Then there exist positive constants C and T_0 depending only on Ω such that the bound*

$$\sup_{0<t<T_0} \|N(v,q)\|_\infty(t) \leqslant C\|v_0\|_\infty \tag{1.6}$$

holds for all $\tilde{L}^r$-solutions (v,q) of (1.1)–(1.4) *with $v_0 \in C^\infty_{c,\sigma}(\Omega)$.*

This estimate together with a density argument enables us to extend the solution semigroup $S(t)$ for (1.1)–(1.4) to $C_{0,\sigma}(\Omega)$ so that it becomes analytic. We thus obtain a general result which includes Theorem 1.1 as a particular case.

THEOREM 1.3. (Analyticity for a general domain) *Let Ω be an admissible, uniformly C^3-domain in $\mathbf{R}^n$. Then the Stokes semigroup $S(t)$ is uniquely extendable to a C_0-analytic semigroup in $C_{0,\sigma}(\Omega)$. Moreover, the estimate* (1.6) *holds with some $C > 0$ and $T_0 > 0$ for $v = S(t)v_0$, $v_0 \in C_{0,\sigma}(\Omega)$, with a suitable choice of pressure q.*

It is natural to extend the Stokes semigroup to L^∞_σ, the solenoidal L^∞ space.

THEOREM 1.4. (Analyticity in L^∞_σ for a bounded domain) *Let Ω be a bounded C^3-domain in $\mathbf{R}^n$. Then the Stokes semigroup $S(t)$ is a (non-C_0-)analytic semigroup in $L^\infty_\sigma(\Omega)$.*

For the Laplace operator or general elliptic operators it is well known that the corresponding semigroup is analytic in L^∞-type spaces. The first pioneering work goes back to K. Yosida [66] for second-order operators on $\mathbf{R}$. Unfortunately, it seems difficult to extend his method to multi-dimensional elliptic operators. K. Masuda [48], [47] (see also [49]) first proved the analyticity of the semigroup generated by a general elliptic operator (including higher-order operators) in $C_0(\mathbf{R}^n)$, the space of continuous functions vanishing at the space-infinity. A key idea is to derive a corresponding resolvent estimate by a localization method together with L^p-estimates and interpolation inequalities. It was extended by H. B. Stewart to Dirichlet problems [61] and for more general boundary conditions [62]. (A complete proof is given in [6, Appendix].) The reader is referred to the book by A. Lunardi [42, Chapter 3] for this Masuda–Stewart method which applies to many other situations. By now, analyticy results in L^∞ spaces are established in various settings [6], [8], [37], [42], [63]. However, their localization argument does not directly apply to the Stokes equations and this may be a reason why the analyticity in $C_{0,\sigma}$ had been left open for a long time. Very recently, M. Hieber and the authors [2] found a way to prove Theorem 1.1 by the Masuda–Stewart type argument based on (1.5).

Although there are several results on analyticity of $S(t)$ in L^r_σ for various domains such as a half space, a bounded domain [27], [55], an exterior domain [12], [35], an aperture domain [20], a layer domain [3], a perturbed half space [19] (even with variable viscosity coefficients [4], [5]), the result corresponding to Theorem 1.3 is available only for a half space [14], [45], [58] (and the whole space, where the Stokes semigroup agrees with the heat semigroup).

We do not touch on the problem of the large time behavior of the Stokes semigroup except in the case when Ω is bounded. In particular, we do not know in general whether or not the Stokes semigroup is bounded in time. This is known for a half space [14], [45], [58]. For a bounded domain it is not difficult to derive even exponential decay as $t\to\infty$. In fact, for a bounded domain we prove that $S(t)$ is a bounded analytic semigroup in $C_{0,\sigma}$ (Remark 5.4 (i)). Moreover, the operator norm $\|S(t)\|$ is bounded in t when Ω is bounded. Such a result is called a maximum modulus result and has been studied in the literature [56], [57], [65] (Remark 5.4 (ii)). Very recently, P. Maremonti [44] proved the boundedness of $\|S(t)\|$ when Ω is an exterior domain using our Theorem 1.1.

To extend analyticity in L^∞_σ to general admissible domains we have to construct

$S(t)$ in L^∞_σ in a unique way, since $\tilde{L}^r_\sigma$ does not contain L^∞_σ. This attempt is so far carried out for a half space in [14], where an explicit solution formula is available. Moreover, it is also shown in [14] that $S(t)$ is a C_0-analytic semigroup in $\mathrm{BUC}_\sigma(\Omega)$ when Ω is a half space; see also [58]. Here $\mathrm{BUC}_\sigma(\Omega)$ denotes the space of all solenoidal, bounded, uniformly continuous vector fields in Ω vanishing on the boundary $\partial\Omega$. Recently, the authors [1] extended Theorem 1.4 (and Theorem 1.1) to the case when Ω is an exterior domain and proved that $S(t)$ is a C_0-analytic semigroup in $\mathrm{BUC}_\sigma(\Omega)$. The analyticity, as well as (1.6), is fundamental to study the Navier–Stokes equations. So far L^∞-type theory is only established when $\Omega=\mathbf{R}^n$ [30], [32] and $\Omega=\mathbf{R}^n_+$ [9], [58]. We shall also discuss the non-linear problem in forthcoming papers.

Our approach to establish (1.6) is completely different from conventional approaches. We appeal to a blow-up argument which is often used in the study of non-linear elliptic and parabolic equations.

We argue by contradiction. Suppose that (1.6) were false for any choice of T_0 and C. Then there would exist a sequence $\{(v_m,q_m)\}_{m=1}^\infty$ of solutions of (1.1)–(1.4) with $v_0=v_{0m}$ and a sequence $\tau_m\downarrow 0$ such that $\|N(v_m,q_m)\|_\infty(\tau_m)>m\|v_{0m}\|_\infty$. There is $t_m\in(0,\tau_m)$ such that $\|N(v_m,q_m)\|_\infty(t_m)\geqslant\frac{1}{2}M_m$ with $M_m=\sup_{0<t<\tau_m}\|N(v_m,q_m)\|_\infty(t)$. We normalize v_m and q_m by dividing by M_m to observe that

$$\sup_{0<t<t_m}\|N(\tilde{v}_m,\tilde{q}_m)\|_\infty(t)\leqslant 1, \tag{1.7}$$

$$\|N(\tilde{v}_m,\tilde{q}_m)\|_\infty(t_m)\geqslant\tfrac{1}{2}, \tag{1.8}$$

$$\|\tilde{v}_{0m}\|_\infty<\frac{1}{m}, \tag{1.9}$$

with $\tilde{v}_m=v_m/M_m$ and $\tilde{q}_m=q_m/M_m$. We rescale $(\tilde{v}_m,\tilde{q}_m)$ around a point $x_m\in\Omega$ satisfying

$$N(\tilde{v}_m,\tilde{q}_m)(x_m,t_m)\geqslant\tfrac{1}{4} \tag{1.10}$$

to get a blow-up sequence of (v_m,q_m) of the form

$$u_m(x,t)=\tilde{v}_m(x_m+t_m^{1/2}x,t_mt),\quad p_m(x,t)=t_m^{1/2}\tilde{q}_m(x_m+t_m^{1/2}x,t_mt).$$

(Such an x_m exists because of (1.8).) Because of the scaling invariance of the equations (1.1) and (1.2), the rescaled function (u_m,p_m) solves (1.1)–(1.2) in a rescaled domain $\Omega_m\times(0,1)$. Note that the time interval is normalized to a unit interval and Ω_m tends to either a half space or the whole space $\mathbf{R}^n$ as $m\to\infty$.

The basic strategy is to prove that the blow-up sequence $\{(u_m,p_m)\}_{m=1}^\infty$ (after taking a subsequence) converges to a solution (u,p) of (1.1)–(1.4) with zero initial deta. If the convergence is strong enough, (1.10) implies that $N(u,p)(0,0)\geqslant\frac{1}{4}$. If the limit (u,p) is unique, it is natural to expect that $u\equiv 0$ and $\nabla p\equiv 0$. This evidently yields a contradiction to $N(u,p)(0,0)\geqslant\frac{1}{4}$. The first part corresponds to "compactness" of a blow-up sequence and the second part corresponds to "uniqueness" of a blow-up limit. (A similar rescaling argument is explained in detail in the recent textbook [26].) When the problem is the heat equation, this strategy is easy to realize. However, for the Stokes equations it turns out that this procedure is highly non-trivial because of the presence of the pressure.

The situation is divided into two cases depending on whether the limit of Ω_m is a half space or the whole space $\mathbf{R}^n$. Let us consider the case when the limit is the whole space. To have necessary compactness for $\{(u_m,p_m)\}_{m=1}^{\infty}$ it is enough to prove that a local space-time Hölder bound for u_m, ∇u_m, $\nabla^2 u_m$ and ∇p_m holds near $(0,1)$ as $m\to\infty$. We are tempted to derive such an interior regularity estimate from (1.7) by localizing the problem. This idea works for the heat equation but for the Stokes equations it does not work (Remark 3.3 (i)). We invoke admissibility of Ω to control the pressure term by (1.5) and derive necessary a-priori estimates from the standard parabolic regularity theory [41]. The uniqueness of the blow-up limit is easy, since the limit equation is the heat equation. Note that the constant in (1.5) is independent of the rescaling procedure, so our Hölder estimate is uniform. The case when Ω_m tends to a half space is more involved. We use Schauder estimates for the Stokes equations developed by V. A. Solonnikov [60] instead of the usual parabolic theory [41]. To show that the blow-up limit (u,p) is trivial, we invoke the uniqueness result for spatially non-decaying velocity in a half space due to Solonnikov [58]. Note that to assert the uniqueness of solutions (u,p) of the Stokes equations (1.1)–(1.4) in a half space with zero initial data and a bound for $\|N(u,p)\|_\infty(t)$, we need to assume some decay for ∇p far from the boundary, since otherwise there is a counterexample (Remark 4.2). We invoke (1.5) to deduce necessary decay for ∇p for the limit.

A blow-up argument was first introduced by E. De Giorgi [13] to study regularity of a minimal surface. B. Gidas and J. Spruck [25] adjusted the blow-up argument to derive a-priori bounds for solutions of a semilinear elliptic problem. It seems that the first application to (semilinear) parabolic problems to get an a-priori bound goes back to [28] (see also [31]). The method has been further developed in recent years to obtain several a-priori bounds; see e.g. [50] and [51]. However, it is quite recent to apply it to the Navier–Stokes equations. For example, a blow-up argument was used to conclude non-existence of type-I blow-up for axisymmetric solutions [38], [52] and solutions having continuously varying vorticity directions [34].

In this paper we use a blow-up argument to prove that a bounded C^3-domain is admissible so that Theorem 1.3 yields Theorem 1.1. It is easy to prove that a half space is admissible. It is possible to prove that an exterior domain (see the recent paper [1]) or a perturbed half space is admissible, but we do not discuss these problems in the present paper. We conjecture that an unbounded domain is admissible if Ω is *not quasicylindrical* (see [7, §6.32]), i.e. $\overline{\lim}_{|x|\to\infty} d_\Omega(x)=\infty$.

This paper is organized as follows. In §2 we define an admissible domain and prove that a bounded C^3-domain is admissible by a blow-up argument. In §3 we derive local Hölder estimates, both interior and up to the boundary, which are key to derive necessary compactness for our blow-up sequence. In §4 we review a uniqueness result for spatially non-decaying solutions for the Stokes equations as well as the heat equation. In §5 we prove key a-priori estimates (Theorem 1.2) by a blow-up argument. As an application we prove Theorem 1.3 (and Theorem 1.1 as a particular case). In §6 we prove Theorem 1.4.

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2. Admissible domains

In this section we introduce the notion of an admissible domain and prove that a bounded domain is admissible by a blow-up argument. We also give a short proof that a half space is admissible. We first recall the Helmholtz decomposition.

2.1. Helmholtz decomposition

Let Ω be an arbitrary domain in $\mathbf{R}^n$ ($n\geqslant 2$). Let $L^r_\sigma(\Omega)$ ($1<r<\infty$) denote the L^r-closure of $C^\infty_{c,\sigma}(\Omega)$, the space of all smooth solenoidal vector fields with compact support in Ω. The Helmholtz decomposition is a topological direct sum decomposition of the form

$$L^r(\Omega)=L^r_\sigma(\Omega)\oplus G^r(\Omega),\quad G^r(\Omega)=\{\nabla p\in L^r(\Omega)\,|\,p\in L^r_{\mathrm{loc}}(\Omega)\}.$$

We do not distinguish between spaces of vector-valued and scalar functions.

Although this decomposition is known to hold (see e.g. [22, §III.1]) for various domains like bounded or exterior domains with smooth boundary, in general there is a domain with (uniformly) smooth boundary such that the L^r-Helmholtz decomposition does not hold (cf. [11] and [46]). Note that this decomposition is an orthogonal decomposition if $r=2$ and that the case $r=2$ is valid for any domain Ω.

In [16] Farwig, Kozono and Sohr introduced the $\tilde{L}^r$ space and proved that the Helmholtz decomposition for $\tilde{L}^r$ is valid for any uniformly C^2-domain for $n=3$. Later, it was generalized to arbitrary uniformly C^1-domains for $n\geqslant 2$ [17]. Let us recall their results. We set

$$\tilde{L}^r(\Omega)=\begin{cases} L^2(\Omega)\cap L^r(\Omega), & 2\leqslant r<\infty, \\ L^2(\Omega)+L^r(\Omega), & 1<r<2. \end{cases}$$

Note that $\tilde{L}^{r_1}\subset\tilde{L}^r$ for $r_1>r$. We define $\tilde{L}^r_\sigma$ and $\widetilde{G}^r$ in a similar way. We then recall the definition of uniformly C^k-domains for $k\geqslant 1$; see e.g. [54, §I.3.2].

Definition 2.1. (Uniformly C^k-domain) Let Ω be a domain in $\mathbf{R}^n$ with $n\geqslant 2$. Assume that there exist $\alpha,\beta,K>0$ such that for each $x_0\in\partial\Omega$ there is a C^k-function h of $n-1$ variables y' such that

$$\sup_{\substack{|l|\leqslant k \\ |y'|<\alpha}} |\partial^l_{y'} h(y')|\leqslant K, \quad \nabla' h(0)=0, \quad h(0)=0,$$

and denote a neighborhood of x_0 by

$$U_{\alpha,\beta,h}(x_0)=\{(y',y_n)\in\mathbf{R}^n \mid h(y')-\beta<y_n<h(y')+\beta \text{ and } |y'|<\alpha\}.$$

Assume that, up to rotation and translation, we have

$$U_{\alpha,\beta,h}(x_0)\cap\Omega=\{(y',y_n)\mid h(y')<y_n<h(y')+\beta \text{ and } |y'|<\alpha\}$$

and

$$U_{\alpha,\beta,h}(x_0)\cap\partial\Omega=\{(y',y_n)\mid y_n=h(y') \text{ and } |y'|<\alpha\}.$$

Then we call Ω a *uniformly C^k-domain* of type α,β,K. Here $\partial^l_x=\partial^{l_1}_{x_1}\dots\partial^{l_n}_{x_n}$ with multi-index $l=(l_1,\dots,l_n)$ and $\partial_{x_j}=\partial/\partial x_j$ as usual and ∇' denotes the gradient in $y'\in\mathbf{R}^{n-1}$.

PROPOSITION 2.2. ([16], [17]) *Let Ω be a uniformly C^1-domain of type $\alpha,\beta,K>0$ and let $1<r<\infty$. Then $\tilde{L}^r(\Omega)$ has a topological direct sum decomposition*

$$\tilde{L}^r(\Omega)=\tilde{L}^r_\sigma(\Omega)\oplus\widetilde{G}^r(\Omega).$$

Let $P(=P_r)$ be the projection to $\tilde{L}^r_\sigma(\Omega)$ associated with this decomposition. Then there is a constant $C=C(r,\alpha,\beta,K)>0$ such that the operator norm of P is bounded by C.

The operator P is often called the Helmholtz projection. In this paper we shall use the $\tilde{L}^r$ space for $r\geqslant 2$, where the $\tilde{L}^r$ norm is given as

$$\|f\|_{\tilde{L}^r}=\max\{\|f\|_{L^r},\|f\|_{L^2}\}.$$

2.2. Definition of an admissible domain

We give a rigorous definition of an admissible domain. Let $d_\Omega(x)$ denote the distance function from $\partial\Omega$, i.e.

$$d_\Omega(x)=\inf\{|x-y| \mid y\in\partial\Omega\}.$$

Let $Q_r=I-P_r$ be the projection to $\widetilde{G}^r(\Omega)$ associated with the Helmholtz decomposition. We shall suppress the subscript r of Q_r.

Definition 2.3. (Admissible domain) Let Ω be a uniformly C^1-domain in $\mathbf{R}^n$ $(n\geqslant 2)$, with $\partial\Omega\neq\varnothing$. We call Ω *admissible* if there exist $r\geqslant n$ and a constant $C=C_\Omega$ such that the bound

$$\sup_{x\in\Omega} d_\Omega(x)|Q[\nabla\cdot f](x)|\leqslant C_\Omega\|f\|_{L^\infty(\partial\Omega)}$$

holds for all matrix-valued functions $f=(f_{ij})_{1\leqslant i,j\leqslant n}\in C^1(\bar{\Omega})$ which satisfy

$$\nabla\cdot f\left(=\sum_{j=1}^n \partial_j f_{ij}\right)\in \tilde{L}^r(\Omega)$$

and

$$\operatorname{tr} f=0 \quad \text{and} \quad \partial_l f_{ij}=\partial_j f_{il} \tag{2.1}$$

for all $i,j,l\in\{1,...,n\}$, where $\partial_j=\partial_{x_j}$.

