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Sharpness of Rickman's Picard theorem in all dimensions

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1. Introduction

By the classical Picard theorem an entire holomorphic map $\mathbb{C}\to\mathbb{C}$ omits at most one point if non-constant. The characteristic example of an entire holomorphic map omitting a point is, of course, the exponential function $z\mapsto e^z$, since every entire holomorphic map $\mathbb{C}\to\mathbb{C}$ omitting a point factors through the exponential map.

Liouville's theorem asserts that all entire conformal maps $\mathbb{R}^n\to\mathbb{R}^n$ are Möbius transformations and, in particular, homeomorphisms for $n\geqslant 3$. This rigidity of spatial conformal geometry no longer persists in quasiconformal geometry. Reshetnyak in the late 1960s and Martio–Rickman–Väisälä in the early 1970s showed that the rich theory of *mappings of bounded distortion*, or so-called *quasiregular mappings*, is a natural replacement for holomorphic functions in higher dimensions. This advancement raised the question of the existence of Picard-type theorems for quasiregular mappings; see e.g. Zorich [22] or Väisälä's survey [20].

Already in his 1967 paper [22] Zorich gave an example of a quasiregular mapping $\mathbb{R}^n\to\mathbb{R}^n$ omitting the origin. This so-called *Zorich map* is the natural higher-dimensional analogue of the exponential function although the mapping is not a local homeomorphism. The branching of the map cannot be avoided by Zorich's *global homeomorphism theorem* from the same article: *For $n\geqslant 3$, quasiregular local homeomorphisms $\mathbb{R}^n\to\mathbb{R}^n$ are homeomorphisms.* Recall that, by Reshetnyak's theorem, quasiregular mappings are (generalized) branched covers, that is, discrete and open mappings and hence local homeomorphisms modulo an exceptional set of (topological) codimension at least 2; we refer to Rickman's monograph [16] for the general theory of quasiregular mappings.

A counterpart of Picard's theorem for quasiregular mappings is due to Rickman [14]: *Given $K>1$ and $n\geqslant 2$ there exists q depending only on K and n so that a non-constant K-quasiregular mapping $\mathbb{R}^n\to\mathbb{R}^n$ omits at most q points.* The sharpness of Rickman's Picard theorem is known in dimension $n=3$ and is also due to Rickman. In [15] he shows the following existence result: *Given any finite set P in $\mathbb{R}^3$ there exists a quasiregular mapping $\mathbb{R}^3 \to \mathbb{R}^3$ omitting exactly P.*

Holopainen and Rickman generalized the Picard theorem to quasiregular mappings into manifolds with many ends in [5] and a fortiori to quasiregular mappings between manifolds in [7]; note also similar results in the sub-Riemannian geometry [6]. These results stem from potential-theoretic proofs of Rickman's Picard theorem due to Lewis [9] and Eremenko–Lewis [3]. It can be said that the ramifications of these methods are now well understood. Recently, Rajala generalized Rickman's Picard theorem to mappings of finite distortion [13]. Whereas the aforementioned potential-theoretic methods are difficult to adapt to this more general class of mappings, Rajala shows that value distribution theory based on modulus methods is still at our disposal.

The sharpness of these theorems, however, is still mostly unknown and Rickman's 3-dimensional construction in [15] provides essentially the only method to produce examples.

In this article we show the precision of Rickman's Picard theorem in all dimensions.

THEOREM 1.1. *Given $n \geqslant 3$, $q \geqslant 2$, and points $y_1, ..., y_q \in \mathbb{R}^n$ there exists a quasiregular mapping $\mathbb{R}^n \to \mathbb{R}^n$ omitting exactly $y_1, ..., y_q$.*

It has already been mentioned that the case of dimension $n=3$ was settled by Rickman. For $n=2$ the number of omitted points is at most 1 by Picard's theorem and the Stoïlow factorization; see e.g. the book by Astala, Iwaniec, and Martin [1, §5.5]. As discussed above, the case $q=1$ is given by the Zorich map for all $n \geqslant 3$. Therefore we may restrict to the cases $n \geqslant 4$ and $q \geqslant 2$. However, it is natural to include $n=3$.

As will become apparent in the following outline of the proof, the proof of Theorem 1.1 is independent of the analytic theory of quasiregular mappings.

The general outline follows the idea of Rickman's construction in [15] and both proofs stem from PL-topology. Rickman's original method relies on a very delicate deformation theory of 2-dimensional branched covers ([15, §5]) which leads to an extension theory of 2-dimensional branched covers; we refer to [2] for an exposition on Rickman's main ideas. These arguments rely essentially on the discrete nature of the branch set in dimension 2. Already when $n=3$, the corresponding deformation theory is much more complicated due to the non-trivial topology of the branch set; see however an application of Piergallini's method in [12] to obtain a quasiregular map $\mathbb{R}^4 \to \mathbb{S}^2 \times \mathbb{S}^2 \# \mathbb{S}^2 \times \mathbb{S}^2$ in [17]. We are not aware of a similar deformation theory, based on a detailed analysis of the branch set, in higher dimensions.

The required extension theory is, however, essentially trivial in all dimensions for BLD-mappings. Recall that a mapping $f: X \to Y$ between metric spaces X and Y is a *mapping of bounded length distortion* (or a BLD-*map*, for short) if f is open and discrete, and there exists a constant $L\geqslant 1$ satisfying

$$\frac{1}{L}\ell(\gamma)\leqslant \ell(f\circ\gamma)\leqslant L\ell(\gamma) \tag{1.1}$$

for all paths γ in X, where $\ell(\gamma)$ is the length of γ. We refer to the seminal paper of Martio and Väisälä [11] for the discussion of the special rôle of BLD-mappings among quasiregular mappings; see also Heinonen–Rickman [4] for the metric theory.

The BLD-theory in the proof of Theorem 1.1 brings forth an alternative, and slightly stronger, formulation. We denote by $\mathbb{S}^n$ and $\mathbb{S}^{n-1}$ the Euclidean unit spheres in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, respectively, and by $B^n(y,\delta)$ the metric ball in $\mathbb{S}^n$ in the inherited metric.

THEOREM 1.2. *Let $n\geqslant 3$, $p\geqslant 2$, and $y_0,...,y_p$ be points in $\mathbb{S}^n$. Let also g be a Riemannian metric on $M:=\mathbb{S}^n\setminus\{y_0,...,y_p\}$ for which $B^n(y_i,\delta)\setminus\{y_i\}$ is isometric, in metric g, to $\mathbb{S}^{n-1}(\delta)\times(0,\infty)$ for some $\delta>0$ and all $0\leqslant i\leqslant p$. Then there exists a surjective* BLD-*mapping $\mathbb{R}^n\to(M,g)$.*

Theorem 1.2 clearly yields Theorem 1.1 as a corollary. Indeed, let $y_1,...,y_q$ be points in $\mathbb{R}^n$. After identifying $\mathbb{R}^n$ with $\mathbb{S}^n\setminus\{e_{n+1}\}$ by stereographic projection, we may fix a Riemannian metric g on $M:=\mathbb{S}^n\setminus\{e_{n+1},y_1,...,y_q\}$ and a BLD-mapping $f:\mathbb{R}^n\to(M,g)$ as in Theorem 1.2. It is now easy to verify that the identity map $(M,g)\to\mathbb{S}^n\setminus\{e_{n+1},y_1,...,y_q\}$ is quasiconformal. Thus $f:\mathbb{R}^n\to\mathbb{R}^n\setminus\{y_1,...,y_q\}$ is quasiregular.

We are not aware of other methods of producing examples of BLD-mappings from $\mathbb{R}^n$ into Riemannian manifolds with many ends.

1.1. Outline of the proof

Using the framework of Theorem 1.2, we outline the construction of a BLD-map $F:\mathbb{R}^n\to\mathbb{S}^n\setminus\{y_0,...,y_p\}$ for $p>2$, and again identify $\mathbb{R}^n$ with $\mathbb{S}^n\setminus\{e_{n+1}\}$ by stereographic projection. It is no restriction to assume that $y_0=e_{n+1}$ and $y_i=(0,t_i)\in\mathbb{R}^{n-1}\times\mathbb{R}\subset\mathbb{S}^n$ for $-1<t_1<t_2<...<t_p<1$ and we will assume so from now on.

Setting aside geometric aspects of the construction, we give first the topological description of $F:\mathbb{R}^n\to\mathbb{S}^n\setminus\{y_0,...,y_p\}$. This description is based on certain essential partitions of $\mathbb{R}^n$ and $\mathbb{S}^n$. Given a closed set X in $\mathbb{R}^n$ (or in $\mathbb{S}^n$), we say that a finite collection of closed sets $X_1,...,X_m$ forms an *essential partition* of X if $X_1\cup...\cup X_m=X$ and the sets X_i have pairwise disjoint interiors.

In the target $\mathbb{S}^n\setminus\{y_0,...,y_p\}$, we fix an essential partition $E_0,...,E_p$ of $\mathbb{S}^n$ into n-cells for which $y_i\in\operatorname{int}E_i$ for each $0\leqslant i\leqslant p$ and so that $E_1\cup...\cup E_p=\overline{B}^n$ and $E_0=\mathbb{S}^n\setminus B^n$.

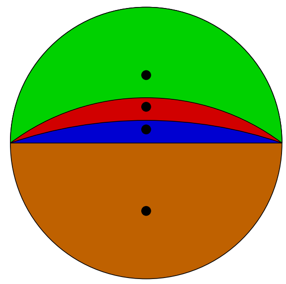

Figure 1. Cells $E_1, ..., E_4$ with (marked) points $y_1, ..., y_4$ for $p=4$ (and $n=2$).

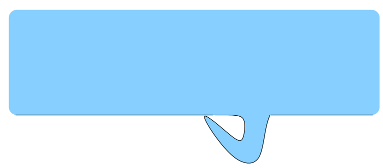

Figure 2. A half-space modulo boundary.

We also assume that, for all i (mod $p+1$), $E_{i-1}\cap E_i\cap E_{i+1}=\mathbb{S}^{n-2}$ and $E_i\cap E_{i+1}$ is an $(n-1)$-cell; see Figure 1. Set $\mathbf{E}=(E_0, ..., E_p)$.

The F-induced essential partition of $\mathbb{R}^n$ is more complicated. Set $\mathbb{R}^n_+=\mathbb{R}^{n-1}\times[0,\infty)$.

Let $E'\subset\mathbb{R}^n$ be a closed set satisfying $E'=\mathrm{cl}(\mathrm{int}\,E')$. A mapping $\varphi:\mathbb{R}^n_+\to E'$ is a *homeomorphism modulo boundary* if $\varphi|_{\mathrm{int}\,\mathbb{R}^n_+}:\mathbb{R}^{n-1}\times(0,\infty)\to\mathrm{int}\,E'$ is a homeomorphism and, for every branched cover $\psi:\partial E'\to\mathbb{S}^{n-1}$, the mapping $\psi\circ\varphi|_{\partial\mathbb{R}^n_+}:\mathbb{R}^{n-1}\times\{0\}\to\mathbb{S}^{n-1}$ is a branched cover. Furthermore, we say that E' is a *half-space modulo boundary* if there exists a homeomorphism modulo boundary $\varphi:\mathbb{R}^n_+\to E'$. Note that $\partial E'$ need not be homeomorphic to $\mathbb{R}^n_+$; see Figure 2.

Suppose, for the sake of argument, there is an essential partition $\Omega_0, ..., \Omega_p$ of $\mathbb{R}^n$ into closed sets and each Ω_i has an essential partition $\Omega_{i,1}, ..., \Omega_{i,j_i}$ into half-spaces modulo boundary. We reduce first the existence of a branched cover $F:\mathbb{R}^n\to\mathbb{S}^n\setminus\{y_0, ..., y_p\}$ to an existence of a branched cover $f:\partial_\cup\mathbf{\Omega}\to\partial_\cup\mathbf{E}$ satisfying $f(\partial\Omega_{i,j})=\partial E_i$. Here, and in what follows, the notation

$$\partial_\cup\mathbf{X}=\bigcup_{i\neq j}X_i\cap X_j$$

is used whenever $\mathbf{X}=(X_0, ..., X_p)$ is an essential partition.

Let $f:\partial_\cup\mathbf{\Omega}\to\partial_\cup\mathbf{E}$ be a branched cover satisfying the additional condition that $f(\partial\Omega_{i,j})=\partial E_i$ for every $i=0, ..., p$ and $1\leqslant j\leqslant j_i$. Since $\Omega_{i,j}$ is a half-space modulo boundary and E_i is an n-cell, we observe that each branched cover $f_{i,j}=f|_{\partial\Omega_{i,j}}$ extends to a branched cover $F_{i,j}:\Omega_{i,j}\to E_i\setminus\{y_i\}$. Indeed, we may fix, for every i and j, a homeomorphism modulo boundary $\varphi_{i,j}:\mathbb{R}^n_+\to\Omega_{i,j}$ as well as a homeomorphism $\psi_i:\mathbb{S}^{n-1}\times[0,\infty)\to E_i\setminus\{y_i\}$. This means that $h_{i,j}=\psi_i^{-1}\circ f_{i,j}\circ\varphi_{i,j}|_{\partial\mathbb{R}^n_+}:\mathbb{R}^{n-1}\times\{0\}\to\mathbb{S}^{n-1}$ is a branched cover. The (trivial) extension $h_{i,j}\times\mathrm{id}:\mathbb{R}^n_+\to\mathbb{S}^{n-1}\times[0,\infty)$ of $h_{i,j}$ now yields the required extension of $f_{i,j}$ after pre- and post-composition with ψ_i and $\varphi_{i,j}^{-1}|_{\mathrm{int}\,\Omega_{i,j}}$, respectively. Thus *$f$ extends to a branched cover $F:\mathbb{R}^n\to\mathbb{S}^n\setminus\{y_1,...,y_p\}$.*

Observe also that in forthcoming constructions we may view $\partial_\cup\mathbf{\Omega}$ and $\partial_\cup\mathbf{E}$ as branched codimension-1 hypersurfaces in $\mathbb{R}^n$ and the map f as a (generalized) Alexander map. In particular, the Zorich map is of this character when $p=2$.

It is crucial that this simple extension is also available for BLD-mappings. It is a simple exercise to observe that the extension $F:\mathbb{R}^n\to\mathbb{S}^n\setminus\{y_0,...,y_p\}$ constructed above will be a BLD-mapping with respect to the Riemannian metric g in $\mathbb{S}^n\setminus\{y_0,...,y_p\}$ if

(i) $f:\partial_\cup\mathbf{\Omega}\to\partial_\cup\mathbf{E}$ is a BLD-map;

(ii) $\varphi_i:\mathbb{R}^n_+\to\Omega_i$ is BLD modulo boundary and $\varphi_i|_{\mathrm{int}\,\mathbb{R}^n_+}$ is an embedding; and

(iii) $\psi_i:\mathbb{S}^{n-1}\times[0,\infty)\to(E_i\setminus\{y_i\},g)$ is bilipschitz.

Here and in what follows, we say that a mapping $\varphi:\mathbb{R}^n_+\to\Omega$, where Ω is a closed set in $\mathbb{R}^n$ with $\Omega=\mathrm{cl}(\mathrm{int}\,\Omega)$, is BLD *modulo boundary* if the restriction $f|_{\mathrm{int}\,\mathbb{R}^n_+}:\mathrm{int}\,\mathbb{R}^n_+\to\mathrm{int}\,\Omega$ is BLD, and for every BLD-map $\psi:\partial\Omega\to\mathbb{S}^{n-1}$, the map $\psi\circ\varphi|_{\partial\mathbb{R}^n_+}:\mathbb{R}^{n-1}\times\{0\}\to\mathbb{S}^{n-1}$ is BLD.

For Riemannian metrics g with cylindrical ends as in Theorem 1.2, it is easy to construct homeomorphisms ψ_i satisfying condition (iii), and so this extension argument reduces the proof of Theorem 1.2 to Theorem 1.3.

A closed set Ω in $\mathbb{R}^n$ is a *Zorich extension domain* if there exists a map $\mathbb{R}^n_+\to\Omega$ which is BLD modulo boundary and a homeomorphism in the interior.

THEOREM 1.3. *Given $n\geqslant 3$ and $p\geqslant 2$ there is an essential partition $\mathbf{\Omega}=(\Omega_0,...,\Omega_p)$ of $\mathbb{R}^n$ for which*

(a) *the sets Ω_i have essential partitions into Zorich extension domains*; *and*

(b) *there exists a* BLD-*map $f:\partial_\cup\mathbf{\Omega}\to\partial_\cup\mathbf{E}$ satisfying $f(\partial\Omega_i)=\partial E_i$ for all $i=0,...,p$.*

Essential partitions $\mathbf{\Omega}$ satisfying both conditions (a) and (b) in Theorem 1.3 are called *Rickman partitions*, since the pairwise common boundary $\partial_\cup\mathbf{\Omega}$ is analogous to the 2-dimensional complex Rickman constructs in [15]. The reader may find it interesting to compare §4 and §5 with [15, §2 and §3].

The partition in Theorem 1.3 is achieved in two stages, with rough Rickman partitions playing an intermediate rôle: an essential partition $\widetilde{\mathbf{\Omega}}=(\widetilde{\Omega}_0,...,\widetilde{\Omega}_p)$ of $\mathbb{R}^n$ is a *rough Rickman partition* if

(a′) each $\widetilde{\Omega}_i$ has an essential partition $(\widetilde{\Omega}_{i,1},...,\widetilde{\Omega}_{i,j_i})$ with each $\widetilde{\Omega}_{i,j}$ being BLD-homeomorphic to $\mathbb{R}^{n-1}\times[0,\infty)$; and (b′) the sets $\partial_\cup\widetilde{\mathbf{\Omega}}$ and $\partial_\cap\widetilde{\mathbf{\Omega}}$ have finite Hausdorff distance, where

$$\partial_\cap\widetilde{\mathbf{\Omega}}=\bigcap_i \widetilde{\Omega}_i$$

is the *common boundary of the partition* $\widetilde{\mathbf{\Omega}}$; $\partial_\cup\widetilde{\mathbf{\Omega}}$ is called the *pairwise common boundary of* $\widetilde{\mathbf{\Omega}}$.

In general, rough Rickman partitions $\widetilde{\mathbf{\Omega}}$ do not admit branched covers $\partial_\cup\widetilde{\mathbf{\Omega}}\to\partial_\cup\mathbf{E}$. To refine our rough Rickman partition $\widetilde{\mathbf{\Omega}}$ to a Rickman partition $\mathbf{\Omega}$, we impose an additional compatibility condition, called the tripod property; see Definition 4.4 for its precise formulation. These particular rough Rickman partitions, together with a modification of Rickman's *sheet construction* in [15, §7], then yield the required global partition $\mathbf{\Omega}$.

In Rickman's original terminology, the construction of rough Rickman partitions is called the *cave construction* and the notion of *cave bases* corresponds to the subdivisions provided by the tripod property.

We summarize the two parts of the proof of Theorem 1.3 as follows. First, we prove the existence of suitable rough Rickman partitions by direct construction.

THEOREM 1.4. *Given $n\geqslant 3$ and $p\geqslant 2$ there exists a rough Rickman partition $\widetilde{\mathbf{\Omega}}=(\widetilde{\Omega}_0,...,\widetilde{\Omega}_p)$ supporting the tripod property.*

As in [15] we begin the proof of Theorem 1.4 by partitioning $\mathbb{R}^n$ with an essential partition $\mathbf{\Omega}'=(\Omega_1',\Omega_2',\Omega_3')$ with Ω_1' and Ω_2' BLD-homeomorphic to $\mathbb{R}^{n-1}\times[0,\infty)$ and Ω_3' having a partition $(\Omega_{3,1}',...,\Omega_{3,2^{n-1}}')$ into pairwise disjoint sets, where each $\Omega_{3,j}'$ is BLD-homeomorphic to $\mathbb{R}^{n-1}\times[0,\infty)$. All sets Ω_i' are unions of unit n-cubes $[0,1]^n+v$, where $v\in\mathbb{Z}^n$, and $(\Omega_1',\Omega_2',\Omega_3')$ satisfies the tripod property. This occupies §5. The final step in the proof of Theorem 1.4 is a generalization of this argument. This step is discussed in §8; see Proposition 8.1.

The essential partition $\widetilde{\mathbf{\Omega}}$ (as well as $\mathbf{\Omega}'$) has the following geometric property. Let X be any of the sets $\widetilde{\Omega}_0$, $\widetilde{\Omega}_1$ or $\widetilde{\Omega}_{2,j}$ for some $1\leqslant j\leqslant 2^{n-1}$, and for each $k\geqslant 0$ write

$$X_k=3^{-k}X.$$

By passing to a subsequence if necessary, the sets X_k and their boundaries $\partial X_k\subset\mathbb{S}^n$ converge in the Hausdorff sense respectively to X_∞ and ∂X_∞, where ∂X_∞ is a "generalized Alexander horned sphere in $\mathbb{S}^n$ with infinitely many horns". Under the normalization $\widetilde{\mathbf{\Omega}}_k=3^{-k}\widetilde{\mathbf{\Omega}}$ for $k\geqslant 0$, in fact $\partial_\cup\mathbf{\Omega}_\infty=\partial_\cap\mathbf{\Omega}_\infty$ for any sublimit $\mathbf{\Omega}_\infty$ of the partitions $\widetilde{\mathbf{\Omega}}_k$, in the Hausdorff sense. This may be considered a *coarse Lakes of Wada* property for the pairwise common boundary of $\widetilde{\mathbf{\Omega}}$. Of course, this observation applies also to Rickman's original cave construction. We do not discuss this feature of $\widetilde{\mathbf{\Omega}}$ in more detail, and leave these details to the interested reader.

The second part of the proof of Theorem 1.3 is the refinement of rough Rickman partitions to Rickman partitions. This formalizes the effect of the sheet construction (called *pillows* in §7) as follows.

PROPOSITION 1.5. *Given a rough Rickman partition* $\widetilde{\mathbf{\Omega}}=(\widetilde{\Omega}_0, ..., \widetilde{\Omega}_p)$ *supporting the tripod property, there exists a Rickman partition* $\mathbf{\Omega}=(\Omega_0, ..., \Omega_p)$ *for which the Hausdorff distance of* $\partial_\cup \mathbf{\Omega}$ *and* $\partial_\cup \widetilde{\mathbf{\Omega}}$ *is at most* 1.

We do not explore the geometry of the domains $\Omega_0, ..., \Omega_p$ further. However, we do observe that the domains in the Rickman partition can be taken to be uniform domains; see Corollaries 5.2 and 8.9.

As discussed in this introduction, Theorem 1.4 and Proposition 1.5 together prove Theorem 1.3, and we obtain Theorem 1.2 from Theorem 1.3 and the observation on the existence of BLD-extensions.

Acknowledgments

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2. Preliminaries

In this section we introduce general metric and combinatorial notions used in the construction. Most of the discussion is in the ambient space $\mathbb{R}^n$ for some fixed $n\geqslant 3$.

2.1. Metric notions

In $\mathbb{R}^n$, let d_∞ be the sup-metric

$$d_\infty(x,y)=\|x-y\|_\infty$$

given by the *supremum norm*

$$\|(x_1, ..., x_n)\|_\infty = \max_i |x_i|.$$

The metric ball $B_\infty(p,r)=\{x\in\mathbb{R}^n:\|p-x\|_\infty<r\}$ of radius $r>0$ about $p\in\mathbb{R}^n$ in this metric then is the open cube

$$B_\infty(p,r)=p+(-r,r)^n.$$

Similarly, $\overline{B}_\infty(p,r)=p+[-r,r]^n$.

Diverting from standard terminology, we apply the term *cube* exclusively to closed n-balls $\overline{B}_\infty(p,r)$. The point p is the center of the cube $\overline{B}_\infty(p,r)$ and of course the *side-length* of $\overline{B}_\infty(p,r)$ is $2r$.

The set E in $\mathbb{R}^n$ is *rectifiably connected* if for every $x,y\in E$ there exists a path $\gamma:[0,1]\to E$ of finite length so that $x,y\in\gamma[0,1]$. In this situation, γ *connects x and y in E*. When E is rectifiably connected in $\mathbb{R}^n$, d_E is its *inner metric* in E; that is, for all $x,y\in E$,

$$d_E(x,y)=\inf_\gamma \ell(\gamma),$$

over all paths γ connecting x and y in E, with $\ell(\gamma)$ being the length of γ. Note that the length of γ is in terms of Euclidean distance. The notion of inner metric gives the following characterization of BLD-homeomorphisms: *A homeomorphism $f:E\to E'$ between rectifiably connected sets E and E' in $\mathbb{R}^n$ is* BLD *if and only if $f:E\to E'$ is bilipschitz in the inner metric.*

2.2. Complexes

For a detailed discussion on simplicial complexes we refer to [8] and [18] and merely recall some notation and terminology. Given a simplicial complex P in $\mathbb{R}^n$, $P^{(k)}$ is its *k-skeleton*, that is, the collection of all k-simplices in P. If m is the largest dimension of simplices in P, then P has *dimension m*, $m=\dim P$. We consider only *homogeneous* simplicial complexes, that is, every simplex in P is contained in a simplex of dimension $\dim P$. We denote by $|P^{(k)}|$ the subset in $\mathbb{R}^n$ which is the union of all simplices in $P^{(k)}$; thus, $|P|=|P^{(m)}|$.

Recall that every k-simplex σ has a standard structure as a simplicial complex having σ as its only k-simplex and the vertices of σ as the 0-skeleton. The i-simplices of this complex form the *i-faces* of σ.

We mainly consider cubical complexes. Much as simplices have a natural structure as a complex, the k-dimensional faces of a cube $Q=\overline{B}_\infty(x,r)$ determine a natural CW complex structure for Q. The k-dimensional faces of Q are *k-cubes*, and a CW complex P is a *cubical complex* if its cells are cubes. Note in particular, that given an i-cube Q and a j-cube Q' the intersection $Q\cap Q'$ is a k-dimensional face of both cubes, $k\leqslant\min\{i,j\}$. The k-skeleton and its realization are defined for cubical complexes in a manner analogous to simplicial complexes.

A homogeneous cubical complex of dimension k is usually referred to as a *cubical k-complex*. A set $E\subset\mathbb{R}^n$ is a *cubical k-set* if there is a cubical k-complex P with $|P|=E$. Cubical k-sets E and E' are *essentially disjoint* if $E\cap E'$ is a cubical set of lower dimension. Given two cubical sets E and E', write

$$E-E'=\mathrm{cl}(E\setminus E'),$$

where $\mathrm{cl}(E\setminus E')$ is the closure of $E\setminus E'$. Clearly, $E-E'=E$ if E' has lower dimension than E.

A cubical k-complex P is *r-fine* if all k-cubes in P have side-length r, i.e. are congruent to $[0,r]^k\subset\mathbb{R}^k\subset\mathbb{R}^n$. Similarly, a set E in $\mathbb{R}^n$ is *r-fine* if $r>0$ is the largest integer for which there exists an r-fine cubical complex P with $E=|P|$, and r is called the *side-length* $\varrho(E)$ of E. In what follows, we assume that all cubical complexes are r-fine for some integer $r>0$. Given an r-fine set $E=|P|$, we tacitly assume that its underlying complex P is also r-fine.

Let P be a $3k$-fine cubical n-complex for $k\geqslant 1$, and $\Omega=|P|$. We denote by Ω^* the subdivision of Ω into cubes of side-length 3. More formally, there exists a unique 3-fine cubical n-complex $\widetilde{P}$ satisfying $\Omega=|\widetilde{P}|$; we set $\Omega^*=\widetilde{P}^{(n)}$ and refer to Ω^* as the *3-fine subdivision* of Ω. We will also need $\Omega^\#$, the *1-fine subdivision* of Ω, i.e. subdivision of Ω into unit cubes, and call $\Omega^\#$ the *unit subdivision* of Ω. In what follows, if $A\subset\mathbb{R}^m$ and $r>0$, we write

$$rA=\{rx\in\mathbb{R}^n : x\in A\}.$$

2.3. Essential partitions

Cubical k-sets $U_1 ..., U_m$ induce the *essential partition* $\{U_1, ..., U_m\}$ of the cubical set U if $U=U_1\cup...\cup U_m$ and the sets U_i are pairwise essentially disjoint. If the sets $U_1, ..., U_m$, and U are n-cells, we usually consider the essential partition ordered and denote it by $\mathbf{U}=(U_1, ..., U_m)$ as in the introduction.

To simplify notation, for $r>0$ we also set $r\mathbf{U}=(rU_1, ..., rU_m)$, and given an n-cell $E\subset U$, write $\mathbf{U}\cap E=(U_1\cap E, ..., U_m\cap E)$ and $\mathbf{U}-E=(U_1-E, ..., U_m-E)$.

2.4. Graphs, forests, and adjacency

The pair $G=(V,E)$ is *a graph* if V is a countable set and E is a collection of unoriented pairs of points in V; V is the set of *vertices* and E the set of *edges* of G. Note that we only allow one edge between two distinct vertices and, in particular, our graphs do not have *loops*, i.e. edges from a vertex to itself.

We use repeatedly the standard fact that a graph contains a maximal tree, that is, given a graph $G=(V,E)$ there is a subtree $T=(V,E')$ containing all vertices of G. The *length* $\ell(G)$ of G is the number of vertices of G, the *valence* of G at v is $\nu(G,v)$ and $\nu(G)=\max_{v\in G}\nu(G,v)$ is the (*maximal*) *valence* of G. We denote by $d_G(v,v')$ the graph distance of v and v' in G, that is, the length of the shortest edge path between v and v' in G.

Given a distinguished vertex $v\in G$, the pair (G,v) is called a *rooted graph* and v the *root* of this graph. The *radius* $r(G,v)$ of G at v is the largest graph distance between v and a leaf of G; a vertex $w\in G$ is a *leaf* if it belongs to exactly one edge, or equivalently, has valence 1. A vertex which is neither a leaf nor the root is an *inner vertex.* A subtree $\Gamma\subset G$ connecting the root v to a leaf w of G is a *branch* when all vertices in Γ other than v and w have valence 2.

Let (G,v) be a finite rooted tree and $v'\neq v$ be a vertex in G. We define the *subtree behind* v' *in* (G,v) as follows. Since G is a tree, there exists a unique $v''\in G$ for which $e=\{v'',v'\}$ is the last edge in the shortest path from v to v'. The graph $(V,E\setminus\{e\})$ has two connected components Γ_v and $\Gamma_{v'}$ containing v and v', respectively. Both components are trees; $\Gamma_{v'}$ is the *subtree behind* v' *in* (G,v).

A graph G is a *forest* if all of its components are trees. A forest $F\subset G$ is *maximal* if components of F are maximal trees in components of G and F contains all vertices of G.

A function $u\colon G\to\mathbb{R}$ on a tree G has the *John property in* G if given v and v' in G there exists $0\leqslant j\leqslant d=d_G(v,v')$ so that u is (strictly) increasing on $v_0,...,v_j$ and (strictly) decreasing on $v_{j+1},...,v_d$, where $v=v_0,v_1,...,v_d=v'$ is the unique shortest edge path from v to v' in G.

Most graphs we consider are adjacency graphs of collections of k-cells in $\mathbb{R}^n$. A set $E\subset\mathbb{R}^n$ is a *k-cell* if E is homeomorphic to the closed cube $[0,1]^k$ in $\mathbb{R}^k$; E is a *cubical k-cell* if for some $r\geqslant 1$ there is an r-fine homogeneous cubical complex P for which $E=|P|$.