Remark 2.4. (i) We note that $\nabla q=Q[\nabla\cdot f]$ is formally obtained by solving the Neumann problem

$$\begin{cases} \Delta q=\operatorname{div}(\nabla\cdot f) & \text{in } \Omega,\\ \dfrac{\partial q}{\partial n_\Omega}=n_\Omega\cdot(\nabla\cdot f) & \text{on } \partial\Omega,\end{cases}$$

where n_Ω is the exterior unit normal of $\partial\Omega$. In particular q (and also ∇q) is harmonic in Ω, since

$$\operatorname{div}(\nabla\cdot f)=\sum_{i,j=1}^n \partial_i\partial_j f_{ij}=\sum_{i,j=1}^n \partial_j\partial_j f_{ii}=0.$$

(ii) The left-hand side of the inequality in Definition 2.3 is always finite. Indeed, since ∇q is harmonic, the mean-value theorem (see e.g. [15, §2.2.2]) implies that

$$\nabla q(x)=\frac{1}{|B_\varrho(x)|}\int_{B_\varrho(x)}\nabla q(y)\,dy \quad \text{for } \varrho<d_\Omega(x),$$

where $B_\varrho(x)$ is the closed ball of radius ϱ centered at x and $|B_\varrho(x)|$ denotes its volume. Applying the Hölder inequality yields

$$|\nabla q(x)|\leqslant|B_\varrho(x)|^{-1/p}\|\nabla q\|_p\leqslant C\varrho^{-n/p}\|\nabla\cdot f\|_{\tilde{L}^r} \quad \text{for } 2\leqslant p\leqslant r,$$

by Proposition 2.2. If $d_\Omega(x)<1$, we take $p=n$. If $d_\Omega(x)\geqslant 1$, we take $p=2$. Since $n\geqslant 2$, this choice implies that $|\nabla q(x)|d_\Omega(x)$ is bounded in Ω. Although $|\nabla q(x)|d_\Omega(x)$ is continuous in Ω, this quantity may not be continuous up to the boundary.

(iii) Although the constant $C=C_\Omega$ in Definition 2.3 depends on the domain, it is independent of dilation and translation. In other words, $C_{\lambda\Omega+x_0}=C_\Omega$ for $x_0\in\mathbf{R}^n$ and $\lambda>0$.

(iv) It is easy to see that the half space $\mathbf{R}^n_+=\{(x',x_n)\,|\,x_n>0\}$ is admissible. In this case

$$Q[\nabla\cdot f]=\nabla q, \quad q(x',x_n)=\int_{x_n}^\infty \mathcal{P}_s[n_\Omega\cdot(\nabla\cdot f)]\,ds,$$

where $\mathcal{P}_s$ denotes the Poisson semigroup, i.e.

$$\mathcal{P}_s[h]=P_s*h \quad \text{with } P_s(x')=\frac{as}{(|x'|^2+s^2)^{n/2}},\ x'\in\mathbf{R}^{n-1},$$

where $2/a$ is the surface area of the $(n-1)$-dimensional unit sphere. Since

$$-n_\Omega\cdot(\nabla\cdot f)=\sum_{j=1}^n \partial_j f_{nj}=\sum_{j=1}^{n-1}\partial_j f_{nj}-\sum_{i=1}^{n-1}\partial_n f_{ii}=\sum_{j=1}^{n-1}\partial_j(f_{nj}-f_{jn})$$

by (2.1), we end up with

$$\nabla q(x)=-\sum_{j=1}^{n-1}\nabla\partial_j\int_{x_n}^\infty \mathcal{P}_s[f_{nj}-f_{jn}]\,ds.$$

By the explicit form of the Poisson semigroup, it is easy to see that

$$\|\partial_j\mathcal{P}_s[h]\|_{L^\infty(\mathbf{R}^{n-1})}(s)\leqslant\frac{c\|h\|_{L^\infty(\mathbf{R}^{n-1})}}{s} \quad \text{for } s>0 \text{ and } 1\leqslant j\leqslant n-1,$$

with $c>0$ independent of s and h. Thus, with $h_j=f_{nj}-f_{jn}$,

$$\begin{aligned}\|\partial_k q\|_{L^\infty(\mathbf{R}^{n-1})}(x_n)&\leqslant\sum_{j=1}^{n-1}\int_{x_n}^\infty\|\partial_k\partial_j\mathcal{P}_s[h_j]\|_{L^\infty(\mathbf{R}^{n-1})}\,ds\\&\leqslant c^2(n-1)\int_{x_n}^\infty\frac{1}{s^2}\,ds\max_{1\leqslant j\leqslant n-1}\|h_j\|_{L^\infty(\mathbf{R}^{n-1})}\leqslant\frac{C'\|f\|_{L^\infty}}{x_n}\end{aligned}$$

for $k\leqslant n-1$. For $k=n$ it is easier to obtain a similar estimate, so we observe that the half space is admissible since $x_n=d_\Omega(x)$.

2.3. Blow-up arguments

Our goal in this subsection is to prove the following result.

THEOREM 2.5. *A bounded domain with C^3 boundary is admissible.*

We shall prove this theorem by an indirect method—a blow-up argument—although it might be possible to prove directly. For this purpose we first derive a weak formulation for $\nabla\Phi=Q[\nabla\cdot f]$.

LEMMA 2.6. *Let Ω be a C^1-domain. Assume that $f=(f_{ij})\in C^1(\bar{\Omega})$ satisfies (2.1) with $\nabla\cdot f\in L^2(\Omega)$ so that $\nabla\Phi=Q[\nabla\cdot f]\in G^2(\Omega)$. Then*

$$-\int_\Omega \Phi\Delta\varphi\,dx=\sum_{i,j=1}^n\int_{\partial\Omega} f_{ij}(x)(n_\Omega^j(x)\partial_i\varphi(x)-n_\Omega^i(x)\partial_j\varphi(x))\,d\mathcal{H}^{n-1} \tag{2.2}$$

for all $\varphi\in C_c^2(\bar{\Omega})$ satisfying $\partial\varphi/\partial n_\Omega=0$ on $\partial\Omega$, where $d\mathcal{H}^{n-1}$ is the surface element of $\partial\Omega$, and $n_\Omega(x)=(n_\Omega^1(x),...,n_\Omega^n(x))$.

Proof. The L^2-Helmholtz decomposition says that for $h=\nabla\cdot f$ there exist unique $h_0\in L^2_\sigma(\Omega)$ and $Q[h]\in G^2(\Omega)$ such that $h=h_0+Q[h]$ with $Q[h]=\nabla\Phi$. Multiply $\nabla\varphi$ with h and use the orthogonality to get

$$\int_\Omega h\cdot\nabla\varphi\,dx=\int_\Omega\nabla\varphi\cdot\nabla\Phi\,dx. \tag{2.3}$$

Since $\partial\varphi/\partial n_\Omega=0$ on $\partial\Omega$, we have

$$\int_\Omega\nabla\varphi\cdot\nabla\Phi\,dx=-\int_\Omega\Phi\Delta\varphi\,dx \tag{2.4}$$

by integration by parts. (Note that $\Phi\in L^2_{\mathrm{loc}}(\bar{\Omega})$ by the Poincaré inequality, see e.g. [15].) We now calculate the left-hand side of (2.3). We observe that

$$\begin{aligned}(\partial_j f_{ij})(\partial_i\varphi)&=\partial_j(f_{ij}\partial_i\varphi)-f_{ij}\partial_i\partial_j\varphi,\\ f_{ij}\partial_i\partial_j\varphi&=\partial_i(f_{ij}\partial_j\varphi)-(\partial_i f_{ij})\partial_j\varphi\end{aligned}$$

for all $1\leqslant i,j\leqslant n$. Since

$$\sum_{i=1}^n\partial_i f_{ij}=\sum_{i=1}^n\partial_j f_{ii}=0$$

by (2.1), we now obtain the identity

$$\int_\Omega h\cdot\nabla\varphi\,dx=\sum_{i,j=1}^n\int_{\partial\Omega} f_{ij}(n_\Omega^i\partial_i\varphi-n_\Omega^i\partial_j\varphi)\,d\mathcal{H}^{n-1}. \tag{2.5}$$

The identities (2.3)–(2.5) yield (2.2). ∎

Proof of Theorem 2.5. We argue by contradiction. Suppose that the condition were false. Then there would exist a sequence $\{\tilde{f}_m\}_{m=1}^{\infty}\subset C^1(\bar{\Omega})$ satisfying (2.1) such that

$$\infty > M_m = \sup_{x\in\Omega} d_\Omega(x)|\nabla\tilde{\Phi}_m(x)| > m\|\tilde{f}_m\|_{L^\infty(\partial\Omega)}$$

with $\nabla\tilde{\Phi}_m = Q[\nabla\cdot\tilde{f}_m]$. (Note that M_m is always finite by Remark 2.4 (ii).) We normalize $\tilde{\Phi}_m$ and $\tilde{f}_m$ by $\Phi_m=\tilde{\Phi}_m/M_m$ and $f_m=\tilde{f}_m/M_m$. There exists a sequence of points $\{x_m\}_{m=1}^{\infty}\subset\Omega$ such that

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla\Phi_m(x)| = 1, \tag{2.6}$$

$$d_\Omega(x_m)|\nabla\Phi_m(x_m)| \geqslant \tfrac{1}{2}, \tag{2.7}$$

$$\|f_m\|_{L^\infty(\partial\Omega)} < \frac{1}{m}. \tag{2.8}$$

Since $\bar{\Omega}$ is compact, a subsequence of $\{x_m\}_{m=1}^{\infty}$ converges to some $x_\infty\in\bar{\Omega}$ as $m\to\infty$.

Case 1. $x_\infty\in\Omega$. We may assume that $\Phi_m(x_\infty)=0$. Since $\nabla\Phi_m$ is harmonic, (2.6) implies that a subsequence of $\{\Phi_m\}_{m=1}^{\infty}$ converges to some function $\Phi\in C^\infty(\Omega)$ locally uniformly in Ω with all its derivatives. By (2.6) the sequence $\{\Phi_m\}_{m=1}^{\infty}$ is bounded in $L^r(\Omega)$ for any $r\in[1,\infty)$ so a subsequence of $\{\Phi_m\}_{m=1}^{\infty}$ converges to Φ weakly in L^r $(1<r<\infty)$. We apply Lemma 2.6 with $\Phi=\Phi_m$ and $f=f_m$ and let $m\to\infty$ to observe that $\Phi\in L^1(\Omega)\cap C^\infty(\Omega)$ satisfies

$$\int_\Omega \Phi(x)\Delta\varphi(x)\,dx = 0$$

for all $\varphi\in C_c^2(\bar{\Omega})(=C^2(\bar{\Omega}))$ satisfying $\partial\varphi/\partial n_\Omega=0$ on $\partial\Omega$ since the right-hand side of (2.2) converges to zero by (2.8). Thus Φ formally solves the homogeneous Neumann problem so that $\nabla\Phi\equiv 0$. (In fact, we apply Lemma 2.8 in the next subsection for a rigorous proof.)

Since a subsequence of $\{\nabla\Phi_m\}_{m=1}^{\infty}$ converges to $\nabla\Phi$ locally uniformly in Ω, (2.7) implies that $d_\Omega(x_\infty)|\nabla\Phi(x_\infty)|\geqslant\frac{1}{2}$. This contradicts the fact that $\nabla\Phi\equiv 0$ so we get a contradiction for the case 1.

Case 2. $x_\infty\in\partial\Omega$. By taking a subsequence, we may assume that $x_m\to x_\infty$. We rescale Φ_m and f_m around x_m so that the distance from the origin to the boundary equals 1. More precisely, we set

$$\Psi_m(x) = \Phi_m(x_m+d_m x) \quad\text{and}\quad g_m(x) = f_m(x_m+d_m x),$$

with $d_m=d_\Omega(x_m)$. It follows from (2.6)–(2.8) that

$$\sup_{x\in\Omega_m} d_{\Omega_m}(x)|\nabla\Psi_m(x)| = 1, \tag{2.9}$$

$$|\nabla\Psi_m(0)| \geqslant \tfrac{1}{2}, \tag{2.10}$$

$$\|g_m\|_{L^\infty(\partial\Omega_m)} < \frac{1}{m}. \tag{2.11}$$

Here Ω_m is the rescaled domain of the form

$$\Omega_m = \left\{ x \in \mathbf{R}^n \;\middle|\; x = \frac{y - x_m}{d_m} \text{ and } y \in \Omega \right\}.$$

We apply (2.2) with Ψ_m, g_m and Ω_m, and let $m \to \infty$. Since the domain is moving, we have to take φ_m satisfying $\partial\varphi_m/\partial n_{\Omega_m} = 0$ so that it converges to some function φ. If $\partial\Omega$ is C^k ($k \geqslant 2$), there exists $\mu > 0$ such that $d_\Omega(x) \in C^k(\Gamma_{\Omega,\mu})$ with a tubular neighborhood $\Gamma_{\Omega,\mu} = \{x \in \bar{\Omega} \mid d_\Omega(x) < \mu\}$ and that, for any $z \in \Gamma_{\Omega,\mu}$, there is a unique projection $z^p \in \partial\Omega$ to $\partial\Omega$, i.e. $|z - z^p| = d_\Omega(z)$; cf. Proposition 3.6 (i). Let $x_m^p \in \partial\Omega$ be the projection of x_m to $\partial\Omega$ for sufficiently large m. The sequence of unit vectors $(x_m - x_m^p)/d_m$ converges to a unit vector e. By translation and rotation we may assume that $e = (0, ..., 0, 1)$. Then Ω_m converges to a half space $\mathbf{R}^n_{+,-1}$, where

$$\mathbf{R}^n_{+,c} = \{(x', x_n) \in \mathbf{R}^n \mid x_n > c\}.$$

More precisely, for any $R > 0$, there is m_0 such that for $m \geqslant m_0$ there is $h_m \in C^2(B_R^{n-1}(0))$ converging to -1 up to third derivatives with the property that

$$\Omega_m \cap (B_R^{n-1}(0) \times [-R, R]) = \{(x', x_n) \in \mathbf{R}^n \mid R > x_n > h_m(x') \text{ and } x' \in B_R^{n-1}(0)\},$$

where $B_R^{n-1}(0)$ denotes the closed ball in $\mathbf{R}^{n-1}$ with radius R centered at the origin. Let $\varphi \in C_c^2(\bar{\mathbf{R}}^n_{+,-1})$ satisfy $\partial\varphi/\partial x_n = 0$ on $\{(x', x_n) \mid x_n = -1\}$. We may assume that $\varphi \in C_c^2(\mathbf{R}^n)$ by a suitable extension. Take $R > 0$ so large that the support of φ is included in the interior of $B_R^{n-1}(0) \times [-R, R]$. We take a normal coordinate associated with Ω_m. Let F_m be the mapping defined by

$$x = (x', x_n) \longmapsto X = z + x_n \nabla d_{\Omega_m}(z), \quad \text{with } z = (x', h_m(x')).$$

We set $\varphi_m(X) = \varphi(F_m^{-1}(X))$. This is well defined for sufficiently large m. We further observe that $\partial\varphi_m/\partial n_{\Omega_m} = 0$ on $\partial\Omega_m$, as $n_{\Omega_m} = -\nabla d_{\Omega_m}$. If $\partial\Omega$ is C^3, then F_m^{-1} is still C^2. Thus $\varphi_m \in C_c^2(\bar{\Omega}_m)$ for sufficiently large m. Here we invoke C^3 regularity.

Since we may assume that $\Psi_m(0) = 0$, by (2.9) the sequence $\{\Psi_m\}_{m=1}^\infty$ is bounded in $L^r(\Omega_m \cap (B_R^{n-1}(0) \times [-R, R]))$, $r \in (1, \infty)$, for any $R > 1$. As $\nabla\Psi_m$ is harmonic in Ω_m, a subsequence of $\{\Psi_m\}_{m=1}^\infty$ converges to some function $\Psi \in C^\infty(\mathbf{R}^n_{+,-1})$ locally uniformly with all its derivatives and weakly in $L^r_{\mathrm{loc}}(\bar{\mathbf{R}}^n_{+,-1})$ ($1 < r < \infty$). Since (2.11) implies that $g_m \to 0$ uniformly, we apply (2.2) with Ψ_m, φ_m and g_m, and let $m \to \infty$ to get

$$\int_{\mathbf{R}^n_{+,-1}} \Psi \Delta\varphi \, dx = 0, \tag{2.12}$$

as F_m^{-1} converges to the identity in C^2 so that $\varphi_m \to \varphi$ in C^2 in a neighborhood of the support $\operatorname{spt}\varphi$. We thus observe that (2.12) is valid for all $\varphi \in C_c^2(\mathbf{R}^n_{+,-1})$ with $\partial\varphi/\partial x_n=0$ on $\{(x', x_n) \mid x_n=-1\}$. We apply a uniqueness result for the Neumann problem with the estimate $\sup x_n |\nabla\Psi|(x', x_n) \leqslant 1$ obtained from (2.9) to get $\nabla\Psi \equiv 0$. (One should apply Lemma 2.9 below for a rigorous proof.)

Since a subsequence of $\{\nabla\Psi_m\}_{m=1}^\infty$ converges to $\nabla\Psi$ locally uniformly in $\mathbf{R}^n_{+,-1}$, (2.10) implies that $|\nabla\Psi(0)| \geqslant \frac{1}{2}$. This contradicts the fact that $\nabla\Phi \equiv 0$, so the proof is now complete. □

Remark 2.7. (i) Even in case 1 the estimate (2.6) does not imply that $\{\nabla\Psi_m\}_{m=1}^\infty$ is uniformly bounded in any Lebesgue spaces on Ω. Thus it is not clear that

$$\int_{\Omega_m} \nabla\Phi_m \cdot \nabla\varphi \, dx \to \int_\Omega \nabla\Phi \cdot \nabla\varphi \, dx,$$

though we know that

$$-\int_{\Omega_m} \Phi_m \Delta\varphi \, dx \to -\int_\Omega \Phi\Delta\varphi \, dx,$$

since Φ_m converges weakly in all L^r spaces $(1<r<\infty)$ as $m\to\infty$ by taking a subsequence. This is the reason we need to assume that φ is at least C^2 and that $\partial\varphi/\partial n_\Omega=0$ on the boundary.

(ii) The proof of Theorem 2.5 actually yields the estimate

$$\sup_{x\in\Omega} d_\Omega(x) |Q[\nabla\cdot f](x)| \leqslant C_\Omega \|n_\Omega \cdot (f - {}^t f)\|_{L^\infty(\partial\Omega)},$$

which is stronger than (1.5). Here,

$$n_\Omega \cdot f = \sum_{j=1}^n n_\Omega^j f_{ij} \quad \text{and} \quad {}^t f_{ij} = f_{ji}.$$

If $f_{ij} = \partial_j v^i$ with $\operatorname{div} v=0$, the quantity $n_\Omega \cdot (f - {}^t f)$ is nothing but the tangential trace of the vorticity, i.e. $\omega \times n_\Omega$ when $n=3$. Moreover, the right-hand side of (2.2) equals

$$\int_{\partial\Omega} (\omega \times n_\Omega) \cdot \nabla\varphi \, d\mathcal{H}^{n-1}.$$

Since $\partial\varphi/\partial n_\Omega=0$ so that $\nabla\varphi = \nabla_{\tan}\varphi$ and since $\omega \times n_\Omega$ is a tangent vector field on $\partial\Omega$, the above quantity equals

$$-\int_{\partial\Omega} (\operatorname{div}_{\partial\Omega}(\omega \times n_\Omega)) \varphi \, d\mathcal{H}^{n-1}.$$

This implies formally that Φ with $f=\partial_j v^i$ solves

$$\begin{cases} -\Delta\Phi=0 & \text{in } \Omega, \\ \dfrac{\partial\Phi}{\partial n_\Omega}=-\operatorname{div}_{\partial\Omega}(\omega\times n_\Omega) & \text{on } \partial\Omega, \end{cases}$$

where $\operatorname{div}_{\partial\Omega}$ denotes the surface divergence; see e.g. [29] and [53]. In general, since the quantity $n_\Omega\cdot(f-{}^t f)$ is tangential, we have

$$\frac{\partial\Phi}{\partial n_\Omega}=-\operatorname{div}_{\partial\Omega}(n_\Omega\cdot(f-{}^t f)) \quad \text{on } \partial\Omega.$$

2.4. Uniqueness of the Neumann problem

We shall state and prove uniqueness results which are used in the proof of Theorem 2.5.

LEMMA 2.8. (Uniqueness for bounded domains) *Let Ω be a bounded domain with C^3 boundary. Assume that $\Phi\in L^1(\Omega)\cap C(\Omega)$ satisfies*

$$\int_\Omega \Phi(x)\Delta\varphi(x)\,dx=0 \tag{2.13}$$

for all $\varphi\in C^2(\bar\Omega)$ satisfying $\partial\varphi/\partial n_\Omega=0$ on $\partial\Omega$. Then Φ is constant.

Proof. We consider the dual problem

$$\begin{cases} -\Delta\varphi=\operatorname{div}\psi & \text{in } \Omega, \\ \dfrac{\partial\varphi}{\partial n_\Omega}=0 & \text{on } \partial\Omega. \end{cases}$$

For arbitrary $\psi\in C_c^\infty(\Omega)$, there exists a solution $\varphi\in W^{3,r}(\Omega)$ for all $r>1$ (see e.g. [35, Lemma 2.3]), where $W^{m,r}(\Omega)$ denotes the L^r-Sobolev space of order m. By the Sobolev embedding we conclude that $\varphi\in C^2(\bar\Omega)$. From (2.13) it follows that

$$\int_\Omega \Phi\operatorname{div}\psi\,dx=0$$

for all $\psi\in C_c^\infty(\Omega)$. This implies that $\nabla\Phi=0$, so Φ is constant. □

LEMMA 2.9. (Uniqueness for the half space) *Let $\Phi\in L^1_{\mathrm{loc}}(\overline{\mathbf{R}}^n_+)$ satisfy*

$$\int_{\mathbf{R}^n_+} \Phi(x)\Delta\varphi(x)\,dx=0$$

for all $\varphi\in C_c^\infty(\overline{\mathbf{R}}^n_+)$ satisfying $\partial\varphi/\partial x_n=0$ on $\{(x',x_n)\,|\,x_n=0\}$. Assume that Φ satisfies

$$\sup_{x\in\mathbf{R}^n_+} x_n|\nabla\Phi(x)|<\infty. \tag{2.14}$$

Then Φ is constant.

Proof. The problem can be reduced to the whole space. Let $\tilde{\Phi}$ be an even extension of Φ to the whole space, i.e. $\tilde{\Phi}(x', x_n) = \Phi(x', -x_n)$ for $x_n < 0$. For arbitrary $\varphi \in C_c^\infty(\mathbf{R}^n)$ let φ_{even} and φ_{odd} be the even and odd parts of φ, i.e.

$$\varphi_{\text{even}}(x) = \frac{\varphi(x', x_n) + \varphi(x', -x_n)}{2} \quad \text{and} \quad \varphi_{\text{odd}}(x) = \frac{\varphi(x', x_n) - \varphi(x', -x_n)}{2}.$$

Integration by parts yields

$$\begin{aligned}\int_{\mathbf{R}^n} \tilde{\Phi}(x)\Delta\varphi(x)\,dx &= \int_{\mathbf{R}^n} \tilde{\Phi}(x)\Delta(\varphi_{\text{even}}(x) + \varphi_{\text{odd}}(x))\,dx \\ &= \int_{\mathbf{R}^n} \tilde{\Phi}(x)\Delta\varphi_{\text{even}}(x)\,dx = 2\int_{\mathbf{R}^n_+} \Phi(x)\Delta\varphi_{\text{even}}(x)\,dx.\end{aligned}$$

Since φ_{even} satisfies $\partial\varphi_{\text{even}}/\partial x_n = 0$ on $\{(x', x_n) \mid x_n = 0\}$, we conclude that

$$\int_{\mathbf{R}^n} \tilde{\Phi}(x)\Delta\varphi(x)\,dx = 0. \tag{2.15}$$

By (2.14) we know that $\tilde{\Phi}$ is locally integrable in $\mathbf{R}^n$. As (2.15) says that $\tilde{\Phi}$ is weakly harmonic, $\tilde{\Phi} = \eta_\varepsilon * \tilde{\Phi}$ by the mean-value theorem if η_ε is a symmetric mollifier, i.e. η_ε is radially symmetric (see e.g. [15, §2.2.3]). Moreover, by integrating $\tilde{\Phi}$ from $x_0 = (0, (x_0)_n) \in \mathbf{R}^n$, with $(x_0)_n \neq 0$, to x, we observe that (2.14) yields

$$|\tilde{\Phi}(x)| \leqslant C\big(1 + \big|\log|x_n|\big| + |x|\big|\log|x_n|\big|\big)$$

for $x' \in \mathbf{R}^n$ and $|x_n| < \frac{1}{2}$ with some constant C independent of x. This implies that

$$\nabla\tilde{\Phi} = \nabla\eta_\varepsilon * \tilde{\Phi}$$

satisfies the estimate

$$|\nabla\tilde{\Phi}(x)| \leqslant C_\varepsilon(1 + |x|) \tag{2.16}$$

for $x' \in \mathbf{R}^{n-1}$ and $|x_n| < 2\varepsilon$ with C_ε independent of x. By (2.14) we conclude that $\nabla\tilde{\Phi}$ satisfies (2.16) for all $x \in \mathbf{R}^n$. As $\tilde{\Phi}$ is weakly harmonic, (2.16) implies that $\nabla\tilde{\Phi}$ is harmonic in $\mathbf{R}^n$. By (2.16), the classical Liouville theorem implies that $\nabla\tilde{\Phi}$ is a polynomial of degree 1. However, by the decay estimate (2.14) for $|x_n| \to \infty$, this polynomial must be zero. Thus $\nabla\tilde{\Phi} = 0$, i.e. Φ is constant. □

Remark 2.10. We actually need only C^2-regularity of the boundary $\partial\Omega$ in case 1 of the proof of Theorem 2.5. Note that identity (2.2) (which is independent of the uniqueness results in this subsection) is still valid for $\varphi \in W^{2,2}(\Omega)$ having compact support in $\bar{\Omega}$. When $\partial\Omega$ is C^2, a slightly modified version of Lemma 2.8 is valid. In fact, for $\Phi \in L^2(\Omega)$ we still assert that $\nabla\Phi \equiv 0$ if (2.13) is satisfied for all $\varphi \in W^{2,2}(\Omega)$ with $\partial\varphi/\partial n_\Omega = 0$ on $\partial\Omega$. (The constructed φ in the proof is now in $W^{2,2}(\Omega)$, but not necessarily in $W^{3,r}(\Omega)$.) Based on these assertions the proof of case 1 goes through with trivial modifications.

3. Uniform Hölder estimates for pressure gradients

The goal of this section is to establish local Hölder estimates for second spatial derivatives and the time derivative of the velocity solving the Stokes equations, both interior and up to boundary. This procedure is a key to derive the necessary compactness for blow-up sequences. Unlike the heat equation the result is not completely local even in the interior case, since we need a uniform Hölder estimate in time for pressure gradients. For this purpose we invoke admissibility of domains.

3.1. Interior Hölder estimates for pressure gradients

We use conventional notation [41] for Hölder (semi)norms for space-time functions. Let $f=f(x,t)$ be a real-valued or an $\mathbf{R}^n$-valued function defined in $Q=\Omega\times(0,T]$, where Ω is a domain in $\mathbf{R}^n$. For $\mu\in(0,1)$ we set several Hölder seminorms

$$[f]^{(\mu)}_{(0,T]}(x)=\sup\left\{\frac{|f(x,t)-f(x,s)|}{|t-s|^{\mu}}\ \middle|\ t,s\in(0,T] \text{ and } t\neq s\right\},$$
$$[f]^{(\mu)}_{\Omega}(t)=\sup\left\{\frac{|f(x,t)-f(y,t)|}{|x-y|^{\mu}}\ \middle|\ x,y\in\Omega \text{ and } x\neq y\right\}$$

and

$$[f]^{(\mu)}_{t,Q}=\sup_{x\in\Omega}[f]^{(\mu)}_{(0,T]}(x) \quad \text{and} \quad [f]^{(\mu)}_{x,Q}=\sup_{t}[f]^{(\mu)}_{\Omega}(t).$$

In the parabolic scale for $\gamma\in(0,1)$ we set

$$[f]^{(\gamma,\gamma/2)}_{Q}=[f]^{(\gamma/2)}_{t,Q}+[f]^{(\gamma)}_{x,Q}.$$

For later convenience we also define the case $\gamma=1$ so that

$$[f]^{(1,1/2)}_{Q}=\|\nabla f\|_{L^\infty(Q)}+[f]^{(1/2)}_{t,Q}.$$

If $l=[l]+\gamma$, where $[l]$ is a non-negative integer and $\gamma\in(0,1)$, we set

$$[f]^{(l,l/2)}_{Q}=\sum_{|\alpha|+2\beta=[l]}[\partial_x^\alpha\partial_t^\beta f]^{(\gamma,\gamma/2)}_{Q}$$

and the parabolic Hölder norm

$$|f|^{(l,l/2)}_{Q}=\sum_{|\alpha|+2\beta\leqslant[l]}\|\partial_x^\alpha\partial_t^\beta f\|_{L^\infty(Q)}+[f]^{(l,l/2)}_{Q}.$$

When f is time-independent, we simply write $[f]^{(\mu)}_{x,Q}$ by $[f]^{(\mu)}_{\Omega}$.

Let Ω be a uniformly C^2-domain in $\mathbf{R}^n$. For a given $v_0 \in \tilde{L}^r_\sigma(\Omega)$, $1<r<\infty$, it is proved in [16] and [18] that there exists a unique solution (v,q) of the Stokes equations (1.1)–(1.4) satisfying $v_t, \nabla q, \nabla^2 v, \nabla v, v \in \tilde{L}^r(\Omega)$ at each $t\in(0,T)$ such that the solution operator $S(t)\colon v_0 \mapsto v(\cdot\,,t)$ is an analytic semigroup in $\tilde{L}^r_\sigma(\Omega)$. Here $T>0$ is taken arbitrarily large. In this paper we simply say that (v,q) is an $\tilde{L}^r$-solution of (1.1)–(1.4). Note that $\nabla q = Q[\Delta v]$ for an $\tilde{L}^r$-solution.

Lemma 3.1. *Let Ω be an admissible, uniformly C^2-domain in $\mathbf{R}^n$ (with $r\geqslant n$). Then there exists a constant $M(\Omega)>0$ such that*

$$[d_\Omega(x)\nabla q]^{(1/2)}_{t,Q_\delta} \leqslant \frac{M}{\delta}\sup\{(\|v_t\|_\infty(t)+\|\nabla^2 v\|_\infty(t))t \mid \delta\leqslant t\leqslant T\}$$

holds for all $\tilde{L}^r$-solutions (v,q) of (1.1)–(1.4) and all $\delta\in(0,T)$, where $Q_\delta=\Omega\times(\delta,T)$. The constant M can be taken uniformly with respect to translation and dilation, i.e.

$$M(\lambda\Omega+x_0)=M(\Omega)$$

for all $\lambda>0$ and $x_0\in\Omega$.

Proof. By an interpolation inequality (see e.g. [64] and [40, §3.2]) there is a dilation-invariant constant C such that for any $\varepsilon>0$ the estimate

$$\|\nabla v\|_\infty(t) \leqslant \varepsilon\|\nabla^2 v\|_\infty(t)+\frac{C}{\varepsilon}\|v\|_\infty(t)$$

holds. Since our solution is an $\tilde{L}^r$-solution, we have

$$\nabla q = Q[\nabla\cdot f], \quad f=(f_{ij})=\partial_j v^i,$$

and moreover

$$\nabla q(x,t)-\nabla q(x,s) = Q[\Delta v(x,t)-\Delta v(x,s)].$$

As Ω is admissible, we have

$$\begin{aligned} &d_\Omega(x)|\nabla q(x,t)-\nabla q(x,s)| \\ &\quad\leqslant C(\Omega)\|\nabla(v(\cdot\,,t)-v(\cdot\,,s))\|_\infty \\ &\quad\leqslant C(\Omega)\left(\varepsilon\max\{\|\nabla^2 v\|_\infty(t),\|\nabla^2 v\|_\infty(s)\}+\frac{C}{\varepsilon}\|v(\cdot\,,t)-v(\cdot\,,s)\|_\infty\right). \end{aligned}$$

Since

$$\begin{aligned} \|v(\cdot\,,t)-v(\cdot\,,s)\|_\infty &\leqslant |t-s|\sup\{\|v_t\|_\infty(\tau) \mid \tau \text{ is between } t \text{ and } s\}, \\ &\leqslant |t-s|\frac{1}{\delta}\sup\{\tau\|v_t\|_\infty(\tau) \mid \delta\leqslant\tau\leqslant T\} \end{aligned}$$

for $t,s\geqslant\delta$, the desired inequality follows by taking $\varepsilon=|t-s|^{1/2}$. As C_Ω is also dilation and translation invariant by Remark 2.4 (iii), so is $M(\Omega)$. □

PROPOSITION 3.2. (Interior Hölder estimates) *Let Ω be an admissible, uniformly C^2-domain in $\mathbf{R}^n$ (with $r\geqslant n$). Take $\gamma\in(0,1)$, $\delta>0$, $T>0$, $R>0$. Then there exists a constant $C=C(M(\Omega),\delta,R,d,\gamma,T)$ such that the estimate*

$$[\nabla^2 v]^{(\gamma,\gamma/2)}_{Q'}+[v_t]^{(\gamma,\gamma/2)}_{Q'}+[\nabla q]^{(\gamma,\gamma/2)}_{Q'}\leqslant CN_T \tag{3.1}$$

holds for all $\tilde{L}^r$-solutions (v,q) of (1.1)–(1.4) provided that $B_R(x_0)\subset\Omega$ and $x_0\in\Omega$, where $Q'=\operatorname{int}B_R(x_0)\times(\delta,T]$ and d denotes the distance of $B_R(x_0)$ and $\partial\Omega$. Here

$$N_T=\sup_{0<t<T}\|N(v,q)\|_\infty(t)<\infty \tag{3.2}$$

and $M(\Omega)$ is the constant in Lemma 3.1.