Two k-cells E and E' are *adjacent* if $E\cap E'$ is a $(k-1)$-cell. We recall from PL-theory that given two adjacent PL k-cells E and E' there exists a PL-homeomorphism $E\cup E'\to E$ which is identity on $\partial(E\cup E')\cap E$, and refer to [8] or [18, Chapter 3] for this and similar results in PL-theory.

A collection $\mathcal{P}$ of k-cells in $\mathbb{R}^n$ has the adjacency graph

$$\Gamma(\mathcal{P})=(\mathcal{P},\{\{E,E'\}:E\in\mathcal{P}\text{ and }E'\in\mathcal{P}\text{ are adjacent}\}).$$

Given a subgraph $\Gamma\subset\Gamma(P)$, we write $|\Gamma|=\bigcup_{E\in\Gamma}E$; in particular, $|\Gamma(P)|=|P|$.

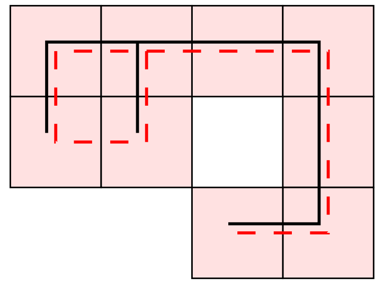

Figure 3. A cubical 2-complex with its adjacency graph and a choice of a maximal tree.

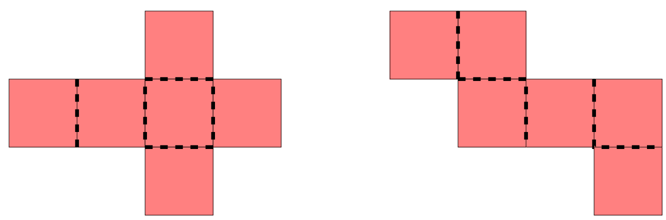

Figure 4. Two different fold-outs of faces of a 3-cube along non-isomorphic maximal trees.

2.5. Remarks on figures

Although we consider n-cells for $n\geqslant 3$, we use 2-dimensional illustrations related to 3-dimensional example configurations, so that often a 3-dimensional situation is seen relative to one of its faces. The figures displayed here often have orientations different than suggested by their coordinates in $\mathbb{R}^3$.

In particular, 'fold-out' diagrams illustrate particular cubical $(n-1)$-complexes. To formalize this, suppose E is a cubical $(n-1)$-cell in $\mathbb{R}^n$ with an essential partition $\{E_1, ..., E_s\}$ into unit $(n-1)$-cubes and let Γ be a maximal tree in $\Gamma(\{E_1, ..., E_s\})$. An $(n-1)$-cell E' in $\mathbb{R}^{n-1}$ then is a *fold-out* of E (*along* Γ) if E' has a partition $\{E'_1, ..., E'_s\}$ with adjacency graph $\Gamma(\{E'_1, ..., E'_s\})$ isomorphic to Γ and there exists a map $\psi: E'\to E$ which sends each cube E'_i isometrically to E_i. We call ψ a *bending* of E'. Sometimes, as in Figure 4, a fold will be indicated by a dashed line.

Fold-out figures, in particular, illustrate 3-cells contained in 3-cubes. Most of our figures of this type, e.g. in §4 and §5, are akin to the following two simple examples.

Consider the cube $Q=[0,3]^3$. Then $F=[0,3]^2\times\{0\}$ is a face of Q and the unit cube $q=[1,2]^2\times[0,1]$ is contained in Q and meets F in the face $f=[1,2]^2\times\{0\}$. We illustrate the fact that q meets F by identifying f in F as in Figure 5.

In our second example, Q and F remain the cube $[0,3]^3$ and its face $[0,3]^2\times\{0\}$ respectively, but $q=[0,1]\times[1,2]\times[0,1]$. Let also F' be the face $\{0\}\times[0,3]^2$ of Q. Then

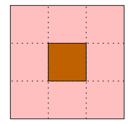

Figure 5. The cube q in Q realized as a square f in F.

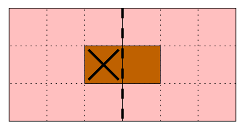

Figure 6. The cube q in Q meeting faces $F \cup F'$.

$q \cap (F \cup F')$ is a union of two faces $f = [0,1] \times [1,2] \times \{0\}$ and $f' = \{0\} \times [1,2] \times [0,1]$ of q. To indicate how q meets $F \cup F'$ in more than one face, we use the symbol '$\times$' to indicate one of the two faces which correspond to q in Q as in Figure 6.

3. Atoms and molecules

In this section we discuss the elementary BLD-theory of certain cubical n-cells. We call these classes of cells *atoms*, *molecules*, *dented atoms* and *dented molecules*.

Definition 3.1. We say that a cubical n-cell $A = |P|$ in $\mathbb{R}^n$ is an *atom* of *length* ℓ if A is r-fine and the adjacency graph $\Gamma(P)$ is a tree of length ℓ.

Given an atom $A = |P|$, we denote its length by $\ell(A)$; i.e. $\ell(A) = \ell(\Gamma(P))$. Note also that every r-fine atom A has a uniquely determined r-fine complex P_A with $A = |P_A|$.

Clearly, by finiteness of adjacency trees, every r-fine atom of length ℓ is uniformly L-bilipschitz to the n-cube $[0,r]^n$, with L depending only on n and ℓ. In what follows, we define more complicated cells, using atoms as building blocks. The hierarchy between atoms in these constructions is given by the notion of proper adjacency. The atoms $A = |P|$ and $A' = |P'|$ are *properly h-adjacent*, for $h > 1$, if

(1) the side-lengths of A and A' satisfy $\varrho(A) \geqslant h\varrho(A')$ or $\varrho(A') \geqslant h\varrho(A)$; and

(2) there exist n-cubes $Q \in P^{(n)}$ and $Q' \in (P')^{(n)}$ for which $A \cap A' = Q \cap Q'$.

Let $\mathcal{A}$ be a finite collection of properly adjacent atoms so that $\Gamma(\mathcal{A})$ is a tree. Suppose also that $|\Gamma(\mathcal{A})|$ is *John*, that is, the function $A \mapsto \varrho(A)$ is a John function on $\Gamma(\mathcal{A})$, so there is a unique $\hat{A} \in \mathcal{A}$ with $\varrho(\hat{A}) = \max_{A \in \mathcal{A}} \varrho(A)$, called the *root* of $\mathcal{A}$.

We exploit the John property to produce bilipschitz mappings from $|\Gamma(\mathcal{A})|$ to proper subdomains of $|\Gamma(\mathcal{A})|$. In particular, we construct bilipschitz maps $|\Gamma(\mathcal{A})| \to \hat{A}$, where $\hat{A}$

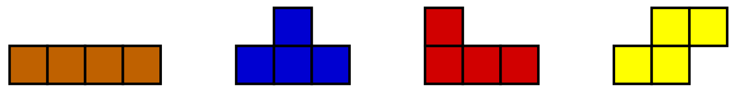

Figure 7. Some atoms of length 4.

is the root of $\mathcal{A}$. To obtain uniform bounds for the bilipschitz constants, we define a collapsibility condition and introduce a class of n-cells called molecules; see Proposition 3.5 for the first bilipschitz contractibility statement for molecules.

Let $A\in\Gamma(\mathcal{A})$ be an inner vertex in $(\Gamma(\mathcal{A}),\hat{A})$ and let $\mathcal{N}(A)$ be neighbors of A in $\Gamma(\mathcal{A})$. For each $a\in\mathcal{N}(A)$, let $q_a\in P_a^{(n-1)}$ be the unique cube satisfying $q_a\cap A=a\cap A$, and denote by $F_A(a)$ the face of q_a containing $q_a\cap A$. Note that, since $|\Gamma(\mathcal{A})|$ is John, there exists a unique $A'\in\mathcal{N}(A)$ so that $\varrho(A')>\varrho(A)$.

Definition 3.2. Let $A\in\Gamma(\mathcal{A})$ be an inner vertex in $(\Gamma(\mathcal{A}),\hat{A})$ and A' be a neighbor of A with $\varrho(A')>\varrho(A)$. The vertex A is λ-*collapsible for* $\lambda>1$ if there exists a collection

$$\{f_a\subset F_A(A'): a\in\mathcal{N}(A)\setminus\{A'\}\}$$

of essentially pairwise disjoint $(n-1)$-cubes with $\varrho(f_a)=\lambda\varrho(F_A(a))$.

Definition 3.3. Let $M=|\Gamma(\mathcal{A})|=\bigcup_{A\in\mathcal{A}}A$ be a cubical n-cell having an essential partition into a finite collection $\mathcal{A}$ of atoms, and let $\nu\geqslant1$ and $\lambda>1$. Then M is a (ν,λ)-*molecule* if

(a) the adjacency graph $\Gamma(\mathcal{A})$ is a tree;
(b) adjacent atoms in $\mathcal{A}$ are properly 3-adjacent;
(c) $\Gamma(\mathcal{A})$ is John;
(d) $\Gamma(\mathcal{A})$ has valence at most ν; and
(e) each inner vertex of $(\Gamma(\mathcal{A}),\hat{A})$ is λ-collapsible, where $\hat{A}$ is the root of $\mathcal{A}$.

Remark 3.4. By (c), $M=|\Gamma(\mathcal{A})|$ is a John domain; see e.g. [10] or [21] for terminology.

Let $M=|\Gamma(\mathcal{A})|$ be a molecule. By (b), the atoms in $\mathcal{A}$ and the tree $\Gamma(\mathcal{A})$ are uniquely determined. The tree $\Gamma(M)=\Gamma(\mathcal{A})$ is the *atom tree* of M, and the root $\hat{A}$ of $\mathcal{A}$ is called the *root* of M. The tree $\Gamma^{\text{int}}(M)=\Gamma\big(\bigcup_{A\in\mathcal{A}}P_A^{(n)}\big)$ is the *internal tree* $\Gamma^{\text{int}}(M)$ of M. In addition,

$$\ell_{\text{atom}}(M)=\max_{A\in\Gamma(\mathcal{A})}\ell(A)$$

is the *atom length* of M, and

$$\ell(M)=\ell(\Gamma(\mathcal{A}))$$

is the (*external*) *length* of M. The (*maximal*) *side-length* of M is

$$\varrho(M)=\max_{A\in\Gamma(\mathcal{A})}\varrho(A).$$

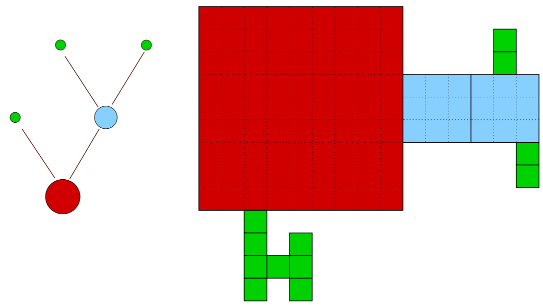

Figure 8. Example of a tree $\Gamma(\mathcal{A})$ and the molecule $M=|\Gamma(\mathcal{A})|$.

The main result on molecules is the following bilipschitz contraction property.

PROPOSITION 3.5. *Let M be a (ν,λ)-molecule with root $\hat{A}$ in $\mathbb{R}^n$. Then there exists an L-bilipschitz homeomorphism*

$$\phi\colon (M,d_M) \longrightarrow (\hat{A},d_{\hat{A}})$$

which is the identity on $\hat{A}\cap\partial M$, where L depends only on n, ν, λ, and $\ell_{\mathrm{atom}}(M)$.

This proposition should not surprise any expert. Its proof is based on the bounded local structure of $\Gamma(M)$ and bilipschitz equivalence of atoms of uniformly bounded length. Due to the specific nature of the statement and its fundamental rôle in our arguments, we discuss its proof in detail. We gratefully acknowledge work of Semmes, especially [19], as the main source of these ideas.

The proof of Proposition 3.5 is by induction on the size of the tree $\Gamma(M)$. We begin with a lemma corresponding to the induction step of this proof. Given sets X and Y in $\mathbb{R}^n$, the set

$$X\star Y=\{tx+(1-t)y\in\mathbb{R}^n : x\in X, y\in Y \text{ and } t\in[0,1]\},$$

is the *join* of X and Y. If Q is an n-cube in $\mathbb{R}^n$, x_Q is its *barycenter*, that is, $Q=\overline{B}_\infty(x_Q,r_Q)$, where $r_Q>0$. For an $(n-1)$-cube F, the *barycenter* x_F is defined as the average of the vertices of F. The definitions coincide for n-cubes.

LEMMA 3.6. *Let Q be an n-cube and let M be a molecule properly adjacent to Q with $\varrho(Q)>\varrho(M)$, and let $\nu\geqslant 1$ and $\lambda>1$.*

Let F be the face of Q containing $M\cap Q$ and let $F_1,...,F_\nu\subset\partial M-Q$ be pairwise disjoint faces of n-cubes $Q_1,...,Q_\nu$ in $\Gamma^{\mathrm{int}}(M)$, respectively. Suppose there exist essentially pairwise disjoint $(n-1)$-cubes $F_1',...,F_\nu'$ in F satisfying $\varrho(F_i')=\lambda\varrho(F_i)$ for every $i=1,...,\nu$.

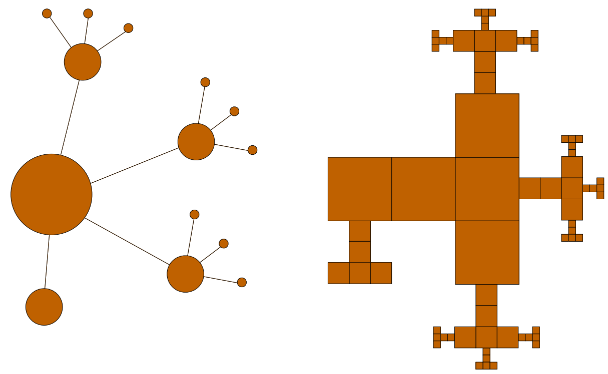

Figure 9. A tree $\Gamma(\mathcal{A})$ and the molecule $M=|\Gamma(\mathcal{A})|$.

Then there exist $L=L(n,\ell_{\text{atom}}(M),\ell(M),\nu,\lambda)\geqslant 1$ *and an* L*-bilipschitz homeomorphism*

$$\phi\colon (M\cup Q, d_{M\cup Q}) \longrightarrow Q,$$

which is the identity on $Q-(F\star\{x_Q\})$ *and an isometry on each* $F_i\star\{x_{Q_i}\}$.

Proof. Let $i\in\{1,...,\nu\}$ and set $F_i''=B_\infty\big(x_{F_i'},\frac{1}{2}\varrho(F_i)\big)\cap F\subset F_i'$. Then F_i'' is an $(n-1)$-cube in F with the same barycenter as F_i' and the same side-length as F_i. We denote by $Q_i''\subset Q$ the n-cube having F_i'' as a face, and set $\Delta_i=F_i\star\{x_{Q_i}\}$ and $\Delta_i''=F_i''\star\{x_{Q_i''}\}$.

By a shelling argument, there exists a PL-homeomorphism $\phi\colon M\cup Q\to Q$ which is the identity in $Q\setminus(F\star\{x_Q\})$ and which restricts to an isometry $\phi|_{\Delta_i}\colon\Delta_i\to\Delta_i''$ for every $i=1,...,\nu$; see e.g. [18, Lemma 3.25]. Since it suffices to consider only a finite number of triangulations and PL-homeomorphisms, ϕ is uniformly bilipschitz with a constant depending only on n, $\ell_{\text{atom}}(M)$, $\ell(M)$, ν, and λ. □

Proof of Proposition 3.5. Let $M=|\Gamma(\mathcal{A})|$ be a (ν,λ)-molecule with root $\hat{A}$; see Figure 9. We may assume that $M\neq\hat{A}$ and, more precisely, that $\Gamma(\mathcal{A})$ has inner vertices, since otherwise the claim follows from Lemma 3.6.

To begin the induction, set $\Gamma_0=\Gamma(\mathcal{A})$, $M_0=M$, and with each leaf $L\in\Gamma_0$ associate a face F_L of an n-cube $Q_L\in\Gamma^{\text{int}}(L)$ with $F_L\subset\partial M_0\cap L$. We denote the set of these chosen faces by $\mathcal{F}_0$, and for every leaf $L\in\Gamma_0$ set $J_L=F_L\star\{x_{Q_L}\}$.

Fix an atom $A_0'\in\Gamma_0$ which is an inner vertex in Γ_0 so that the rooted subtree $\Gamma_0'=\Gamma_{A_0'}$ behind A_0' in $(\Gamma_0,\hat{A})$ consists of leaves of Γ_0. Also choose an atom $A_0\in\Gamma_0\setminus\Gamma_0'$ adjacent to A_0' in Γ_0. Let Q_0 be the unique n-cube in A_0 and F_0 be the unique face of Q_0 which contains $A_0\cap A_0'$; set $J_0=F_0\star\{x_{Q_0}\}$ and $\mathcal{F}_0'=\{F_L:L\in\Gamma_0'\}$.

Since $M=|\Gamma(\mathcal{A})|$ is a (ν,λ)-molecule and A_0' is an inner vertex in $\Gamma(\mathcal{A})$, A_0' is λ-collapsible. Thus there exists a collection $\{F_L':L\in\Gamma_0'\}$ of pairwise disjoint $(n-1)$-cubes

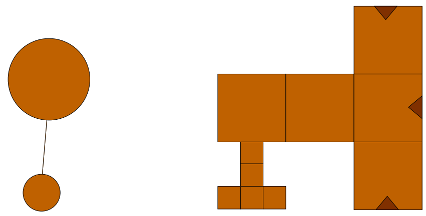

Figure 10. An intermediate tree Γ_i and the cell $|\Gamma_i|$.

satisfying $\varrho(F'_L)=\lambda\varrho(F_L)$ for every $L\in\Gamma'_0$.

By Lemma 3.6, there exist a constant $L\geqslant 1$, depending only on n, ν, δ, $\ell_{\mathrm{atom}}(M)$, and $\ell(M)$, and an L-bilipschitz homeomorphism

$$\phi_0\colon (|\Gamma'_0|\cup Q_0, d_{|\Gamma'_0|\cup Q_0}) \longrightarrow (Q_0, d_{Q_0}),$$

which is the identity on $Q-(F_0\star\{x_{Q_0}\})$ and an isometry on each join J_L for $L\in\Gamma'_0$.

We now define $\Gamma_1=\Gamma_0\setminus\Gamma'_0$ and $\mathcal{F}_1=(\mathcal{F}_0\setminus\mathcal{F}'_0)\cup\{F_0\}$. Then $M_1=|\Gamma_1|$ is a (ν,λ)-molecule with root $\hat{A}$. In terms of this notation, ϕ_0 extends, by identity, to an L-bilipschitz homeomorphism

$$\phi_0\colon (M_0, d_{M_0}) \longrightarrow (M_1, d_{M_1}),$$

which is an isometry on each join $J_L=F_L\star\{x_{Q_L}\}$ for $L\in\Gamma'_0$.

Clearly, $\ell(M_1)<\ell(M_0)$. We iterate this step to obtain a descending sequence of subgraphs $\Gamma_0,...,\Gamma_i$ of $\Gamma(\mathcal{A})$ so that every Γ_j has at least one vertex fewer than Γ_{j-1} for $j=1,...,i$; see Figure 10. Since $\Gamma(\mathcal{A})$ is a finite tree, there exists $i_0\geqslant 1$ depending on $r(\Gamma(\mathcal{A}),\hat{A})$ so that Γ_{i_0} consists of only $\hat{A}$.

For $i=0,...,i_0$, we also obtain collections of faces $\mathcal{F}_0,...,\mathcal{F}_i$ on leaves of graphs $\Gamma_0,...,\Gamma_i$, and L-bilipschitz homeomorphisms

$$\phi_{j-1}\colon (|\Gamma_{j-1}|, d_{|\Gamma_{j-1}|}) \longrightarrow (|\Gamma_j|, d_{|\Gamma_j|})$$

which are isometries on the joins over the faces in $\mathcal{F}_{j-1}$ for every $j=1,...,i_0$. As in the construction above, $\phi_i(|\Gamma_{i-1}|)$ is contained in a join over a face in $\mathcal{F}_i$. Thus

$$\phi_i\circ...\circ\phi_0\colon (|\Gamma_0|, d_{|\Gamma_0|}) \longrightarrow (|\Gamma_i|, d_{|\Gamma_i|})$$

is L-bilipschitz for every $i=0,...,i_0$, where L depends only on n, ν, λ, and $\ell_{\mathrm{atom}}(M)$, and so

$$\phi_{i_0}\circ...\circ\phi_0\colon (|\Gamma_0|, d_{|\Gamma_0|}) \longrightarrow (\hat{A}, d_{\hat{A}})$$

satisfies the conditions of the claim. This concludes the proof. □

Corollary 3.7. *Let $M=|\Gamma(\mathcal{A})|$ be a (ν,λ)-molecule and let $\Gamma\subset\Gamma(\mathcal{A})$ be a subtree containing the root $\hat{A}$ of M. Then there exist an $L\geqslant 1$ depending only on n, ν, λ, and $\ell_{\mathrm{atom}}(M)$, and an L-bilipschitz homeomorphism $\phi\colon(M,d_M)\to(|\Gamma|,d_{|\Gamma|})$ which is the identity on $|\Gamma|\cap\partial M$.*

Proof. Let Γ' be a component of $\Gamma(\mathcal{A})\setminus\Gamma$. Then $|\Gamma|$ is an (ν,λ)-molecule. Thus the claim follows by applying Proposition 3.5 to the components of $\Gamma(\mathcal{A})\setminus\Gamma$ followed by Lemma 3.6 on the roots of these trees. □

Before introducing dented atoms, we record a uniform bilipschitz equivalence result in spirit of Proposition 3.5. A half-space in $\mathbb{R}^n$ appears as the normalized target; full details of the proof are left to the interested reader.

Proposition 3.8. *Let $\nu\geqslant 1$, $\lambda>1$, $\ell\geqslant 1$, and let $\{M_m\}_{m\geqslant 0}$ be an increasing sequence of (ν,λ)-molecules so that, for every $m\geqslant 1$,*

(1) M_m-M_{m-1} is connected and contains the root of M_m;

(2) $\ell_{\mathrm{atom}}(M_m)\leqslant\ell$; and

(3) if A and A' are adjacent in $\Gamma(M_m)$ with $\varrho(A)<\varrho(A')$ then $\varrho(A')=3\varrho(A)$.

Let $M=\bigcup_{m\geqslant 0}M_m$. Then (M,d_M) is L-bilipschitz equivalent to $\mathbb{R}^{n-1}\times[0,\infty)$, where L depends only on n, ν, λ, and ℓ.

Sketch of proof. Let Γ be the tree $\bigcup_{m\geqslant 0}\Gamma(M_m)$, and let Γ' be the unique branch passing through all roots $\widehat{M}_m$ of M_m for $m\geqslant 0$. We may consider Γ' as a sequence of atoms with increasing side-length, and for every $m\geqslant 0$ denote by Γ'_m the part of Γ' contained in $\Gamma(M_m)$.

Following the idea of Corollary 3.7, we obtain a sequence $\{\psi_m\}_{m\geqslant 0}$ of L'-bilipschitz contractions $\psi_m\colon(M_m,d_{M_m})\to(|\Gamma'_m|,d_{|\Gamma'_m|})$ so that $\psi_{m+1}|_{M_m}=\psi_m$ for every $m\geqslant 0$, where L' depends only on n, ν, λ, and ℓ. This produces an L-bilipschitz map $\psi\colon(M,d_M)\to(|\Gamma'|,d_{|\Gamma'|})$.

It remains now to show that $(|\Gamma'|,d_{|\Gamma'|})$ is L''-bilipschitz equivalent to $\mathbb{R}^{n-1}\times[0,\infty)$, where L'' depends only on n and ℓ.

Let A be the unique vertex in Γ' with valence 1. Since Γ' is a branch, we may now enumerate the vertices in Γ' as $A=a_0,a_1,a_2,\dots$ with a_k adjacent to a_{k+1}. By (3), $\varrho(a_{k+1})=3\varrho(a_k)$ for every $k\geqslant 0$. Thus $(|\Gamma'|,d_{|\Gamma'|})$ is L'''-bilipschitz equivalent, $L'''=L'''(n,\ell)$, to a cone

$$\{(x_1,\dots,x_n)\in\mathbb{R}^n: x_n^2\leqslant x_1^2+\dots x_{n-1}^2\},$$

and hence L''-bilipschitz equivalent to $\mathbb{R}^{n-1}\times[0,\infty)$, where L'' depends only on n and ℓ. □

3.1. Dented atoms

Definition 3.9. Let A be an atom in $\mathbb{R}^n$. A molecule M contained in A is *on the boundary* of A if $A-M$ is an n-cell and for each $Q\in\Gamma^{\text{int}}(M)$,

(i) Q is contained in a strictly larger cube of $\Gamma^{\text{int}}(A)$; and

(ii) $Q\cap\partial A$ contains a face of Q.

Definition 3.10. Let A be an atom in $\mathbb{R}^n$ and let $M_1, ..., M_\nu\subset A$ be pairwise disjoint molecules on the boundary of A each having side-length at most $3^{-2}\varrho(A)$. The n-cell $D=A-\bigcup_{i=1}^{\nu} M_i$ is a *dented atom* if

(i) each M_i is contained in an n-cube in $\Gamma(A)$; and

(ii) $\text{dist}(Q,Q')\geqslant\min\{\varrho(Q),\varrho(Q')\}$ for all $Q\in\Gamma^{\text{int}}(M_i)$ and $Q'\in\Gamma^{\text{int}}(M_j)$ for $i\neq j$.

The molecules $M_1, ..., M_\nu$ are called *dents* of A, and A is the *hull* of D, $\text{hull}(D)$.

Remark 3.11. The reader may find the constant 3^{-2} curious, but this explicit constant is chosen to be compatible with the constructions in §5, more specifically, §5.3.1. These constructions also have the property that each cube in $\Gamma(A)$ has at most 2 dents.

By (ii), the hull and the dents of a dented atom are unique. Given a dented atom

$$D=A-\bigcup_{i=1}^{\nu} M_i,$$

we write $\Sigma(D)=\bigcup_{i=1}^{\nu}\Gamma(M_i)$, $\Sigma^{\text{int}}(D)=\bigcup_{i=1}^{\nu}\Gamma^{\text{int}}(M_i)$, and $\varrho(D)=\varrho(\text{hull}(D))$. For notational consistency, we consider every atom as a (trivially) dented atom and define $\text{hull}(A)=A$ for every molecule A. When $\text{hull}(D)$ is a cube, D is a *dented cube*.

The main result on dented atoms is the following uniform bilipschitz restoration result. We note that neither the internal geometry of the hull nor the geometry of dents have a rôle in the statement. This is a consequence of confining the dents to be in cubes of the hull and the local nature of the construction of the homeomorphism.

PROPOSITION 3.12. *Suppose D is a dented atom with hull A. Then there exists $L=L(n)$ and an L-bilipschitz homeomorphism $\phi\colon(D,d_D)\to(A,d_A)$ which is the identity on $D\cap\partial A$.*

For the proof, we introduce a useful neighborhood for cubes contained in the dents. Let Q and $q=B_\infty(x_q,r_q)$ be n-cubes in $\mathbb{R}^n$ so that $q\subset Q$ and q has a face in ∂Q. The set

$$\text{Cone}(q,Q)=\left\{x\in B_\infty\left(x_q,\tfrac{7}{6}r_q\right)\cap Q : 2\,\text{dist}(x,q)\leqslant\text{dist}(x,\partial Q)\right\}$$

is the *truncated conical neighborhood* of q in Q.

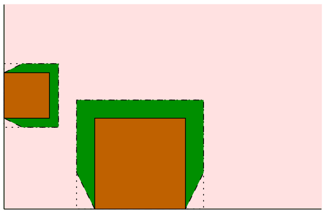

Figure 11. Two cubes and their (truncated) conical neighborhoods in a larger cube.

LEMMA 3.13. *Let* $D=A-\bigcup_i M_i$ *be a dented atom in* $\mathbb{R}^n$. *Then there exists* $\mu>0$ *depending only on* n *such that*

$$\#\{q'\in\Sigma(D):\text{Cone}(q',Q)\cap\text{Cone}(q,Q)\neq\varnothing\}\leqslant\mu$$

for all $q\in\Sigma(D)$.

Proof. Let q and q' be pairwise disjoint n-cubes in an n-cube Q so that q and q' have a face in ∂Q. If either $\varrho(q)=\varrho(q')$ or $\varrho(q)\geqslant 3\varrho(q')$ and $\text{dist}_\infty(q,q')\geqslant\varrho(q')$, then Definitions 3.9 and 3.10 show that $\text{Cone}(q,Q)\cap\text{Cone}(q',Q)=\varnothing$.

Suppose now that $D=A-\bigcup_i M_i$ is a dented atom, and that the n-cubes q and q' in $\Sigma(D)$ are contained in $Q\in\Gamma(A)$. Then, by the definition of dented atom and the first observation, $\text{Cone}(q,Q)\cap\text{Cone}(q',Q)\neq\varnothing$ if and only if $q\cap q'\neq\varnothing$. Hence it suffices that μ be larger than the number of neighbors of q of the same side-length, so we may take $\mu=3^n$. □

Remark 3.14. Let D be a dented atom and consider cubes $Q,Q'\in\Gamma(\text{hull}(D))$, $Q\neq Q'$. Then, if $q,q'\in\Sigma(D)$ with $q\subset Q$ and $q'\subset Q'$, we have that $\text{Cone}(q,Q)\cap\text{Cone}(q',Q')=\varnothing$.

Proof of Proposition 3.12. Recall that the atom A is the hull of D and that $A-D$ is a pairwise disjoint union of molecules. The proof is an inductive collapsing of $A-D$ along the forest $\Sigma(D)$ removing leaves one by one. Let $m=\#\Sigma(D)$.

Let Σ be a subforest of $\Sigma(D)$, $q\in\Sigma$ be a leaf, $Q\in\Gamma(A)$ be the cube containing q, and let $\Sigma'=\Sigma\setminus\{q\}$. Then there exists a PL-homeomorphism $\phi_{\Sigma,q}\colon A-|\Sigma|\to A-|\Sigma'|$ having support in $\text{Cone}(q,Q)$; that is $\phi_{\Sigma,q}(x)=x$ for $x\notin\text{Cone}(q,Q)$. Clearly, we may take $\phi_{\Sigma,q}$ L-bilipschitz with L depending only on n.

Using this observation, we find a sequence $\Sigma(D)=\Sigma_0\supset\ldots\supset\Sigma_m=\varnothing$ of forests and L-bilipschitz PL-homeomorphisms $\phi_i\colon A-|\Sigma_{i-1}|\to A-|\Sigma_i|$ having support in the conical neighborhood of the leaf $\Sigma_{i-1}\setminus\Sigma_i$ for every $i=1,\ldots,m$.

Lemma 3.13 shows that the number of cones over cubes in Σ is locally bounded, and thus

$$\phi=\phi_m\circ\ldots\circ\phi_0\colon(D,d_D)\longrightarrow(A,d_A)$$

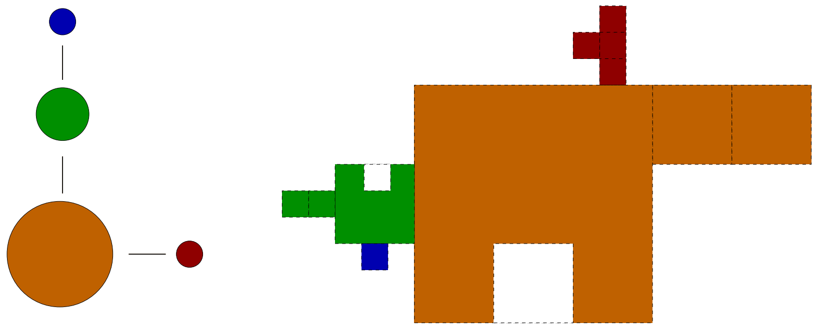

Figure 12. A dented molecule U with a tree $\Gamma(U)$.

is a bilipschitz homeomorphism with a bilipschitz constant depending only on n. □

3.2. Dented molecules

We end this section by defining dented molecules, which relate to dented atoms as molecules relate to atoms.