Proof. Since ∇q is harmonic in Ω, the Cauchy-type estimate implies that

$$\sup_{x\in B_{R+d/2}(x_0)}|\nabla^2 q(x,t)|\leqslant\frac{C_0}{d}\|\nabla q\|_{L^\infty(\Omega)}(t)\quad\text{and}\quad B_{R+d/2}(x_0)\subset\Omega,$$

where C_0 depends only on n. This together with Lemma 3.1 implies

$$[\nabla q]^{(1,1/2)}_{Q''}\leqslant\left(\frac{C_0}{d}+\frac{4M}{d}\right)\frac{1}{\delta}N_T$$

for any $x_0\in\Omega$, $R>0$ and $\delta>0$, where $Q''=\operatorname{int}B_{R+d/2}(x_0)\times\left(\frac{1}{2}\delta,T\right]$. By the standard local Hölder estimate for the heat equation

$$v_t-\Delta v=-\nabla q\quad\text{in } Q'',$$

this pressure gradient estimate implies estimates for $\nabla^2 v$ and v_t in Q' [41, Chapter IV, Theorem 10.1]. □

Remark 3.3. (i) We are tempted to claim that if (v,q) solves the Stokes system (1.1)–(1.2) without boundary and initial condition, then the desired interior Hölder estimate would be valid. Such a type of estimate is in fact true for the heat equation [41, Chapter IV, Theorem 10.1]. However, for the Stokes equations this is no longer true. In fact, if we take $v(x,t)=g(t)$ and $p(x,t)=-g'(t)\cdot x$ with $g\in C^1[0,\infty)$, this is always a solution of (1.1)–(1.2) satisfying $N_{T_1}<\infty$ for any $T_1>0$. However, evidently v_t may not be Hölder continuous in time unless ∇p is Hölder continuous in time. This is why we use a global setting with admissibility of the domain.

(ii) In the constant C the dependence of Ω is through $M(\Omega)$ so it is invariant under dilation provided that d and R are taken independent of the dilation.

3.2. Local Hölder estimates up to the boundary

The regularity up to the boundary is more involved. We begin with the statement and give a proof in subsequent sections.

THEOREM 3.4. (Estimates near the boundary) *Let* Ω *be an admissible, uniformly* C^3*-domain of type* (α,β,K) *in* $\mathbf{R}^n$ *(with* $r\geqslant n$*). Then there exists* $R_0=R_0(\alpha,\beta,K)>0$ *such that for any* $\gamma\in(0,1)$, $\delta\in(0,T)$ *and* $R\leqslant\frac{1}{2}R_0$ *there exists a constant*

$$C=C(M(\Omega),\delta,\gamma,T,R,\alpha,\beta,K)$$

such that (3.1) *is valid for all* $\tilde{L}^r$*-solutions* (v,q) *of* (1.1)–(1.4) *with*

$$Q'=Q'_{x_0,R,\delta}=\Omega_{x_0,R}\times(\delta,T],\quad \Omega_{x_0,R}=\operatorname{int}B_R(x_0)\cap\Omega,$$

provided that $x_0\in\partial\Omega$.

The proof is more involved. We first localize the Stokes equations near the boundary by using the cut-off technique and the Bogovskiĭ operator [22, §III.3] to recover the divergence-free property. Then we apply a global Schauder estimate for the Stokes equations in a localized domain. As in the interior case we use the admissibility of the domain to obtain the Hölder estimate for the pressure in time.

We begin with Hölder estimates for q in time since we are not able to control the Hölder norm of ∇q up to the boundary.

LEMMA 3.5. *Assume the same hypotheses as in Lemma* 3.1. *Then there exists* $R_0=R_0(\alpha,\beta,K)>0$ *such that for* $\nu\in(0,1)$ *and* $R\in(0,R_0]$ *there exists a constant*

$$C_0=C_0(M(\Omega),\nu,\alpha,R,\beta,K)$$

such that

$$[q]^{(\nu,\nu/2)}_{Q'}\leqslant\frac{C_0N_T}{\delta} \tag{3.3}$$

is valid for all $\tilde{L}^r$*-solutions* (v,q) *of* (1.1)–(1.4) *and* $Q'=Q'_{x_0,R,\delta}$ *for* $x_0\in\partial\Omega$.

For this purpose we prepare a basic fact for the distance function.

PROPOSITION 3.6. *Let* Ω *be a uniformly* C^2*-domain of type* (α,β,K).

(i) *There is a constant* $R_*=R_*(\alpha,\beta,K)>0$ *such that every*

$$x\in\Gamma_{\Omega,R_*}=\{x\in\Omega\mid d_\Omega(x)<R_*\}$$

has a unique projection $x_p\in\partial\Omega$ *(i.e.* $|x-x_p|=d_\Omega(x)$*) and* x *is represented as*

$$x=x_p-dn_\Omega(x_p)$$

with $d{=}d_\Omega(x)$. *The mapping* $x\mapsto(x_p,d)$ *is* C^1 *in* Γ_{Ω,R_*}.

(ii) *There is a positive constant* $R_1{=}R_1(\alpha,\beta,K)\leqslant R_*$ *such that* $\Omega_{x_0,R_1}\subset U_{\alpha,\beta,h}(x_0)$ *and the projection* x_p *of* $x\in\Omega_{x_0,R_1}$ *is on* $x_0+\mathrm{graph}(h)$.

(iii) *For each* $R\in(0,R_1)$ *and* $\nu\in[0,1)$ *there is a constant* $C{=}C(\alpha,\beta,K,R,\nu)$ *such that*

$$|\tilde q(x)-\tilde q(y)|\leqslant C\|d_\Omega^\nu\nabla\tilde q\|_{L^\infty(\Omega)}\left(|d_\Omega(y)^{1-\nu}-d_\Omega(x)^{1-\nu}|+\frac{|x_p-y_p|}{\max\{d_\Omega(x)^\nu,d_\Omega(y)^\nu\}}\right)$$

for $x,y\in\Omega_{x_0,R}$ *and for all* $\tilde q\in C^1(\Omega)$ *and* $x_0\in\partial\Omega$.

Proof. (i) This is non-trivial but well known. See e.g. [36] or [39, §4.4].

(ii) This is easy by taking R small. The smallness depends on a bound for the second fundamental form of $\partial\Omega$.

(iii) For $x\in\Omega_{x_0,R}$ ($R\leqslant R_1$) we consider its normal coordinate (x_p,d). Since $\Omega_{x_0,R}\subset U_{\alpha,\beta,h}(x_0)$, there is a unique $x_p'\in\mathbf{R}^{n-1}$ such that $x_p{=}(x_p',h(x_p'))$. Moreover, we are able to use (x_p',d) as a coodinate system. For $x,y\in\Omega_{x_0,R}$ with $x{=}(x_p',d_\Omega(x))$, $y{=}(y_p',d_\Omega(y))$ and $d_\Omega(y){>}d_\Omega(x)$, we estimate

$$|\tilde q(x)-\tilde q(y)|\leqslant|\tilde q(x)-\tilde q(z)|+|\tilde q(z)-\tilde q(y)|$$

with $z{=}(x_p',d_\Omega(y))$. Thus we connect x and z by a straight line which is parallel to $n_\Omega(x_p)$ and observe that, with $x_\tau{=}x(1-\tau)+\tau z$ $(0\leqslant\tau\leqslant1)$,

$$\begin{aligned}|\tilde q(x)-\tilde q(z)|&\leqslant|z-x|\int_0^1\frac{1}{d_\Omega^\nu(x_\tau)}|d_\Omega^\nu\nabla\tilde q(x_\tau)|\,d\tau\leqslant\int_{d_\Omega(x)}^{d_\Omega(y)}\frac{1}{s^\nu}\,ds\,\|d_\Omega^\nu\nabla\tilde q\|_{L^\infty(\Omega)}\\&\leqslant\frac{d_\Omega(z)^{1-\nu}-d_\Omega(x)^{1-\nu}}{1-\nu}\|d_\Omega^\nu\nabla\tilde q\|_{L^\infty(\Omega)}.\end{aligned}$$

It remains to estimate $|\tilde q(z)-\tilde q(y)|$. We connect z and y by a curve $C_{z,y}$ of the form

$$C_{z,y}=\{x(\tau)\,|\,0\leqslant\tau\leqslant1,x_p'(\tau)=x_p'(1-\tau)+\tau y_p'\text{ and }d_\Omega(x(\tau))=d_\Omega(y)\}$$

so that the projection in $\mathbf{R}^{n-1}$ is a straight line connecting x_p' and y_p'. We now estimate

$$|\tilde q(z)-\tilde q(y)|\leqslant\int_{C_{z,y}}\frac{1}{d_\Omega(y)^\nu}d_\Omega^\nu(y)|\nabla\tilde q|(x)\,d\mathcal{H}^1(x)=\frac{1}{d_\Omega(y)^\nu}\mathcal{H}^1(C_{z,y})\|d_\Omega^\nu\nabla\tilde q\|_{L^\infty(\Omega)}.$$

Since $\mathcal{H}^1(C_{z,y})\leqslant C|x_p-y_p|$, the proof is now complete. □

Proof of Lemma 3.5. We take $R_1{>}0$ as in Proposition 3.6. For $x_0\in\partial\Omega$ we take $\tilde x_0{=}x_0-\frac12R_1n_\Omega(x_0)$. We may assume that $q(\tilde x_0,t){=}0$ for all $t\in(0,T)$. Since

$$[d_\Omega(x)^\nu\nabla q]_{t,Q_\delta}^{(\nu/2)}\leqslant([d_\Omega(x)\nabla q]_{t,Q_\delta}^{(1/2)})^\nu(2\|\nabla q\|_{L^\infty(Q_\delta)})^{1-\nu},$$

Lemma 3.1 implies that

$$\|d_\Omega(x)^\nu \nabla \tilde{q}(x,\cdot)\|_{L^\infty(\Omega)}(t,s) \leqslant \frac{M^\nu N_T 2^{1-\nu}}{\delta}|t-s|^{\nu/2} \quad \text{for } t,s\in(\delta,T],$$

with $\tilde{q}(x,t,s)=q(x,t)-q(x,s)$. We now apply Proposition 3.6 (iii) with $y=\tilde{x}_0$ to get

$$|q(x,t)-q(x,s)| \leqslant C(d_\Omega(\tilde{x}_0)^{1-\nu}+|x_p-x_0|d_\Omega(\tilde{x}_0)^{-\nu})\frac{M^\nu N_T 2^{1-\nu}}{\delta}|t-s|^{\nu/2}$$

for $t,s\in(\delta,T]$ and all $x\in\Omega_{x_0,R}$, $R\leqslant R_0=\frac{1}{4}R_1$. Since $d_\Omega(\tilde{x}_0)=2R_0$ and $|x_p-x_0|<R$, the above inequality implies that

$$[q]^{(\nu/2)}_{t,Q'} \leqslant \frac{C_0 N_T}{\delta}, \quad C_0=C((2R_0)^{1-\nu}+R(2R_0)^{-\nu})M^\nu 2^{1-\nu}.$$

For the Hölder estimate in space we simply apply Proposition 3.6 (iii) with $\nu=0$ to get

$$|q(x,t)-q(y,t)| \leqslant C\|\nabla q\|_{L^\infty(\Omega)}(t)(|d_\Omega(y)-d_\Omega(x)|+|x_p-y_p|) \leqslant C\|\nabla q\|_{L^\infty(\Omega)}(t)|x-y|$$

for $x,y\in\Omega_{x_0,R}$, $R\leqslant R_0$, and $t\in(0,T)$. This implies that

$$[q]^{(\nu)}_{x,Q'} \leqslant \frac{C_0 N_T}{\delta}$$

so the proof is now complete. □

3.3. Helmholtz decomposition and the Stokes equations in Hölder spaces

To prove local Hölder estimates up to the boundary (Theorem 3.4) we recall several known Hölder estimates for the Helmholtz decomposition and the Stokes equations established in [55] and [60] via a potential-theoretic approach; see also [59]. We recall notions for the spaces of Hölder continuous functions. By $C^\gamma(\bar{\Omega})$ with $\gamma\in(0,1)$ we mean the space of all continuous functions in $\bar{\Omega}$ with $[f]^{(\gamma)}_\Omega<\infty$. Similarly, we use $C^{\gamma,\gamma/2}(\overline{Q})$ for the space of all continuous functions in $\overline{Q}$ with $[f]^{(\gamma,\gamma/2)}_Q<\infty$.

Proposition 3.7. (Helmholtz decomposition) *Let Ω be a bounded $C^{2+\gamma}$-domain in $\mathbf{R}^n$ with $\gamma\in(0,1)$.*

(i) *For $f\in C^\gamma(\bar{\Omega})$ there is a (unique) decomposition $f=f_0+\nabla\Phi$ with $f_0,\nabla\Phi\in C^\gamma(\bar{\Omega})$ such that*

$$\int_\Omega f_0\cdot\nabla\varphi\,dx=0 \quad \text{for all } \varphi\in C^\infty(\bar{\Omega}). \tag{3.4}$$

(ii) *There is a constant* $C_H>0$ *depending only on* γ *and* Ω *only through its* $C^{2+\gamma}$ *regularity such that*

$$|f_0|_\Omega^{(\gamma)}+|\nabla\Phi|_\Omega^{(\gamma)}\leqslant C_H|f|_\Omega^{(\gamma)} \quad \textit{for all } f\in C^\gamma(\bar{\Omega}), \tag{3.5}$$

where $|f|_\Omega^{(\gamma)}$ *denotes the Hölder norm of* f, *i.e.* $|f|_\Omega^{(\gamma)}=[f]_\Omega^{(\gamma)}+\|f\|_{L^\infty(\Omega)}$.

(iii) *For each* $\varepsilon\in(0,1-\gamma)$ *there is a constant* $C'_H>0$ *depending only on* γ, ε *and* Ω *only through its* $C^{2+\gamma}$ *regularity such that*

$$|f_0|_Q^{(\gamma,\gamma/2)}+|\nabla\Phi|_Q^{(\gamma,\gamma/2)}\leqslant C'_H|f|_Q^{(\gamma+\varepsilon,(\gamma+\varepsilon)/2)} \quad \textit{for all } f\in C^{\gamma+\varepsilon,(\gamma+\varepsilon)/2}(\overline{Q}). \tag{3.6}$$

Proof. The parts (i) and (ii) are established in [55] and [60]; the dependence of the constant is not explicit but can be seen from the proof.

In [60, Corollary on p. 175] it is proved that the left-hand side of (3.6) is dominated by a (similar type) constant multiple of

$$|f|_Q^{(\gamma,\gamma/2)}+\sup_{\substack{x,y\in\Omega\\ t,s\in(0,T]}}\frac{|(f(x,t)-f(x,s))-(f(y,t)-f(y,s))|}{|x-y|^\mu|t-s|^{\gamma/2}} \tag{3.7}$$

for arbitrary $\mu\in(0,1)$. By the Young inequality we get

$$\frac{1}{|x-y|^\varepsilon|t-s|^{\gamma/2}}\leqslant\frac{\varepsilon}{\gamma+\varepsilon}\frac{1}{|x-y|^{\gamma+\varepsilon}}+\frac{\gamma}{\gamma+\varepsilon}\frac{1}{|t-s|^{(\gamma+\varepsilon)/2}}.$$

Thus we take $\mu=\varepsilon$ to see that the second term of (3.7) is dominated by

$$\frac{2\varepsilon}{\gamma+\varepsilon}\sup_{t\in(0,T]}[f]_\Omega^{(\gamma+\varepsilon)}(t)+\frac{2\gamma}{\gamma+\varepsilon}\sup_{x\in\Omega}[f]_{(0,T]}^{((\gamma+\varepsilon)/2)}(x).$$

Hence the estimate (3.6) follows and (iii) is proved. □

Remark 3.8. The operator $f\mapsto f_0$ is essentially the Helmholtz projection P for Hölder vector fields, since (3.4) implies that $\operatorname{div} f=0$ in Ω and $f\cdot n_\Omega=0$ on $\partial\Omega$. The estimate (3.5) shows the continuity of P in the Hölder space $C^\gamma(\bar{\Omega})$. However, it is mentioned in [60] (without a proof) that P is not continuous in $C^{\gamma,\gamma/2}(\overline{Q})$. In other words, one cannot take $\varepsilon=0$ in the estimate (3.6).

We next recall the Schauder-type estimates for the Stokes system

$$v_t-\Delta v+\nabla q=f_0 \quad \text{in } \Omega\times(0,T), \tag{3.8}$$

$$\operatorname{div} v=0 \quad \text{in } \Omega\times(0,T), \tag{3.9}$$

$$v=0 \quad \text{on } \partial\Omega\times(0,T), \tag{3.10}$$

$$v=0 \quad \text{on } \Omega\times\{0\}. \tag{3.11}$$

PROPOSITION 3.9. *Let Ω be a bounded $C^{2+\gamma}$-domain in $\mathbf{R}^n$ with $\gamma\in(0,1)$ and $T>0$. Then, for each $f_0\in C^{\gamma,\gamma/2}(\overline{Q})$ satisfying* (3.4), *there is a unique solution*

$$(v,\nabla q)\in C^{2+\gamma,1+\gamma/2}(\overline{Q})\times C^{\gamma,\gamma/2}(\overline{Q})$$

(*up to an additive constant for q*) *of* (3.8)–(3.11). *Moreover, there is a constant C_S dependeng only on γ, T and Ω only through its $C^{2+\gamma}$ regularity such that*

$$|v|_Q^{(2+\gamma,1+\gamma/2)}+|\nabla q|_Q^{(\gamma,\gamma/2)}\leqslant C_S|f_0|_Q^{(\gamma,\gamma/2)}. \tag{3.12}$$

Remark 3.10. (i) This result is a special case of a very general result [60, Theorem 1.1], where the viscosity constant in front of Δ in (3.8) depends on space and time and the boundary and initial data are inhomogeneous. Note that the divergence-free condition (3.4) for f_0 is assumed in order to establish (3.12).

(ii) If the domain is a bounded C^3-domain, clearly it is a uniformly C^3-domain of type (α,β,K) for some (α,β,K). The constants C_H, C_H' and C_S in Propositions 3.7 and 3.9 depend on Ω only through (α,β,K) when Ω is a bounded C^3-domain (which is of course a $C^{2+\gamma}$-domain for all $\gamma\in(0,1)$).

3.4. Localization procedure

We shall prove Theorem 3.4 using Lemma 3.5 and a localization procedure with necessary Hölder estimates (Propositions 3.7 and 3.9). We first recall the Bogovskiĭ operator B_E in [10]. Let E be a bounded subdomain in Ω with Lipschitz boundary. The Bogovskiĭ operator B_E is a rather explicit operator but here we only need a few properties. This linear operator B_E is well defined for average-zero functions, i.e. such that $\int_E g\,dx=0$. Moreover, $\operatorname{div} B_E(g)=g$ in E and if the support $\operatorname{spt} g\subset E$, then $\operatorname{spt} B_E(g)\subset E$.

The operator B_E satisfies the estimates

$$\|B_E(g)\|_{W^{1,p}(E)}\leqslant C_B\|g\|_{L^p(E)} \qquad \text{for } g\in L^p(E) \text{ satisfying } \textstyle\int_E g\,dx=0, \tag{3.13}$$

$$\|B_E(g)\|_{L^p(E)}\leqslant C_B\|g\|_{W_0^{-1,p}(E)} \qquad \text{for } g\in W_0^{-1,p}(E)=W^{1,p'}(E)^*, \tag{3.14}$$

with some constant C_B independent of g, where $1/p'+1/p=1$ and $1<p<\infty$. In particular B_E is bounded from $L^p_{\mathrm{av}}=\{g\in L^p(E)\,|\int_E g\,dx=0\}$ to the Sobolev space $W^{1,p}(E)$. The estimate (3.14) is a special case of [23, Theorem 2.5] which asserts that B_E is bounded from $W_0^{s,p}(\Omega)$ to $W_0^{s+1,p}(\Omega)$ for $s>-2+1/p$. The bound C_B depends on p but its dependence on E is through the Lipschitz regularity constant of ∂E.

Proof of Theorem 3.4. We take R_0 as in Lemma 3.5 and take $R \leqslant \frac{1}{2}R_0$. For $x_0 \in \partial\Omega$ we take a bounded C^3-domain Ω' such that $\Omega_{x_0,3R/2} \subset \Omega' \subset \Omega_{x_0,2R}$. Evidently $\partial\Omega_{x_0,R} \cap \partial\Omega$ is strictly included in $\partial\Omega' \cap \partial\Omega$. Moreover, one can arrange that Ω' is of type (α', β', K'), where (α', β', K') depends on (α, β, K) and R. Such an Ω' is constructed for example by considering $\Omega'' = \Omega_{x_0,7R/4}$ and mollifying near the set $\partial B_{7R/4}(x_0) \cap \partial\Omega$ in a suitable way to get Ω'.

Let θ be a smooth cut-off function of [0,1] supported in $\left[0, \frac{3}{2}\right)$, i.e. $\theta \in C^\infty[0,\infty)$ such that $\theta \equiv 1$ on $[0,1]$ and $0 \leqslant \theta \leqslant 1$ with $\operatorname{spt}\theta \subset \left[0, \frac{3}{2}\right)$. We set $\theta_R(x) = \theta(|x-x_0|/R)$ which is a cut-off function of $\Omega_{x_0,R}$ supported in Ω'. By construction, its derivatives depend only on R. We also take a cut-off function ϱ_δ in the time variable. Let $\varrho \in C^\infty[0,\infty)$ satisfy $\varrho \equiv 1$ on $[1,\infty)$ and $\varrho = 0$ on $\left[0, \frac{1}{2}\right)$ with $0 \leqslant \varrho \leqslant 1$. For $\delta > 0$ we set $\varrho_\delta(t) = \varrho(t/\delta)$. We set $\xi = \theta_R \varrho_\delta$ and observe that $u = v\xi$ and $p = q\xi$ solves

$$\begin{cases} u_t - \Delta u + \nabla p = f, \\ \operatorname{div} u = g, \end{cases}$$

in $U = \Omega' \times (0,T)$ with

$$f = v\xi_t - 2\nabla v \cdot \nabla\xi - v\Delta\xi + q\nabla\xi \quad \text{and} \quad g = v \cdot \nabla\xi (= \operatorname{div}(v\xi)).$$

We next use the Bogovskiĭ operator $B_{\Omega'}$ to make the vector field solenoidal. We set $u^* = B_{\Omega'}(g)$ and $\tilde{u} = u - u^*$. Then $(\tilde{u}, p)$ solve

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} + \nabla p = \tilde{f}, \\ \operatorname{div} \tilde{u} = 0, \end{cases}$$

in U with $\tilde{f} = f + u_t^* - \Delta u^*$. We shall fix Ω' so that C_H' in (3.6) and C_S in (3.12) depends on Ω' only through (α, β, K) and R. If we know that $\tilde{f} \in C^{\gamma+\varepsilon,(\gamma+\varepsilon)/2}(\overline{U})$ with $\varepsilon \in (0, 1-\gamma)$ then by the Helmholtz decomposition in Hölder spaces (Proposition 3.7), one obtains $\tilde{f} = f_0 + \nabla\Phi$ with $f_0 \in C^{\gamma,\gamma/2}(\overline{U})$ satisfying (3.4) and

$$|f_0|_{(\gamma)} + |\nabla\Phi|_{(\gamma)} \leqslant C_H' |\tilde{f}|_{(\gamma+\varepsilon)}, \tag{3.15}$$

where we use the shorthand notation $|f|_{(\gamma)} = |f|_U^{(\gamma,\gamma/2)}$. If we set $\tilde{p} = p - \Phi$, then $(\tilde{u}, \tilde{p})$ solves (3.8)–(3.11) with $\Omega = \Omega'$, where f_0 satisfies the solenoidal condition (3.4). Applying the Schauder estimate (3.12) yields

$$|\tilde{u}|_{(2+\gamma)} + |\nabla\tilde{p}|_{(\gamma)} \leqslant C_S |f_0|_{(\gamma)}. \tag{3.16}$$

By the definition of $\tilde{f}$ we observe that

$$\begin{aligned} |\tilde{f}|_{(\gamma+\varepsilon)} &\leqslant |f|_{(\gamma+\varepsilon)} + |u_t^*|_{(\gamma+\varepsilon)} + |\Delta u^*|_{(\gamma+\varepsilon)} \\ &\leqslant c_0 \big(|v|_{\Omega' \times (\delta/2,T]}^{(\gamma+\varepsilon,(\gamma+\varepsilon)/2)} + |\nabla v|_{\Omega' \times (\delta/2,T]}^{(\gamma+\varepsilon,(\gamma+\varepsilon)/2)} + |q|_{\Omega' \times (\delta/2,T]}^{(\gamma+\varepsilon,(\gamma+\varepsilon)/2)} \big) + |u^*|_{(2+\gamma+\varepsilon)}, \end{aligned}$$

where c_0 depends only on R, T, δ and $\gamma+\varepsilon$. Since N_T in (3.2) is finite, by an interpolation inequality as in the proof of Lemma 3.1, we have $[\nabla v]_{t,Q_\delta}^{(1/2)} \leqslant CN_T/\delta$ with C depending only on (α,β,K). We now apply this estimate together with estimate (3.3) for q in Lemma 3.5 to get

$$|\tilde{f}|_{(\gamma+\varepsilon)} \leqslant CN_T + |u^*|_{(2+\gamma+\varepsilon)} \tag{3.17}$$

with a constant $C=C(M(\Omega),\gamma+\varepsilon,\alpha,\beta,K,R,\delta)$. Since

$$|v|_{Q'}^{(2+\gamma,1+\gamma/2)} \leqslant |u|_{(2+\gamma)} \leqslant |\tilde{u}|_{(2+\gamma)} + |u^*|_{(2+\gamma)} \quad \text{and} \quad |\nabla q|_{Q'}^{(\gamma,\gamma/2)} \leqslant |\nabla\tilde{p}|_{(\gamma)} + |\nabla\Phi|_{(\gamma)},$$

the desired estimates follow from (3.15)–(3.17) once we have established that

$$|u^*|_{(2+\gamma+\varepsilon)} \leqslant CN_T$$

with $C=C(M(\Omega),\gamma+\varepsilon,\alpha,\beta,K,R,\delta)$.

We shall present a proof for

$$[u_t^*]_{t,U}^{(\mu/2)} \leqslant CN_T \tag{3.18}$$

for $\mu\in(0,1)$ since the other quantities can be estimated in a similar way and are even easier. By (3.13) and (3.14) we have

$$\|u_t^*\|_{L^p(\Omega')} \leqslant C_B \|\operatorname{div} u_t\|_{W_0^{-1,p}(\Omega')}, \tag{3.19}$$

$$\|u_t^*\|_{W^{1,p}(\Omega')} \leqslant C_B \|\operatorname{div} u_t\|_{L^p(\Omega')}. \tag{3.20}$$

To estimate $\|\operatorname{div} u_t\|_{W_0^{-1,p}(\Omega')}$ we use the equations $v_t-\Delta v+\nabla q=0$ and $\operatorname{div} v=0$. For an arbitrary $\varphi\in W^{1,p'}(\Omega')$ we have

$$\begin{aligned}
\int_{\Omega'} \varphi \operatorname{div} u_t \, dx &= \int_{\Omega'} (\varphi v_t\cdot\nabla\xi + \varphi\nabla\xi_t\cdot v)\, dx \\
&= \int_{\Omega'} (\varphi\nabla\xi\cdot(\Delta v-\nabla q) + \varphi\nabla\xi_t\cdot v)\, dx \\
&= \int_{\Omega'} \left(-\sum_{i=1}^n \partial_{x_i}(\varphi\nabla\xi)\cdot\partial_{x_i} v + q\operatorname{div}(\varphi\nabla\xi) + \varphi\nabla\xi_t\cdot v \right) dx \\
&\qquad + \int_{\partial\Omega'} \left(\varphi\nabla\xi\cdot\frac{\partial v}{\partial n_{\Omega'}} - q\varphi\frac{\partial\xi}{\partial n_{\Omega'}} \right) d\mathcal{H}^{n-1}.
\end{aligned}$$

This implies that

$$\left| \int_{\Omega'} \varphi \operatorname{div} u_t \, dx \right| \leqslant C_\xi (\|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty)(\|\varphi\|_{W^{1,1}(\Omega')} + \|\varphi\|_{L^1(\partial\Omega')}) \tag{3.21}$$

with C_ξ depending only on R and δ (independent of t), where the L^∞-norm is taken on Ω'. By a trace theorem (e.g. [15, §5.5, Theorem 1]) there is a constant C (depending only on the Lipschitz regularity of the domain) such that

$$\|\varphi\|_{L^1(\partial\Omega')} \leqslant C\|\varphi\|_{W^{1,1}(\Omega')}.$$

By the Hölder inequality, $\|\varphi\|_{W^{1,1}(\Omega')} \leqslant C'\|\varphi\|_{W^{1,p}(\Omega')}$ with C' depending on the volume of Ω'. Thus (3.21) yields

$$\|\operatorname{div} u_t\|_{W_0^{-1,p}(\Omega')} \leqslant C_0(\|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty)$$

with C_0 depending only on δ, R and Ω' through (α, β, K). By (3.19) this yields

$$\|u_t^*\|_{L^p(\Omega')} \leqslant C_B C_0(\|\nabla v\|_\infty + \|q\|_\infty + \|v\|_\infty). \tag{3.22}$$

We next estimate $\|u_t^*\|_{W^{1,p}}$. By (3.20) a direct computation shows that

$$\|u_t^*\|_{W^{1,p}(\Omega')} \leqslant C_0 C_B(\|v\|_\infty + \|v_t\|_\infty) \tag{3.23}$$

since $\operatorname{div} u_t = \operatorname{div} \partial_t(\xi v) = \partial_t(\nabla\xi \cdot v)$ as $\operatorname{div} v = 0$.

We now apply the Gagliardo–Nirenberg inequality (see e.g. [26])

$$\|u_t^*\|_\infty \leqslant c\|u_t^*\|_{L^p(\Omega')}^{1-\sigma}\|u_t^*\|_{W^{1,p}(\Omega')}^{\sigma}, \quad \sigma = \frac{n}{p},$$

to (3.22) and (3.23) to get

$$\|u_t^*\|_\infty \leqslant C_1 C_B(\|v\|_\infty + \|v_t\|_\infty)^\sigma(\|\nabla v\|_\infty + \|v\|_\infty + \|q\|_\infty)^{1-\sigma}$$

with C_1 depending only on δ, R and Ω' through (α, β, K). We replace u^* by

$$u^*(\cdot, t) - u^*(\cdot, s)$$

and observe that

$$\begin{aligned} \|u_t^*(\cdot, t) - u_t^*(\cdot, s)\|_\infty \leqslant C_1 C_B\big(&\|\nabla v(\cdot, t) - \nabla v(\cdot, s)\|_\infty + \|q(\cdot, t) - q(\cdot, s)\|_\infty \\ &+ \|v(\cdot, t) - v(\cdot, s)\|_\infty\big)^{1-\sigma}\left(\frac{2N_T}{\min\{t, s\}}\right)^{\sigma}, \end{aligned} \tag{3.24}$$

for $t, s > 0$. As observed at the end of the proof of Lemma 3.1, we have

$$[\nabla v]_{t,Q_\delta}^{(1/2)} \leqslant \frac{CN_T}{\delta}.$$

By (3.3) we now conclude that

$$\sup_{x\in\Omega'}[\nabla v]_{t,\Omega'\times(\delta/2,T]}^{(\mu')} + \sup_{x\in\Omega'}[q]_{t,\Omega'\times(\delta/2,T]}^{(\mu')} \leqslant \frac{CN_T}{\delta}, \quad \mu' = \frac{\mu}{2(1-\sigma)},$$

provided that $\mu' < \frac{1}{2}$ (i.e. $p > n/(1-\mu)$). Dividing both sides of (3.24) by $|t-s|^{\mu/2}$ and taking the supremum for $s, t \geqslant \frac{1}{2}\delta$ we get (3.18) since $u^* = 0$ for $t \leqslant \frac{1}{2}\delta$. □

4. Uniqueness for the Stokes equations in a half space

The goal of this section is to establish a uniqueness theorem for the Stokes equations in a half space $\mathbf{R}^n_+=\{(x',x_n)\,|\,x_n>0\}$ to be able to characterize the limit of rescaled limits in our blow-up argument. The result presented below is by no means optimal but convenient to apply.

THEOREM 4.1. (Uniqueness) *Assume that* (v,q) *satisfies*

$$v\in C(\bar{\mathbf{R}}^n_+\times(0,T))\cap C^{2,1}(\mathbf{R}^n_+\times(0,T)),\quad \nabla q\in C(\mathbf{R}^n_+\times(0,T)) \tag{4.1}$$

and

$$\int_0^T\int_{\mathbf{R}^n_+}(v\cdot(\varphi_t+\Delta\varphi)-\varphi\cdot\nabla q)\,dx\,dt=0 \tag{4.2}$$

for all $\varphi\in C_c^\infty(\mathbf{R}^n_+\times[0,T))$ *with* (1.2)–(1.3) *for* $\Omega=\mathbf{R}^n_+$. *Also assume that*

$$\sup_{0<t<T}\|N(v,q)\|_\infty(t)<\infty \tag{4.3}$$

and

$$\sup_{\substack{x\in\mathbf{R}^n_+\\ 0<t<T}} t^{1/2}x_n|\nabla q(x,t)|<\infty. \tag{4.4}$$

Then $v\equiv0$ *and* $\nabla q\equiv0$.

Remark 4.2. Without decay condition (4.4) for the pressure gradient there is a non-trivial solution. In fact, let $v^i=v^i(x_n,t)$ be the solution of the heat equation

$$\begin{aligned}
v^i_t-\partial^2_{x_n}v^i&=a^i &&\text{in } \{x_n\,|\,x_n>0\}\times(0,T),\\
v^i&=0 &&\text{on } \{0\}\times(0,T),\\
v^i&=0 &&\text{on } \{x_n\,|\,x_n>0\}\times\{0\},
\end{aligned}$$

for $i=1,\dots,n-1$ with $a^i\in C^1[0,T]$ (independent of x). We set $v=(v^1,\dots,v^{n-1},0)$ and

$$q(x,t)=-\sum_{i=1}^{n-1}a^i(t)x_i.$$

Then (v,q) solves the Stokes equations (1.1)–(1.4) with $\Omega=\mathbf{R}^n_+$ and $v_0=0$. It satisfies (4.3) but it does not satify (4.4). This is a non-trivial solution unless $a^i\equiv0$ for all $i=1,\dots,n-1$. Note that (4.2) is satisfied for this (v,q), since (v,q) satisfies (1.1)–(1.4) with $v_0=0$. So this example shows that the uniqueness of Theorem 4.1 is no longer true without (4.4).