Definition 3.15. A dented atom D' is *properly adjacent to a dented atom* D if $\mathrm{hull}(D')\cup\mathrm{hull}(D)$ is a molecule and either

(1) $\mathrm{hull}(D')\subset\mathrm{hull}(D)$ and $D'\cap D=\mathrm{hull}(D')\cap D$; or

(2) $\mathrm{hull}(D')\cap\mathrm{hull}(D)=D'\cap D$.

Note from (1) that proper adjacency is not a symmetric relation. However, we symmetrize this relation by saying that dented atoms D and D' are *properly adjacent* if D' is properly adjacent to D or D is properly adjacent to D'.

Let $\mathcal{D}$ be a finite collection of dented atoms so that each pair of atoms in $\mathcal{D}$ is either properly adjacent or pairwise disjoint. Since dented atoms are n-cells, the adjacency tree $\Gamma(\mathcal{D})$ is well defined. Let $U=|\Gamma(\mathcal{D})|$ and $M=\bigcup_{D\in\mathcal{D}}\mathrm{hull}(D)$. By proper adjacency of the dented atoms, M is a molecule.

Definition 3.16. An n-cell U is a *dented molecule* if there exists a finite collection $\mathcal{D}$ of pairwise properly adjacent dented atoms so that $\Gamma(\mathcal{D})$ is a tree with $U=|\Gamma(\mathcal{D})|$. The n-cell $\mathrm{hull}(U)=\bigcup_{D\in\mathcal{D}}\mathrm{hull}(D)$ is the *hull of* U. The vertex $\widehat{D}\in\mathcal{D}$ is the *root* of U if $\mathrm{hull}(\widehat{D})$ is the root of $\mathrm{hull}(U)$.

Remark 3.17. Note that, given a dented molecule $U=|\Gamma(\mathcal{D})|$, the collection $\mathcal{D}$ is uniquely determined. We call elements of $\mathcal{D}$ the *dented atoms* of U and set $\Gamma(U)=\Gamma(\mathcal{D})$.

Let U be a dented molecule. We define internal and external vertices of $\Gamma(U)$ as follows.

Definition 3.18. A dented atom $D\in\Gamma(U)$ is *internal* if there exists strictly larger $D'\in\Gamma(U)$ whose hull contains D, i.e. $\varrho(D')>\varrho(D)$ and $D\subset\mathrm{hull}(D')$. A dented atom in $\Gamma(U)$ is *external* if it is not internal. Denote by $\Gamma_I(U)$ the set of internal vertices of $\Gamma(U)$ and by $\Gamma_E(U)$ the set of external vertices.

The motivation for this dichotomy is the following easy observation, which we record as a lemma.

LEMMA 3.19. *Let U be a dented molecule. Then $D\mapsto\mathrm{hull}(D)$ is a tree isomorphism $\Gamma_E(U)\to\Gamma(\mathrm{hull}(U))$. In particular,*

$$\mathrm{hull}(U)=\bigcup_{D\in\Gamma_E(U)}\mathrm{hull}(D).$$

We finish this section by introducing terminology related to dented molecules. Let D be a dented molecule.

Definition 3.20. A vertex $d\in\Gamma(D)$ is *expanding in D* if the subtree $\Gamma(D)_d$ behind d in $\Gamma(D)$ consists of atoms.

Note that, if d is expanding in D then d is an atom, since $d\in\Gamma(D)_d$.

Definition 3.21. A dented molecule D' is a *partial hull* of D if there exist vertices $d_1,...,d_m$ of $\Gamma(D)$ for which

$$D'=D\cup\bigcup_{k=1}^{m}\mathrm{hull}(d_k).$$

Remark 3.22. In §5 (e.g. in §5.3.1), we consider a sequence of dented molecules $\{U_i\}_{i\geqslant1}$ for which $\mathrm{hull}(U_i)$ is a (ν,λ)-molecule with ν and λ depending only on n, although the adjacency tree $\Gamma(U_i)$ no longer has uniformly bounded valence.

We show there exist L-bilipschitz maps $U_i\to\mathrm{hull}(U_i)$ with L depending only on n. This proof is based on a sequence of partial hulls from U_i to $\mathrm{hull}(U_i)$.

Since we prove this statement only for particular dented molecules based on notions in the following section, we postpone this statement to §5. Nevertheless, we invite the interested reader to consider a general statement along the lines of Propositions 3.5 and 3.12.

4. Local rearrangements and the tripod property

In this section we develop tools to produce rough Rickman partitions; recall §1. Throughout this section we consider different kinds of repartitions in a single cube. These rearrangements are only tangentially related to the final essential partitions introduced in

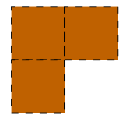

Figure 13. The essential partition $\mathbf{D}$.

§5, so the reader may find these constructions unmotivated. Our aim is to simplify these later discussions by introducing these local modifications and their properties here before exploiting them later. Thus the reader should consider this section as preparation for §5.

To motivate the rôle of our tools, consider the following example. Let D_1, D_2, and D_3 be the cubes $[0,1]^{n-1}\times[0,1]$, $[0,1]^{n-1}\times[-1,0]$, and $[1,2]\times[0,1]^{n-2}\times[0,1]$, respectively, and $\mathbf{D}$ be the essential partition (D_1,D_2,D_3) of their union.

The Hausdorff distance of the common boundary $\partial_\cap\mathbf{D}$ and the pairwise common boundary $\partial_\cup\mathbf{D}$ satisfy

$$\operatorname*{dist}_{\mathcal{H}}(\partial_\cup\mathbf{D},\partial_\cap\mathbf{D})=1 \tag{4.1}$$

in the sup-metric.

Let $k>0$ and consider now the sets $V_i=3^kD_i$ for $i=1,2,3$, and the associated essential partition $\mathbf{V}=(V_1,V_2,V_3)$. Of course, topological properties and bilipschitz equivalence of the cubes remain invariant under this scaling. The Hausdorff distances in (4.1) scale accordingly, and so

$$\operatorname*{dist}_{\mathcal{H}}(\partial_\cup\mathbf{V},\partial_\cap\mathbf{V})=3^k. \tag{4.2}$$

We will show that in this case, as well as in more general situations, there exists an essential partition $\mathbf{W}=(W_1,W_2,W_3)$ of $\bigcup_i V_i$ into n-cells (W_i,d_{W_i}) uniformly bilipschitz to $[0,3^k]^n$ with

$$\operatorname*{dist}_{\mathcal{H}}(\partial_\cup\mathbf{W},\partial_\cap\mathbf{W})\leqslant 6 \tag{4.3}$$

in the sup-metric.

Property (4.3) is a consequence of the so-called tripod property, informally mentioned in the introduction, which we now formally define. In later sections, we discuss other structures related to partitions.

We first need an equivalence relation. Let U be a 3-fine cubical n-set in $\mathbb{R}^n$ and let U^* be a 3-fine subdivision of U. Suppose $\mathbf{U}=(U_1,U_2,U_3)$ is an essential partition of U, and let $(\partial_\cup\mathbf{U})^\#$ be the unit subdivision of $\partial_\cup\mathbf{U}$ as defined in §2.2. Let $\Gamma_\cup(\mathbf{U})$ be the subgraph of the adjacency graph $\Gamma((\partial_\cup\mathbf{U})^\#)$ composed of vertices of $\Gamma((\partial_\cup\mathbf{U})^\#)$ and all edges $\{q,q'\}\in\Gamma((\partial_\cup\mathbf{U})^\#)$ for which $q\cup q'\subset U_i\cap U_j$ for a pair $i\neq j$.

Example 4.1. In the discussion accompanying Figure 13, $\Gamma_\cup(\mathbf{D})$ consists of two vertices $\{[0,1]^{n-1}\times\{0\},\{1\}\times[0,1]^{n-1}\}$ and has no edges, whereas $\Gamma_\cup(3^k\mathbf{D})$, for $k\geqslant1$, is a pairwise disjoint union of two connected subgraphs.

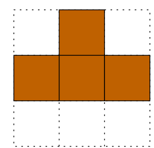

Figure 14. Profile of q_1, q_2, q_3, and q_4 on the face common to Q and Q'.

Definition 4.2. The cubes q and q' in $(\partial_\cup \mathbf{U})^\#$ are $\mathbf{U}$-*equivalent* if

(a) q and q' are in the same component of $\Gamma_\cup(\mathbf{U})$; and

(b) $q \cup q' \subset Q$ for some $Q \in U^*$.

Denote by $[q]$ the $\mathbf{U}$-equivalence class of $q \in (\partial_\cup \mathbf{U})^\#$ and by $|[q]|$ the union $\bigcup_{q' \in [q]} q'$. For each pair (i,j), $i \neq j$, the $\mathbf{U}$-equivalence class $[q]$ of $q \in (\partial_\cup \mathbf{U})^\#$ is said to be *between* U_i *and* U_j when $q \subset U_i \cap U_j$.

Remark 4.3. Condition (b) in Definition 4.2 implies that the equivalence class $[q]$ of $q \in (\partial_\cup \mathbf{U})^\#$ has diameter at most 3 in the sup-metric. Note that equivalence classes are cubical 1-fine sets of dimension $n-1$, and that the number of $(n-1)$-cubes in $[q]$ is uniformly bounded by a constant depending only on n.

Definition 4.4. An essential partition $\mathbf{U}$ of U has the *tripod property* if there exists an essential partition Δ of $\partial_\cup \mathbf{U}$ into cubical $(n-1)$-cells satisfying

(Δ1) each $c \in \Delta$ is contained in a $\mathbf{U}$-equivalence class; and

(Δ2) to each $c_1 \in \Delta$ corresponds a unique pair $c_2, c_3 \in \Delta$ for which $c_1 \cap c_2 \cap c_3$ contains an $(n-2)$-cell in $\partial_\cap \mathbf{U}$ with c_1, c_2, and c_3 contained in different $\mathbf{U}$-equivalence classes.

The tripod property of an essential partition is most conveniently verified using the following local tripod property.

Definition 4.5. Given an essential partition $\mathbf{U}$ and a cube $Q \subset |\mathbf{U}|$ of side-length at least 3, we say that $\mathbf{U}$ *has the tripod property relative to* Q if there exists an essential partition Δ of $Q \cap \partial_\cup \mathbf{U}$ into $(n-1)$-cells satisfying (Δ1) and (Δ2).

Example 4.6. To give a simple example of an essential partition $\mathbf{U}$ satisfying the tripod property we consider $\mathbf{U} = (Q-A, A, Q')$, where $Q = [0,3]^3$, $Q' = [0,3]^2 \times [-3,0]$, and A is the atom $A = \bigcup_{r=1}^4 q_r$, where $q_r = [r-1, r] \times [1,2] \times [0,1]$ for $r = 1,2,3$ and $q_4 = [1,2] \times [2,3] \times [0,1]$; see Figure 14.

Note first that $(Q-A) \cap Q'$ has three components $f_1 = [0,1] \times [2,3] \times \{0\}$, $f_2 = [0,3] \times [0,1] \times \{0\}$, and $f_3 = [2,3] \times [2,3] \times \{0\}$, whereas $A \cap (Q-A)$ and $A \cap Q'$ are 2-cells. We organize the essential partition Δ of $\partial_\cup \mathbf{U}$ into three triples Δ_1, Δ_2, and Δ_3 by subdividing the cells $A \cap (Q-A)$ and $A \cap Q'$ as follows.

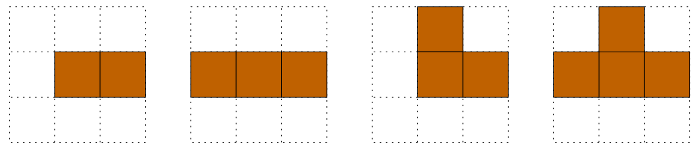

Figure 15. Congruence classes of building blocks for $n=3$.

For $r=1,3$, we set

$$\Delta_r = \{f_r, q_r \cap (Q-A), q_r \cap Q'\}.$$

Let

$$\Delta_2 = \{f_2, (q_2 \cup q_4) \cap (Q-A), (q_2 \cup q_4) \cap Q'\}.$$

For each r, we directly check that Δ_r is a triple of $(n-1)$-cells. In addition, $\bigcap_{c\in\Delta_r} c$ is an $(n-2)$-cell for every $r=1,2,3$. Hence $\Delta = \bigcup_{r=1}^{3} \Delta_r$ is an essential partition of $\partial_\cup \mathbf{U}$ satisfying conditions (Δ1) and (Δ2).

4.1. Building blocks

We introduce the elementary atoms which generate rough Rickman partitions.

An $(n-1)$-cell F in $\mathbb{R}^n$ is *planar* if F is congruent to an $(n-1)$-cell in $\mathbb{R}^{n-1}$. Suppose P is an r-fine n-cell and F is a planar $(n-1)$-cell. Then P is *F-based* if there exists an $(n-1)$-cell F' in $\mathbb{R}^{n-1}$ and a cubical $(n-1)$-cell $P' \subset F'$ so that $P \cup F$ is congruent to $(P' \times [0,r]) \cup F' \subset \mathbb{R}^n$.

Let $T_n = \{0, \pm e_1, ..., \pm e_n\}$ and let $\mathcal{T}_n$ be the graph with vertices T_n and edges $\{0, e_i\}$ and $\{0, -e_i\}$ for $i=1, ..., n$.

Definition 4.7. An atom A is an (n-dimensional) *building block* if $\Gamma(A)$ is isomorphic to a proper subtree of $\mathcal{T}_{n-1}$ having at least two vertices.

The fundamental property used in what follows is that an n-dimensional building block is an n-cell. We record now some observations based on the combinatorial structure of building blocks.

Let B be a building block in $\mathbb{R}^n$. Since $\Gamma(B)$ is a proper subtree of $\mathcal{T}_{n-1}$, we observe that, for all $q \in \Gamma(B)$, the cubical set $q \cap \partial B$ is an $(n-1)$-cell which induces an essential partition of the faces of q, and the adjacency graph $\Gamma(q \cap \partial B)$ of these faces is connected. Moreover, $\Gamma(B)$ has valence less than $2(n-1)$ and contains at most one vertex $q \in \Gamma(B)$ having valence greater than 1. Further, if $\#\Gamma(B) > 2$ there exists a unique n-cube q_B in B which is a vertex in $\Gamma(B)$ with valence greater than 1: this unique cube q_B is the *center* of B.

A building block B in $\mathbb{R}^n$ is *r-fine* if B is an r-fine atom for some $r>0$.

Suppose Q is a cube of side-length $3r$ containing an r-fine building block B along a face F of Q. Then, for every cube $q\in\Gamma(B)$, $q\cap F$ is an $(n-1)$-cube and a face of q. For the following definition, recall (as in §3) that a *barycenter of a k-cube C* is the unique point in C equidistant from all vertices of C.

Definition 4.8. Suppose $Q\subset\mathbb{R}^n$ is an n-cube of side-length $3r$ containing an F-based r-fine building block $B\subset Q$, where F is a face of Q. Let x_F be the barycenter of F. The building block B is *centered* in Q if either of the following conditions is satisfied:

(1) if B has a center q_B then x_F is the barycenter of $q_B\cap F$; or

(2) if $\#\Gamma(B)=2$, then $\Gamma(B)$ contains the cube q with x_F being the barycenter of $q\cap F$.

The significance of centered building blocks is motivated by the following observation.

Remark 4.9. Let $Q\subset\mathbb{R}^n$ be a cube of side-length 3 and B be a 1-fine centered building block contained in Q along the face F of Q. Since B is centered, the barycenter x_F of F is the barycenter x_{f_0} of a face f_0 of a unique cube q_0 in $\Gamma(B)$. Suppose that $q\in\Gamma(B)$ is a cube adjacent to q_0. Since Q has side-length 3 and the barycenter of q_0 is x_F, we have that $q\cap(\partial Q-F)$ is a face of q. In particular, the components of $B\cap(\partial Q-F)$ are unit $(n-1)$-cubes, which are in one-to-one correspondence with cubes in $B-q_0$, cf. Figure 15.

Convention. We assume from now on that every r-fine building block B in a cube Q is centered and based on a face of Q whenever Q has side-length $3r$. We extend the notion of *center* by defining the unique cube in B containing the barycenter of F on its boundary to be the *center* of B.

Building blocks give rise to a local tripod property of the following form.

Proposition 4.10. *Let $n\geqslant 3$, and let Q and Q' be n-cubes of side-length 3 with a common face $F=Q\cap Q'$, and let B be an F-based building block in Q. Then $\mathbf{U}=(Q-B,B,Q')$ has the tripod property.*

We begin the proof of Proposition 4.10 with a partitioning lemma.

Lemma 4.11. *Let $n\geqslant 2$, and let A be a 1-fine atom in $Q=[0,3]^n$ containing the cube $[1,2]^n$, with $\Gamma(A)$ isomorphic to a subgraph of $\mathcal{T}_n$ and $1<\#\Gamma(A)\leqslant 2n$. Then $Q-A$ has an essential partition $\mathcal{P}$ into n-cells. Moreover, there exist cubes $\mathcal{C}_\mathcal{P}=\{q_C\in A^\#:C\in\mathcal{P}\}$ so that $q_C\neq q_{C'}$ for cells $C\neq C'$ in $\mathcal{P}$ and $q_C\cap C$ contains an $(n-1)$-cube for every $C\in\mathcal{P}$.*

Proof. In the special case $\#\Gamma(A)=2$, we may take $\mathcal{P}=\{Q-A\}$ and $\mathcal{C}_\mathcal{P}=\{[1,2]^n\}$.

The proof in the general case is by induction on the dimension n. The claim clearly holds for $n=2$; consider e.g variations of Example 4.6. Suppose that $n\geqslant 3$ is a dimension for which the claim holds for $n-1$.

Let A be a 1-fine atom in $Q=[0,3]^n$ containing $[1,2]^n$ with $\Gamma(A)$ isomorphic to a subtree of $\mathcal{T}_n$ and $1<\#\Gamma(A)\leqslant 2n$. By rotation, we may assume that $[1,2]^n+e_1\in\Gamma(A)$. Let $F=[0,3]^{n-1}$. Then $A\cap(F\times[1,2])=A'\times[1,2]$, where A' is an $(n-1)$-dimensional atom in F where $1<\#\Gamma(A')\leqslant 2(n-1)$. The adjacency graph $\Gamma(A')$ is isomorphic to a subgraph of $\mathcal{T}_{n-1}$. By induction, $F-A'$ has an essential partition $\mathcal{P}'$ into $(n-1)$-cells and, for each $C'\in\mathcal{P}'$, we may fix $q_{C'}\in\mathcal{C}_{\mathcal{P}'}\subset(A')^\#$ so that each $C'\cap q_{C'}$ contains an $(n-2)$-cube.

Let $\mathcal{P}''=\{C'\times[0,3]:C'\in\mathcal{P}'\}$. We observe that $Q-(|\mathcal{P}''|\cup A)$ consists of unit cubes in $(A'\times[0,3]-A)^\#$. It is now easy to find, for each $C'\in\mathcal{P}'$ a cubical n-cell $\Omega_{C'}$ so that $C'\times[0,3]\subset\Omega_{C'}$, $\bigcup_{C'\in\mathcal{P}'}\Omega_{C'}=Q-A$, and that the sets $\Omega_{C'}$ are pairwise essentially disjoint. We set $\mathcal{P}=\{\Omega_{C'}:C'\in\mathcal{P}'\}$ and $\mathcal{C}_\mathcal{P}=\{q_{C'}\times[1,2]:C'\in\mathcal{P}'\}$. □

The following corollary encapsulates the key consequence of Lemma 4.11.

Corollary 4.12. *Let $n\geqslant 3$, Q be an n-cube of side-length 3 and F be a face of Q. Given an F-based building block B in Q, the set $F-B$ has an essential partition $\mathcal{P}$ into cubical $(n-1)$-cells and there exists a collection $\mathcal{C}_\mathcal{P}=\{q_C\in B^\#:C\in\mathcal{P}\}$ of pairwise essentially disjoint unit n-cubes so that $C\cap q_C$ contains an $(n-2)$-cube for every $C\in\mathcal{P}$.*

Proof. We may assume that $Q=[0,3]^n$ and $F=[0,3]^{n-1}$. Since $F\cap B$ is an $(n-1)$-dimensional atom containing $[1,2]^{n-1}$ and having an adjacency tree isomorphic to a (proper) subtree of $\mathcal{T}_{n-1}$ with at least two vertices, the claim follows from Lemma 4.11. □

Proof of Proposition 4.10. Clearly the pairwise common boundary $\partial_\cup\mathbf{U}$ consists of $\mathbf{U}$-equivalence classes $(Q-B)\cap B$, $B\cap Q'$, and $(Q-B)\cap Q'$. The classes $(Q-B)\cap B$ and $B\cap Q'$ are $(n-1)$-cells meeting $\partial_\cap\mathbf{U}$ in an $(n-2)$-cell. We construct now an essential partition of $\partial_\cup\mathbf{U}$ into $(n-1)$-cells as required.

Let $\mathcal{P}$ and $\mathcal{C}_\mathcal{P}$ be sets as in Corollary 4.12. Then there exists an essential partition $\{A_C:C\in\mathcal{P}\}$ of B into atoms A_C satisfying $q_C\subset A_C$; consider, for example, the components of the graph $\Gamma(B^\#\setminus\mathcal{P})$. For every $C\in\mathcal{P}$, take $\Delta_C=\{A_C\cap(Q-B), A_C\cap Q', C\}$. Then $\Delta=\bigcup_{C\in\mathcal{P}}\Delta_C$ is the required partition of $\partial_\cup\mathbf{U}$. □

In what follows, Proposition 4.10 is used to verify the tripod property for essential partitions obtained by rearrangements based on building blocks.

4.2. Flat (planar) rearrangements

Although the notion of atom admits a large variety of possible constructions, we restrict ourselves to only a few basic constructions, all of which appear in this section. These choices yield a double edged sword: we avoid self-intersections and thus preserve the topology of the original essential partition after rearrangement, as a penalty we create neglected faces (discussed in §4.4).

In the next two sections we discuss local rearrangements, based on centered building blocks in a single n-cube. This section concerns *flat rearrangements*, in that atoms are extended across a single face of a cube. The following section considers the case that atoms are extended across several faces of a cube.

With this objective in mind, we say that an atom A, which is a pairwise essentially disjoint union of building blocks, *consists of building blocks*. Note that planar atoms admit unique partitions into building blocks, but essential partitions of non-planar atoms into building blocks are not unique. Indeed, in each corner where two planar parts of a non-planar atom meet, there are two possible partitions if one of the building blocks consists of two cubes. This ambiguity is, however, not significant in our considerations, since in these cases we may take any possible partition. Keeping this ambiguity in mind, we give the following definition.

Definition 4.13. Given an atom A consisting of building blocks, we denote by $\widetilde{\Gamma}(A)$ the adjacency graph $\Gamma(\mathcal{B})$, where $\mathcal{B}$ is an essential partition of A into building blocks. We also set $\ell_{\mathrm{bb}}(A)=\ell(\Gamma(\mathcal{B}))$.

Thus, when the essential partition of A into building blocks is clear from the context, we denote this adjacency graph by $\widetilde{\Gamma}(A)$. Note that $\widetilde{\Gamma}(A)$ is always a tree.

We consider different cases, starting from simple and heading to more complicated constructions.

Let Q be an n-cube of side-length 9 and F be a face of Q. We subdivide Q into 3^n congruent n-cubes of side-length 3, i.e., we consider Q^*. Then Q^* induces a subdivision of F into 3^{n-1} congruent $(n-1)$-cubes of side-length 3. The collection of these $(n-1)$-cubes is F^*. Let $\mathcal{Q}(Q;F)$ be the subset of cubes in Q^* with a face in F^*.

Definition 4.14. A quadruple $(Q,F,\mathcal{Q}_0',q_0)$ forms *initial data* if

(a) q_0 is an n-cube of side-length 3 so that $q_0\cap Q$ is a face of q_0 and $q_0\cap F$ is an $(n-2)$-cube; and

(b) $\mathcal{Q}_0'\subset\mathcal{Q}(Q;F)$ is a collection with

(i) $\Gamma(\mathcal{Q}_0')$ connected, and

(ii) $q_0\cap|\mathcal{Q}_0'|=q_0\cap Q$.

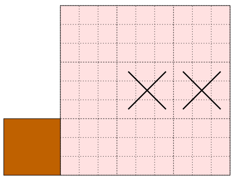

Figure 16. An example of an initial data $(Q, F, \mathcal{Q}_0', q_0)$. The face F (side-length 9) and cube q_0 (side-length 3) are viewed from above, cubes in $\mathcal{Q}(Q;F)\setminus\mathcal{Q}_0'$ are marked with 'x'; $n{=}3$.

Definition 4.15. Let $(Q, F, \mathcal{Q}_0', q_0)$ be initial data. A maximal tree $\Gamma\subset\Gamma(\mathcal{Q}_0'\cup\{q_0\})$ is a *spanning tree associated with this initial data* if Γ has valence less than $2(n-1)$.

The valence bound $2(n-1)$ in Definition 4.15 was already anticipated in our valence bound for building blocks, recall Definition 4.7.

The following simple lemma shows the existence of spanning trees in the configurations we consider here. Let q_F be the unique cube of side-length 3 in $\mathcal{Q}(Q;F)$ having valence $2(n-1)$ in $\Gamma(\mathcal{Q}(Q;F))$; thus the barycenter of $q_F\cap F$ is the barycenter of F.

LEMMA 4.16. *Suppose $(Q, F, \mathcal{Q}_0', q_0)$ forms initial data and $\Gamma(\mathcal{Q}_0'\setminus\{q_F\})$ is connected. Then there exists a spanning tree $\Gamma\subset\Gamma(\mathcal{Q}_0')$.*

Proof. Let Γ' be a maximal tree in $\Gamma(\mathcal{Q}_0'\setminus\{q_F\})$. Since $\Gamma(\mathcal{Q}_0'\setminus\{q_F\})\subset\Gamma(\mathcal{Q}(Q;F))$ and q_F is the unique vertex in $\Gamma(\mathcal{Q}(Q;F))$ having valence $2(n-1)$, Γ' is a spanning tree of $\Gamma(\mathcal{Q}_0'\setminus\{q_F\})$. If $q_F\notin\mathcal{Q}_0'$, we may take $\Gamma=\Gamma'$.

If $q_F\in\mathcal{Q}_0'$, let $q'\in\Gamma(\mathcal{Q}_0')$ be a vertex adjacent to q_F. We extend Γ' to a tree Γ containing q_F by adding the edge $\{q', q_F\}$. Since the valence of q' in Γ' is less than $2(n-1)-1$, the claim follows. □

Spanning trees repartition Q using atoms.

LEMMA 4.17. *Given initial data $(Q, F, \mathcal{Q}_0', q_0)$ and a spanning tree Γ, there exists a 1-fine atom A_Γ in Q with the following properties*:

(1) *$A_\Gamma\cap q'$ is an F-based building block for every $q'\in\mathcal{Q}_0'$*;
(2) *the adjacency graph $\widetilde{\Gamma}(A_\Gamma)$ of building blocks is $\Gamma\setminus\{q_0\}$*;
(3) *$A_\Gamma\cup q_0$ is an n-cell*; *and*
(4) *$A_\Gamma\cap\partial Q\subset F\cup q_0$.*

We call A_Γ the (*unique*) *atom associated with the spanning tree* Γ (and *initial data* $(Q, F, \mathcal{Q}_0', q_0)$).

Remark 4.18. Note that the atom A_Γ in Lemma 4.17 is on the boundary of Q as defined in §3.1. Thus $Q-A_\Gamma$ is a dented cube and, in particular, an n-cell.

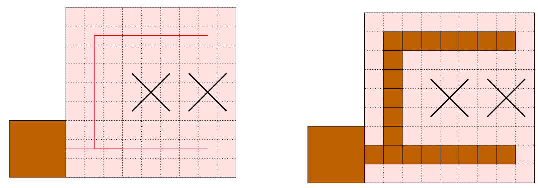

Figure 17. A spanning tree (left) and the corresponding atom (right) associated with the initial data in Figure 16.

Proof of Lemma 4.17. To obtain the building blocks, we make the following observation.

Suppose $q'\in\Gamma$ is a vertex other than q_0. Let $\Gamma_{q'}$ be the star of q' in Γ, that is, the subgraph of Γ containing only edges connecting to q' and all vertices on these edges. We let $E_{q'}=|\Gamma_{q'}|$. Then $E_{q'}$ is a building block.

To each $q'\in\mathcal{Q}'_0$ corresponds a unique F-based centered building block $B_{q'}\subset q'$ which is a translation of $\frac{1}{3}E_{q'}$. These building blocks form an essential partition of the atom $A_\Gamma=\bigcup_{q'\in\mathcal{Q}'_0} B_{q'}$, whose adjacency graph $\widetilde{\Gamma}(A_\Gamma)=\Gamma(\{B_{q'}:q'\in\mathcal{Q}'_0\})$ is isomorphic to Γ.

Conditions (1), (2), and (4) are clearly satisfied by the construction. Since Γ is a tree, A_Γ is an atom. As q_0 is a leaf in Γ and $A_\Gamma\cap q_0$ is an $(n-1)$-cube, $A_\Gamma\cup q_0$ is an n-cell and (3) holds. □

Atoms associated with initial data and spanning trees immediately yield a local tripod property.

Lemma 4.19. *Let Q and Q' be n-cubes of side-length 9 sharing the face F. Suppose that $(Q,F,\mathcal{Q}(Q;F),q_0)$ forms initial data with spanning tree Γ. Let A_Γ be the atom associated with Γ and $(Q,F,\mathcal{Q}(Q;F),q_0)$. Then the essential partition $\mathbf{U}=(Q-A_\Gamma,A_\Gamma,Q')$ of $Q\cup Q'$ has the tripod property.*

Proof. Let q be a cube in $\mathcal{Q}(Q;F)$ and let q_- be the unique cube in Q' sharing a face with q. Denote by B_q the building block $q\cap A_\Gamma$. By Proposition 4.10, $(q-B_q,B_q,q_-)$ satisfies the tripod property. Let Δ_q be an essential partition of $(\partial_\cup\mathbf{U})\cap q$ as in Definition 4.4. Since $\mathcal{Q}(Q;F)$ is an essential partition of a cubical set having $\partial_\cup\mathbf{U}$ (essentially) in its interior, $\Delta=\bigcup_{q\in\mathcal{Q}(Q;F)}\Delta_q$ is a required essential partition of $\partial_\cup\mathbf{U}$. □

More generally, we may consider initial data $(Q,F,\mathcal{Q}'_0,q_0)$, where $q_0\in\mathcal{Q}(Q;F)$; this means that $q_0\subset Q$ with $q_0\cap F$ being a face of q_0. Initial data of this type is called *internal initial data*. This notion of initial data is especially useful for extending a 3-fine building block inside a cube of side-length 9. We formulate now this rearrangement procedure.