This result is easily reduced to a uniqueness theorem which is essentially due to Solonnikov [58]. Although it is stated in a different way [58, Theorem 1.1], his proof, based on the duality argument (proving the solvability of the dual problem), yields the following uniqueness result (Lemma 4.3). Note that for a half space the Stokes semigroup is not bounded in L^1 (for each $t>0$) [14] although the derivative satisfies the usual regularizing effect

$$\|\nabla S(t)v_0\|_{L^1(\mathbf{R}^n_+)} \leqslant Ct^{-1/2}\|v_0\|_{L^1(\mathbf{R}^n_+)}$$

as proved in [33].

Lemma 4.3. *Assume that* (v,q) *satisfies* (4.1)–(4.2) *and* (1.2)–(1.3) *with* $\Omega=\mathbf{R}^n_+$. *Also assume that*

$$\sup_{\delta<t<T} \|N(v,q)\|_\infty(t) < \infty \tag{4.5}$$

for any $\delta\in(0,T)$. *Also assume that* $|\nabla q(x,t)|\to 0$ *as* $x_n\to\infty$ *for* $t\in(0,T)$. *If* $v(\cdot,t)$ *converges* $*$*-weakly to* 0 *in* $L^\infty(\mathbf{R}^n_+)$ *as* $t\downarrow 0$, *then* $v\equiv 0$ *and* $\nabla q\equiv 0$.

Proof of Theorem 4.1. To apply this uniqueness result it suffices to prove that

$$v(\,\cdot\,,t)\to 0 \quad (\text{*-weakly in } L^\infty) \text{ as } t\downarrow 0.$$

Since (v,q) solves (1.1), multiplying by $\varphi\in C_c^\infty(\mathbf{R}^n_+\times[0,T))$ and integration by parts yield

$$\int_\delta^T\int_{\mathbf{R}^n_+} (v\cdot(\varphi_t+\Delta\varphi)-\varphi\cdot\nabla q)\,dx\,dt+\int_{\mathbf{R}^n_+} v(x,\delta)\cdot\varphi(x,\delta)\,dx=0.$$

By (4.2) we easily observe that

$$\int_{\mathbf{R}^n_+} v(x,\delta)\cdot\varphi(x,\delta)\,dx\to 0$$

as $\delta\to 0$. In particular,

$$\int_{\mathbf{R}^n_+} v(x,\delta)\cdot\psi\,dx\to 0$$

for all $\psi\in C_c^\infty(\mathbf{R}^n_+)$. Since v is bounded by (4.3) and $C_c^\infty(\mathbf{R}^n_+)$ is dense in $L^1(\mathbf{R}^n_+)$, this implies that $v(\,\cdot\,,t)\to 0$ ($*$-weakly in L^∞). □

Remark 4.4. (i) The continuity assumption (in Theorem 4.1 and Lemma 4.3)

$$v\in C(\overline{\mathbf{R}}^n_+\times(0,T))$$

in (4.1) is redundant if one assumes (4.3) or (4.5).

(ii) Without the decay condition on the pressure gradient ∇q as $x_n\to\infty$, one still concludes that v depends only on x_n and t; see [58, proof of Theorem 1.1]. Since $\operatorname{div} v=0$ and v vanishes on the boundary, this implies that the normal component v^n (of v) vanishes identically so that $\partial q/\partial x_n=0$. Thus v^i $(1\leqslant i\leqslant n-1)$ solves the heat equation with a spatially constant external source term a^i which agrees with the counterexample for uniqueness without decay of ∇q as $x_n\to\infty$. This observation shows that to establish uniqueness it suffices to assume the decay of $\partial q/\partial x_j$ $(j=1,...,n-1)$ as $x_n\to\infty$.

We conclude this section by giving a uniqueness result for the heat equation which is very easy to prove.

LEMMA 4.5. *Assume that* $u\in L^1_{\mathrm{loc}}(\mathbf{R}^n\times[0,T))$ *satisfies*

$$\int_0^T\int_{\mathbf{R}^n} u(x,t)(\varphi_t(x,t)+\Delta\varphi(x,t))\,dx\,dt=0 \tag{4.6}$$

for all $\varphi\in C_c^\infty(\mathbf{R}^n\times[0,T))$*. Also assume that*

$$\sup_{t\in(0,T)}\|u\|_\infty(t)<\infty. \tag{4.7}$$

Then $u\equiv0$.

Proof. We prove this statement by a duality argument. We first observe that (4.6) holds for

$$\psi\in C^\infty(\mathbf{R}^n\times[0,T))\quad\text{with } \psi,\nabla\psi,\nabla^2\psi,\psi_t\in L^1(\mathbf{R}^n\times[0,T)) \tag{4.8}$$

and $\operatorname{spt}\psi\subset\mathbf{R}^n\times[0,T)$. This is easily proved by setting $\varphi=\theta_R\psi$ in (4.6) and by letting $R\to\infty$, where θ_R is the cut-off function defined in the proof of Theorem 3.4. The procedure is justified by (4.7).

For an arbitrary $f\in C_c^\infty(\mathbf{R}^n\times[0,T))$, we solve

$$\begin{cases}\psi_t+\Delta\psi=f & \text{in } \mathbf{R}^n\times[0,T),\\ \psi(x,T)=0 & \text{for } x\in\mathbf{R}^n.\end{cases}$$

It is not difficult to see that $\psi\in C^\infty(\mathbf{R}^n\times[0,T))$ satisfies (4.8), so we conclude that

$$\int_0^T\int_{\mathbf{R}^n} uf\,dx\,dt=0$$

for all $f\in C_c^\infty(\mathbf{R}^n\times[0,T))$. This implies that $u\equiv0$. □

5. Blow-up arguments—a-priori L^∞ estimates

In this section we shall prove Theorem 1.2 by a blow-up argument. We then derive Theorem 1.3 which implies Theorem 1.1 since a bounded domain is admissible (Theorem 2.5).

5.1. A-priori estimates under stronger regularity assumption

Proposition 5.1. *The assertion of Theorem 1.2 holds under the extra restrictions that $v(\cdot,t)\in C^2(\bar{\Omega})$ for $t\in(0,1)$ and $\|N(v,q)\|_\infty(t)$ is bounded in $(0,1)$ as a function of t.*

Proof. We argue by contradiction. Suppose that (1.6) were false for any choice of T_0 and C. Then there would exist an $\tilde{L}^r$-solution (v_m,q_m) of (1.1)–(1.4) with $v_0=v_{0m}\in C^\infty_{c,\sigma}(\Omega)$ and a sequence $\tau_m\downarrow 0$ (as $m\to\infty$) such that $\|N(v_m,q_m)\|_\infty(\tau_m)>m\|v_{0m}\|_\infty$. There is $t_m\in(0,\tau_m)$ such that

$$\|N(v_m,q_n)\|_\infty(t_m)\geqslant \tfrac{1}{2}M_m,\quad M_m=\sup_{0<t<\tau_m}\|N(v_n,q_m)\|_\infty(t).$$

Note that, due to our extra assumption, M_m is finite. We normalize v_m and q_m by defining $\tilde{v}_m=v_m/M_m$ and $\tilde{q}_m=q_m/M_m$. Then $(\tilde{v}_m,\tilde{q}_m)$ satisfies estimates (1.7)–(1.9). Since $(\tilde{v}_m,\tilde{q}_m)$ is an $\tilde{L}^r$-solution, we have $\nabla\tilde{q}_m=Q[\Delta\tilde{v}_m]$. As Ω is admissible so that (1.5) holds, (1.7) implies that there is a dilation- and translation-invariant constant C_Ω independent of m such that

$$\sup\{t^{1/2}d_\Omega(x)|\nabla\tilde{q}_m(x,t)|\,|\,x\in\Omega_m \text{ and } t\in(0,t_m)\}\leqslant C_\Omega. \tag{5.1}$$

Here we have invoked the assumption $v(\cdot,t)\in C^2(\bar{\Omega})$ to be able to apply the estimate for Q. We rescale $(\tilde{v}_m,\tilde{q}_m)$ around a point $x_m\in\Omega$ satisfying (1.10) to get a blow-up sequence (u_m,p_m) of the form

$$u_m(x,t)=\tilde{v}_m(x_m+t_m^{1/2}x,t_mt),\quad p_m(x,t)=t_m^{1/2}\tilde{q}_m(x_m+t_m^{1/2}x,t_mt).$$

By the scaling invariance of the Stokes equations (1.1)–(1.2), this (u_m,p_m) solves the Stokes equations in the rescaled domain $\Omega_m\times(0,1]$, where

$$\Omega_m=\left\{x\in\mathbf{R}^n\,\middle|\, x=\frac{y-x_m}{t_m^{1/2}} \text{ and } y\in\Omega\right\}.$$

It follows from (1.7), (5.1) and (1.10) that

$$\sup_{0<t<1}\|N(u_m,p_m)\|_{L^\infty(\Omega_m)}\leqslant 1, \tag{5.2}$$

$$\sup\{t^{1/2}d_{\Omega_m}(x)|\nabla p_m(x,t)|\,|\,x\in\Omega_m \text{ and } 0<t<1\}\leqslant C_\Omega, \tag{5.3}$$

$$N(u_m,p_m)(0,1)\geqslant\tfrac{1}{4}. \tag{5.4}$$

Moreover, for initial data v_{0m} the condition (1.9) implies that $\|u_{0m}\|_{L^\infty(\Omega_m)}\to 0$ (as $m\to\infty$). The situation is divided into two cases depending on whether or not

$$c_m=\frac{d_\Omega(x_m)}{t_m^{1/2}}$$

tends to infinity as $m\to\infty$. This c_m is the distance from zero to $\partial\Omega_m$, i.e. $c_m=d_{\Omega_m}(0)$.

Case 1. $\overline{\lim}_{m\to\infty} c_m=\infty$. We may assume that $\lim_{m\to\infty} c_m=\infty$ by taking a subsequence. In this case the rescaled domain Ω_m expands to $\mathbf{R}^n$. Therefore, for any $\varphi\in C_c^\infty(\mathbf{R}^n\times[0,1))$, the blow-up sequence (u_m,p_m) satisfies

$$\int_0^1\int_{\mathbf{R}^n}(u_m\cdot(\varphi_t+\Delta\varphi)-\nabla p_m\cdot\varphi)\,dx\,dt=-\int_{\mathbf{R}^n}u_m(x,0)\cdot\varphi(x,0)\,dx$$

for sufficiently large $m>0$. By (5.2) and Proposition 3.2 we have a necessary compactness to conclude that there exists a subsequence of solutions still denoted by (u_m,p_m) such that (u_m,p_m) converges to some (u,p) locally uniformly in $\mathbf{R}^n\times(0,1]$ together with ∇u_m, $\nabla^2 u_m$, u_{mt} and ∇p_m. (Note that the constant C in (3.1) is invariant under dilation and translation of Ω so (3.1) for (u_m,p_m) gives equicontinuity of $\nabla^2 u_m$, u_{mt} and ∇p_m.) As, for each $R>0$,

$$\inf\{d_{\Omega_m}(x)\,|\,|x|\leqslant R\}\to\infty\quad\text{as } m\to\infty,$$

the estimate (5.3) implies that $\nabla p=0$. Thus the limit $u\in C(\mathbf{R}^n\times(0,1])$ solves

$$\int_0^1\int_{\mathbf{R}^n}u\cdot(\varphi_t+\Delta\varphi)\,dx\,dt=0$$

for all $\varphi\in C_c^\infty(\mathbf{R}^n\times[0,1))$ since $\|u_{0m}\|_{L^\infty(\Omega_m)}\to 0$ as $m\to\infty$. Since u is bounded by (5.2), applying the uniqueness of the heat equation (Lemma 4.5) we conclude that $u\equiv 0$. However, (5.4) implies that $N(u,p)(0,1)\geqslant\frac{1}{4}$ which is a contradiction, so case 1 does not occur.

Case 2. $\overline{\lim}_{m\to\infty} c_m<\infty$. By taking a subsequence, we may assume that c_m converges to some $c_0\geqslant 0$. We may also assume that x_m converges to a boundary point $\hat{x}\in\partial\Omega$. By rotation and translation of coordinates, we may assume that $\hat{x}=0$ and that the exterior normal $n_\Omega(\hat{x})=(0,...,0,-1)$. Since Ω is a uniformly C^3-domain of type (α,β,K), the domain Ω is represented locally near $\hat{x}$ on the form

$$\Omega_{\text{loc}}=\{(x',x_n)\in\mathbf{R}^n\mid h(x')<x_n<h(x')+\beta\text{ and }|x'|<\alpha\}$$

with a C^3-function h such that $\nabla' h(0)=0$ and $h(0)=0$, where derivatives up to third order of h are bounded by K. If one rescales with respect to x_m, Ω_{loc} is expanded as

$$(\Omega_m)_{\text{loc}}$$
$$=\{(y',y_n)\in\mathbf{R}^n\mid h(t_m^{1/2}y'+x'_m)<t_m^{1/2}y_n+(x_m)_n<h(t_m^{1/2}y+x'_m)+\beta\text{ and }|t_m^{1/2}y'|<\alpha\}.$$

Since $d_\Omega(x_m)/(x_m)_n \to 1$ as $m\to\infty$ and $x'_m \to 0$, the domain $(\Omega_m)_{\mathrm{loc}}$ converges to

$$\mathbf{R}^n_{+,-c_0} = \{(x', x_n) \in \mathbf{R}^n \mid x_n > -c_0\}.$$

In fact, if one expresses

$$(\Omega_m)_{\mathrm{loc}} = \{(y', y_n) \in \mathbf{R}^n \mid h_m(y') < y_n < \beta_m + h_m(y') \text{ and } |y'| < \alpha_m\}$$

with $\alpha_m = \alpha/t_m^{1/2}$, $\beta_m = \beta/t_m^{1/2}$, $h_m(y') = h(t_m^{1/2} y' + x'_m)/t_m^{1/2} - (x_m)_n/t_m^{1/2}$, then $h_m \to -c_0$ locally uniformly up to third derivatives and $\alpha_m, \beta_m \to \infty$. Note that $|\partial_x^\mu h_m|$ for μ, $1 \leqslant |\mu| \leqslant 3$, is uniformly bounded by K.

Thus, (u_m, p_m) solves (1.1)–(1.4) in $(\Omega_m)_{\mathrm{loc}} \times (0,1]$. By (5.2) and Theorem 3.4 we have the necessary compactness to conclude that there exists a subsequence (u_m, p_m) converging to some (u,p) locally uniformly in $\overline{\mathbf{R}}^n_{+,-c_0} \times (0,1]$ together with ∇u_m, $\nabla^2 u_m$, u_{mt} and ∇p_m as in the interior case. (Note that Ω_m is still of type (α, β, K) which is uniform in m.)

Now we observe that the limit (u,p) solves the Stokes equations (1.1)–(1.4) in a half space with zero initial data in a weak sense. In fact, as (u_m, p_m) satisfies

$$\int_0^1 \int_{\mathbf{R}^n_{+,-c_0}} (u_m \cdot (\varphi_t + \Delta\varphi) - \varphi \cdot \nabla p_m)\, dx\, dt = -\int_{\mathbf{R}^n_{+,-c_0}} u_m(x,0) \cdot \varphi(x,0)\, dx$$

for all $\varphi \in C_c^\infty(\mathbf{R}^n_{+,-c_0} \times [0,1))$, we note that (5.2) and (5.3) are inherited by (u,p), and in particular

$$\sup\{t^{1/2}(x_n + c_0)|\nabla p(x,t)| \mid x' \in \mathbf{R}^{n-1}, x_n > -c_0 \text{ and } t \in (0,1)\} \leqslant C_\Omega.$$

Since the convergence of u_m is up to the boundary, the boundary condition is also preserved. The limit $(u,p) \in C(\mathbf{R}^n_{+,-c_0} \times [0,1])$ solves a weak form of the Stokes equations with zero initial data:

$$\int_0^1 \int_{\mathbf{R}^n_{+,-c_0}} (u \cdot (\varphi_t + \Delta\varphi) - \varphi \cdot \nabla p)\, dx\, dt = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbf{R}^n_{+,-c_0} \times [0,1)).$$

We thus apply the uniqueness to the Stokes equations in a half space (Theorem 4.1) to conclude that $u \equiv 0$ and $\nabla p \equiv 0$.

However, (5.4) implies that $N(u,p)(0,0) \geqslant \frac{1}{4}$ which is a contradiction, so case 2 does not occur either.

We have thus proved (1.6). □

5.2. Regularity for $\tilde{L}^r$-solutions

We shall prove that the extra conditions for v in Proposition 5.1 can be removed. For example we have the following result.