Corollary 4.20. *Let Q be a cube of side-length 9 and F be a face of Q. Let also $q_1,...,q_p$ be pairwise essentially disjoint cubes in $\mathcal{Q}(Q;F)$. Suppose, for $1\leqslant r\leqslant p$, each*

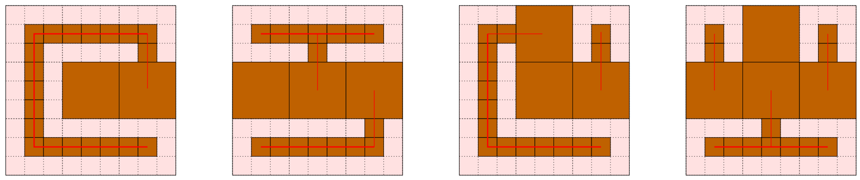

Figure 18. Some examples of atoms A_r for $r=1, ..., p$ each associated with an internal initial data; here $p=1, 2, 2, 3$.

$(Q, F, \mathcal{Q}'_r, q_r)$ forms internal initial data with $\mathcal{Q}'_r \subset \mathcal{Q}(Q; F)$ and $\mathcal{Q}'_t \cap \mathcal{Q}'_s = \varnothing$ for $t \neq s$. Suppose $\Gamma_1, ..., \Gamma_p$, respectively, are spanning trees for these initial data. Then there exist pairwise disjoint 1-fine atoms A_r associated with initial data $(Q, F, \mathcal{Q}'_r, q_r)$ for $r=1, ..., p$.

It is easy to obtain a local tripod property for these repartitions. We leave the details, similar to those of the proof of Lemma 4.19, to the interested reader.

Corollary 4.21. *Let Q and Q' be n-cubes of side-length 9 sharing the face F, and suppose that, for each $1 \leqslant r \leqslant p$, $(Q, F, \mathcal{Q}'_r, q_r)$ forms internal initial data as in Corollary 4.20 so that in addition*

$$B := |\mathcal{Q}(Q; F)| - \bigcup_{r=1}^{p} |\mathcal{Q}'_r|$$

is a building block of side-length 3. For each $1 \leqslant r \leqslant p$, let Γ_r be a spanning tree for $(Q, F, \mathcal{Q}'_r, q_r)$, associate an atom A_r with Γ_r as in Corollary 4.20 and define A as the (disjoint) union of the atoms A_r. Then the essential partition

$$\mathbf{U} = (Q - (B \cup A), B \cup A, Q')$$

of $Q \cup Q'$ has the tripod property.

Convention. Henceforth we do not differentiate between initial data and internal initial data, and refer to both as initial data.

4.3. Non-flat (non-planar) rearrangements

We consider now local rearrangements in the non-flat case. For our purposes it suffices to consider rearrangements which occur in a single cube.

Let Q be an n-cube of side-length 9 and $\mathcal{F}$ be a subset of the collection of all faces of Q. Let $\mathcal{F}$ be partitioned into the sets $\mathcal{F}^1$ and $\mathcal{F}^2$ so that $|\mathcal{F}^r|$ is an $(n-1)$-cell for $r=1, 2$. Note that each $|\mathcal{F}^r|$, in particular, is a union of faces of Q.

Let $\mathcal{Q}(Q; \mathcal{F}) \subset Q^*$ be the cubes having a face in $|\mathcal{F}|$; we denote by $\mathcal{Q}(Q; \mathcal{F}^r) \subset \mathcal{Q}(Q; \mathcal{F})$ those with a face in $|\mathcal{F}^r|$. Note that $\{\mathcal{Q}(Q; \mathcal{F}^1), \mathcal{Q}(Q; \mathcal{F}^2)\}$ is not (necessarily) a partition of $\mathcal{Q}(Q; \mathcal{F})$. The following definition generalizes Definition 4.14.

Definition 4.22. A triple

$$(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$$

forms *non-flat initial data* if the following conditions are satisfied:

(a) for every $r=1,2$, $q_r \subset \mathbb{R}^n - Q$ is an n-cube of side-length 3 with $Q \cap q_r$ being a face of q_r and $q_r \cap |\mathcal{F}^r|$ being an $(n-2)$-cube;

(b) $\{\mathcal{Q}_1'', \mathcal{Q}_2''\}$ is a partition of $\mathcal{Q}(Q;\mathcal{F})$ which, for $r=1,2$, satisfies

(0) $\mathcal{Q}_r'' \subset \mathcal{Q}(Q;\mathcal{F}^r)$,

(1) $\Gamma(\mathcal{Q}_r'')$ is connected,

(2) $q_r \cap |\mathcal{Q}_r''|$ is a face of q_r, and

(3) $q_r \cap |\mathcal{F}^r| \cap |\mathcal{Q}_r''|$ is an $(n-2)$-cube.

Remark 4.23. Let $(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$ be as in Definition 4.22, and let q_1^+ and q_2^+ be the n-cubes in Q^* sharing a face with q_1 and q_2, respectively. Since $q_1 \cap Q$ is a face of q_1, condition (2) in (b) shows that $q_1^+ \in \mathcal{Q}_1''$. Clearly, the same argument holds for q_2 and we also have $q_2^+ \in \mathcal{Q}_2''$.

Let $r \in \{1,2\}$, $\widehat{\Gamma} \subset \Gamma(\mathcal{F}^r)$ be a maximal tree and $\mathcal{Q}' \subset \mathcal{Q}(Q;\mathcal{F}) \cup \{q_r\}$. A subgraph $\Gamma \subset \Gamma(\mathcal{Q}')$ is *dominated* by $\widehat{\Gamma}$ if, for each vertex $q \in \Gamma$ and the star Γ_q of q in Γ, either there exists a vertex $F_q \in \Gamma(\mathcal{F}^r)$ satisfying $\Gamma_q \setminus \{q_r\} \subset \mathcal{Q}(Q;F_q)$ or there exists an edge $\{F_q, F_q'\} \in \widehat{\Gamma}$ satisfying $\Gamma_q \setminus \{q_r\} \subset \mathcal{Q}(Q;F_q) \cup \mathcal{Q}(Q;F_q')$.

Definition 4.24. Let $(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$ form non-planar initial data. A maximal forest $\Sigma = \Gamma_1 \cup \Gamma_2 \subset \Gamma(\mathcal{Q}'' \cup \{q_1, q_2\})$ is a *spanning forest associated with this data* if

(i) Σ has valence less than $2(n-1)$; and

(ii) for $r=1,2$, Γ_r is a maximal tree in $\Gamma(\mathcal{Q}_r'' \cup \{q_r\})$ dominated by a maximal tree of $\Gamma(\mathcal{F}^r)$.

The proof of the following existence result for spanning forests is analogous to Lemma 4.16, and we omit the details. Let $\mathcal{Q}_c'(Q;\mathcal{F})$ be the collection of all cubes in $\mathcal{Q}(Q;\mathcal{F})$ having valence $2(n-1)$.

Lemma 4.25. *Suppose that* $(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$ *forms non-planar initial data for which* $\Gamma(\mathcal{Q}_r'' \setminus \mathcal{Q}_c(Q;\mathcal{F}))$ *is connected for* $r=1,2$. *Then there exists a spanning forest* Σ *associated with* $(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$.

Lemma 4.26. *Let* $(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$ *form non-planar initial data, and let* $\Sigma = \Gamma_1 \cup \Gamma_2 \subset \Gamma(\mathcal{Q}' \cup \{q_1, q_2\})$ *be a spanning forest.*

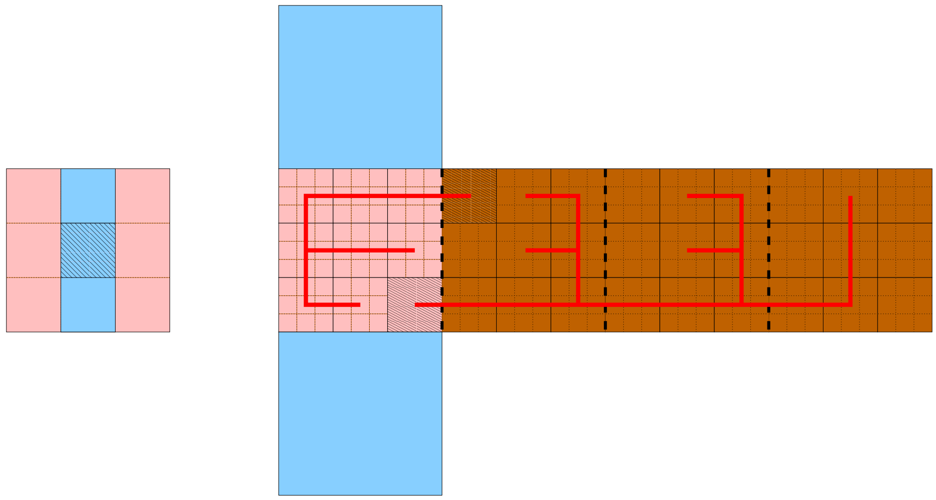

Figure 19. A spanning forest on four faces $\mathcal{F}$ of a cube Q; note that the forest enters each cube q in $\mathcal{Q}(Q;\mathcal{F})$. Here Q is the center cube in a building block consisting of 3 cubes (left figure).

Then there exist a 1*-fine cubical set* A_Σ *in* Q *composed of pairwise disjoint* 1*-fine atoms* A_1 *and* A_2 *which, for* $r=1,2$, *satisfies the following properties*:

(1) *each* A_r *is composed of building blocks*;

(2) *for every* $q''\in\mathcal{Q}''_r$, $A_r\cap q''$ *is an atom having an essential partition into at most two building blocks*;

(3) *every building block in* A_r *is* F*-based with* $F\in\mathcal{F}^r$;

(4) $A_r\cup q_r$ *is an* n*-cell*;

(5) $A_r\cap\partial Q\subset|\mathcal{F}^r|\cup q_r$; *and*

(6) *the adjacency graph of the cells* $\{A_r\cap Q'':Q''\in\mathcal{Q}''_r\}$ *is isomorphic to* Γ_r.

The set A_Σ in Lemma 4.26 is said to be *associated with this initial data and the spanning forest* Σ. Property (2) is a consequence of the trees Γ_1 and Γ_2 being dominated by $\Gamma(\mathcal{F}^1)$ and $\Gamma(\mathcal{F}^2)$ respectively. Property (3) asserts that A_Σ is *on the boundary of* Q.

Remark 4.27. As in Remark 4.18, the components A_1 and A_2 of A_Σ in Lemma 4.26 are atoms on the boundary of Q. In particularly, $Q-A_\Sigma$ is a dented cube.

Proof of Lemma 4.26. Consider first the tree Γ_1. Let $q'\in\Gamma_1$ be an F-based cube, where $F\in\mathcal{F}^1$, and let $\Gamma_{q'}$ be the star of q' in Γ_1.

If $|\Gamma_{q'}|$ is F-based, we fix a building block $B_{q'}$ as in Lemma 4.17. Suppose, however, that $|\Gamma_{q'}|$ is not F-based. Then, by (ii) in Definition 4.24, there exists a face $F'\in\mathcal{F}^1$ so that each cube in $\Gamma_{q'}$ is either F-based or F'-based. Thus there exist an F-based building

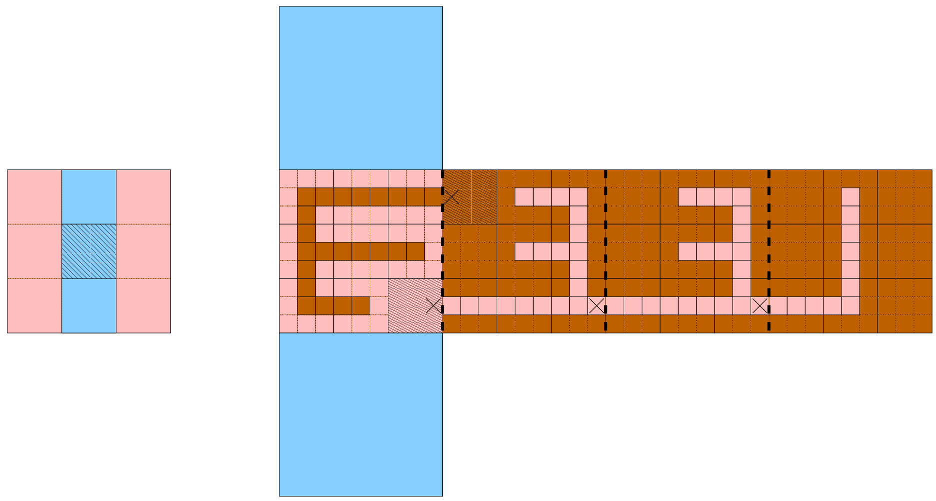

Figure 20. The atoms A_1 and A_2 associated with the initial data in Figure 19.

block B_F and an F'-based building block $B_{F'}$ in q' with the following properties:

(i) $B_F \cap B_{F'}$ is an $(n-1)$-cube; and

(ii) $B_F \cup B_{F'}$ meets the neighbors of q' in Γ_1 in $(n-1)$-cubes.

In this case, we take $B_{q'} = B_F \cup B_{F'}$, and define $A_1 = \bigcup_{q' \in \Gamma} B_{q'}$. The atom A_2 is defined similarly. It is easy to check that A_1 and A_2 satisfy properties (1)–(6). □

These non-planar rearrangements satisfy the tripod property.

Lemma 4.28. *Let* $\mathbf{U} = (U_1, U_2, U_3)$ *be an essential partition and* $Q \subset U_3$ *be an* n*-cube of side-length* 9 *sharing a face with both* U_1 *and* U_2*. Let*

$$(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$$

form non-planar initial data for which

(i) $q_r \subset U_r$ *for* $r = 1, 2$;

(ii) $|\mathcal{F}^r| \subset Q \cap U_{j_r}$, *where* $\{j_r, r\} = \{1, 2\}$; *and*

(iii) $|\mathcal{F}^1| \cup |\mathcal{F}^2| = Q \cap \partial_\cup \mathbf{U}$.

Let Σ *be a spanning forest for this initial data and let* $A_\Sigma = A_1 \cup A_2$ *be the union of the atoms associated with this initial data and spanning forest.*

Then the essential partition

$$\mathbf{V} = (U_1 \cup A_1, U_2 \cup A_2, U_3 - A_\Sigma)$$

has the tripod property in Q.

Proof. It suffices to verify that $\partial_\cup \mathbf{V}$ satisfies the tripod property in every cube in $\mathcal{Q}(Q;\mathcal{F})$.

Let $q \in \mathcal{Q}(Q;\mathcal{F})$. We consider two cases. Suppose first that $b = q \cap A_\Sigma$ is a building block, with A_Σ from Lemma 4.26. Let q' be the unique n-cube in $U_2 \cup U_3$ sharing a side with q. By Proposition 4.10, the essential partition $(q-b, b, q')$ of $q \cup q'$ satisfies the tripod property.

Suppose next that $A = q \cap A_\Sigma$ has an essential partition into two building blocks, say b_1 and b_2. By (ii), there are exactly two n-cubes q_1 and q_2 in $U_2 \cup U_3$ sharing a side with q. Let $f_1 = q \cap q_1$ and $f_2 = q \cap q_2$. By relabeling, we may assume that b_r is f_r-based for $r = 1, 2$. Since the building blocks b_1 and b_2 are centered and do not contain common n-cubes, we may assume, by relabeling again if necessary, that $b_2 \cap f_1 = \varnothing$. Since $b_1 \cup b_2$ is connected, it follows that $c_{bf} = b_1 \cap f_2$ must be an $(n-1)$-cube. We also note that the set $c_{bb} = (\partial b_1) \cap b_2$ is a unit $(n-1)$-cube and $(\partial b_1) \cap b_2 = b_1 \cap \partial b_2$. Define $E_1 = (\partial_\cup (q, q-b_1, q_1) - c_{bb}) \cup c_{bf}$ and $E_2 = \partial_\cup (q, q-b_2, q_2) - (c_{bb} \cup c_{bf})$.

Thus, by elementary modifications of the proof of Proposition 4.10, there exists, for $r = 1, 2$, an essential partition Δ_r of E_r satisfying the conditions of Definition 4.4, so that $\Delta = \Delta_1 \cup \Delta_2$ is an essential partition of $\partial_\cup (q, q-A, q_1 \cup q_2)$ satisfying the conditions of Definition 4.4. The claim follows. □

4.4. Neglected faces in $\mathcal{Q}(Q;\mathcal{F})$

We finish this section by a slight modification of our analysis for non-flat initial data. This is to compensate for the fact that while the spanning forest contains every subcube $q \in \mathcal{Q}(Q;\mathcal{F})$, some cubes q will have faces, contained in ∂Q, disjoint from atoms in $A_\Sigma = A_1 \cup A_2$. For example, consider Figure 20. It is easy to find a cube q in $\mathcal{Q}(Q;\mathcal{F})$ which meets more faces of ∂Q than $q \cap A_\Sigma$. Such cubes q are only of side-length 3, but this will create a problem in satisfying the tripod property when, in §5, we scale these configurations, and so preparations are given here. We make a formal definition.

Definition 4.29. Let $(Q, (\mathcal{F}^1, \mathcal{Q}_1'', q_1), (\mathcal{F}^2, \mathcal{Q}_2'', q_2))$ form non-flat initial data, Σ be a spanning forest, and $A_\Sigma = A_1 \cup A_2$ be the cubical set associated with Σ from Lemma 4.26. A cube $q \in \mathcal{Q}(Q;\mathcal{F})$ has an A_Σ*-neglected face* if q has more faces contained in ∂Q than $q \cap A_\Sigma$ has building blocks.

Remark 4.30. Note that, for each $q \in \mathcal{Q}(Q, \mathcal{F})$, $q \cap A_\Sigma$ is either a building block or a union of two building blocks.

Let $\mathcal{N}(Q; A_\Sigma)$ denote the collection of all A_Σ-neglected faces in cubes in $\mathcal{Q}(Q;\mathcal{F})$.

Definition 4.31. Suppose $q\in\mathcal{Q}(Q;\mathcal{F})$ has an A_Σ-neglected face f and let $p\in\{1,2\}$ be such that $f\subset|\mathcal{F}^p|$. Then f *admits a flat extension of* A_Σ if there exist $q'\in\mathcal{Q}(Q;\mathcal{F})$ adjacent to q and a face f' of q' contained in $|\mathcal{F}^p|$ so that $q'\cap A_p$ contains an f'-based atom and $f\cap f'$ is an $(n-2)$-cube. We call f' a *link of* A_Σ *into* f.

To motivate this terminology, consider a cube $q\in\mathcal{Q}(Q,\mathcal{F})$ having a neglected face f and let $q'\in\mathcal{Q}(Q;\mathcal{F})$ be the cube adjacent to q as in Definition 4.31. Then $q\cap A_\Sigma=q\cap A_r$ and $q'\cap A_\Sigma=q\cap A_p$, where $\{r,p\}=\{1,2\}$. Moreover, both cubes q and q' are F-based for $F\in\mathcal{F}^p$. Thus using a flat rearrangement, the atom $q'\cap A_\Sigma$ may be extended to a molecule by adding an atom which enters the cube q and is $f\cup f'$ based. This heuristics is made precise in §5.

Note that, in Figure 20, all neglected faces admit a flat extension of A_Σ. In general this is, however, not the case; see Figure 35. For this reason, we partition the neglected faces into collections, called pre-basins, so that each collection contains at least one neglected face admitting a flat extension. Note that pre-basins are always flat, in the sense that each pre-basin is contained in a single face in $\mathcal{F}^1\cup\mathcal{F}^2$.

Let $\mathcal{N}_{\mathrm{ext}}(Q;A_\Sigma)$ be the collection of all faces in $\mathcal{N}(Q;A_\Sigma)$ admitting a flat extension of A_Σ.

Definition 4.32. Given $p\in\{1,2\}$, a collection $C\subset\mathcal{N}(Q;A_\Sigma)$ is a *pre-basin on* $|\mathcal{F}^p|$ if

(PB1) $|C|\subset F$ for some $F\in\mathcal{F}^p$;

(PB2) $\Gamma(C)$ is connected; and

(PB3) $C\cap\mathcal{N}_{\mathrm{ext}}(Q;A_\Sigma)\neq\varnothing$,

Remark 4.33. It is easy to observe that the components of the graph $\mathcal{N}(Q;A_\Sigma)$ are pre-basins. Indeed, given a component $C\subset\mathcal{N}(Q;A_\Sigma)$, by the definitions of spanning forest and connected component, there exists a pair $\{f,f'\}$ where $f\in C$ and f' is a link of A_Σ to f.

This formulation of pre-basins is sufficient for all forthcoming constructions in dimensions $n>3$. In §5.3.4, when $n=3$, we will also need to subdivide pre-basins. We formalize this with the notion of a system of basins; however this procedure is (quite) general and need not be restricted only to dimension $n=3$. Note that, whereas a pre-basin always consists of neglected faces, a basin need not contain a neglected face; see Figure 22.

Given a pre-basin $C\subset\mathcal{N}(Q;A_\Sigma)$, we introduce a cell σ_C, called a *connecting cell*, as follows. By (PB3), we may fix $f_C\in C\cap\mathcal{N}_{\mathrm{ext}}(Q;A_\Sigma)$. Let f'_C be a link into f_C, and let q_C and q'_C denote the unique cubes in $\mathcal{Q}(Q;\mathcal{F})$ having f_C and f'_C as faces, respectively. Let σ_C be the connected component of f'_C-A_Σ meeting f_C in an $(n-2)$-cell, and set $\Omega_C=|C|\cup\sigma_C$. The cell Ω_C is called an *extension of* $|C|$ *to* f'_C.

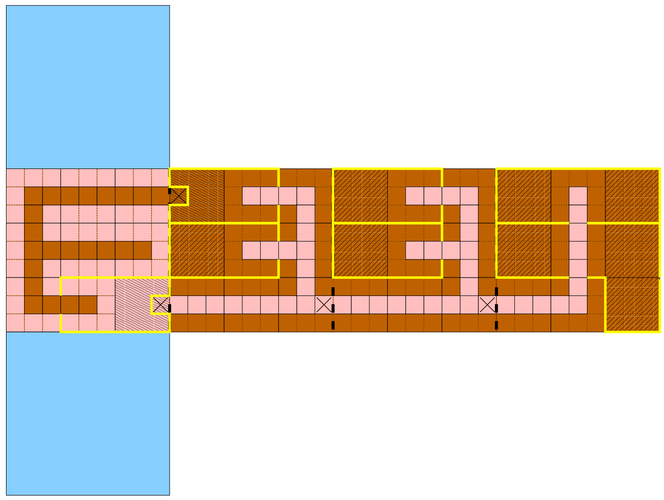

Figure 21. Extended pre-basins for a partition of $\mathcal{N}(Q;A_\Sigma)$ into 8 pre-basins given the data in Figure 20; the neglected faces are shaded.

Let $\mathfrak{P}$ be a partition of $\mathcal{N}(Q;A_\Sigma)$ into pre-basins, and suppose we have fixed, for each $C\in\mathfrak{P}$, an extension Ω_C of $|C|$; see Figure 21. Let $\Omega_{\mathfrak{P}}=\bigcup_{C\in\mathfrak{P}}\Omega_C$.

Definition 4.34. An essential partition $\mathcal{B}$ of $\Omega_{\mathfrak{P}}$ is a *system of basins* (*associated with* $\Omega_{\mathfrak{P}}$) if

(B1) each $B\in\mathcal{B}$ is a subset of $F\in\mathcal{F}^1\cup\mathcal{F}^2$;

(B2) $\Gamma(B^{\#})$ is connected for every $B\in\mathcal{B}$;

(B3) $\Gamma(B^{\#})$ admits a spanning tree;

(B4) $B\cap A_\Sigma$ contains a unit $(n-2)$-cube for every $B\in\mathcal{B}$; and

(B5) for every $B\in\mathcal{B}$ there exists $C\in\mathfrak{P}$ so that $B-|\mathcal{N}(Q;A_\Sigma)|$ is contained in a connecting cell σ_C.

The elements of $\mathcal{B}$ are called *basins*.

Note that the condition (B5) is more flexible than requiring that $B-|C|\subset\sigma_C$, as can be observed by contrasting Figure 22 with Figure 21.

Remark 4.35. The existence of a system of basins is straightforward given a partition $\mathfrak{P}$ of $\mathcal{N}(Q;A_\Sigma)$. Indeed, for every $C\in\mathfrak{P}$, fix $f_C\in C\cap\mathcal{N}_{\mathrm{ext}}(Q;A_\Sigma)$. Let f'_C be a link into f_C and let σ_C be a connecting cell. We then subdivide $\bigcup_{C\in\mathfrak{P}}\sigma_C$ into pairwise disjoint 1-fine sets σ'_C with connected graphs $\Gamma(\sigma'_C)$ so that the sets $B_C=|C|\cup\sigma'_C$ satisfy conditions (B2) and (B4) for every $C\in\mathfrak{B}$. Since $\Gamma(\sigma'_C)$ has valence less than $2(n-1)-1$ and $|C|$ is 3-fine, it is also straightforward to show that $\Gamma(B_C^{\#})$ admits a spanning tree. Clearly conditions (B1) and (B5) are satisfied. Thus $\mathcal{B}=\{B_C:C\in\mathfrak{P}\}$ is a system of basins.

Finally, we introduce a (flat) rearrangement along a system of basins. Let $\mathcal{B}$ be a

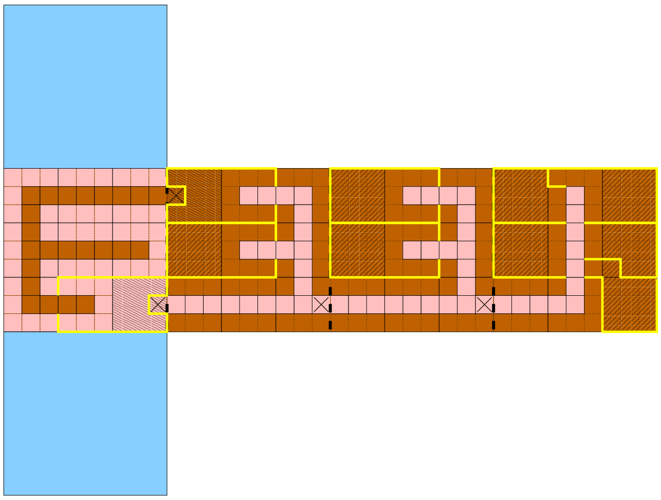

Figure 22. A partition of $\mathcal{N}(Q;A_\Sigma)$ into 10 basins associated with the data of Figure 21.

system of basins associated with $\Omega_{\mathfrak{P}}$, and let $B\in\mathcal{B}$. By (B5) we may fix $q_B\in A_\Sigma$ so that $B\cap q_B$ is an $(n-2)$-cube.

Let $F_B\in\mathcal{F}^1\cup\mathcal{F}^2$ be the unique face of Q satisfying (B1). Then the quadruple $(3Q,3F_B,3B^\#,3q_B)$ satisfies the conditions for flat initial data. The only modification is that $3Q$ and F_B now have side-length 27. We call $(3Q,3F_B,3B^\#,3q_B)$ *scaled flat initial data.*

By (B3), we may fix, for every $B\in\mathcal{B}$, a spanning tree Γ_B of $\Gamma(3B^\#\cup\{3q_B\})$. Similarly as in the proof of Lemma 4.17, we find a 1-fine atom A_{Γ_B} associated with the initial data $(3Q,3F_B,3B^\#,3q_B)$ and the spanning tree Γ_B. This observation is formalized as the next lemma, with the details left to the interested reader.

LEMMA 4.36. *Let Q be a cube of side-length 9 and $A_\Sigma\subset Q$ be a union of two atoms as in Lemma 4.26. Suppose that $\mathcal{B}$ is a system of basins associated with $\Omega_{\mathfrak{P}}$, where $\mathfrak{P}$ is a partition of $\mathcal{N}(Q;A_\Sigma)$ into pre-basins. For every $B\in\mathcal{B}$, let $(3Q,3F_B,3B^\#,3q_B)$ be a scaled flat initial data and Γ_B be a spanning tree of $\Gamma(3B^\#\cup\{3q_B\})$.*

Then there exist 1-fine pairwise disjoint atoms A_{Γ_B}, $B\in\mathcal{B}$, satisfying conditions (1)–(4) in Lemma 4.17 and so that $3A_\Sigma\cup\bigcup_{B\in\mathcal{B}}A_{\Gamma_B}$ is a pairwise disjoint union of two molecules.

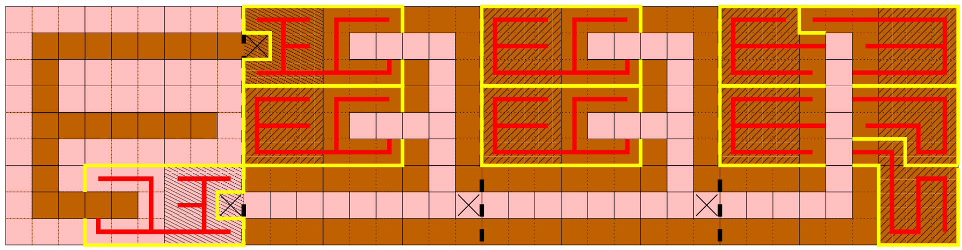

Figure 23. A selection of spanning trees associated the configuration in Figure 22.

5. Rough Rickman partitions

This section applies the elementary constructions from §4 to produce domains Ω_1, Ω_2, and Ω_3 which form a rough Rickman partition of $\mathbb{R}^n$, and proves Theorem 1.4 for $p{=}2$. The proof is based on the existence of uniform essential partitions associated with the exhaustion of $[0,\infty)^{n-1}\times\mathbb{R}=\bigcup_{k\geqslant 0} 3^k Q_0$, where $Q_0=[0,3]^{n-1}\times[-3,3]$.

THEOREM 5.1. *For $m\geqslant 0$, there exist essential partitions*

$$\mathbf{\Omega}_m=(\Omega_{m,1},\Omega_{m,2},\Omega_{m,3})$$

of n-cells $3^m(Q_0\cup([3,6]\times[0,3]^{n-1}))$ contained in $[0,\infty)^{n-1}\times\mathbb{R}$ with the following properties:

(1) *the sequence $(\mathbf{\Omega}_m)$ is stable, i.e.*

(1a) $\Omega_{m,p}\cap 3^{m-2}Q_0=\Omega_{m',p}\cap 3^{m-2}Q_0$ *for $m'>m>2$ and $p=1,2,3$,*

(1b) $\Omega_{m,3}\subset(\operatorname{int}[0,\infty)^{n-1})\times\mathbb{R}=\bigcup_{m\geqslant 0}\mathbf{\Omega}_m$;

(2) *each $\Omega_{m,p}$ is a dented molecule satisfying the following properties*:

(2a) *there exist $\nu\geqslant 1$, $\lambda>1$, and $\ell_0\geqslant 1$ depending only on n so that each* $\operatorname{hull}(\Omega_{m,p})$ *is a (ν,λ)-molecule with atom length at most ℓ_0, and*

(2b) *there exist $L\geqslant 1$ depending only on n and an L-bilipschitz homeomorphism* $(\Omega_{m,p},d_{\Omega_{m,p}})\to(\operatorname{hull}(\Omega_{m,p}),d_{\operatorname{hull}(\Omega_{m,p})})$ *which is the identity on* $\partial\operatorname{hull}(\Omega_{m,p})\cap\Omega_{m,p}$;

(3) *each $\mathbf{\Omega}_m$, $m\geqslant 1$, satisfies the tripod property.*

For $p=1,2,3$, each domain $\Omega_p=\bigcup_{m\geqslant 0}\Omega_{m,p}$ in its inner metric d_{Ω_p} is bilipschitz equivalent to $\mathbb{R}^{n-1}\times[0,\infty)$. Moreover, there exist bilipschitz homeomorphisms

$$\phi_1\colon[0,\infty)^{n-1}\times[0,\infty)\longrightarrow(\Omega_1,d_{\Omega_1})\quad\textit{and}\quad\phi_2\colon[0,\infty)^{n-1}\times(-\infty,0]\longrightarrow(\Omega_2,d_{\Omega_2})$$

which restrict to the identity mappings on $\partial[0,\infty)^{n-1}\times[0,\infty)$ and $\partial[0,\infty)^{n-1}\times(-\infty,0]$, respectively; *the boundary $\partial[0,\infty)^{n-1}$ is understood relative to $\mathbb{R}^{n-1}$.*

Conditions (1)–(3) have the following interpretations. Condition (1) refers to an induction process, which consists of two main steps: scaling and rearranging, and allows us to paste the essential partitions $\mathbf{\Omega}_m$ together. Condition (2) yields that the domains $\Omega_{m,j}$ are uniformly bilipschitz equivalent to cubes $[0,3^m]^n$. Finally, (3) ensures that $\mathrm{dist}_{\mathcal{H}}(\partial_\cup \mathbf{\Omega}_m, \partial_\cap \mathbf{\Omega}_m) \leqslant 6$ in the sup-metric; compare with (4.3). We also observe the following corollary; see §5.4.