PROPOSITION 5.2. *Let Ω be a uniformly C^3-domain in $\mathbf{R}^n$. Let (v,q) be an $\tilde{L}^r$-solution of (1.1)–(1.4) for $r>n$. Assume that $v_0\in D(\tilde{A}_r)$, where $\tilde{A}_r$ is the Stokes operator in $\tilde{L}^r_\sigma(\Omega)$, i.e. $-\tilde{A}_r$ is the generator of the Stokes semigroup in $\tilde{L}^r_\sigma(\Omega)$. Then $v(\cdot,t)\in C^2(\bar{\Omega})$ for all $t>0$. Moreover, for each $T>0$, we have*

$$\sup_{0<t<T}\|N(v,q)\|_\infty(t)<\infty. \tag{5.5}$$

Proof. We shall claim a stronger statement

$$\sup_{0<t<T}(\|v\|_{W^{1,r}_{\mathrm{ul}}}(t)+t^{1/2}\|\nabla v\|_{W^{1,r}_{\mathrm{ul}}}(t)+t(\|\nabla^2 v\|_{W^{1,r}_{\mathrm{ul}}}(t)+\|\partial_t v\|_{W^{1,r}_{\mathrm{ul}}}(t)+\|\nabla q\|_{W^{1,r}_{\mathrm{ul}}}(t))) \leqslant C\|v_0\|_{D(\tilde{A}_r)} \tag{5.6}$$

with $C=C(T,\Omega,r)$. Here $W^{1,r}_{\mathrm{ul}}$ is a uniformly local $W^{1,r}$ space defined by

$$W^{1,r}_{\mathrm{ul}}(\Omega)=\{f\in L^r_{\mathrm{ul}}(\Omega)\mid \nabla f\in L^r_{\mathrm{ul}}(\Omega)\},\quad \|f\|_{W^{1,r}_{\mathrm{ul}}}=\|f\|_{L^r_{\mathrm{ul}}}+\|\nabla f\|_{L^r_{\mathrm{ul}}},$$

and

$$L^r_{\mathrm{ul}}(\Omega)=\left\{f\in L^r_{\mathrm{loc}}(\Omega)\;\middle|\;\|f\|_{L^r_{\mathrm{ul}}}=\sup_{x\in\Omega}\left(\int_{\Omega_{x,R}}|f(y)|^r\,dy\right)^{1/r}\right\},$$

where $\Omega_{x,R}=\operatorname{int}B_R(x)\cap\Omega$ and R is a fixed positive number. The norm depends on R but the topology defined by the norm is independent of the choice of R. The norm of the domain $D(\tilde{A}_r)$ is defined by

$$\|u\|_{D(\tilde{A}_r)}=\|u\|_{\tilde{L}^r(\Omega)}+\|\tilde{A}_r u\|_{\tilde{L}^r(\Omega)},\quad \|u\|_{\tilde{L}^r(\Omega)}=\max\{\|u\|_{L^r(\Omega)},\|u\|_{L^2(\Omega)}\},$$

when $r\geqslant 2$. As proved in [16] and [18], this norm is equivelent to the norm

$$\|u\|_{\widetilde{W}^{2,r}(\Omega)}=\sum_{|l|\leqslant 2}\|\partial_x^l u\|_{\tilde{L}^r(\Omega)}.$$

Note that once we have proved (5.6), the inequality and $v(\cdot,t)\in C^2(\bar{\Omega})$ follows from the Sobolev embedding. (One can even claim that $\nabla^2 v(\cdot,t)$ is Hölder continuous with exponent $\gamma=1-n/r$.)

We shall prove (5.6). We first observe that by the analyticity of the semigroup

$$S(t)=e^{-t\tilde{A}_r},$$

we have

$$\sup_{0<t<T} t\|v_t\|_{D(\tilde{A}_r)}(t) \leqslant C_1\|v_0\|_{D(\tilde{A}_r)},$$

since $\tilde{A}_r v_t = -\tilde{A}_r e^{-t\tilde{A}_r}\tilde{A}_r v_0$. It is easy to see that

$$\sup_{0<t<T} \|v\|_{D(\tilde{A}_r)}(t) \leqslant C_2\|v_0\|_{D(\tilde{A}_r)} \tag{5.7}$$

with C_j depending only on T, Ω and r. Thus we have proved that

$$\sup_{0<t<T} (\|v\|_{\widetilde{W}^{1,r}(\Omega)}(t) + \|\nabla v\|_{\widetilde{W}^{1,r}(\Omega)}(t) + t\|v_t\|_{\widetilde{W}^{2,r}(\Omega)}(t)) \leqslant C_3\|v_0\|_{D(\tilde{A}_r)}, \tag{5.8}$$

as the $D(\tilde{A}_r)$-norm and the $\widetilde{W}^{2,r}$-norm are equivalent. The estimate (5.8) controls the terms

$$\|v\|_{W^{1,r}_{\mathrm{ul}}}, \quad t^{1/2}\|\nabla v\|_{W^{1,r}_{\mathrm{ul}}} \quad \text{and} \quad t\|v_t\|_{W^{1,r}_{\mathrm{ul}}}$$

in (5.6).

To show (5.6) it remains to prove that

$$\sup_{0<t<T} t(\|\nabla^2 v\|_{W^{1,r}_{\mathrm{ul}}}(t) + \|\nabla q\|_{W^{1,r}_{\mathrm{ul}}}(t)) \leqslant C_4\|v_0\|_{D(\tilde{A}_r)}. \tag{5.9}$$

We take R sufficiently small so that $\Omega_{x,3R} \subset U_{\alpha,\beta,h}(x_0)$ for any $x_0 \in \partial\Omega$. We normalize q by taking

$$\hat{q}(x) = q(x) - \frac{1}{|\Omega''|}\int_{\Omega''} q(x)\,dx, \quad \Omega'' = \Omega_{x_0,3R}.$$

It follows from the Poincaré inequality [15, §5.8.1] that

$$\|\hat{q}\|_{L^r(\Omega'')} \leqslant c\|\nabla q\|_{L^r(\Omega'')} \tag{5.10}$$

with c independent of x_0. Since Ω is C^3 and (v,q) solves

$$-\Delta v + \nabla q = -v_t \ \text{ and } \ \operatorname{div} v = 0 \quad \text{in } \Omega''$$

with

$$v = 0 \quad \text{on } \partial\Omega'' \cap \partial\Omega,$$

the local higher regularity theory for elliptic systems (see [22, Chapter V]) shows that

$$\|\nabla^3 v\|_{L^r(\Omega')} + \|\nabla^2 q\|_{L^r(\Omega')} \leqslant C(\|v_t\|_{W^{1,r}(\Omega'')} + \|v\|_{W^{1,r}(\Omega'')} + \|\hat{q}\|_{L^r(\Omega'')})$$

with $\Omega' = \Omega_{x_0,2R}$. Here the dependence with respect to t is suppressed. The last term is estimated by (5.10), so we observe that

$$\|\nabla^3 v\|_{L^r(\Omega')} + \|\nabla^2 q\|_{L^r(\Omega')} \leqslant C(\|v_t\|_{W^{1,r}(\Omega)} + \|v\|_{W^{1,r}(\Omega)} + \|\nabla q\|_{L^r(\Omega)}) \tag{5.11}$$

with C depending only on Ω, R and r, but independent of $x_0\in\partial\Omega$. If $x_0\in\Omega$ is taken so that $B_{2R}(x_0)\subset\Omega$, then interior higher regularity theory yields (5.11) with $\Omega'=B_R(x_0)$ (by taking $\Omega''=B_{2R}(x_0)$). As Ω is covered by $\Omega_{x_0,2R}$, $x_0\in\partial\Omega$, and $B_R(x_0)$, with $x_0\in\Omega$ such that $B_{2R}(x_0)\subset\Omega$, the estimate (5.11) implies that

$$\|\nabla^3 v\|_{L^r_{\mathrm{ul}}(\Omega)}+\|\nabla^2 q\|_{L^r_{\mathrm{ul}}(\Omega)}\leqslant C(\|v_t\|_{W^{1,r}(\Omega)}+\|v\|_{W^{1,r}(\Omega)}+\|\nabla q\|_{L^r(\Omega)}). \tag{5.12}$$

Since $\nabla q=Q[\Delta v]$ implies that

$$\|\nabla q\|_{\tilde{L}^r(\Omega)}\leqslant C'\|\Delta v\|_{\tilde{L}^r(\Omega)},$$

with $C'=C'(\Omega,r)$, the estimate (5.12) together with (5.8) now yields (5.9). □

Proof of Theorem 1.2. Combining Propositions 5.1 and 5.2 yields Theorem 1.2, as $C^\infty_{c,\sigma}(\Omega)$ is included in $D(\tilde{A}_r)$. □

5.3. Analyticity of the Stokes semigroup in $C_{0,\sigma}$

We shall prove Theorem 1.3. To show the C_0-property of the semigroup, we start with the following result.

PROPOSITION 5.3. *Let Ω be a uniformly C^2-domain in $\mathbf{R}^n$. Also let (v,q) be an $\tilde{L}^r$-solution of* (1.1)–(1.4) *with $r>n$ and $v_0\in D(\tilde{A}_r)$. Then*

$$\lim_{t\downarrow 0}\|v(\cdot,t)-v_0\|_\infty=0. \tag{5.13}$$

In other words,

$$\lim_{t\downarrow 0}\|e^{-t\tilde{A}_r}v_0-v_0\|_\infty=0.$$

Proof. By the Gagliardo–Nirenberg inequality, we have

$$\|v(t)-v_0\|_{L^\infty(\Omega)}\leqslant C\|v(t)-v_0\|^{1-\theta}_{L^r(\Omega)}\|v(t)-v_0\|^{\theta}_{W^{1,r}(\Omega)} \tag{5.14}$$

with $\theta=1-n/r$, where $v(t)=v(\cdot,t)$. Since

$$\|f\|_{W^{1,r}(\Omega)}\leqslant\|f\|_{W^{2,r}(\Omega)}\leqslant\|f\|_{\widetilde{W}^{2,r}(\Omega)}\leqslant C'\|f\|_{D(\tilde{A}_r)},$$

we have by (5.7) that

$$\|v(t)-v_0\|_{W^{1,r}(\Omega)}\leqslant C'(\|v(t)\|_{D(\tilde{A}_r)}+\|v_0\|_{D(\tilde{A}_r)})\leqslant C''\|v_0\|_{D(\tilde{A}_r)}. \tag{5.15}$$

As $e^{-t\tilde{A}^r}$ is strongly continuous in $\tilde{L}^r$, (5.14) with (5.15) yields (5.13). □

Proof of Theorem 1.3. By the a-priori estimate (1.6), the operator $S(t)$ is uniquely extended to a bounded operator $\widetilde{S}(t)$ in $C_{0,\sigma}$ at least for small t, say $t\in[0,T_0)$. Since $S(t)$ is a semigroup in $\tilde{L}^r$, we have

$$\widetilde{S}(t_1)\widetilde{S}(t_2)=\widetilde{S}(t_1+t_2) \quad \text{as long as } t_1+t_2<T_0. \tag{5.16}$$

We extend $\widetilde{S}(t)$ to $t\geqslant T_0$ by $\widetilde{S}(t)=\widetilde{S}(t_1)\dots\widetilde{S}(t_m)$ so that $t_i\in(0,T_0)$ and $t_1+\dots+t_m=t$. This is well-defined in the sense that $\widetilde{S}(t)$ is independent of the division of t by the semigroup property (5.16). Thus we are able to define the Stokes semigroup $\widetilde{S}(t)$ for all $t\geqslant 0$ which we simply write by $S(t)$ (since it agrees with $S(t)$ on $C_{0,\sigma}\cap\tilde{L}^r$). Our estimate (1.6) is inherited by $S(t)$. Moreover, by the semigroup property, the estimate (1.6) yields $\|S(t)v_0\|_\infty\leqslant C_T\|v_0\|_\infty$ with C_T independent of $v_0\in C_{0,\sigma}(\Omega)$ and $t\in(0,T)$ for arbitrary $T>0$. As $dS(t)/dt=S(t-s)\,dS(s)/ds$ for $s\in(0,t)$, the estimate (1.6) together with an L^∞ bound for $S(t)$ yields

$$\sup_{0<t<T} t\left\|\frac{d}{dt}S(t)v_0\right\|_\infty\leqslant C_T'\|v_0\|_\infty,$$

with a constant C_T' independent of $v_0\in C_{0,\sigma}(\Omega)$. This implies that $S(t)$ is an analytic semigroup in $C_{0,\sigma}(\Omega)$.

It remains to prove that $S(t)$ is a C_0-semigroup in $C_{0,\sigma}(\Omega)$. Since $C_{c,\sigma}^\infty(\Omega)$ is dense in $C_{0,\sigma}(\Omega)$, for each $v_0\in C_{0,\sigma}(\Omega)$ there is $v_{0m}\in C_{c,\sigma}^\infty(\Omega)$ such that $v_{0m}\to v_0$ in $L^\infty(\Omega)$. Since $\|S(t)v_0\|_\infty\leqslant C_T\|v_0\|_\infty$ for $0<t<T$, we have

$$\begin{aligned}\|S(t)v_0-v_0\|_\infty &\leqslant \|S(t)v_0-S(t)v_{0m}\|_\infty+\|S(t)v_{0m}-v_{0m}\|_\infty+\|v_{0m}-v_0\|_\infty\\ &\leqslant (C_T+1)\|v_{0m}-v_0\|_\infty+\|S(t)v_{0m}-v_{0m}\|_\infty.\end{aligned}$$

By Proposition 5.3, letting $t\downarrow 0$ yields

$$\varlimsup_{t\downarrow 0}\|S(t)v_0-v_0\|_\infty\leqslant (C_T+1)\|v_{0m}-v_0\|_\infty.$$

Letting $m\to\infty$, we conclude that $S(t)$ is a C_0-semigroup in $C_{0,\sigma}(\Omega)$. □

As a bounded domain is admissible, Theorem 1.3 yields Theorem 1.1.

Remark 5.4. (i) In general, we do not know whether or not $S(t)$ is a bounded analytic semigroup in the sense that

$$\left\|\frac{d}{dt}S(t)v_0\right\|_\infty\leqslant\frac{C}{t}\|v_0\|_\infty \tag{5.17}$$

for some C independent of $t>0$. When Ω is bounded, one can claim such boundedness. In fact, multiplying v with (1.1) and integrating by parts we obtain the energy equality

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2(t)+\|\nabla v\|_{L^2}^2(t)=0.$$

Since Ω is bounded, the Poincaré inequality implies that

$$\|\nabla v\|_{L^2}^2 \geqslant \nu \|v\|_{L^2}^2$$

with some $\nu>0$. Thus

$$\|S(t)v_0\|_{L^2}^2 \leqslant e^{-2\nu t}\|v_0\|_{L^2}^2.$$

If Ω is sufficiently smooth, by the Sobolev inequality and the property of the Stokes semigroup in L^2 (see [54, §III.2.1]), we have

$$\|S(t)v_0\|_{L^\infty} \leqslant C_1\|S(t)v_0\|_{W^{2k,2}} \leqslant C_2\|A_2^k S(t)v_0\|_{L^2}$$

for an integer $k>\frac{1}{4}n$ with C_j $(j=1,2,\dots)$ independent of t and $v_0\in L^2_\sigma(\Omega)$. As $S(t)$ is an analytic semigroup in L^2_σ, this yields

$$\|S(t)v_0\|_{L^\infty} \leqslant C_3\|S(t-1)v_0\|_{L^2} \quad \text{for } t\geqslant 1.$$

We have thus proved that

$$\|S(t)v_0\|_{L^\infty} \leqslant C_4 e^{-\nu t}\|v_0\|_{L^2} \leqslant C_5 e^{-\nu t}\|v_0\|_{L^\infty}, \quad t\geqslant 1. \tag{5.18}$$

Similarly,

$$\left\|\frac{d}{dt}S(t)v_0\right\|_{L^\infty} \leqslant C_1\left\|\frac{d}{dt}S(t)v_0\right\|_{W^{2k,2}} \leqslant C_2\|A_2^{k+1}S(t)v_0\|_{L^2} \leqslant C_6 e^{-\nu t}\|v_0\|_{L^\infty} \quad \text{for } t\geqslant 1.$$

Since

$$\left\|\frac{d}{dt}S(t)v_0\right\|_\infty \leqslant \frac{C_7}{t}\|v_0\|_\infty \quad \text{for } t\leqslant 1,$$

this yields (5.17). Thus $S(t)$ is a bounded analytic semigroup in $C_{0,\sigma}(\Omega)$ and $L^\infty_\sigma(\Omega)$ (see the next section) when Ω is a smoothly bounded domain. If one uses L^r-theory $(r>n)$ instead of L^2-theory, the result is still valid for a bounded domain with C^3 boundary.

(ii) Since we have (5.18) for $t\geqslant T_0>0$, our a-priori estimate (1.6) in particular implies that

$$\|S(t)v_0\|_\infty \leqslant C\|v_0\|_\infty \quad \text{for all } t>0 \text{ and } v_0\in C_{0,\sigma}(\Omega),$$

with C depending only on Ω when Ω is bounded. This type of result is often called a maximum modulus result in the literature.

The maximum modulus theorem was first stated in [65] when Ω is a bounded, convex domain with smooth boundary for $v_0\in C_{c,\sigma}^\infty(\Omega)$. Later a full proof was given in [56]. It was extended by [57] to a general bounded domain with C^2 boundary. It was also extended by [43] to $v_0\in C_{0,\sigma}(\Omega)$, but with $\partial\Omega$ assumed to be $C^{2+\gamma}$ with $\gamma\in(0,1)$.

By our extension to the L_σ^∞ space in the next section, we conclude that

$$\|S(t)v_0\|_\infty \leqslant C\|v_0\|_\infty, \quad v_0 \in L_\sigma^\infty(\Omega),$$

for all $t>0$, with C depending only on Ω when Ω is bounded and of C^3 boundary.

(iii) It is interesting to discuss whether or not our semigroup $S(t)$ is a $\frac{1}{2}\pi$-type analytic semigroup (i.e. it is extendable to a holomorphic semigroup in $\operatorname{Re} t>0$). Our results say that $S(t)$ is an ε-type analytic semigroup for some $\varepsilon>0$. If we are able to prove (1.6) for $\operatorname{Re} t\in(0,T_0)$ with $|\arg t|<\alpha$ for $\alpha\in\left(0,\frac{1}{2}\pi\right)$ where analyticity is valid, then we conclude that $S(t)$ is a $\frac{1}{2}\pi$-analytic semigroup. This idea would work provided that the Schauder-type estimate for complex t with $|\arg t|<\varepsilon$ would be available. It is of course likely but there seems to be no explicit reference. Very recently, M. Hieber and the authors [2] proved a necessary resolvent estimate to conclude that $S(t)$ is a $\frac{1}{2}\pi$-type analytic semigroup (without proving (1.6) for complex t).

(iv) A closer examination of the proof of Proposition 5.1 shows that it suffices to apply the estimate

$$\sup_{x\in\Omega} d_\Omega(x)|Q[\nabla\cdot f](x)| \leqslant C\|f\|_{L^\infty(\Omega)},$$

which is weaker than (1.5) in the sense that the norm in the right-hand side is over Ω, not only over $\partial\Omega$.

6. Results for L_σ^∞

In this section we shall prove that the Stokes semigroup is a (non-C_0-)analytic semigroup in $L_\sigma^\infty(\Omega)$ when Ω is bounded, as stated in Theorem 1.4. The space $L_\sigma^\infty(\Omega)$ is defined by

$$L_\sigma^\infty(\Omega) = \left\{ f \in L^\infty(\Omega) \;\middle|\; \int_\Omega f\cdot\nabla\varphi\,dx = 0 \text{ for all } \varphi\in \widehat{W}^{1,1}(\Omega)\right\},$$

where $\widehat{W}^{1,1}(\Omega)$ is the homogeneous Sobolev space of the form

$$\widehat{W}^{1,1}(\Omega) = \{\varphi\in L^1_{\mathrm{loc}}(\Omega) \mid \nabla\varphi\in L^1(\Omega)\}.$$

6.1. Approximation

We begin with an approximation result when Ω is star-shaped (with respect to some point $a\in\mathbf{R}^n$, i.e. $\lambda\overline{(\Omega-a)}\subset\Omega-a$ for all $\lambda\in(0,1)$).

LEMMA 6.1. (Approximation) *Let Ω be a bounded, star-shaped domain in $\mathbf{R}^n$. There exists a constant $C=C_\Omega$ such that for any $v\in L^\infty_\sigma(\Omega)$ there exists a sequence*

$$\{v_m\}_{m=1}^\infty \subset C^\infty_{c,\sigma}(\Omega)$$

such that

$$\|v_m\|_\infty \leqslant C\|v\|_\infty \tag{6.1}$$

and

$$v_m \to v \quad a.e.\ in\ \Omega \tag{6.2}$$

as $m\to\infty$. If in addition $v\in C(\overline{\Omega})$, then the convergence is locally uniform in Ω. And if furthermore $v=0$ on $\partial\Omega$, then the convergence is uniform in $\overline{\Omega}$.

Proof. Since Ω is star-shaped, we may assume that

$$\lambda\overline{\Omega} \subset \Omega \quad \text{for all } \lambda \in [0,1)$$

after a translation. We extend $v\in L^\infty_\sigma(\Omega)$ by zero outside Ω and observe that the extension (still denoted by v) is in $L^\infty_\sigma(\mathbf{R}^n)$ with $\operatorname{spt} v\subset\overline{\Omega}$. We set $v_\lambda(x)=v(x/\lambda)$ and observe that $\operatorname{spt} v_\lambda\subset\lambda\overline{\Omega}\subset\Omega$. Since $v_\lambda\to v$ a.e. as $\lambda\uparrow 1$, it is easy to find the desired sequence by mollifying v_λ i.e. considering $v_\lambda * \eta_\varepsilon$. Here C in (6.1) can be taken to be 1. □

To establish the above approximation result for a general bounded domain we need a localization lemma.

LEMMA 6.2. (Localization) *Let Ω be a bounded domain with Lipschitz boundary in $\mathbf{R}^n$. Also let $\{G_k\}_{k=1}^N$ be an open covering of $\overline{\Omega}$ in $\mathbf{R}^n$ and let $\Omega_k=G_k\cap\Omega$. Then there exists a family of bounded linear operators $\{\pi_k\}_{k=1}^N$ from $L^\infty_\sigma(\Omega)$ into itself satisfying $u=\sum_{k=1}^N \pi_k u$ and, for each $k=1,...,N$,*

(i) $\pi_k u|_{\Omega_k}\in L^\infty_\sigma(\Omega_k)$ *and* $\pi_k u|_{\Omega\setminus\Omega_k}=0$ *for* $u\in L^\infty_\sigma(\Omega)$;

(ii) $\pi_k u\in C(\overline{\Omega}_k)$ *and* $\pi_k u|_{\partial\Omega_k\setminus\partial\Omega}=0$ *for* $u\in C(\overline{\Omega})\cap L^\infty_\sigma(\Omega)$;

(iii) $\pi_k u|_{\partial\Omega_k}=0$ *if* $u|_{\partial\Omega}=0$, *for* $u\in C(\overline{\Omega})\cap L^\infty_\sigma(\Omega)$.

Proof. We proceed by induction on N. If $N=1$, the result is trivial by taking π_1 as the identity.

Assume that the result is valid for N. We shall prove the assertion when the number of operators is $N+1$. We set

$$D=\bigcup_{k=2}^{N+1}\Omega_k \quad \text{and} \quad U=\bigcup_{k=2}^{N+1} G_k$$

and observe that $\Omega=\Omega_1\cup D$ and $\{G_1,U\}$ is a covering of $\overline{\Omega}$.

Let $\{\xi_1,\xi_2\}$ be a partition of unity of Ω associated with $\{G_1,U\}$, i.e. $\xi_j\in C_c^\infty(\mathbf{R}^n)$ with $0\leqslant\xi_j\leqslant 1$, $\operatorname{spt}\xi_1\subset G_1$, $\operatorname{spt}\xi_2\subset U$ and $\xi_1+\xi_2=1$ in $\bar\Omega$. For $E=\Omega_1\cap D$ let B_E denote the Bogovskiĭ operator. We set

$$\pi_1 u=\begin{cases} u\xi_1-B_E(u\cdot\nabla\xi_1) & \text{in } E,\\ u\xi_1 & \text{in } \Omega_1\backslash D,\\ 0 & \text{in } \Omega\backslash\Omega_1.\end{cases}$$

Since $u\in L^\infty_\sigma(\Omega)$, $\xi_1=0$ in $\Omega\backslash\Omega_1$ and $\nabla\xi_1=0$ in $\Omega_1\backslash D$, we see that

$$\int_E u\cdot\nabla\xi_1\,dx=\int_\Omega u\cdot\nabla\xi_1\,dx=0. \tag{6.3}$$

By the Sobolev inequality and (3.13), we observe that, with $p>n$,

$$\begin{aligned}\|B_E(u\cdot\nabla\xi_1)\|_{L^\infty(E)}&\leqslant C\|B_E(u\cdot\nabla\xi_1)\|_{W^{1,p}(E)}\leqslant CC_B\|u\cdot\nabla\xi_1\|_{L^p(E)}\\ &\leqslant CC_B\|\nabla\xi_1\|_{L^p(E)}\|u\|_{L^\infty(\Omega)},\end{aligned}$$

with a constant C independent of u and ξ_1. We thus observe that

$$\|\pi_1 u\|_{L^\infty(\Omega_1)}\leqslant C_1\|u\|_{L^\infty(\Omega)}\quad\text{for all } u\in L^\infty_\sigma(\Omega),$$

with C_1 independent of u.

By (6.3), we see that $\operatorname{div} B_E(u\cdot\nabla\xi_1)=u\cdot\nabla\xi_1$ in E. Moreover, $B_E(u\cdot\nabla\xi_1)=0$ on $\partial(\Omega_1\cap D)$. Thus, for each $\varphi\in L^1_{\mathrm{loc}}(\bar\Omega_1)$ with $\nabla\varphi\in L^1(\Omega_1)$, we have

$$\begin{aligned}\int_{\Omega_1}\pi_1 u\cdot\nabla\varphi\,dx&=\int_{\Omega_1}u\,\xi_1\cdot\nabla\varphi\,dx-\int_E B_E(u\cdot\nabla\xi_1)\cdot\nabla\varphi\,dx\\ &=\int_{\Omega_1}u\,\xi_1\cdot\nabla\varphi\,dx+\int_E(u\cdot\nabla\xi_1)\varphi\,dx=\int_\Omega u\cdot\nabla(\xi_1\varphi)\,dx=0.\end{aligned}$$

By the Poincaré inequality, if $\varphi\in\widehat{W}^{1,1}(\Omega_1)$ then $\varphi\in L^1_{\mathrm{loc}}(\bar\Omega_1)$ (not only $\varphi\in L^1_{\mathrm{loc}}(\Omega_1)$). Thus the above identity implies that $\pi_1 u|_{\Omega_1}\in L^\infty_\sigma(\Omega_1)$. By definition, $\pi_1 u=0$ in $\Omega\backslash\Omega_1$. If $u\in C(\bar\Omega)$, it is easy to see that the term $B_E(u\cdot\nabla\xi_1)$ is always Hölder continuous by the Sobolev embeddings.

For $u\in L^\infty_\sigma(\Omega)$ we set

$$\pi_D u=\begin{cases} u\xi_2-B_E(u\cdot\nabla\xi_2) & \text{in } E,\\ u\xi_2 & \text{in } D\backslash\Omega_1,\\ 0 & \text{in } \Omega\backslash D.\end{cases}$$

By definition,

$$u=\pi_1 u+\pi_D u$$

and as for π_1 this π_D satisfies all properties of π_k in (i)–(iii) with Ω_k replaced by D. Since $\bar{D}$ is covered by $\{G_k\}_{k=2}^{N+1}$, by our induction assumption there is a family of bounded linear operators $\{\hat{\pi}_k\}_{k=2}^{N+2}$ in $L^\infty_\sigma(D)$ satisfying $v=\sum_{k=2}^{N+1}\hat{\pi}_k v$ and (i)–(iii) with u replaced by v and with π_k replaced by $\hat{\pi}_k$ for $k=2,...,N+1$. If we set

$$\pi_1=\pi_1 \quad \text{and} \quad \pi_k=\hat{\pi}_k\circ\pi_D \ \ (k=2,...,N+1),$$

then it is rather clear that this π_k satisfies all desired properties. □

LEMMA 6.3. (Approximation) *The assertion of Lemma 6.1 is still valid when Ω is a bounded domain with Lipschitz boundary in $\mathbf{R}^n$.*

Proof. If Ω is a bounded domain with Lipschitz boundary, then it is known that there is an open covering $\{G_k\}_{k=1}^N$ of $\bar{\Omega}$ such that $\Omega_k=G_k\cap\Omega$ is bounded, star-shaped with respect to an open ball B_k, with $\overline{B}_k\subset\Omega$ (i.e. star-shaped with respect to any point of B_k) and G_k has a Lipschitz boundary; see [22, §III.3, Lemma 4.3]. In the sequel we only need the property that G_k is bounded and star-shaped with respect to a point.

We apply Lemma 6.2 and set $u_k=\pi_k u$ to observe that $u_k|_{\Omega_k}\in L^\infty_\sigma(\Omega_k)$ and that $u_k|_{\Omega\backslash\Omega_k}=0$. Since Ω_k is star-shaped, by Lemma 6.1 there is $\{u_{k,j}\}_{j=1}^\infty\subset C^\infty_{c,\sigma}(\Omega_k)$ such that

$$\|u_{k,j}\|_{L^\infty(\Omega_k)}\leqslant\|u_k\|_{L^\infty(\Omega_k)} \quad \text{and} \quad u_{k,j}\to u_k \text{ a.e. in } \Omega.$$

(The constant C in (6.1) can be taken to be 1.) We still denote the zero extension of $u_{k,j}$ on $\Omega\backslash\Omega_k$ by $u_{k,j}$.

If we set $u_m=\sum_{k=1}^N u_{k,m}$, then by construction $u_j\in C^\infty_{c,\sigma}(\Omega)$,

$$u_m\to\sum_{k=1}^N u_k=u \quad \text{a.e. in } \Omega$$

and

$$\|u_m\|_{L^\infty(\Omega)}\leqslant\sum_{k=1}^N\|u_{k,m}\|_{L^\infty(\Omega)}\leqslant\sum_{k=1}^N\|u_k\|_{L^\infty(\Omega)}\leqslant\left(\sum_{k=1}^N\|\pi_k\|\right)\|u\|_{L^\infty(\Omega)},$$

where $\|\pi_k\|$ denotes the operator norm of π_k in $L^\infty_\sigma(\Omega)$. We thus conclude that there is a desired approximating sequence $\{u_m\}_{m=1}^\infty$ for $u\in L^\infty_\sigma(\Omega)$.

If $u\in C(\bar{\Omega})(\cap L^\infty_\sigma(\Omega))$, then $u_k\in C(\bar{\Omega}_k)$ and $u_k|_{\partial\Omega_k\backslash\partial\Omega}=0$. Thus for any compact set $K_k\subset\Omega_k$ such that $d(K_k)=\inf_{x\in K_k}d_\Omega(x)>0$, we see that $u_{k,m}$ converges to u_k uniformly in K_k by Lemma 6.1 as $m\to\infty$. Let K be a compact set in Ω. Then $d(K_k)\geqslant d(K)>0$ for $K_k=\bar{\Omega}_k\cap K$. Hence

$$\|u-u_m\|_{L^\infty(K)}\leqslant\sum_{k=1}^N\|u_k-u_{k,m}\|_{L^\infty(K)}=\sum_{k=1}^N\|u_k-u_{k,m}\|_{L^\infty(K_k)}\to 0\quad\text{as } m\to\infty.$$

Thus we have proved that u_m converges to u locally uniformly in Ω. If $u|_{\partial\Omega}=0$ so that $u_k|_{\partial\Omega_k}=0$, then $u_{k,m}$ converges to u_k uniformly in $\bar{\Omega}_k$ by Lemma 6.1. Arguing in the same way by replacing K by $\bar{\Omega}$, we conclude that u_m converges to u uniformly in $\bar{\Omega}$. □

Remark 6.4. This lemma in particular implies that

$$C_{0,\sigma}(\Omega)=\{v\in C(\bar{\Omega})\cap L^\infty(\bar{\Omega})\mid \operatorname{div} v=0 \text{ in } \Omega \text{ and } v=0 \text{ on } \partial\Omega\}$$

when Ω is bounded. This gives an alternative and direct proof of a result of [43], where the maximum modulus result for the stationary problem is invoked.

Proof of Theorem 1.4. Since Ω is bounded so that $L^\infty_\sigma\subset L^r_\sigma$ for any $r>1$, $S(t)$ is well defined from L^∞_σ to L^r_σ. It suffices to transfer the estimate for $v=S(t)v_0$ in (1.6) to the case $v_0\in L^\infty_\sigma(\Omega)$. By Lemma 6.3, there is a sequence $v_{0m}\in C^\infty_{c,\sigma}(\Omega)$ approximating v_0. Our estimate (1.6) implies that

$$\sup_{0<t<T_0}(\|v_m\|_\infty(t)+t(\|v_{mt}\|_\infty+\|\nabla^2 v_m\|_\infty)(t))\leqslant C\|v_{0m}\|_\infty$$

is valid for such v_{0m}, by Theorem 1.2. Here T_0 and C are independent of m. Since $v_{0m}\to v_0$ in L^r, by (6.2) and the Lebesgue dominated convergence theorem, we see that $v_m\to v$ in L^r uniformly in $t\in[0,T]$; note that $S(t)$ is a semigroup in L^r_σ. Thus we obtain

$$\sup_{0<t<T_0}(\|v\|_\infty(t)+t(\|v_t\|_\infty+\|\nabla^2 v\|_\infty)(t))\leqslant\varlimsup_{m\to\infty}\|v_{0m}\|_\infty.$$

By (6.2), one is able to replace the right-hand side by a constant multiple of $\|v_0\|_\infty$, so we obtain the desired estimate for claiming the analyticity of $S(t)$ in $L^\infty_\sigma(\Omega)$.

This semigroup $S(t)$ is a non-C_0-semigroup. Indeed, suppose the contrary to get

$$S(t)v_0\to v_0\quad\text{in } L^\infty \text{ as } t\downarrow 0$$

for all $v_0\in L^\infty_\sigma(\Omega)$. Our estimate for $\nabla^2 v$ implies that $S(t)v_0$ (for $t>0$) is at least continuous in $\bar{\Omega}$. However, if $S(t)v_0$ converges uniformly, then v_0 must be (uniformly) continuous, which is a contradiction. □

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