Corollary 5.2. *Let $p=1,2,3$ and $m\geqslant 1$. Then $\Omega_{m,p}$ are John domains with John constant depending only on n. Furthermore, each Ω_p is a uniform domain.*

Proof of Theorem 1.4 (for $p=2$) given Theorem 5.1. Let $\mathbf{\Omega}'=(\Omega_1',\Omega_2',\Omega_3')$ be the essential partition of $[0,\infty)^{n-1}\times\mathbb{R}$ from Theorem 5.1. By (1a) and (3) in Theorem 5.1, $\mathbf{\Omega}'$ satisfies the tripod property.

We subdivide $\mathbb{R}^n$ into 2^{n-1} congruent subsets $W_1,...,W_{2^{n-1}}$, where $W_1=[0,\infty)^{n-1}\times\mathbb{R}$. Since $\Omega_3'\subset \operatorname{int} W_1$, by reflecting Ω_3' with respect to the common sides of $W_1,...,W_{2^{n-1}}$ we obtain pairwise disjoint domains $\Omega_4',...,\Omega_{2^{n-1}+2}'$. The unions of the corresponding reflections of Ω_1' and Ω_2' are the domains Ω_1 and Ω_2 claimed in Theorem 1.4. Thus Ω_1 and Ω_2 are connected.

Let $\phi_1: [0,\infty)^{n-1}\times[0,\infty)\to(\Omega_1',d_{\Omega_1'})$ and $\phi_2: [0,\infty)^{n-1}\times(-\infty,0]\to(\Omega_2',d_{\Omega_2'})$ be two bilipschitz homeomorphisms which reduce to the identity mapping on the boundary, a consequence of Theorem 5.1. Reflections across the pairwise common sides of the domains $W_1,...,W_{2^{n-1}}$ extend ϕ_1 and ϕ_2 to bilipschitz homeomorphisms

$$\psi_1: \mathbb{R}^{n-1}\times[0,\infty)\longrightarrow(\Omega_1,d_{\Omega_1}) \quad\text{and}\quad \psi_2: \mathbb{R}^{n-1}\times(-\infty,0]\longrightarrow(\Omega_2,d_{\Omega_2}).$$

Finally, if

$$\Omega_3=\Omega_3'\cup...\cup\Omega_{2^{n-1}+2}' \quad\text{and}\quad \mathbf{\Omega}=(\Omega_1,\Omega_2,\Omega_3),$$

condition (3) in Theorem 5.1 ensures that $\mathbf{\Omega}$ is a rough Rickman partition satisfying the tripod property. □

5.1. Proof of Theorem 5.1 – First steps

We begin the proof of Theorem 5.1 in this section by explicitly giving the initial steps of the inductive construction of the partitions $\mathbf{\Omega}_m$. The general induction is based on rearrangements in three types of cubes and their successive scalings, and we consider these rearrangements in detail in §5.2. We complete the proof finally in §5.4.

Let Ω be a 3-fine n-cell and suppose that $\mathbf{U}=(U_1,U_2,U_3)$ is an essential partition of Ω into n-cells. A cube $Q\in\Omega^*$ of side-length 3 is a $\mathbf{U}$-*cube* if there exists $i\in\{1,2,3\}$ for which $Q\subset U_i$. The index i is the *color of Q in* $\mathbf{U}$, and the indices $\{1,2,3\}\setminus\{i\}$ are *complementary indices* (*of the color of Q*). Let also

$$\mathcal{Q}_\partial(\mathbf{U})=\{Q\in|\mathbf{U}|^*: Q\cap\partial_\cup\mathbf{U} \text{ contains an } (n-1)\text{-cell}\}.$$

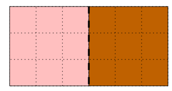

Figure 24. The faces of Ω_1 in $\partial_\cup \mathbf{\Omega}$

5.1.1. The initial step (step 0)

We begin with the n-cubes

$$\Omega_1 = [0,3]^n, \quad \Omega_2 = [0,3]^{n-1} \times [-3,0], \quad \text{and} \quad \Omega_3 = [3,6] \times [0,3]^{n-1}$$

of side-length 3, and set

$$\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$$

and $\Omega = |\mathbf{\Omega}| = \Omega_1 \cup \Omega_2 \cup \Omega_3$; see Figure 24.

For consistency, let also

$$\mathbf{\Omega}_0 = (\Omega_{0,1}, \Omega_{0,2}, \Omega_{0,3}) = (\Omega_1, \Omega_2, \Omega_3).$$

Note that $\mathbf{\Omega}_0$ does not satisfy the tripod property for the (trivial) reason that $\Omega_2 \cap \Omega_3$ is not $(n-1)$-dimensional. However, we note that $\partial_\cap \mathbf{\Omega} = \Omega_1 \cap \Omega_2 \cap \Omega_3$ is an $(n-2)$-cube.

In anticipation of the forthcoming induction step, we note that $\partial_\cup \mathbf{\Omega}_0 \subset [0,3]^n$. Furthermore, the cube $[0,3]^n$ is contained in the domain $\Omega_{0,1}$ but has one $(n-1)$-dimensional face contained in $\partial \Omega_{0,2}$ and one in $\Omega_{0,3}$. The cube $[0,3]^n$ will therefore be an example of a $\mathcal{C}$-cube. This is one of the three general categories we will use $\mathcal{C}$-cubes ($\mathcal{C}$ for color), $\mathcal{D}$-cubes ($\mathcal{D}$ for dent), and $\mathcal{N}$-cubes ($\mathcal{N}$ for neglected). They are formally introduced in §5.2, but have clear antecedents from various local rearrangements in §4.

5.1.2. First rearrangement

First scale $\mathbf{\Omega}_0$ by 3, and let

$$\mathbf{\Omega}'_1 = 3\mathbf{\Omega}_0 = (3\Omega_{0,1}, 3\Omega_{0,2}, 3\Omega_{0,3}).$$

To rearrange $\mathbf{\Omega}'_1$ to achieve the tripod property and properties (1)–(3) in Theorem 5.1, we modify $\mathbf{\Omega}'_1$ using atoms which allow respectively $3\Omega_{0,2}$ and $3\Omega_{0,3}$ to penetrate $3\Omega_{0,1}$; this will produce $\mathbf{\Omega}_1$. We apply Lemma 4.26 to $C = 3\Omega_{0,1}$ and thus obtain the essential partition

$$\mathbf{\Omega}_1 = (\Omega_{1,1}, \Omega_{1,2}, \Omega_{1,3}) = (3\Omega_{0,1} - (A_2 \cup A_3), 3\Omega_{0,2} \cup A_2, 3\Omega_{0,3} \cup A_3)$$

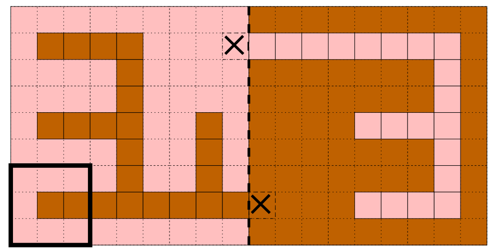

Figure 25. An example of the evolution of $\partial_\cup \mathbf{\Omega}_1$ with the cube $[0,3]^3$ emphasized.

of $|\mathbf{\Omega}'_1|$ into n-cells satisfying the tripod property, where A_2 and A_3 are atoms from the process of Lemma 4.26, see Figure 25; rearrangements of this type will be called $\mathcal{C}$-modifications (Lemma 5.15) in the inductive construction, since they are performed in a scaled copy of a $\mathcal{C}$-cube.

In the proof of Lemma 4.26, we are free to use any maximal forest Σ. In particular, we may assume that $[0,3]^n$ is a leaf of Σ as in Figure 25; this choice is used to obtain stability condition (1a). Thus we arrive at $\mathbf{\Omega}_1$ in accord with the conditions of Theorem 5.1.

As orientation toward the general induction step, we note that $\partial_\cup \mathbf{\Omega}_1$ is contained in a union of n-cubes of side-length 3 contained in $3\Omega_{0,1}$. Indeed, let

$$\mathcal{Q} = \{Q \in \mathcal{Q}_\partial(\mathbf{\Omega}'_1) : Q \subset 3[0,3]^n = 3\Omega_{0,1}\}.$$

Then $\partial_\cup \mathbf{\Omega}_1 \subset |\mathcal{Q}|$.

Moreover, for all $Q \in \mathcal{Q}$, there exists exactly one $j_Q \in \{2,3\}$ so that $\mathrm{cl}(\mathrm{int}\, Q \cap \Omega_{1,j_Q})$ is a building block. If $Q \cap \Omega_{1,j_Q} = \mathrm{cl}(\mathrm{int}\, Q \cap \Omega_{1,j_Q})$, then Q is a $\mathcal{D}$-cube. Otherwise, Q is an $\mathcal{N}$-cube.

5.1.3. The second step

Whereas the essential partition $\mathbf{\Omega}_0$ was explicitly chosen and $\mathbf{\Omega}_1$ was described using Lemma 4.26, at this point we only give a heuristic description for $\mathbf{\Omega}_2$.

The essential partition $\mathbf{\Omega}_2$ is obtained from $\mathbf{\Omega}_1$ by first defining $\mathbf{\Omega}'_2 = 3\mathbf{\Omega}_1$ and rearranging $3\Omega_{1,1}$, $3\Omega_{1,2}$, and $3\Omega_{1,3}$ with flat rearrangements (Lemma 4.17) and by flat rearrangements in basins (Lemma 4.36); we attach atoms of side-length 1 to the atoms $3A_2$ and $3A_3$ and, correspondingly, remove them from $3\Omega_{1,1}$. Figure 26 illustrates this step. This modification will be called a secondary $\mathcal{C}$-modification and will be formalized in Lemma 5.20. Note, however, that in order to satisfy the stability requirement (1a), we also impose the additional condition that $\mathbf{\Omega}_2 \cap [0,3]^n = \mathbf{\Omega}_1 \cap [0,3]^n$. This is possible, since $\Omega_{1,3} \cap [0,3]^n$ is a leaf.

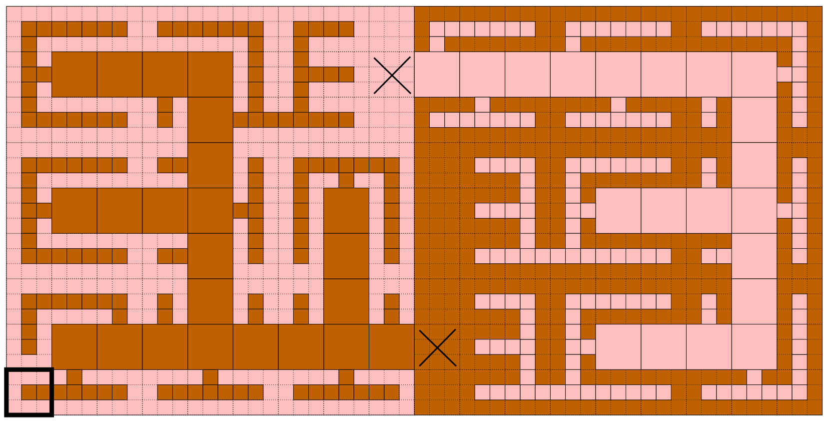

Figure 26. An example of $\mathbf{\Omega}_2$ with the cube $[0,3]^3$ highlighted.

5.2. $\mathcal{C}$-, $\mathcal{D}$-, and $\mathcal{N}$-cubes

With this preparation, we formally define $\mathcal{C}$-, $\mathcal{D}$-, and $\mathcal{N}$-cubes; primary cubes have side-length 3 and secondary cubes side-length 9. The corresponding rearrangements, based on flat and non-flat rearrangements in §4, are then discussed in the following sections. Note that, if Q is a primary or a secondary cube, the corresponding rearrangement is performed in $3Q$.

Restricting exclusively to these cubes in the iteration process provides a systematic rearrangement process. We obtain the sequence $\{\mathbf{\Omega}_m\}_{m\geqslant 0}$, using scalings and rearrangements, in such a way that for each $m\geqslant 0$ there exists an essentially disjoint collection $\mathfrak{L}_m$ of primary and secondary cubes (of different types) which covers $\partial_\cup \mathbf{\Omega}_m$. After scaling $\mathbf{\Omega}_m$ by 3, we perform appropriate rearrangements of the right type in each cube in $3\mathfrak{L}_m$. This yields a new essential partition $\mathbf{\Omega}_{m+1}$ and a new list $\mathfrak{L}_{m+1}$ of essentially disjoint cubes which also have the property $\partial_\cup \mathbf{\Omega}_{m+1} \subset |\mathfrak{L}_{m+1}|$. The rearrangements in these cubes are mutually independent, and it follows from the properties of rearrangements in §4 that $\mathbf{\Omega}_{m+1}$ satisfies the tripod property. We discuss the list $\mathfrak{L}_m$ and this inductive step in §5.2.6.

Although there are a priori six different types of cubes, only four types of rearrangements occur here. The reason is that all $\mathcal{N}$-cubes are contained in secondary $\mathcal{C}$-cubes and secondary $\mathcal{N}$-cubes, and secondary $\mathcal{D}$-cubes never appear. In Table 27 we list the four types of rearrangements and descendants they produce; we use the subscript 2 to denote secondary cubes.

Q	modification in $3Q$	descendant(s) in $3Q$
$\mathcal{C}$-cube	$\mathcal{C}$	$\mathcal{C}_2$
$\mathcal{D}$-cube	$\mathcal{D}$	$\mathcal{C}$, $\mathcal{D}$
$\mathcal{C}_2$	$\mathcal{C}_2$	$\mathcal{C}$, $\mathcal{D}$, $\mathcal{N}_2$
$\mathcal{N}_2$	$\mathcal{N}_2$	$\mathcal{C}$, $\mathcal{D}$, $\mathcal{N}_2$

Table 27. Cubes, modifications, and descendants.

5.2.1. $\mathcal{C}$-cubes

Let U be an n-cell and $\mathbf{U}=(U_1,U_2,U_3)$ be an essential partition of U. Recall that $\mathcal{Q}_\partial(\mathbf{U})$ is the collection of cubes Q in $|\mathbf{U}|^*$ for which $Q\cap\partial_\cup\mathbf{U}$ is an $(n-1)$-cell.

Let $Q\in\mathcal{Q}_\partial(\mathbf{U})$ be a $\mathbf{U}$-cube of color $i\in\{1,2,3\}$, and j and k be complementary indices. For $p=j,k$, let $\mathcal{Q}'_p(Q)$ be the collection of all unit n-cubes in $Q^\#$ meeting U_p in an $(n-1)$-cube, and let $\mathcal{Q}'(Q)=\mathcal{Q}'_j(Q)\cup\mathcal{Q}'_k(Q)$. Let $\mathcal{Q}'_c(Q)$ be the cubes in $\mathcal{Q}'(Q)$ having (maximal) valence $2(n-1)$ in the adjacency graph $\Gamma(\mathcal{Q}'(Q))$ as in §4.3.

Definition 5.3. Let $\mathbf{U}=(U_1,U_2,U_3)$ be an essential partition. A $\mathbf{U}$-cube $Q\in\mathcal{Q}_\partial(\mathbf{U})$ of color i is a *$\mathcal{C}$-cube in* $\mathbf{U}$ if, for complimentary colors j and k,

(i) there are unit n-cubes $q_j\subset U_j$ and $q_k\subset U_k$ with $q_j\cap q_k=\varnothing$ such that both cubes q_j and q_k have a face contained in ∂Q; let q'_j and q'_k be the unique cubes in $\mathcal{Q}'(Q)$ which share a face with q_j and q_k, respectively; and

(ii) the adjacency graph $\Gamma(\{q_k\}\cup(\mathcal{Q}'_j(Q)\setminus(\mathcal{Q}'_c(Q)\cup\{q'_j\})))$ is connected.

The collection of $\mathcal{C}$-cubes in $\mathbf{U}$ is denoted by $\mathcal{C}(\mathbf{U})$. Note that each $Q\in\mathcal{C}(\mathbf{U})$ satisfies $Q\cap\partial_\cup\mathbf{U}\subset\partial Q$, since $\mathcal{C}$-cubes are $\mathbf{U}$-cubes.

Remark 5.4. Definition 5.3 formalizes the heuristic properties of $\mathcal{C}$-cubes discussed in §5.1.1. First, a $\mathcal{C}$-cube Q is contained in one element of the essential partition, and, second, Q meets the other two elements in a codimension-1 set (item (i)). Item (ii) formalizes a necessary condition for a rearrangement to extend color k between i and j in the scaled copy of Q. For $\mathbf{\Omega}_0$ and $\mathbf{\Omega}_1$ this condition could be simplified to the condition that $Q\cap\Omega_j$ is a union of faces of Q.

Definition 5.5. Let $\mathbf{U}$ and $\mathbf{V}$ be essential partitions satisfying $|\mathbf{U}|=3|\mathbf{V}|$. A cube Q of side-length 9 is a *secondary $\mathcal{C}$-cube in* $\mathbf{U}$ *with respect to* $\mathbf{V}$ if $\frac{1}{3}Q$ is a $\mathcal{C}$-cube with respect to $\mathbf{V}$.

The collection of secondary $\mathcal{C}$-cubes in $\mathbf{U}$ with respect to $\mathbf{V}$ is denoted by $\mathcal{C}_2(\mathbf{U};\mathbf{V})$.

Remark 5.6. Note that in Definition 5.5 we do not require $\mathbf{U}\cap Q=3\big(\mathbf{V}\cap\frac{1}{3}Q\big)$. In fact, if $\mathbf{U}\cap Q$ is obtained by a $\mathcal{C}$-modification in Q (see §5.2.4), then $\mathbf{U}\cap Q\neq 3\mathbf{V}\cap Q$.

5.2.2. $\mathcal{D}$- and $\mathcal{N}$-cubes

Let $\mathbf{U}=(U_1,U_2,U_3)$ and $\mathbf{V}=(V_1,V_2,V_3)$ be essential partitions satisfying $|\mathbf{U}|=3|\mathbf{V}|$. We first discuss $\mathcal{D}$-cubes.

Definition 5.7. A cube $Q\in\mathcal{Q}_\partial(\mathbf{U})$ of side-length 3 is a *$\mathcal{D}$-cube in* $\mathbf{U}$ *relative to* $\mathbf{V}$ if

(1) Q is a $3\mathbf{V}$-cube of color i;

(2) Q is not a $\mathbf{U}$-cube;

(3) there exists complementary colors j and k for which $A:=Q\cap U_j$ is an n-cell and Q has no neglected faces, and $(\operatorname{int} Q)\cap U_k=\varnothing$;

(4) A is either a $(Q\cap\partial 3V_i)$-based building block in Q or a union of two building blocks based on different faces of Q; and

(5) $(Q-A,A,\Omega)$ has the tripod property, where Ω is the smallest n-cell consisting of n-cubes of side-length 3 for which $A\cap\partial Q\subset\Omega$.

The collection of all $\mathcal{D}$-cubes in $\mathbf{U}$ with respect to $\mathbf{V}$ is denoted $\mathcal{D}(\mathbf{U};\mathbf{V})$. Note that in Definition 5.7, A in (3) is always a 1-fine atom. If A in (3) and (4) is a building block, we say that Q is a *$\mathcal{D}$-cube of type* 1. Otherwise, Q is a *$\mathcal{D}$-cube of type* 2.

Since by definition $\mathcal{D}$-cubes have no neglected faces, we also need $\mathcal{N}$-cubes.

Definition 5.8. A cube $Q\in\mathcal{Q}_\partial(\mathbf{U})$ is an *$\mathcal{N}$-cube in* $\mathbf{U}$ *relative to* $\mathbf{V}$ if

(1) Q is a $3\mathbf{V}$-cube of color i;

(2) Q is not a $\mathbf{U}$-cube;

(3) there exists a unique complementary color j such that $A:=\operatorname{cl}(\operatorname{int} Q\cap U_j)$ is a $(Q\cap\partial 3V_i)$-based building block in Q while $(\operatorname{int} Q)\cap U_k=\varnothing$;

(4) Q has a neglected face contained in $3V_j$; and

(5) $(Q-A,A,\Omega)$ has the tripod property, where Ω is the smallest n-cell consisting of n-cubes of side-length 3 for which $A\cap\partial Q\subset\Omega$.

The collection of all $\mathcal{N}$-cubes in $\mathbf{U}$ with respect to $\mathbf{V}$ is denoted $\mathcal{N}(\mathbf{U};\mathbf{V})$. We define secondary $\mathcal{N}$-cubes as follows.

Definition 5.9. Let $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ be essential partitions satisfying $|\mathbf{U}|=3|\mathbf{V}|=9|\mathbf{W}|$. A cube Q of side-length 9 is a *secondary $\mathcal{N}$-cube in* $\mathbf{U}$ (*with respect to* $\mathcal{N}(\mathbf{V};\mathbf{W})$) if $\frac{1}{3}Q$ is an $\mathcal{N}$-cube in $\mathbf{V}$ relative to $\mathbf{W}$.

We let $\mathcal{N}_2(\mathbf{U};\mathbf{V},\mathbf{W})$ denote the collection of all secondary $\mathcal{N}$-cubes in $\mathbf{U}$ with respect to $\mathcal{N}(\mathbf{V};\mathbf{W})$.

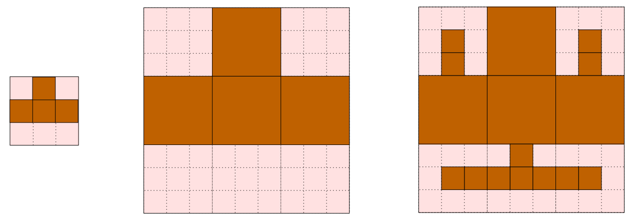

Figure 28. An example of an essential partition $\mathbf{V}$ in Q, and essential partitions $3\mathbf{V}$ and $\mathbf{U}$ in $D=3Q$ for a building block in Figure 18.

5.2.3. $\mathcal{D}$-modifications

We consider first $\mathcal{D}$-cubes of type 1. The first rearrangement is called a *$\mathcal{D}$-modification*, which has already been anticipated by Lemma 4.17 and Corollary 4.20.

LEMMA 5.10. ($\mathcal{D}$-modification of type 1) *Let V be an n-cell, $\mathbf{V}=(V_1,V_2,V_3)$ and $\mathbf{W}=(W_1,W_2,W_3)$ be essential partitions satisfying $V=|\mathbf{V}|=3|\mathbf{W}|$, and let $Q\in\mathcal{D}(\mathbf{V};\mathbf{W})$ be a $\mathcal{D}$-cube of type 1; let i be the color of $\frac{1}{3}Q$ in $\mathbf{W}$ and let j be such that $A:=Q\cap V_j$ is an F-based building block, where F is a face of Q. Then there exists a pairwise disjoint union of atoms $B_\Sigma\subset 3Q$ composed of 1-fine $3F$-based building blocks on the boundary of $3Q$, so that the n-cells $U_i=3V_i-B_\Sigma$, $U_j=3V_j\cup B_\Sigma$, and $U_k=3V_k$, where k is the other complementary index, form an essential partition*

$$\mathbf{U}=(U_1,U_2,U_3)$$

of $|3\mathbf{V}|$ satisfying

(1) $B_\Sigma\cap\partial(3Q)\subset\partial_\cup 3\mathbf{V}$;
(2) $\partial_\cup\mathbf{U}\cap 3Q\subset|\mathcal{C}(\mathbf{U})|\cup|\mathcal{D}(\mathbf{U};\mathbf{V})|$; *and*
(3) B_Σ *is an atom for $n>3$ and consists of at most 3 components for $n=3$.*

Furthermore, $\mathbf{U}$ has the tripod property in $3Q$.

Convention. Before giving the proof of Lemma 5.10, we emphasize that the figures in this section (e.g. in Figure 28) use only the two complementary colors j and k. The third color, the color i of the cube itself, never appears.

Proof of Lemma 5.10. It suffices to find planar initial data for Corollary 4.20, the claim then follows from Lemma 4.19. Note that Corollary 4.20 is necessary only for $n=3$, since for $n>3$ we may use Lemma 4.17.

Define $F'=3F\cap 3V_i$, that is, $F'=3F-3A$. For $n>3$, F' is connected. For $n=3$, F' is at most three 2-cells. Let $D=3Q$.

Let D^* be the 3-regular subdivision of D and let $(D^*)'\subset D^*$ be the subset of cubes having a face contained in F'. For $n>3$ we fix a unit cube Q_1 in A. Then $Q_1\cap F'$ is

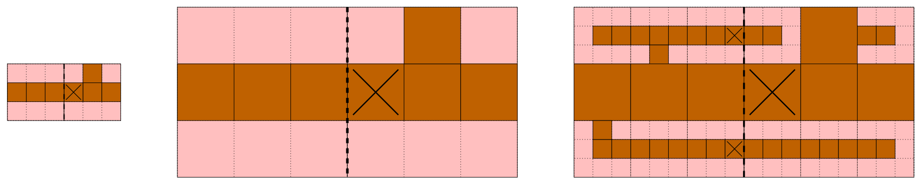

Figure 29. Analogue of Figure 28 for $\mathcal{D}$-cube of type 2.

an $(n-2)$-cell. For $n=3$, we fix unit cubes $Q_1, ..., Q_p$ in A, where p is the number of components of F'.

When $n>3$ we choose a maximal tree $\Sigma \in \Gamma((D^*)' \cup \{Q_1\})$, and for $n=3$, we fix a maximal forest $\Sigma = \Gamma_1 \cup ... \cup \Gamma_p$ in $\Gamma((D^*)' \cup \{Q_1, ..., Q_p\})$. The vertex sets of the trees Γ_i now give the required partition for $(D^*)'$. Corollary 4.20 yields a 1-fine set B_Σ whose components are $3F$-based atoms. The tripod property in D for $\mathbf{U}$ follows from Lemma 4.19, and condition (4) in Lemma 4.17 shows that $B_\Sigma \subset F \cup \operatorname{int} D$.

Assertions (1) and (2) follow from Corollary 4.20, the fact that cubes in $(D^*)'$ are $\mathcal{D}$-cubes in $\mathcal{D}(\mathbf{U}; \mathbf{V})$ and the observation that $3A = |\mathcal{C}(\mathbf{U})| \cap D$. □

For $\mathcal{D}$-cubes of type 2, the corresponding arrangement is also called a *$\mathcal{D}$-modification.*

LEMMA 5.11. ($\mathcal{D}$-modification of type 2) *Let V be an n-cell, let $\mathbf{V} = (V_1, V_2, V_3)$ and $\mathbf{W} = (W_1, W_2, W_3)$ be essential partitions satisfying $V = |\mathbf{V}| = 3|\mathbf{W}|$, and $Q \in \mathcal{D}(\mathbf{V}; \mathbf{W})$ be a $\mathcal{D}$-cube of type 2; let i be the color of $\frac{1}{3}Q$ in $\mathbf{W}$ and take j so that $A := Q \cap V_j = B \cup B'$ is an atom, where B and B' are essentially disjoint building blocks. Then there exists a pairwise disjoint union $A_\Sigma \subset 3Q$ of 1-fine atoms on the boundary of $3Q$ consisting of building blocks with*

$$\mathbf{U} = (U_1, U_2, U_3)$$

being an essential partition of $3|\mathbf{V}|$ by n-cells satisfying the tripod property in $3Q$. Here $U_i = 3V_i - A_\Sigma$, $U_j = 3V_j \cup A_\Sigma$, and $U_k = 3V_k$, where k is the remaining complementary index. Moreover,

(1) $A_\Sigma \cap \partial(3Q) \subset \partial_\cup 3\mathbf{V}$;

(2) $\partial_\cup \mathbf{U} \cap 3Q \subset |\mathcal{C}(\mathbf{U})| \cup |\mathcal{D}(\mathbf{U}; \mathbf{V})|$; *and*

(3) A_Σ *is an atom for $n>3$ and consists of at most four components for $n=3$.*

Finally, $\mathbf{U}$ has the tripod property in $3Q$.

Proof. This case uses Lemma 4.26 in place of Corollary 4.20, and Lemma 4.28 in place of Lemma 4.19.

We may assume that $i=1$, $j=2$, and $k=3$, and that B and B' are f- and f'-based, respectively. Let $D = 3Q$.

Let $\mathcal{Q}'$ be the collection of the cubes in D^* meeting $f\cup f'$ and not contained in $3A^\#$. Recall that $\Gamma(\mathcal{Q}')$ is the adjacency graph of the cubes in $\mathcal{Q}'$. For $n>3$, $\Gamma(\mathcal{Q}')$ is connected, and we may fix a cube $q\in 3A^\#$ of side-length 3 and a maximal tree $\Sigma\subset\Gamma(\mathcal{Q}'\cup\{q\})$. It is a simple observation that we may now apply Lemma 4.26 directly to Σ and obtain a 1-fine atom A_Σ satisfying (1)–(6) therein.

For $n=3$, we observe that $\Gamma(\mathcal{Q}')$ has at most four components $\Gamma_1, ..., \Gamma_p$, $p\leqslant 4$; Figure 29 illustrates $p=3$. It is easy to observe that we may fix pairwise essentially disjoint cubes $q_1, ..., q_p$ in $3A^*$ so that $\Gamma(\Gamma_i\cup\{q_i\})$ is connected, and thus create a maximal forest $\Sigma=\Sigma_1\cup...\cup\Sigma_p$ with $\Sigma_i\subset\Gamma(\Gamma_i\cup\{q_i\})$. A slight modification of Lemma 4.26 yields $A_\Sigma=A_{\Sigma_1}\cup...\cup A_{\Sigma_p}$, where A_{Σ_i} is a 1-fine atom satisfying (1)–(6) therein.

In both cases, $\mathbf{U}=(3V_1-A_\Sigma, 3V_2\cup A_\Sigma, 3V_3)$ satisfies the required conditions. □

The essential properties of $\mathcal{D}$-modifications are summarized in the next two corollaries.

Corollary 5.12. *Let* $\mathbf{V}$, Q, A, $\mathbf{U}$, *and* $\{i,j,k\}=\{1,2,3\}$ *be as in Lemma* 5.10 *or as in Lemma* 5.11. *Then* $\mathbf{U}\cap 3Q$ *satisfies the tripod property and in addition*

(a) $\partial_\cup \mathbf{U}\cap 3Q\subset|\mathcal{C}(\mathbf{U})|\cup|\mathcal{D}(\mathbf{U};\mathbf{V})|$; *and*

(b) $\mathcal{C}(\mathbf{U}\cap 3Q)=(3A)^*$.

Moreover, to each $f\in(((\partial_\cup\mathbf{V})\cap Q)-A)^\#$ *corresponds one* $3f$*-based building block in* U_j.

By condition (1) in Lemmas 5.10 and 5.11, $\mathcal{D}$-modifications are performed independently in each cube of $3\mathcal{D}(\mathbf{V};\mathbf{W})$ in the sense that, given two adjacent $\mathcal{D}$-cubes Q and Q' in $\mathcal{D}(\mathbf{V};\mathbf{W})$, all $\mathcal{D}$-modifications (of types 1 and 2) in $3Q$ and $3Q'$ leave the essential partition $3\mathbf{V}$ unmodified on the common face $3Q\cap 3Q'$. This is summarized in the following definition and corollary; see Definition 4.13 for the meaning of $\widetilde{\Gamma}(\cdot)$ and $\ell_{\mathrm{bb}}(\cdot)$.

Let $\mathbf{V}=(V_1, V_2, V_3)$ be an essential partition of an n-cell by n-cells so that V_p is a dented molecule for $p=1,2,3$. Let $\mathbf{W}=(W_1, W_2, W_3)$ be an essential partition satisfying $|\mathbf{V}|=3|\mathbf{W}|$ and let $\mathcal{D}'\subset\mathcal{D}(\mathbf{V},\mathbf{W})$ be a non-empty subfamily.

Definition 5.13. An essential partition $\mathbf{U}$ of $3|\mathbf{V}|$ into n-cells is *obtained by $\mathcal{D}$-modifications in $\mathcal{D}'$ from essential partitions* $\mathbf{V}$ *relative to* $\mathbf{W}$ if $\mathbf{U}$ satisfies the tripod property in each cube in $3\mathcal{D}'$ and

(a) $\mathbf{U}-|3\mathcal{D}'|=3\mathbf{V}-|3\mathcal{D}'|$;

(b) for every cube $C\in 3\mathcal{D}'$, the essential partition $\mathbf{U}\cap C$ is obtained by a $\mathcal{D}$-modification, that is, $\mathbf{U}$ has the properties (1)–(3) of Lemma 5.10 or 5.11 relative to C;

(c) each leaf $A\in\Gamma(U_i)$ is a 1-fine atom adjacent to a 3-fine atom $A'=3a'$, where a' is a leaf in $\Gamma(V_i)$; and

(d) to each leaf $a\in\Gamma(V_i)$ correspond at most $3\max_{a'}\ell_{\mathrm{bb}}(a')$ leaves in $\Gamma(U_i)$ adjacent to $3a\in\Gamma(U_i)$, where the maximum is taken over the leaves a' in $\Gamma(V_i)$.

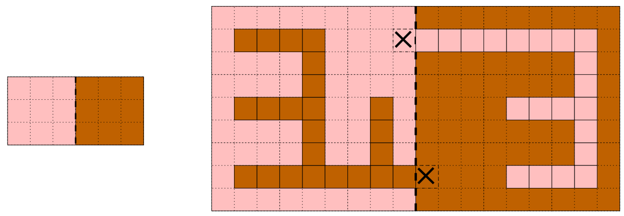

Figure 30. A cube Q and an essential partition $\mathbf{U}$ in $3Q$.

For $\mathcal{D}'=\mathcal{D}(\mathbf{V};\mathbf{W})$, we say that $\mathbf{U}$ is *obtained by $\mathcal{D}$-modification from $\mathbf{V}$ relative to $\mathbf{W}$*.

COROLLARY 5.14. *Let $\mathbf{V}=(V_1,V_2,V_3)$ be an essential partition of an n-cell by n-cells so that V_p is a dented molecule for $p=1,2,3$, and let $\mathbf{W}=(W_1,W_2,W_3)$ be an essential partition satisfying $|\mathbf{V}|=3|\mathbf{W}|$. Given a non-empty subfamily $\mathcal{D}'\subset\mathcal{D}(\mathbf{V};\mathbf{W})$, there exists an essential partition $\mathbf{U}$ which is obtained by $\mathcal{D}$-modification in $\mathcal{D}'$ from $\mathbf{V}$ relative to $\mathbf{W}$.*

5.2.4. $\mathcal{C}$-modification

The following rearrangement is a *$\mathcal{C}$-modification*.

LEMMA 5.15. *Let V be an n-cell and $\mathbf{V}=(V_1,V_2,V_3)$ be an essential partition of V. Suppose $Q\in\mathcal{C}(\mathbf{V})$ has color i in $\mathbf{V}$, and let j and k be complementary colors. Then there exist atoms A_j and A_k in $3Q$ which are composed of building blocks along $\partial(3Q)$ so that $U_i=3V_i-(A_j\cup A_k)$, $U_j=3V_j\cup A_j$, and $U_k=3V_k\cup A_k$ are n-cells and*

$$\mathbf{U}=(U_1,U_2,U_3) \tag{5.1}$$

is an essential partition of $3V$ into n-cells having the tripod property in $3Q$. Moreover,

(1) *$(A_j\cup A_k)\cap\partial(3Q)\subset\partial_\cup 3\mathbf{V}$; and*

(2) *$(\partial_\cup\mathbf{U})\cap 3Q\subset|\mathcal{D}(\mathbf{U};\mathbf{V})|\cup|\mathcal{N}(\mathbf{U};\mathbf{V})|$.*

Proof. The proof is a straightforward application of Lemma 4.28 to appropriate non-planar initial data.

For notational convenience, take $i=3$. Let $q_1\subset V_1$ and $q_2\subset V_2$ be unit cubes as in Definition 5.3. For $p=1,2$, let $\mathcal{F}^p$ be the collection of faces of Q which meet V_p in an $(n-1)$-cell. Then

$$(Q,(\mathcal{F}^1,\mathcal{Q}(Q;\mathcal{F}^2),q_1),(\mathcal{F}^2,\mathcal{Q}(Q;\mathcal{F}^1),q_2))$$

are non-planar initial data; cf. Definition 4.24.

Let Σ be a spanning forest as in Lemma 4.25 and A_1 and A_2 be atoms associated with Σ as in Lemma 4.17. By Lemma 4.28, the essential partition

$$(V_1 \cup A_1, V_2 \cup A_2, V_3 - (A_1 \cup A_2))$$

satisfies the tripod property in $3Q$.

Property (1) follows immediately from (5) in Lemma 4.26, and (2) from the observation that every cube in $3(\mathcal{Q}(Q;\mathcal{F}^1) \cup \mathcal{Q}(Q;\mathcal{F}^2))$ is either a $\mathcal{D}$- or $\mathcal{N}$-cube. □

It is obvious that $\mathcal{C}$-modifications are performed independently. We formalize this in the following definition and corollary. Let $\mathbf{V}$ be an essential partition of an n-cell into n-cells and let $\mathcal{C}' \subset \mathcal{C}(\mathbf{V})$ be non-empty.

Definition 5.16. An essential partition $\mathbf{U}$ of $3|\mathbf{V}|$ into n-cells is *obtained by $\mathcal{C}$-modification in $\mathcal{C}'$ from* $\mathbf{V}$ if $\mathbf{U}$ satisfies the tripod property in each cube in $3\mathcal{C}'$ and

(a) $\mathbf{U} - |3\mathcal{C}'| = 3\mathbf{V} - |3\mathcal{C}'|$;

(b) if $C \in 3\mathcal{C}'$, then $\mathbf{U} \cap C$ is obtained from $3\mathbf{V}$ by a $\mathcal{C}$-modification, that is, $\mathbf{U}$ satisfies the properties of Lemma 5.15 relative to C; and

(c) $\partial_\cup \mathbf{U} \cap |3\mathcal{C}'| \subset |\mathcal{D}(\mathbf{U};\mathbf{V}) \cup \mathcal{N}(\mathbf{U};\mathbf{V})|$.

Corollary 5.17. *Let $\mathbf{V} = (V_1, V_2, V_3)$ be an essential partition of an n-cell so that V_p is a dented molecule for $p = 1, 2, 3$. Let $\mathcal{C}' \subset \mathcal{C}(\mathbf{V})$ be a non-empty subfamily. Then there exists an essential partition $\mathbf{U} = (U_1, U_2, U_3)$ of $|3\mathbf{V}|$ which is obtained by $\mathcal{C}$-modification in $\mathcal{C}'$.*

5.2.5. Secondary $\mathcal{C}$- and $\mathcal{N}$-modifications

The $\mathcal{C}$-modification in Lemma 5.15 is a 'primary' $\mathcal{C}$-modification. To illustrate the necessity of 'secondary' $\mathcal{C}$- and $\mathcal{N}$-modifications, consider the following example. This is necessitated by the presence of neglected faces (Definition 4.29).

Example 5.18. Let

$$\mathbf{W} = (W_1, W_2, W_3) = ([0,3]^3, [0,3]^2 \times [-3,0], [3,6] \times [0,3]^2).$$

The cube $Q = [0,3]^3$ is a $\mathcal{C}$-cube of color 1 in $\mathbf{W}$.

Using Lemma 5.15 we perform a $\mathcal{C}$-modification in $C = 3Q$, that is, obtain the essential partition $\mathbf{V}$ relative to C as in Lemma 5.15; see Figure 30.

Consider now the essential partition $3\mathbf{V}$. One possible essential partition $\mathbf{U}''$ of $|3\mathbf{V}|$ is in Figure 31. Notice in Figure 30 that $\mathbf{V}$ already has three neglected faces each of which meets the vertical fold. Thus $\mathbf{U}''$ cannot satisfy the tripod property. A glance at

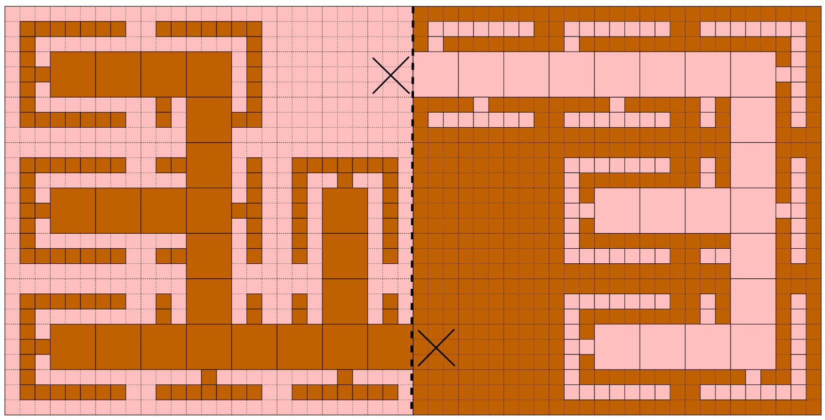

Figure 31. An essential partition $\mathbf{U}''$ in $9Q$.

Figure 31 also shows that in addition there are now 3 cubes, in fact cubes in $3\mathcal{N}(\mathbf{V};\mathbf{W})$, of side-length 9 in $\mathbf{U}''$ which cannot be partitioned into $\mathcal{C}$- and $\mathcal{D}$-cubes.

In this situation, we achieve the tripod property with a 'secondary' $\mathcal{C}$-modification in $3\mathbf{V}$. The procedure imitates that of Lemma 5.10, and takes into account both neglected faces and $\mathcal{D}$-cubes; see Figure 32.

To be more precise, the essential partition $\mathbf{U}$ is obtained by a secondary $\mathcal{C}$-modification as follows. We observe first that $\partial_\cup \mathbf{V}\cap 9Q\subset|\mathcal{D}(\mathbf{V};\mathbf{W})|\cup|\mathcal{N}(\mathbf{V};\mathbf{W})|$. Let also $A_j=C\cap V_j$ for each color $j=1,2$.

We perform now an independent $\mathcal{D}$-modification in each cube in $3\mathcal{D}(\mathbf{V};\mathbf{W})$. For each cube $Q'\in 3\mathcal{N}(\mathbf{V};\mathbf{W})$, we extend either $3A_1$ or $3A_2$ from outside Q' into Q' using a 1-fine atom; after this extension Q' meets each color in its interior. This extension uses Lemma 4.36. Let $\mathbf{U}$ be the essential partition obtained this way. Note that $3\mathcal{N}(\mathbf{V};\mathbf{W})=\mathcal{N}_2(\mathbf{U};\mathbf{V},\mathbf{W})$.

Regarding the construction of the sequence $\{\mathbf{\Omega}_m\}_{m\geqslant 0}$, we may take $\mathbf{\Omega}_2=\mathbf{U}$ if we have $\mathbf{V}=\mathbf{\Omega}_1$ and $\mathbf{W}=\mathbf{\Omega}_0$, since the tripod property is now satisfied.

The secondary $\mathcal{N}$-modification in the cubes $3\mathcal{N}_2(\mathbf{U};\mathbf{V},\mathbf{W})$ is defined as follows. Let $Q'\in\mathcal{N}_2(\mathbf{U};\mathbf{V},\mathbf{W})$. By Lemma 4.36, $U_j\cap Q'$ is a disjoint union of a molecule and a 1-fine atom, each of a different complementary color. Moreover, there are three types of cubes in Q', that is, the sets $\mathcal{C}(\mathbf{U}\cap Q')$, $\mathcal{D}\big(\mathbf{U}\cap Q';\mathbf{V}\cap\frac{1}{3}Q'\big)$ and $\mathcal{N}\big(\mathbf{U}\cap Q';\mathbf{V}\cap\frac{1}{3}Q'\big)$ are all non-empty; note that all cubes of $\mathcal{N}\big(\mathbf{U}\cap Q';\mathbf{V}\cap\frac{1}{3}Q'\big)$ and none of $\mathcal{D}\big(\mathbf{U}\cap Q';\mathbf{V}\cap\frac{1}{3}Q'\big)$ meet the two distinguished faces of Q'. Secondary $\mathcal{N}$-modification in $3Q'$ consists of independent $\mathcal{C}$- and $\mathcal{D}$-modifications and applications of Lemma 4.36 and is similar to the case of a secondary $\mathcal{C}$-modification.

We formalize now the secondary modifications in the form of lemmas. The proofs of

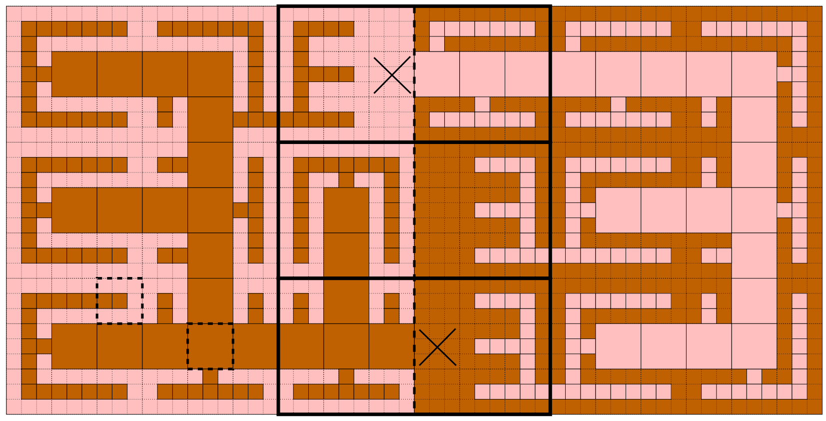

Figure 32. An essential partition $\mathbf{U}$ obtained by a secondary $\mathcal{C}$-modification from $3\mathbf{V}$. The three cubes of $\mathcal{N}_2(\mathbf{U};\mathbf{V},\mathbf{W})$ are highlighted with solid lines and a $\mathcal{C}$-cube and a $\mathcal{D}$-cube highlighted with dashed lines.

those lemmas follow the proofs of Lemmas 5.10, 5.11, and 5.15. Since the main difficulty is in their formulation, we leave the details of the proofs to the interested reader. For the statement, we give the following definition.

Definition 5.19. An essential partition $\mathbf{U}$ is *obtained by a secondary modification from* $\mathbf{V}$ *with respect to* $\mathbf{W}$ *in a cube* $C=3Q$ *of side-length* 27 if $\mathbf{U}$ satisfies the tripod property in C, $\mathbf{V}$ and $\mathbf{W}$ are essential partitions satisfying $|\mathbf{U}|=3|\mathbf{V}|=9|\mathbf{W}|$, and there are colors j and k and molecules M_j and M_k in $3Q$ such that $U_i=3V_i-(M_j\cup M_k)$, $U_j=3V_j\cup M_j$, and $U_k=3V_k\cup M_k$ are n-cells, and these molecules satisfy the following conditions:

(1) $(M_j\cup M_k)\cap\partial(3Q)\subset\partial_\cup 3\mathbf{V}$;

(2) $(\partial_\cup\mathbf{U})\cap 3Q\subset|\mathcal{C}(\mathbf{U})|\cup|\mathcal{D}(\mathbf{U};\mathbf{V})|\cup|\mathcal{N}_2(\mathbf{U};\mathbf{V},\mathbf{W})|$;

(3) $3A_j\subset M_j$ and $3A_k\subset M_k$;

(4) for $p=j,k$, M_p-3A_p consists of pairwise disjoint 1-fine atoms made of building blocks; and

(5) when $n>3$ and $p=1,2,3$, each building block in $\widetilde{\Gamma}(3A_p)$ meets at most one atom in M_p-3A_p,

where A_j and A_k are atoms in Q satisfying $V_i=3W_i-(A_j\cup A_k)$, $V_j=3W_j\cup A_j$, and $V_k=3W_k\cup A_k$; here $\mathbf{V}=(V_1,V_2,V_3)$ and $\mathbf{W}=(W_1,W_2,W_3)$.

Lemma 5.20. (Secondary $\mathcal{C}$-modification) *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be essential partitions satisfying* $|\mathbf{V}|=3|\mathbf{W}|$, *and suppose that* $\mathbf{V}$ *has been obtained by a* $\mathcal{C}$*-modification in* $\frac{1}{3}Q$ *from* $\mathbf{W}$. *Then there exists an essential partition* $\mathbf{U}$ *of* $3|\mathbf{V}|$ *which is obtained by a secondary modification from* $\mathbf{V}$ *with respect to* $\mathbf{W}$ *in* $3Q$.

LEMMA 5.21. (Secondary $\mathcal{N}$-modification) *Let* $\mathbf{V}$, $\mathbf{W}$, *and* $\mathbf{W}'$ *be essential partitions satisfying* $|\mathbf{V}|=3|\mathbf{W}|=9|\mathbf{W}'|$, *and let* $Q\in\mathcal{N}_2(\mathbf{V};\mathbf{W},\mathbf{W}')$ *be a secondary* $\mathcal{N}$*-cube so that* $\mathbf{V}$ *is obtained by a secondary modification in* $\frac{1}{3}Q$ *from* $\mathbf{W}$ *with respect to* $\mathbf{W}'$. *Then there exists an essential partition* $\mathbf{U}$ *of* $3|\mathbf{V}|$ *so that* $\mathbf{U}$ *is obtained by a secondary modification from* $\mathbf{V}$ *with respect to* $\mathbf{W}$ *in* $3Q$.

5.2.6. The Machine

Having all necessary modifications now at our disposal, we introduce the main induction step. Corollary 5.24 will summarize the process.

Let $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{W}'$ be essential partitions satisfying $|\mathbf{V}|=3|\mathbf{W}|=9|\mathbf{W}'|$. Suppose that all secondary $\mathcal{C}$-cubes in $\mathcal{C}_2(\mathbf{V};\mathbf{W})$ and all secondary $\mathcal{N}$-cubes Q in $\mathcal{N}_2(\mathbf{V};\mathbf{W},\mathbf{W}')$ are obtained by secondary $\mathcal{C}$- and $\mathcal{N}$-modifications from $\frac{1}{3}Q$, respectively. Suppose also that $|\mathbf{V}|\cap|3\mathcal{C}(\mathbf{W})|$ is obtained by a series of independent $\mathcal{C}$-modifications and $|\mathbf{V}|\cap|3\mathcal{D}(\mathbf{W};\mathbf{W}')|$ by a series of independent $\mathcal{D}$-modifications.

Based on the modifications introduced in this section (Lemmas 5.10, 5.15, 5.20, and 5.21) we note:

(a) the sets $|\mathcal{C}(\mathbf{V})|$ and $|\mathcal{D}(\mathbf{V};\mathbf{W})|$ are essentially disjoint; and

(b) the sets $|\mathcal{C}_2(\mathbf{V};\mathbf{W})|$ and $|\mathcal{N}_2(\mathbf{V};\mathbf{W},\mathbf{W}')|$ are essentially disjoint.

Indeed, by definition of $\mathcal{C}$- and $\mathcal{D}$-cubes, $\mathcal{C}(\mathbf{V})\cap\mathcal{D}(\mathbf{V};\mathbf{W})=\varnothing$. The claim (a) now follows from the observation that cubes in $\frac{1}{3}(\mathcal{C}(\mathbf{V})\cup\mathcal{D}(\mathbf{V};\mathbf{W}))$ are unit cubes. The claim (b) now follows with a similar argument.

It is essential to notice that cubes in $\mathcal{C}_2(\mathbf{V};\mathbf{W})\cup\mathcal{N}_2(\mathbf{V};\mathbf{W},\mathbf{W}')$ contain cubes in $\mathcal{C}(\mathbf{V})\cup\mathcal{D}(\mathbf{V};\mathbf{W})$, that is, the intersection $|\mathcal{C}(\mathbf{V})\cup\mathcal{D}(\mathbf{V};\mathbf{W})|\cap|\mathcal{C}_2(\mathbf{V};\mathbf{W})\cup\mathcal{N}_2(\mathbf{V};\mathbf{W},\mathbf{W}')|$ has non-empty interior; see e.g. Figure 32. For this reason, and to have a well-defined order of independent rearrangements, we set

$$\mathfrak{R}(\mathbf{V};\mathbf{W},\mathbf{W}')=\{Q\in\mathcal{C}(\mathbf{V})\cup\mathcal{D}(\mathbf{V};\mathbf{W}):Q\not\subset|\mathcal{C}_2(\mathbf{V};\mathbf{W})\cup\mathcal{N}_2(\mathbf{V};\mathbf{W},\mathbf{W}')|\};$$

here $\mathfrak{R}$ stands for remaining.

The inductive process can now be organized in a form of a list of operations to be performed. Given a collection $\mathfrak{L}_{\mathbf{W}}$ of essentially disjoint cubes in $|\mathbf{W}|$, we define the *list* $\mathfrak{L}_{\mathbf{V}}$ *with respect to the history* $(\mathbf{W},\mathbf{W}',\mathfrak{L}_{\mathbf{W}})$ by

$$\begin{aligned}\mathfrak{L}_{\mathbf{V}}&=\mathfrak{L}(\mathbf{V};\mathbf{W},\mathbf{W}',\mathfrak{L}_{\mathbf{W}})\\&=\{Q\in\mathcal{C}_2(\mathbf{V};\mathbf{W})\cup\mathcal{N}_2(\mathbf{V};\mathbf{W},\mathbf{W}')\cup\mathfrak{R}(\mathbf{V};\mathbf{W},\mathbf{W}'):Q\subset 3|\mathfrak{L}_{\mathbf{W}}|\}.\end{aligned}\tag{5.2}$$

Note that cubes in $\mathfrak{L}(\mathbf{V};\mathbf{W},\mathbf{W}',\mathfrak{L}_{\mathbf{W}})$ are pairwise essentially disjoint and have side-length either 3 or 9.

Remark 5.22. To see how the collection $\mathfrak{L}_{\mathbf{W}}$ organizes the modification process, consider the essential partitions $\mathbf{\Omega}_0$ and $\mathbf{\Omega}_1$. The essential partition $\mathbf{\Omega}_0$ has one $\mathcal{C}$-cube Q and, after scaling, we perform the (only possible) rearrangement in $3Q$. Thus the essential partition $\mathbf{\Omega}_1$ has one secondary $\mathcal{C}$-cube $3Q$. There are also several $\mathcal{C}$-cubes in $|\mathbf{\Omega}_1|-3Q$, contained in $\Omega_{1,2}\cup\Omega_{1,3}$, but all these $\mathcal{C}$-cubes have one face in $3Q$. Since it suffices to perform a rearrangement on one side of $\partial_\cup 3\mathbf{\Omega}_1$, we perform the secondary $\mathcal{C}$-modification in $9Q$ and ignore the other $\mathcal{C}$-cubes. Thus, if we set $\mathfrak{L}_{\mathbf{\Omega}_0}=\{Q\}$ and $\mathfrak{L}_{\mathbf{\Omega}_1}=\{3Q\}$, the rearrangement to obtain $\mathbf{\Omega}_2$ is performed in $3|\mathfrak{L}_{\mathbf{\Omega}_1}|$. We follow this general principle of nested rearrangements throughout the construction. For example, for the next rearrangement we define $\mathfrak{L}_{\mathbf{\Omega}_2}=\mathfrak{L}(\mathbf{\Omega}_2;\mathbf{\Omega}_1,\mathbf{\Omega}_0,\mathfrak{L}_{\mathbf{\Omega}_1})$, and rearrangements take place in $3|\mathfrak{L}_{\mathbf{\Omega}_2}|$; note that $\mathfrak{L}_{\mathbf{\Omega}_2}$ consists of $\mathcal{C}$- and $\mathcal{D}$-cubes and secondary $\mathcal{N}$-cubes as mentioned in Example 5.18. In particular, we have $9|\mathfrak{L}_{\Omega_0}|=3|\mathfrak{L}_{\mathbf{\Omega}_1}|\supsetneq|\mathfrak{L}_{\mathbf{\Omega}_2}|$.

Definition 5.23. Given essential partitions $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{W}'$ satisfying $|\mathbf{V}|=3|\mathbf{W}|=9|\mathbf{W}'|$ and a list $\mathfrak{L}_{\mathbf{W}}$ of cubes in $\mathbf{W}$, an essential partition $\mathbf{U}$ is *properly obtained* (*following* $\mathfrak{L}_{\mathbf{V}}=\mathfrak{L}(\mathbf{V};\mathbf{W},\mathbf{W}',\mathfrak{L}_{\mathbf{W}})$) if $\mathbf{U}$ is obtained

(i) by $\mathcal{C}$-modification in $3(\mathcal{C}(\mathbf{V})\cap\mathfrak{L}_{\mathbf{V}})$;

(ii) by $\mathcal{D}$-modification in $3(\mathcal{D}(\mathbf{V},\mathbf{W})\cap\mathfrak{L}_{\mathbf{V}})$; and

(iii) by secondary modification in $3((\mathcal{C}_2(\mathbf{V};\mathbf{W})\cup\mathcal{N}_2(\mathbf{V};\mathbf{W},\mathbf{W}'))\cap\mathfrak{L}_{\mathbf{V}})$.

These modifications now yield the following corollary, which can be viewed as the induction step in the construction; the specific sequence $\{\mathbf{\Omega}_m\}_{m\geqslant 0}$ satisfying Theorem 5.1 appears in the next section.

Corollary 5.24. *Let* $\mathbf{V}$, $\mathbf{W}$, *and* $\mathbf{W}'$ *be essential partitions for which* $|\mathbf{V}|=3|\mathbf{W}|=9|\mathbf{W}'|$ *and let* $\mathfrak{L}_{\mathbf{W}}$ *be the list of cubes in* $|\mathbf{W}|$. *Suppose* $\mathbf{V}$ *is properly obtained following* $\mathfrak{L}_{\mathbf{V}}=\mathfrak{L}(\mathbf{V};\mathbf{W},\mathbf{W}',\mathfrak{L}_{\mathbf{W}})$ *and suppose that* $\mathbf{V}$ *satisfies the tripod property and* $\partial_\cup\mathbf{V}\subset|\mathfrak{L}_{\mathbf{V}}|$.

Then there exists a properly obtained essential partition $\mathbf{U}$ *satisfying* $|\mathbf{U}|=3|\mathbf{V}|$ *and* $\partial_\cup\mathbf{U}\subset|\mathfrak{L}_{\mathbf{U}}|\subset 3|\mathfrak{L}_{\mathbf{V}}|$, *where* $\mathfrak{L}_{\mathbf{U}}=\mathfrak{L}(\mathbf{U};\mathbf{V},\mathbf{W},\mathfrak{L}_{\mathbf{V}})$. *In particular*, $\mathbf{U}$ *satisfies the tripod property.*

Proof. It suffices to note that $\mathbf{U}$ is obtained by independent modifications in each cube $3Q$ for $Q\in\mathfrak{L}(\mathbf{V};\mathbf{W},\mathbf{W}',\mathfrak{L}_{\mathbf{W}})$, and is hence properly obtained. These modifications also yield a new list $\mathfrak{L}(\mathbf{U};\mathbf{V},\mathbf{W},\mathfrak{L}_{\mathbf{V}})$; Lemmas 5.10, 5.11, 5.15, 5.20, and 5.21 cover the possible situations of different modifications. Thus $\partial_\cup\mathbf{U}\subset|\mathfrak{L}(\mathbf{U};\mathbf{V},\mathbf{W},\mathfrak{L}_{\mathbf{V}})|$ and $\mathbf{U}$ satisfies the tripod property. □

5.3. Inductive construction

Throughout this section, $\mathbf{\Omega}_0$ and $\mathbf{\Omega}_1$ are the essential partitions defined in §5.1.1 and §5.1.2.

PROPOSITION 5.25. *Let* $n\geqslant 3$, $\mathbf{\Omega}_0=([0,3]^n, [0,3]^{n-1}\times[-3,0], [3,6]\times[0,3]^{n-1})$ *and let* $\mathbf{\Omega}_1$ *be an essential partition as in* §5.1.2. *Then there exist essential partitions* $\mathbf{\Omega}_m=(\Omega_{m,1},\Omega_{m,2},\Omega_{m,3})$ *for* $m\geqslant 1$ *satisfying the tripod property and the following conditions*:

(a) $|\mathbf{\Omega}_m|=3|\mathbf{\Omega}_{m-1}|$;

(b) $\partial_\cup \mathbf{V}_m\subset|\mathfrak{L}(\mathbf{\Omega}_m;\mathbf{\Omega}_{m-1},\mathbf{\Omega}_{m-2},\mathfrak{L}_{\mathbf{\Omega}_{m-1}})|$;

(c) *all cubes in* $\mathfrak{L}_{\mathbf{\Omega}_m}=\mathcal{L}(\mathbf{\Omega}_m;\mathbf{\Omega}_{m-1},\mathbf{\Omega}_{m-2},\mathfrak{L}_{\mathbf{\Omega}_{m-1}})$ *are properly obtained*; *and*

(d) $\mathbf{\Omega}_\ell\cap 3^{m-2}|\mathbf{\Omega}_0|=\mathbf{\Omega}_m\cap 3^{m-2}|\mathbf{\Omega}_0|$ *for all* $\ell>m$.

In addition, there exist $\nu\geqslant 1$ *and* $\lambda>1$, *depending only on* n, *so that for all* $m\geqslant 0$ *and each* $p=1,2,3$,

(e) *every* $\mathrm{hull}(\Omega_{m,p})$ *is a* (ν,λ)*-molecule with the atom length of* $\Gamma(\mathrm{hull}(\Omega_{m,p}))$ *bounded by a constant depending only on* n; *and*

(f) *there exists* $L=L(n)\geqslant 1$ *and an* L*-bilipschitz map*

$$\psi_{m,p}\colon(\Omega_{m,p},d_{\Omega_{m,p}})\longrightarrow(\mathrm{hull}(\Omega_{m,p}),d_{\mathrm{hull}(\Omega_{m,p})})$$

which is the identity on $\Omega_{m,p}\cap\partial\mathrm{hull}(\Omega_{m,p})$.

We first prove Proposition 5.25 in dimensions $n>3$ and then consider the more complicated dimension $n=3$ separately; see §5.3.4. Proposition 5.25 is obtained in three steps. In higher dimensions, we first construct the sequence $\mathbf{\Omega}_3,\mathbf{\Omega}_4,\ldots$ by induction using Corollary 5.24 and then check conditions (a)–(d) and the tripod property. Property (e) is more subtle and considered separately in §5.3.2. Finally, we prove Property (f), the most demanding part, in §5.3.3. For $n=3$, the steps are similar with the exception that we use specific $\mathcal{C}$- and secondary $\mathcal{C}$-modifications to meet condition (e).

5.3.1. Proof of Proposition 5.25 in dimension $n>3$

Consider the essential partitions $\mathbf{\Omega}_1$, $\mathbf{\Omega}_0$, and $\frac{1}{3}\mathbf{\Omega}_0$ in the rôle of the essential partitions $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{W}'$ of §5.2.6.

We obtain $\mathbf{\Omega}_1$ by one $\mathcal{C}$-modification from $3\mathbf{\Omega}_0$ and take $\mathbf{\Omega}_2$ to be either the essential partition $\mathbf{U}$ in Lemma 5.20, or directly apply Corollary 5.24. In particularly, $\mathbf{\Omega}_2$ satisfies the tripod property and is properly obtained following $\mathfrak{L}_{\mathbf{\Omega}_1}=\mathfrak{L}\big(\mathbf{\Omega}_1;\mathbf{\Omega}_0,\frac{1}{3}\mathbf{\Omega}_0,\mathfrak{L}_{\mathbf{\Omega}_0}\big)$, where $\mathfrak{L}_{\mathbf{\Omega}_0}=\{[0,3]^n\}$; cf. Remark 5.22. We take $\mathfrak{L}_{\mathbf{\Omega}_2}=\mathfrak{L}(\mathbf{\Omega}_2;\mathbf{\Omega}_1,\mathbf{\Omega}_0,\mathfrak{L}_{\mathbf{\Omega}_1})$.

To meet the stability requirement (d) of the proposition, let $Q_0=[0,1]^{n-1}\cup[-1,1]$, and note that $\mathbf{\Omega}_2\cap 9Q_0$ is obtained by a single $\mathcal{D}$-modification from $3\mathbf{\Omega}_1\cap 9Q_0$. Thus, by making proper choices in the $\mathcal{C}$-modification yielding $\mathbf{\Omega}_1$ and in the secondary $\mathcal{C}$-modification yielding $\mathbf{\Omega}_2$, we may assume that $\mathbf{\Omega}_2 \cap 3Q_0 = \mathbf{\Omega}_1 \cap 3Q_0$; compare with Figures 25 and 26 and the discussion in §5.1.2 and §5.1.3. Indeed, using the notation from §5.1.2, $\Omega_{1,3} = 3\Omega_{0,3} \cup A_3$ and we may assume as in Figure 25 that the building block $\Omega_{1,3} \cap 3Q_0$ is a leaf in $\Gamma(A_3)$. Let $\mathbf{U}$ be an essential partition given by Corollary 5.24, and define $\mathbf{\Omega}_2$ by $\mathbf{\Omega}_2 - 3Q_0 = \mathbf{U} - 3Q_0$ and $\mathbf{\Omega}_2 \cap 3Q_0 = \mathbf{\Omega}_1 \cap 3Q_0$.

Since $\mathbf{\Omega}_2 \cap 3Q_0 = \mathbf{\Omega}_1 \cap 3Q_0$, it is easy to obtain the rest of the sequence $\mathbf{\Omega}_0, \mathbf{\Omega}_1, \mathbf{\Omega}_2, ...$ by applying Corollary 5.24 to the essential partitions $\mathbf{\Omega}_{m-1}$, $\mathbf{\Omega}_{m-2}$, and $3\mathbf{\Omega}_{m-2}$ and modifying the obtained essential partitions as for $m=2$.

Corollary 5.24 yields that the essential partitions in the sequence $\{\mathbf{\Omega}_m\}_{m \geqslant 0}$ satisfy the tripod property and conditions (a)–(c).

Remark 5.26. Recall that, as a direct consequence of the definition, each dented molecule $\Omega_{m,p}$ has a unique essential partition into dented atoms. Recall that the adjacency graph of this essential partition of $\Omega_{m,p}$ into dented atoms is $\Gamma(\Omega_{m,p})$.

Remark 5.27. Whereas there is no simple inclusion relation between the domains $\Omega_{m,p}$ and $\Omega_{m+1,p}$, the trees $\Gamma(\Omega_{m,p})$ and $\Gamma(\Omega_{m+1,p})$ are closely related. Indeed, formally, $\Gamma(\Omega_{m+1,p})$ is obtained by adding leaves to $\Gamma(\Omega_{m,p})$. At the same time, however, a vertex representing an atom of side-length at least 3 in $\Gamma(\Omega_{m,p})$ becomes a dented atom in $\Gamma(\Omega_{m+1,p})$.

Finally, the tree $\Gamma(\Omega_p)$ of the limit $\Omega_p = \bigcup_{m \geqslant 1} \Omega_{m,p}$ is an inverse limit of the trees $\Gamma(\Omega_{m,p})$.

5.3.2. Condition (e)

We consider first some general properties of dented molecules $\Omega_{m,p}$ and their hulls $\mathrm{hull}(\Omega_{m,p})$ (cf. §3.2), and then obtain condition (e) in Proposition 5.25.

Remark 5.28. The trees $\Gamma(\Omega_{m,p})$ and $\Gamma(\mathrm{hull}(\Omega_{m,p}))$ are related since $\Gamma(\mathrm{hull}(\Omega_{m,p}))$ is obtained by removing those (dented) atoms from $\Gamma(\Omega_{m,p})$ which are contained in $\mathrm{hull}(\mathrm{hull}(\Omega_{m,p}) - \Omega_{m,p})$. Thus $\Gamma(\mathrm{hull}(\Omega_{m,p}))$ can be viewed as a subtree of $\Gamma(\Omega_{m,p})$ where the remaining vertices are (undented) atoms instead of dented atoms.

Recall that a vertex $D \in \Gamma(\Omega_{m,p})$ is internal if there exists a vertex $D' \in \Gamma(\Omega_{m,p})$ so that $\varrho(D') > \varrho(D)$ and $D \subset \mathrm{hull}(D')$, and that a vertex is external if not internal (Definition 3.18).

Remark 5.29. Although we focus on one of the domains in the following lemma, it should be noted that the other two domains also have a rôle, since they create the dents. This is crucial for the combinatorics to settle (e) and (f). Suppose $D\in\Gamma(\Omega_{m,p})$ is an internal vertex and $D'\in\Gamma(\Omega_{m,p})$ is a vertex closest to D in $\Gamma(\Omega_{m,p})$ satisfying $D\subset\text{hull}(D')$.

Then D is contained in a dent M' of D'. This dent is a vertex in $\Gamma(\Omega_{m,r})$ for $r\neq p$. Note also that since D is contained in a dent of M', we have

$$\varrho(\text{hull}(D'))\geqslant 3^2\varrho(\text{hull}(M'))\geqslant 3^4\varrho(\text{hull}(D)).$$

LEMMA 5.30. *Let $m>1$. Suppose A is a leaf in $\Gamma(\Omega_{m,p})$ and let $3^k\in\{1,3\}$ be the side-length of A. Let D be the vertex adjacent to A in $\Gamma(\Omega_{m,p})$ satisfying $\varrho(D)>\varrho(A)$. Then $3^{-k}A$ is a leaf of $\Omega_{m-k,p}$.*

If $\varrho(D)>3\varrho(A)$, then the atom $3^{-k}A$ arose from a $\mathcal{C}$-modification and A is an internal vertex of $\Gamma(\Omega_{m,p})$,

Otherwise, $\varrho(D)=3\varrho(A)$ and $3^{-k}A$ came from a $\mathcal{D}$-modification or a secondary modification. Furthermore, in this case, A is an external vertex of $\Gamma(\Omega_{m,p})$ if and only if D is an external vertex of $\Gamma(\Omega_{m,p})$.

Remark 5.31. Whereas the number of atoms A attached to D in Lemma 5.30 satisfying $\varrho(D)=3\varrho(A)$ is uniformly bounded, there will be no upper bound for the number of atoms A attached to D in general. This follows from the observation that there is no upper bound for the size of a dent of a dented molecule and each cube in a dent is penetrated by a (dented) molecule which is an internal vertex attached to D. Note that trees $\Gamma(\Omega_{m,p})$ have internal vertices only for $m\geqslant 3$.

Note also that the essential partition $\mathbf{\Omega}_1$ is exceptional in the following sense. The molecule $\Omega_{1,p}$, for $p=2,3$, is obtained from $3\Omega_{0,p}$ by a $\mathcal{C}$-modification but the leaf $\Omega_{1,p}-3\Omega_{0,p}$ is not contained in a dent of $\Omega_{1,p}$. This is the one case in the construction of the sequence $\{\mathbf{\Omega}_m\}_{m\geqslant 0}$ when a $\mathcal{C}$-modification does not produce an internal vertex.

Proof of Lemma 5.30. Since A is a leaf, it is an atom. Moreover, $\varrho(D)\geqslant 3\varrho(A)$ by construction.

First observe that if $3^{-k}A$ is obtained by a $\mathcal{C}$-modification, there exists a dent M of $3^{-k}D$ with $3^{-k}A\subset M$. Since $M\subset\text{hull}(3^{-k}D)$, it follows that $A\subset\text{hull}(D)$. Thus in this case A is internal and $\varrho(A)\leqslant 3^{-4}\varrho(D)$.

Since the ratio of side-lengths in a $\mathcal{D}$-modification and in secondary modifications is 3, the atom $3^{-k}A$ is obtained by a $\mathcal{D}$-modification or a secondary modification if and only if $\varrho(D)=3\varrho(A)$.

Suppose now that $\varrho(D)=3\varrho(A)$. We show that A is an internal vertex if and only if D is an internal vertex.

Suppose first that A is an internal vertex. Then there exists $D'\in\Gamma(\Omega_{m,p})$ containing A in its hull. Let M' be the dent of D' containing A. By properties of modifications, we have either $D \subset M'$ or $M' \subset \text{hull}(D)$, since D is adjacent to A and $A \subset M'$. Since $\varrho(M') \geqslant 9\varrho(A) = 3\varrho(D)$, we have $D \subset M'$. Thus D is internal.

Suppose now that D is an internal vertex. Then there exists $D_0 \in \Gamma(\Omega_{m,p})$ and a dent M_0 of D_0 satisfying $D \subset M_0 \subset \text{hull}(D_0)$. We may assume that D_0 is minimal in the sense that, for every $D' \in \Gamma(\Omega_{m,p})$ between D and D_0 in $\Gamma(\Omega_{m,p})$, $D \not\subset \text{hull}(D')$.

Let $D_1, ..., D_\ell$ be the shortest path in $\Gamma(\Omega_{m,p})$ from D_0 to D so that D_1 is adjacent to D_0. Then $D_1 \subset M_0$ and we note that $\varrho(\text{hull}(D_1))^{-1}\text{hull}(D_1)$ has been obtained by a $\mathcal{C}$-modification in a cube Q of side-length 9.

Furthermore, by properties of modifications, we observe that all modifications to obtain dented molecules $D_1, ... D_\ell$ take place in cubes $3^j Q$ where $3^j \leqslant \varrho(\text{hull}(D_1))$. Thus all dented atoms $D_1, ..., D_\ell$ are contained in the cube $Q' := \varrho(\text{hull}(D_1))Q \subset M_0$. In particular, $D \subset Q'$. Since A is obtained from D by either a $\mathcal{D}$-modification or a secondary modification, we also have $A \subset Q' \subset M_0$. Thus A is internal. □

Lemma 5.32. (Property (e)) *Let $n>3$. There exist $\nu \geqslant 1$ and $\lambda > 1$ depending only on n so that the adjacency tree $\Gamma(\text{hull}(\Omega_{m,p}))$ is a (ν, λ)-molecule for every $m \geqslant 2$ and each p.*

Proof. By Lemma 3.19, $\Gamma(\text{hull}(\Omega_{m,p}))$ is isomorphic to the tree $\Gamma_E(\Omega_{m,p})$ of external vertices of $\Gamma(\Omega_{m,p})$, and Lemma 5.30 shows that external vertices arise from $\mathcal{D}$-modifications or secondary modifications. Thus it suffices to estimate the number of atoms created by a $\mathcal{D}$-modification or secondary modification for $m>2$.

Let $1<k<m$, and let A be an atom in $\Gamma(\Omega_{k,p})$ created by a $\mathcal{D}$-modification or a secondary modification. Then A has side-length 1 and is contained in a union of at most two cubes of side-length 9. Since there exist 3^n essentially disjoint cubes of side-length 3 in a cube of side-length 9, the atom A consists of strictly less than $2\cdot 3^n$ building blocks; see Remark 5.33 below. Since we attach at most one atom to a building block of $3A$, this modification of $3A$ attaches strictly less than $2\cdot 3^n$ atoms. We conclude that the tree $\Gamma(\text{hull}(\Omega_{m,p}))$ is at most $2\cdot 3^n$-valent.

To show that $\text{hull}(\Omega_{m.p})$ is λ-collapsible for some $\lambda>1$, let $M \in \Gamma(\text{hull}(\Omega_{m,p}))$ be a molecule of side-length 3^k. Then M is attached to at most $2\cdot 3^n$ molecules of side-length 3^{k-1} and to one molecule M' of side-length 3^{k+1}. Let F' be the face of a cube in M' where M and M' meet.

Let $\varepsilon>0$, to be fixed in a moment, and take ℓ with

$$(1+\varepsilon)3^{k-1}\ell \leqslant 3^{k+1} < (1+\varepsilon)3^{k-1}(\ell+1).$$

Then there exist at least ℓ^{n-1} pairwise disjoint $(n-1)$-cubes of side-length $(1+\varepsilon)\cdot 3^{k-1}$ on F. Since

$$\ell^{n-1} > \left(\frac{9}{1+\varepsilon} - 1\right)^{n-1}, \tag{5.3}$$

we may fix $\varepsilon>0$ small enough, depending on n, so that

$$\ell^{n-1} > 2\cdot 3^n \tag{5.4}$$

when $n\geqslant 4$. We conclude that there exists $\lambda>1$, depending only on n, for which M, and hence $\mathrm{hull}(\Omega_{m,p})$, is λ-collapsible. □

Remark 5.33. Note that, although estimates (5.3) and (5.4) hold also for $n=3$, the number of building blocks in an atom A no longer is an upper bound for atoms attached to $3A$. In fact, $\mathcal{D}$-modification in dimension 3 may attach as many as three atoms to a single building block; cf. Figure 18.

5.3.3. Condition (f)

It suffices to consider $m\geqslant 4$. Let $p\in\{1,2,3\}$. To simplify notation, set $V=\Omega_{m,p}$.

LEMMA 5.34. *There exist $L=L(n)\geqslant 1$ and an L-bilipschitz map*

$$\varphi\colon (V, d_V) \longrightarrow (\mathrm{hull}(V), d_{\mathrm{hull}(V)})$$

which is the identity on $V\cap\partial\mathrm{hull}(V)$.

We begin the proof of Lemma 5.34 with two auxiliary lemmas. For the statements, we need some new notation and also use terms from §3.2.

Let $d\in\Gamma(V)$ be a dented atom and let $D\in\Gamma(V)$ be the unique dented atom adjacent to D satisfying $\varrho(D)>\varrho(d)$. Let Q_d and Q'_d be the unique (dented) cubes of side-length $\varrho(d)$ in d and in D, respectively, having a common face F'_d. Note that $F'_d\subset Q_d\cap Q_D=d\cap D$.

Let F_d be a face of Q_d contained in ∂d sharing an $(n-2)$-cube with F'_d. We call $J_d=F_d\star\{x_{Q_d}\}$ and $J'_d=F'_d\star\{x_{Q'_d}\}$ the *internal* and the *external join* of D, respectively. Note that $J_d\subset d$ and $J'_d\subset D$.

The first key ingredient in the proof of Lemma 5.34 is a bilipschitz equivalence property for expanding descendants; recall Definitions 3.20 and 3.21 of expanding descendants and partial hulls, respectively, in §3.2.

LEMMA 5.35. *Let P be a partial hull of V and let $D\in\Gamma(P)$ be a dented atom having only expanding descendants. Then there exist $L=L(n)\geqslant 1$ and an L-bilipschitz map $\varphi_D\colon(|\Gamma(P)_D|, d_{|\Gamma(P)_D|})\to(\mathrm{hull}(D), d_{\mathrm{hull}(D)})$ which is the identity on $D\cap\partial\mathrm{hull}(D)$. Moreover, for any descendant $d\in\Gamma(P)$ of D, $\varphi_D(|\Gamma(P)_d|)\subset J_d$.*

Proof. By Lemma 5.32, for every descendant $d\in\Gamma(P)$, $|\Gamma(P)_d|$ is a collapsible (ν,λ)-molecule with ν and λ depending only on n. Thus, by Proposition 3.5, there exist

$L'=L'(n)\geqslant 1$ and an L'-bilipschitz mapping $\psi_D\colon(|\Gamma(P)_D|, d_{|\Gamma(P)_D|})\to(D,d_D)$ which is the identity on $D-\bigcup_d J_d$, where the union is over descendants of D.

Proposition 3.12 then produces $L''=L''(n)\geqslant 1$ and an L''-bilipschitz map

$$\phi_D\colon(D,d_D)\longrightarrow(\mathrm{hull}(D), d_{\mathrm{hull}}(D))$$

which is the identity on $D\cap\partial\mathrm{hull}(D)$. Furthermore, by a simple modification of the proof of Proposition 3.12, we may assume that ϕ_D is an isometry from J_d to J'_d for each descendant d of D. Thus $\varphi_D=\phi_D\circ\psi_D$ is the desired map. □

The second key ingredient in the proof of Lemma 5.34 is a regrouping of joins associated with expanding descendants of large relative side-length. We begin by counting the number of descendants, and again need some notation.

Let P be a partial hull of $V=\Omega_{m,p}$. Let $D\in\Gamma(P)$ be a dented atom and $B\in\widetilde{\Gamma}(\mathrm{hull}(D))$ be a building block. Let $\mathcal{A}(P,D;B)$ denote the vertices of $\Gamma(P)$ adjacent to D which have side-length $3^{-4}\varrho(D)$ and are contained in B. Note that there are no vertices adjacent to D and contained in B with side-length greater than $3^{-4}\varrho(D)$; recall Remark 5.29.

LEMMA 5.36. *Let* $n>3$, $D\in\Gamma(P)$, *and* $B\in\widetilde{\Gamma}(\mathrm{hull}(D))$. *Then*

$$\#\mathcal{A}(P,D;B)\leqslant 8n^2 3^n. \tag{5.5}$$

Proof. The argument is similar to the counting argument in the proof of Lemma 5.32. Let $\varrho(D)=3^k$. Let $M_B=B\cap\mathrm{hull}(\mathrm{hull}(D)-D)$. We note first that given a cube $Q\in\Gamma^{\mathrm{int}}(B)$, $Q\cap M_B$ is a pairwise disjoint union of two molecules, since $Q\cap M_B$ stems from a $\mathcal{C}$-modification performed in $3^{k-2}Q$. We also have that $\varrho(M_B)=3^{k-2}$.

Let $U_B\subset M_B$ be the union of the atoms of side-length 3^{k-2} in $\Gamma(M_B)$. The dented molecules in $\mathcal{A}(P,D;B)$ are in one-to-one correspondence with $\Gamma^{\mathrm{int}}(U_B)$. Indeed, other cubes in $\Gamma^{\mathrm{int}}(M_D)$ have side-length at most 3^{k-3} and the dented molecules adjacent to D which they contain have side-length at most 3^{k-5}.

Since an atom of side-length 3^{k-2} in a cube of side-length 3^k has at most $2n3^n$ cubes, we have for each $Q\in\Gamma^{\mathrm{int}}(B)$ the estimate

$$\#\Gamma^{\mathrm{int}}(U_B\cap Q)\leqslant 2\cdot 2n3^n=4n3^n.$$

As $\#\Gamma^{\mathrm{int}}(B)<2n$, i.e an n-dimensional building block consists of less than $2n$ cubes, we have

$$\#\Gamma^{\mathrm{int}}(U_B)\leqslant 2n4n3^n=8n^2 3^n.$$ □

In the proof of Lemma 5.34 we construct a sequence of partial hulls from the dented molecule V to the molecule $\mathrm{hull}(V)$. The crux of the proof is to inductively contract the leaves into joins associated with building blocks and then isometrically move these joins further. The estimate in Lemma 5.36 is used to obtain the necessary collapsibility properties of the partial hulls. We formalize this step in the following lemma.

Let $D\in\Gamma(P)$ be a dented molecule and $B\in\widetilde{\Gamma}(\mathrm{hull}(D))$ be a building block. Let $Q'_B\in\Gamma(B)$ denote the center of B, and F'_B the unique face of Q'_B contained in $\partial\mathrm{hull}(D)$. Let $Q_B\subset 3^{k-1}(3^{-k}Q'_B)^{\#}$ be the unique cube of side-length $3^{-1}\varrho(B)$ having $F_B=Q_B\cap F'_B$ as a face of Q_B with the same barycenter as F'_B. We call $J_B=F_B\star\{x_{Q_B}\}$ the *join associated with* B.

LEMMA 5.37. *Let P be a partial hull of V. Suppose $D\in\Gamma(P)$ is a dented atom. Then there exist $L=L(n)\geqslant 1$ and an L-bilipschitz map*

$$\psi_D\colon(D,d_D)\longrightarrow(\mathrm{hull}(D),d_{\mathrm{hull}(D)})$$

which is the identity on $\partial D-\mathrm{hull}(D)$, and which for every $B\in\widetilde{\Gamma}(\mathrm{hull}(D))$ satisfies

(1) *$\psi_D(B\cap D)=B$; and*

(2) *for each $d\in\mathcal{A}(P,D;B)$, $\psi_D|_{J_d}$ is an isometric embedding from J_d into J_B.*

In addition, suppose Q is the smallest cube having D on the boundary, and let for every $B\in\widetilde{\Gamma}(\mathrm{hull}(D))$, f_B be an $(n-1)$-cube of side-length $3^{-4}\varrho(B)$ in $B\cap\partial Q$ having distance at least 3^{-4} to $\partial B-\partial Q$ and to each J_d. Then $\varphi_D|_{f_B\star\{x_{q_B}\}}$ is an isometry into J_B, where x_{q_B} is the barycenter of the unique cube q_B in Q having f_B as a face.

Proof. The argument is similar to the collapsing argument in Lemma 5.32. Let $\varrho(D)=3^k$ and $B\in\widetilde{\Gamma}(\mathrm{hull}(D))$.

As by definition $\varrho(F_B)=3^{k-1}$, we may fix 26^{n-1} $(n-1)$-cubes of side-length $\frac{27}{26}3^{k-4}$ in F_B. Since Lemma 5.36 yields that

$$\#\mathcal{A}(P,D;B)<26^{n-1}$$

and $\varrho(J_d)=3^{-4}\varrho(D)=3^{k-4}$, there exists for each $d\in\mathcal{A}(P,D;B)$ an $(n-1)$-cube $F''_d\subset F_B$ of side-length 3^{k-4} so that the pairwise distances of these $(n-1)$-cubes are at least $\frac{1}{26}3^{k-4}$. Thus there exist $L=L(n)\geqslant 1$ and an L-bilipschitz map $\psi_B\colon B\to B$ which is the identity on $B-\partial Q$ and which is an isometric embedding from J_d to $F''_d\star\{x_{q''_d}\}$, where q''_d is the unique n-cube in Q having F''_d as a face.

The required mapping φ_D is now the composition of the extensions of the various maps ψ_B to all of D. We leave the modification of the argument in the case of additional $(n-1)$-cubes f_B for the interested reader. □

Proof of Lemma 5.34. We construct a sequence $P_0, ..., P_k$ of partial hulls of V, where $P_0=V$ and $P_k=\mathrm{hull}(V)$. In each stage, we remove one dented atom of smallest side-length.

Let $P_0=V$ and $\mathcal{J}_0=\varnothing$. Suppose that, for $k\geqslant 0$, we have constructed

(a) partial hulls $P_0, ..., P_k$ of V so that $P_{\ell+1}$ is a partial hull of P_ℓ for $0\leqslant\ell<k-1$;

(b) collections $\mathcal{J}_0, ..., \mathcal{J}_k$ of joins associated with building blocks in these partial hulls so that the joins $\mathcal{J}_\ell$ are contained in atoms of $\Gamma(P_\ell)$ which are hulls of the dented atoms D in $\Gamma(P_{\ell-1})$ for $1\leqslant\ell\leqslant k$, and for such D, the number of joins contained in $|\Gamma(P_\ell)_D-D|$ is at most 3^n, recall that $\Gamma(P_\ell)_D$ is the subtree in $\Gamma(P_\ell)$ behind vertex D;

(c) for every $1\leqslant\ell<k$, an L-bilipschitz map $\psi_\ell\colon(P_\ell, d_{P_\ell})\to(P_{\ell+1}, d_{P_{\ell+1}})$, which is the identity on those atoms of $\Gamma(P_\ell)$ which are atoms of $\Gamma(P_{\ell-1})$, where L is at most the product of bilipschitz constants in Lemmas 5.35 and 5.37.

If $P_k\neq\mathrm{hull}(V)$, we construct P_{k+1} as follows. As $\Gamma(V)$ is finite, this process terminates.

Since $P_k\neq\mathrm{hull}(V)$, there exist dented atoms in $\Gamma(P_k)$. Let $D_k\in\Gamma(P_k)$ be the dented atom having smallest side-length, and $d\in\Gamma(P_k)$ be an atom adjacent to D_k in $\mathrm{hull}(D_k)$. By minimality of D_k, d is an expanding vertex (Definition 3.20) in $\Gamma(P_k)$.

Let $\mathcal{J}_k(D_k)$ be the joins in $\mathcal{J}_k$ which are contained in $|\Gamma(P_k)_{D_k}-D_k|$. We treat these joins as (virtual) adjacent atoms. Thus each join $J\in\mathcal{J}_k(D_k)$ increases (virtually) the valence of $\Gamma(P_k)_{D_k}$ by 1 at the dented atom containing it, and so when $n\geqslant 4$, the valence of $\Gamma(P_k)_{D_k}$ increases at each vertex by at most 3^n.

We leave it to the interested reader to verify that $\Gamma(P_k)_{D_k}$ remains λ-collapsible with λ depending only on n even when the joins $\mathcal{J}_k(D_k)$ are understood as (virtual) adjacent atoms; compare with Lemma 5.32.

Let $\varphi_k\colon(|\Gamma(P_k)_{D_k}|, d_{|\Gamma(P_k)_{D_k}|})\to(D_k, d_{D_k})$ be a bilipschitz map as in Lemma 5.35 with the property that, for each $J\in\mathcal{J}_k(D_k)$, $\varphi_k|_J$ is an isometry.

Let $\phi_k\colon(D_k, d_{D_k})\to(\mathrm{hull}(D_k), d_{\mathrm{hull}(D_k)})$ be a bilipschitz map as in Lemma 5.37 with the property that, for each descendant d of D_k, ϕ_k is an isometry from J_d into some J_B for $B\in\widetilde{\Gamma}(\mathrm{hull}(D_k))$.

Let ψ_k be the composition of $\phi_k\circ\varphi_k$ and $P_{k+1}=P_k\cup\mathrm{hull}(D_k)$. To obtain $\mathcal{J}_{k+1}$, we remove the joins $\mathcal{J}_k(D_k)$ from $\mathcal{J}_k$ and add the joins associated with the building blocks in $\mathrm{hull}(D_k)$. This completes the induction step and the proof. □

5.3.4. Proposition 5.25 in dimension $n=3$

The essential partitions $\mathbf{\Omega}_0$ and $\mathbf{\Omega}_1$ fixed in §5.1.1 and §5.1.2 are the starting point for the induction also in dimension $n=3$. To obtain the partitions $\mathbf{\Omega}_m$ for $m\geqslant 2$, we use

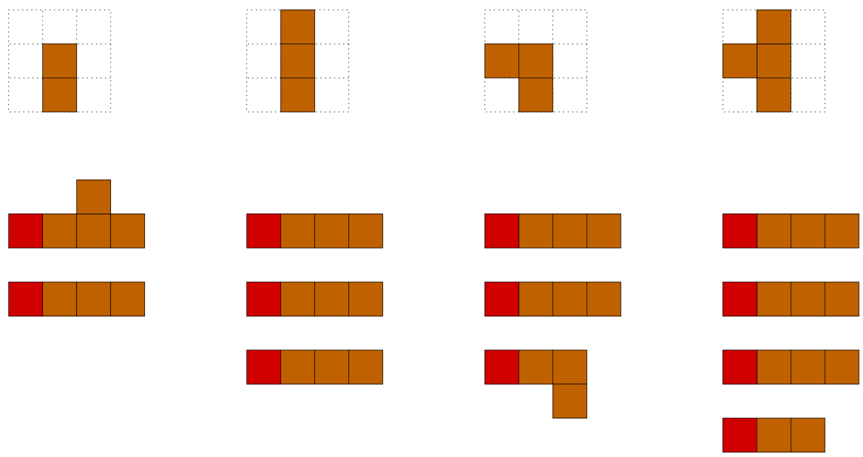

Figure 33. Visible faces of building blocks.

explicit configurations of atoms in order to obtain branching estimates for the adjacency trees. We begin this section by introducing the particular modifications we use in the induction.

When $n=3$ it is easy to exhibit an explicit catalog of $\mathcal{C}$-modifications associated with building blocks. Similarly, the secondary modifications can be explicitly illustrated. These configurations are exhibited in figures and the estimates are obtained simply by counting building blocks and cubes in these configurations.

Visible faces. Suppose Q is a cube of side-length 3 in $\mathbb{R}^3$, F is a face of Q, and B is an F-based building block in Q. Having Figure 15 at our disposal, we observe that for every $q\in B^{\#}$, $q\cap F$ is a unit square and $B\cap(Q-B)$ is a 2-cell consisting of at most four faces of q.

Figure 33 displays foldouts of faces of all (unit) cubes q in building blocks B which may occur in Q. Note that the foldout pictures show only faces of cubes q contained in F or $Q-B$. These faces are the *visible* faces of ∂q; only these are in $\partial_\cup \mathbf{U}$.

$\mathcal{C}$-modification. Based on the catalog in Figure 33, we observe that in dimension $n=3$ it suffices to fix four $\mathcal{C}$-modifications which can be applied in all cubes in all building blocks of side-length 9. The case of five visible faces is illustrated in Figure 34. A comprehensive list of examples of $\mathcal{C}$-modifications to cubes with three or four visible faces is given in Figure 35.

Summary. Let $3B$ be a building block of side-length 9 and suppose that in each $Q'\in 3B^*$ we have performed one of the $\mathcal{C}$-modifications illustrated in Figures 34 and 35,

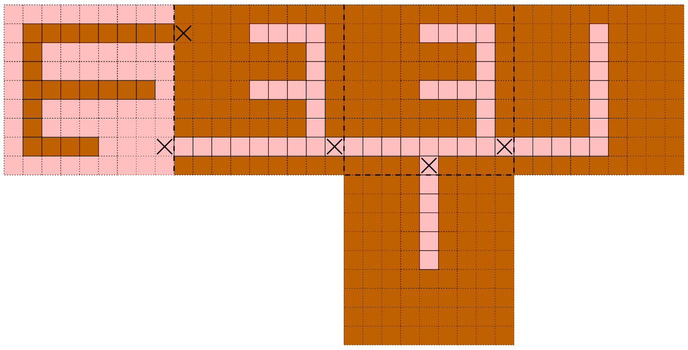

Figure 34. A cube q in B with five visible faces.

and let $A_{Q',i}\subset Q'$, $i=1,2$, be the corresponding atoms; $\varrho(A_{Q',i})=1$. Then

(i) each atom $A_{Q',i}$ consists of at most 20 building blocks and 56 cubes;

(ii) in each cube $Q'\in 3B^*$, $A_{Q',1}\cup A_{Q',2}$ consists of at most 28 building blocks and 79 cubes; and

(iii) $\bigcup_{Q'}(A_{Q',1}\cup A_{Q',2})$ consists of at most 100 building blocks and 285 cubes.

Secondary modifications. Observe first that, while a secondary $\mathcal{C}$-modification may occupy as many as four faces of a cube of side-length 27, a secondary $\mathcal{N}$-modification is confined to two faces. Thus the upper bounds for unit cubes and building blocks are achieved by secondary $\mathcal{C}$-cubes, and so there is no need to consider explicitly secondary $\mathcal{N}$-modifications.

Let Q and B be as above and let Q'' be the unique cube sharing the face F with Q. Let $\mathbf{V}=(Q-B,Q'',B)$ and let $\mathbf{U}=(U_1,U_2,U_3)$ be the essential partition of $|3\mathbf{V}|$ obtained after $\mathcal{C}$-modifications, based on Figures 34 and 35. Note that components of

$$3A_1=U_1-3(Q-B)$$

are atoms having 8 building blocks.

Let $Q'\in 3B^*$. Figure 36 presents an example of a system of basins in Q' when Q' has five visible faces. For cubes with fewer visible faces, similar systems of basins can be found; these systems have fewer basins. Figure 37 illustrates a $\mathcal{C}$-modification in the largest basin in Figure 36. The systems of basins for cubes in $3B^*$ with fewer visible faces can be chosen to have basins not larger than this basin in terms of the number of unit cubes in added atoms. We encourage the interested reader to verify these statements by illustrations.

Summary. Let $\mathbf{\Omega}=(\Omega_1,\Omega_2,\Omega_3)$ be an essential partition of $3Q$ obtained from $\mathbf{U}$ by a secondary modification. For $Q'\in 3B^*$, let $M_{Q',j}=\Omega_j\cap Q'$, $j=1,2$. Then $M_{Q',j}$ is a

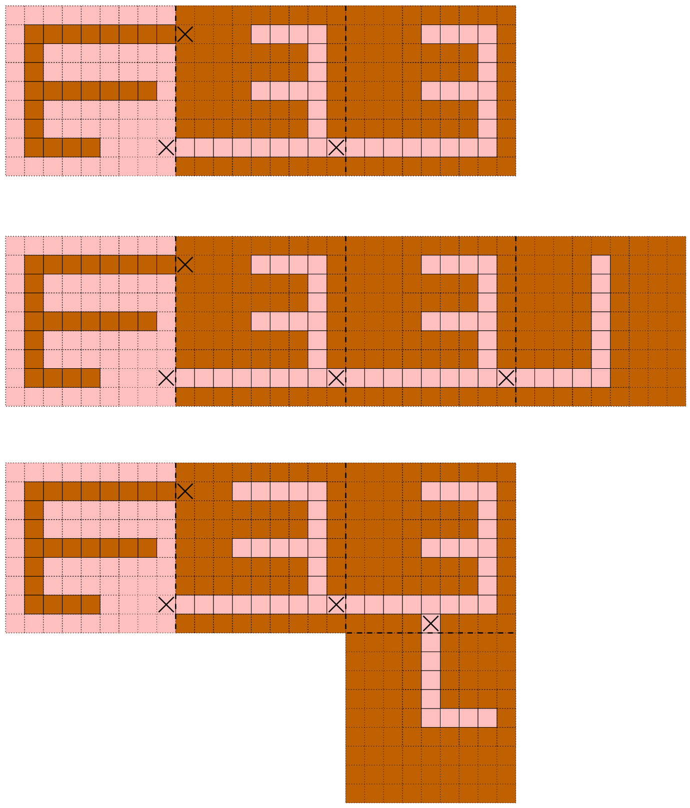

Figure 35.

molecule having $3A_{Q',j}$ as its root. Let $L_{Q',j}=M_{Q',j}-3A_{Q',j}$ be the union of leaves of $M_{Q',j}$. Then

(i) each component of $L_{Q',j}$ consists of at most 16 building blocks and 47 cubes;

(ii) for each cube $Q'\in 3B^*$, $L_{Q',1}\cup L_{Q',2}$ has at most 31 components and consists of at most 243 building blocks; and

(iii) the union $\bigcup_{Q'}(L_{Q',1}\cup L_{Q',2})$ consists of at most 829 building blocks.

Furthermore, $\Gamma(M_{Q',j})$ has valence at most 45.

5.3.5. Completion of the proof of Proposition 5.25 for $n=3$

We construct the sequence $\{\mathbf{\Omega}_m\}_{m\geqslant 0}$ of essential partitions using Corollary 5.24 iteratively as in §5.3.1 with the only exception that for $\mathcal{C}$- and secondary $\mathcal{C}$-modifications, we use the explicit configurations illustrated in §5.3.4. Thus, again, the essential partitions $\mathbf{\Omega}_m$ satisfy conditions (a)–(d) and the tripod property.

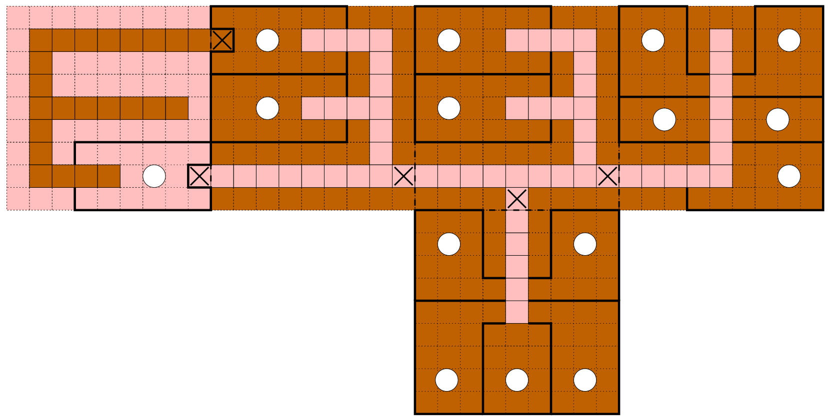

Figure 36. An example of a system of basins. Basins indicated with (large) dots.

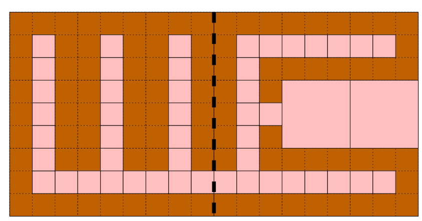

Figure 37.

To verify condition (e), we note first that Lemma 5.30 has no dimensional restrictions, and so it applies also for $n=3$.

Regarding Lemma 5.32, we note that, when $n=3$, the statistics in §5.3.4 imply that $\Gamma(\mathrm{hull}(\Omega_{m,p}))$ has valence at most 20 and every atom in $\Gamma(\mathrm{hull}(\Omega_{m,p}))$ consists of at most 56 cubes. Since $9^2>8^2>56$, we may take $\varepsilon=\frac{8}{9}$ in the proof. Thus the claim of Lemma 5.32 holds also for $n=3$, and so $\{\boldsymbol{\Omega}_m\}_{m\geqslant 0}$ satisfies condition (e).

To verify condition (f), we observe that, using the configurations in §5.3.4 now show that $\#\mathcal{A}(P,D;B)\leqslant 285$ in Lemma 5.36. Since $285<26^2$, the process of Lemma 5.37 is therefore also at our disposal, and thus it suffices to discuss the proof of Lemma 5.34 for $n=3$. The bounds we need here are already available from §5.3.4.

Concerning item (b) in the proof of Lemma 5.34, the number of joins in our construction is now 16. Furthermore, the maximal (virtual) valence of $\Gamma(P_k)_{D_k}$ is at most $31+16=47$, since the atoms in $\Gamma(P_k)_{D_k}-D_k$ are expanding and hence obtained by a $\mathcal{D}$-modification, a secondary modification or by a $\mathcal{C}$-modification over one face of a cube. Finally, since D_k has at most 16 descendants for $n=3$, the result of Lemma 5.34 therefore holds also for $n=3$. This concludes the verification of condition (f) and the proof of

Proposition 5.25 in dimension $n=3$.

5.4. Proof of Theorem 5.1

Let $\{\mathbf{\Omega}_m\}_{m\geqslant 0}$ be a sequence of essential partitions as in Proposition 5.25 and observe that conditions (2) and (3) in the claim of Theorem 5.1 are satisfied.

We conclude the proof of Theorem 5.1 by showing that, for $p=1,2,3$,

$$\Omega_p = \bigcup_{m\geqslant 0} \Omega_{m,p}$$

is bilipschitz equivalent to $\mathbb{R}^{n-1}\times[0,\infty)$ in its inner geometry.

By (2b), (Ω_p, d_{Ω_p}) is bilipschitz equivalent to $(\operatorname{hull}(\Omega_p), d_{\operatorname{hull}(\Omega_p)})$ for each p. Since $\operatorname{hull}(\Omega_3)$ is a monotone union of (ν,λ)-molecules, where ν and λ depend only on n, (Ω_3, d_{Ω_3}) is bilipschitz equivalent to $\mathbb{R}^{n-1}\times[0,\infty)$ by Proposition 3.8.

Concerning $\operatorname{hull}(\Omega_2)$, we observe first that $\operatorname{hull}(\Omega_2)\cap[0,\infty)^{n-1}\times[0,\infty)$ consists of an infinite collection of pairwise disjoint (ν,δ)-molecules. Thus $(\operatorname{hull}(\Omega_2), d_{\operatorname{hull}(\Omega_2)})$ is bilipschitz equivalent to $[0,\infty)^{n-1}\times(-\infty,0]$ as we may apply Proposition 3.5 to these molecules separately. Since the components of $\operatorname{hull}(\Omega_2)\cap[0,\infty)^{n-1}\times[0,\infty)$ do not meet $\partial[0,\infty)^{n-2}\times\mathbb{R}$, we obtain a bilipschitz homeomorphism

$$[0,\infty)^{n-1}\times(-\infty,0] \longrightarrow (\operatorname{hull}(\Omega_2), d_{\operatorname{hull}(\Omega_2)})$$

which is the identity on the boundary $\partial[0,\infty)^{n-1}\times(-\infty,0]$.

We are left with $\operatorname{hull}(\Omega_1)$. Since $\operatorname{hull}(\Omega_{1,m})=[0,3^{m+1}]^n$ for every $m\geqslant 1$, we have $\operatorname{hull}(\Omega_1)=[0,\infty)^n$.

This completes the construction of a rough Rickman partition of $[0,\infty)^{n-1}\times\mathbb{R}$ and the proof of Theorem 5.1.

Proof of Corollary 5.2. The domains $\Omega_{m,p}$ are John domains with a John constant depending only on n. This can be seen for example as follows. Let $a,b\in\operatorname{int}\Omega_p$ be points and let $A,B\in\Gamma(\Omega_p)$ be the (dented) atoms containing a and b, respectively. Let $D_1,...,D_r$ be the geodesic in $\Gamma(\Omega_p)$ connecting A and B. Since the atoms $\operatorname{hull}(D_r)$ have uniformly bounded length and the function $r\mapsto\varrho(\operatorname{hull}(D_r))$ satisfies the combinatorial John condition as noted in §3, we observe that there exists $C>1$ depending only on n so that a and b can be connected with a path $\gamma\colon[0,1]\to\Omega_{m,p}$ satisfying

$$\min\{|a-\gamma(t))|, |\gamma(t)-b|\}\leqslant C\operatorname{dist}(\gamma(t),\partial\Omega_{m,p})$$

for $0\leqslant t\leqslant 1$.

The domains Ω_p, for $p=1,2,3$, are uniform domains by the same argument. □

6. From cubes to simplices

In this section we introduce a particular triangulation of the pairwise common boundary $\partial_\cup \mathbf{\Omega}$ of a rough Rickman partition $\mathbf{\Omega}=(\Omega_1,\Omega_2,\Omega_3)$. While the construction of the domains Ω_p is facilitated by using cubes as fundamental units, an Alexander-type mapping is more naturally described using simplices. We wish to remind the reader that the rough Rickman partition $\mathbf{\Omega}$ must be modified once more to obtain a Rickman partition $\widetilde{\mathbf{\Omega}}$ supporting a suitable BLD-mapping on $\partial_\cup \widetilde{\mathbf{\Omega}}$. The triangulation of $\partial_\cup \mathbf{\Omega}$ and a parity function carried by it have important roles in the construction of $\widetilde{\mathbf{\Omega}}$ in the next section.

The space $\mathbb{R}^n$ has a natural structure as a CW-complex with unit cubes $[0,1]^n+v$, $v\in\mathbb{Z}^n$, as n-cells, and the k-dimensional faces of these cubes as k-cells. Every $(n-1)$-cube Q of this complex has a natural subdivision into $(n-1)$-simplices. In what follows the convex hull of points $v_0,...,v_k$ in $\mathbb{R}^k$, $0\leqslant k\leqslant n-1$, is

$$[v_0,...,v_k].$$

Let Q be an $(n-1)$-cube in $\mathbb{R}^n$ and, for $k=0,...,n-1$, let Q_k be a k-dimensional face of Q. The n-tuple $\mathcal{Q}=(Q_0,...,Q_{n-1})$ is a *flag in* Q if

$$Q_0\subset Q_1\subset ... \subset Q_{n-1}=Q. \tag{6.1}$$

Each k-cell Q_k has a uniquely defined barycenter c_{Q_k} and, by the arrangement (6.1), the vectors $c_{Q_0}-c_{Q_{n-1}},...,c_{Q_{n-2}}-c_{Q_{n-1}}$ are linearly independent with

$$S_{\mathcal{Q}}=[c_{Q_0},...,c_{Q_{n-1}}]$$

being an n-simplex contained in Q. We say that $S_{\mathcal{Q}}$ is the *simplex induced by the flag* $\mathcal{Q}$. Furthermore,

$$Q=\bigcup_{\mathcal{Q}} S_{\mathcal{Q}},$$

the union over all flags $(Q_0,...,Q_n)$ in Q. Two $(n-1)$-simplices $S_{\mathcal{Q}}$ and $S_{\mathcal{Q}'}$ determined by different flags $\mathcal{Q}$ and $\mathcal{Q}'$, may intersect but they have no common interior. Thus simplices induced by flags triangulate Q.

As simplices induced by flags are determined by the barycenters of lower-dimensional faces of $(n-1)$-cubes, every $(n-1)$-dimensional subcomplex $\mathbb{X}$ of $\mathbb{R}^{n-1}$, which is a union of its $(n-1)$-cells, admits a triangulation with simplices induced by flags. We call the simplicial complex associated with such a triangulation the *standard simplicial structure of* $\mathbb{X}$. Note that, since simplices in the standard simplicial structure arise as a subdivision of unit cubes in $\mathbb{R}^n$, the k-simplices $(0<k\leqslant n)$ in the standard simplicial structure have diameter between $\frac{1}{2}$ and $\frac{1}{2}\sqrt{n}$.

Convention. From now on we tacitly assume that a given $(n-1)$-simplex σ in an $(n-1)$-dimensional cubical complex $\mathbb{X}$ has the standard simplicial structure of $\mathbb{X}$.

In particular, the pairwise common boundary $\partial_\cup \mathbf{\Omega}$ of a Rickman partition $\mathbf{\Omega}$ admits this standard simplicial structure.

There is an elementary labeling function associated with the standard simplicial structure. Let $\mathbb{X}$ be an $(n-1)$-dimensional subcomplex of $\mathbb{R}^n$ so that $\mathbb{X}$ is a union of its $(n-1)$-cells and let $\mathbb{X}^{(0)}$ be the vertices of the standard simplicial structure. Since every vertex v in $\mathbb{X}$ is a barycenter of a unique unit cube Q_v in the cubical structure of $\mathbb{R}^n$, the map

$$\begin{aligned}\vartheta_{\mathbb{X}}: \mathbb{X}^{(0)} &\longrightarrow \{0, ..., n-1\},\\ v &\longmapsto \dim Q_v,\end{aligned}$$

is well defined. Moreover, $\vartheta_{\mathbb{X}}(\sigma)=\{0, ..., n-1\}$ for every $(n-1)$-simplex σ in the standard simplicial structure of $\mathbb{X}$. We call $\vartheta_{\mathbb{X}}$ the *labeling function of* $\mathbb{X}$.

6.1. Parity functions

Let $\mathbf{\Omega}=(\Omega_1, \Omega_2, \Omega_3)$ be a rough Rickman partition of $\mathbb{R}^n$ and let σ be an $(n-1)$-simplex in $(\partial_\cup \mathbf{\Omega})^{(n-1)}$. Then $\sigma=[v_0, ..., v_{n-1}]$, where $0 \leqslant k \leqslant n-1$ and v_k is a barycenter of a k-cube in $\partial_\cup \mathbf{\Omega}$. Since $\partial_\cup \mathbf{\Omega}$ is the pairwise common boundary, σ lies on the boundary of exactly two domains in $\mathbf{\Omega}$. We say that σ is $\mathbf{\Omega}$*-positive* if there exist i and j with $\sigma \subset \Omega_i \cap \Omega_j$ and

(1) $j=i+1 \pmod 3$; and

(2) there exists a unit vector $v \in \mathbb{R}^n$ with $v_{n-1}+v \in \Omega_i$ and

$$\det(v_0-v_{n-1}, ..., v_{n-2}-v_{n-1}, v) > 0. \tag{6.2}$$

Otherwise, σ is $\mathbf{\Omega}$*-negative.* A vector v satisfying (6.2) is called an *oriented normal of* σ if v is orthogonal to v_k-v_{n-1} for every $0 \leqslant k \leqslant n-1$.

The *parity function* of $\mathbf{\Omega}$ is the function $\nu_{\mathbf{\Omega}}: (\partial_\cup \mathbf{\Omega})^{(n-1)} \to \{\pm 1\}$ defined by

$$\nu_{\mathbf{\Omega}}(\sigma) = \begin{cases} 1, & \text{if } \sigma \text{ is } \mathbf{\Omega}\text{-positive},\\ -1, & \text{if } \sigma \text{ is } \mathbf{\Omega}\text{-negative}.\end{cases}$$

The next lemma describes the change of the parity on adjacent simplices.

LEMMA 6.1. *Let $\mathbf{\Omega}=(\Omega_1, \Omega_2, \Omega_3)$ be a rough Rickman partition of $\mathbb{R}^n$. Let σ and σ' be adjacent $(n-1)$-simplices in $\partial \Omega_i$. Then $\nu_{\mathbf{\Omega}}(\sigma)=-\nu_{\mathbf{\Omega}}(\sigma')$ if there exists $j \neq i$ such that $\sigma \cup \sigma' \subset \partial \Omega_j$, and $\nu_{\mathbf{\Omega}}(\sigma)=\nu_{\mathbf{\Omega}}(\sigma')$ otherwise.*

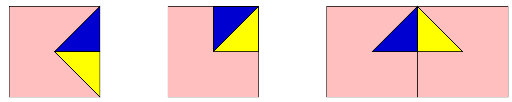

Figure 38. Congruence classes of planar $\sigma\cup\sigma'$ for $n=3$ and $k=0,1,2$.

Proof. Let $\sigma=[v_0,...,v_{n-1}]$ and $\sigma'=[v'_0,...,v'_{n-1}]$. Suppose first that σ and σ' are contained in an $(n-1)$-dimensional plane P. We claim that

$$(v'_0-v'_{n-1})\wedge...\wedge(v'_{n-2}-v'_{n-1}) = -(v_0-v_{n-1})\wedge...\wedge(v_{n-2}-v_{n-1}). \tag{6.3}$$

It is then easy to verify the claim of the lemma, as the oriented normal vectors of σ and σ' will be opposite normals of P.

Let $\mathcal{Q}=(Q_0,...,Q_n)$ and $\mathcal{Q}'=(Q'_0,...,Q'_n)$ be flags defining $\sigma=S_{\mathcal{Q}}$ and $\sigma'=S_{\mathcal{Q}'}$, respectively. Since σ and σ' have a common side, there exists $0\leqslant k\leqslant n-1$ such that $v_i=v'_i$ for $i\neq k$.

Suppose first that $0<k<n-1$. Then Q'_k and Q_k have a common face Q_{k-1} and are contained in Q_{k+1}. Since

$$c_{Q_{k-1}}-c_{Q_{k+1}} = (c_{Q_{k-1}}-c_{Q'_k})+(c_{Q'_k}-c_{Q_{k+1}}) = (c_{Q_k}-c_{Q_{k+1}})+(c_{Q'_k}-c_{Q_{k+1}}),$$

it follows that

$$\begin{aligned} v'_k-v_{n-1} &= v'_k-v_{k+1}+(v_{k+1}-v_{n-1}) \\ &= v_{k-1}-v_{k+1}-(v_k-v_{k+1})+(v_{k+1}-v_{n-1}) \\ &= -(v_k-v_{n-1})+(v_{k-1}-v_{n-1})+(v_{k+1}-v_{n-1}), \end{aligned}$$

and so

$$\begin{aligned} &(v'_0-v'_{n-1})\wedge...\wedge(v'_k-v'_{n-1})\wedge...\wedge(v'_{n-2}-v'_{n-1}) \\ &\quad= (v_0-v_{n-1})\wedge...\wedge(v'_k-v_{n-1})\wedge...\wedge(v_{n-2}-v_{n-1}) \\ &\quad= -(v_0-v_{n-1})\wedge...\wedge(v_k-v_{n-1})\wedge...\wedge(v_{n-2}-v_{n-1}). \end{aligned}$$

Thus (6.3) holds. The cases $k=0$ and $k=n-1$ are similar.

Suppose now that σ and σ' are not contained in an $(n-1)$-dimensional hyperplane. In this case, using the notation above, $v'_{n-1}\neq v_{n-1}$ and $v'_k=v_k$ for $0\leqslant k<n-1$. By the construction of $\mathbf{\Omega}$, there also exists an n-cube Q having σ and σ' on its boundary. In particular, $w=c_Q-v_{n-1}$ and $w'=c_Q-v'_{n-1}$ are orthogonal to σ and σ', respectively.

As the n-simplices $[v_0,...,v_{n-1},c_Q]$ and $[v'_0,...,v'_{n-1},c_Q]$ are planar in $\mathbb{R}^{n+1}$ and share an $(n-1)$-dimensional face, we have, by the previous argument,

$$(v'_0-c_Q)\wedge...\wedge(v'_{n-1}-c_Q) = -(v_0-c_Q)\wedge...\wedge(v_{n-1}-c_Q),$$

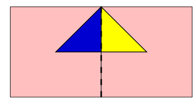

Figure 39. Fold-out of the congruence class of $\sigma\cup\sigma'$ for $n=3$.

so that

$$(v_0'-v_{n-1}')\wedge\ldots\wedge(v_{n-2}'-v_{n-1}')\wedge w' = -(v_0-v_{n-1})\wedge\ldots\wedge(v_{n-2}-v_{n-1})\wedge w.$$

Since Q is contained in one of the elements of the partition $\mathbf{\Omega}$, the claim now follows by considering separately the cases $Q\subset Q_i$ and $Q\subset Q_j$, where $j=i+1$ (mod 3); in both cases the oriented normals for σ and σ' are either w and $-w'$, or $-w$ and w', respectively. □

7. Pillows and pillow covers

In this section we establish the most significant case, $p=3$, of Proposition 1.5. Using the ideas of Rickman [15, §7] we prove the following proposition.

PROPOSITION 7.1. *Let $\mathbf{\Omega}=(\Omega_1,\Omega_2,\Omega_3)$ be a rough Rickman partition of $\mathbb{R}^n$ supporting the tripod property. Then there exists a Rickman partition $\widetilde{\mathbf{\Omega}}=(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)$ of $\mathbb{R}^n$ for which the Hausdorff distance of $\partial_\cup\mathbf{\Omega}$ and $\partial_\cup\widetilde{\mathbf{\Omega}}$ is at most* 1.

We would like to recall that the construction in §5 yields a rough Rickman partition $\mathbf{\Omega}=(\Omega_1,\Omega_2,\Omega_3)$, where Ω_1 and Ω_2 are connected and Ω_3 has 2^{n-1} components. It should, however, be noted that we may construct the essential partition $\widetilde{\mathbf{\Omega}}$ in Proposition 7.1 from any rough Rickman partition. Indeed, the construction of $\widetilde{\mathbf{\Omega}}$ is local and the number of components of the sets Ω_i has no rôle in the argument.

The proof of Proposition 1.5 is based on a construction of what we call a pillow cover of $\partial_\cup\mathbf{\Omega}$, and yields the final essential partition $\widetilde{\mathbf{\Omega}}$. The labeling and parity functions of $\mathbf{\Omega}$ lead at once to a BLD-map $\partial_\cup\widetilde{\mathbf{\Omega}}\to\widehat{\mathbb{S}}^{n-1}$, where $\widehat{\mathbb{S}}^{n-1}=\mathbb{S}^{n-1}\cup\mathbb{B}^{n-1}$. The bound on the Hausdorff distances of $\partial_\cup\mathbf{\Omega}$ and $\partial_\cup\widetilde{\mathbf{\Omega}}$ is immediate from the pillow construction.

Remark 7.2. (Idea of the pillow cover) We summarize the idea of the pillow cover construction as follows. Let $\mathbf{\Omega}$ be a rough Rickman partition of $\mathbb{R}^n$ and consider a triangulation on $\partial_\cup\mathbf{\Omega}$ as in §6.

To construct a pillow cover of $\partial_\cup\mathbf{\Omega}$ we (locally) replace each pair of adjacent $(n-1)$-simplices with a sextuplet of adjacent (Lipschitz) $(n-1)$-simplices. This sextuplet can be seen as a (branched) double cover of $\widehat{\mathbb{S}}^{n-1}$; note that $\widehat{\mathbb{S}}^{n-1}$ consists of three $(n-1)$-cells.

We use the tripod property of $\partial_\cup\mathbf{\Omega}$, organize the adjacent $(n-1)$-simplices into directed trees, and modify the domains Ω_k by modifying their boundaries via this local

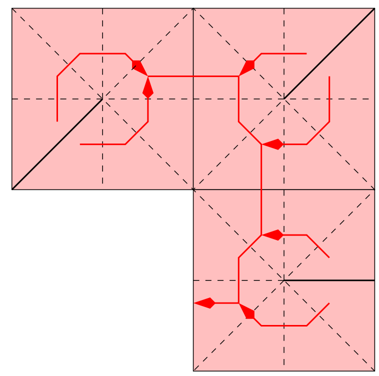

Figure 40.

replacement procedure of $(n-1)$-simplices. This process extends Ω_k between Ω_i and Ω_j for $\{i,j,k\}=\{1,2,3\}$ and, as a consequence, we obtain a new essential partition $\widetilde{\mathbf{\Omega}}=(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)$ of $\mathbb{R}^n$.

Finally, the local (combinatorial) properties of $\partial_\cup\widetilde{\mathbf{\Omega}}$ allow us to construct a BLD-map $\partial_\cup\widetilde{\mathbf{\Omega}}\to\partial_\cup\mathbf{E}$ which shows that domains in $\widetilde{\mathbf{\Omega}}$ are Zorich extension domains.

We discuss first the pillow construction locally for planar $(n-1)$-cells contained in $\partial_\cup\mathbf{\Omega}$. For notational convenience let $E\subset\partial_\cup\mathbf{\Omega}$ be a cubical $(n-1)$-cell contained in a hyperplane $P=\mathbb{R}^{n-1}\times\{0\}$ of $\mathbb{R}^n$ so that $E\subset\Omega_i\cap\Omega_j$ for some $i\neq j$. Throughout §§7.1–7.4 we consider E fixed but arbitrary and E inherits a standard simplicial structure from $\partial_\cup\mathbf{\Omega}$. We denote by $\nu=\nu_{E,\mathbf{\Omega}}\colon E^{(n-1)}\to\{\pm1\}$ the restriction of the parity function ν_Ω to E. Similarly, $\vartheta=\vartheta_{E,\mathbf{\Omega}}\colon E^{(0)}\to\{0,...,n-1\}$ is the restriction of the labeling function $\vartheta_{\partial_\cup\mathbf{\Omega}}$ to E.

Let $\mathcal{E}$ be the adjacency graph $\Gamma(E^{(n-1)})$ and fix a maximal tree $\widehat{\mathcal{E}}$ in $\mathcal{E}$. Contrary to the case of maximal trees of adjacency graphs of cubical complexes, we consider $\widehat{\mathcal{E}}$ as a directed tree, and fix orientation on $\widehat{\mathcal{E}}$ so that $\widehat{\mathcal{E}}$ is connected and all simplices in $\widehat{\mathcal{E}}$ have at most one outgoing edge and (possibly several or no) incoming edges.

Suppose σ is an $(n-1)$-simplex σ of $\widehat{\mathcal{E}}$ and the $(n-2)$-simplex τ is a face of σ. Let σ' be an $(n-1)$-simplex in $\widehat{\mathcal{E}}$ adjacent to σ, that is, $\sigma'\cap\sigma=\tau$. Then τ is an *entry face* of σ if the edge between σ and σ' is an incoming edge to σ, and τ is an *exit face* of σ if it is the (unique) outgoing edge from σ. If τ is an entry or an exit face of a simplex, τ is considered *open*, otherwise τ is a *closed face* of σ; in the configuration of Figure 40 the open faces are marked with dashed lines.

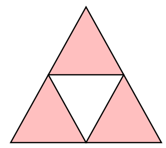

Figure 41. The 2-simplices τ_1, τ_2, and τ_3 surrounding τ_0 in a subdivision of τ; $n=4$.

7.1. Pillow of a simplex

As a preparatory step, let $\tau=[v_1, ..., v_{n-1}]$ be an $(n-2)$-simplex in $\mathbb{R}^{n-1}$, and consider τ as a face of an $(n-1)$-simplex σ in E. We define a subdivision $\tau_0, ..., \tau_{n-1}$ of τ as follows.

Suppose first that $n\geqslant 4$. For $i=1, ..., n-1$, let

$$\tau_i = \left[\tfrac{1}{2}(v_1+v_i), ..., \tfrac{1}{2}(v_{n-1}+v_i)\right] \subset \tau.$$

Then τ_i is an $(n-2)$-simplex congruent to τ having diameter $\frac{1}{2}\operatorname{diam}\tau$ and having v_i as a vertex; see Figure 41. Finally, let $\tau_0=\tau-\bigcup_{i=1}^{n-1}\tau_i$; we use here and from now on the notation $\alpha-\beta=\overline{\alpha\setminus\beta}$ also for simplices. For $n=4$, τ_0 is a 2-simplex, while when $n>4$, τ_0 is a more general polyhedron.

When $n=3$, τ is a line segment $[v_1, v_2]$. In this case, we set $\tau_1=\left[v_1, v_1+\frac{1}{3}(v_2-v_1)\right]$, $\tau_2=\left[v_2, v_2+\frac{1}{3}(v_1-v_2)\right]$, and $\tau_0=\tau-(\tau_1\cup\tau_2)$; thus τ_0 is the 'middle third' of τ.

Definition 7.3. Let $u:\tau\to[-1,1]$ be a continuous function on τ. Then u is an *opening* if $u|_{\operatorname{int}\tau_0}>0$ and $u|_{\tau\setminus\operatorname{int}\tau_0}=0$. Similarly, u is a *shuffle* if

(1) $u|_{\operatorname{int}\tau_0}>0$;

(2) there exist $i\neq j$ in $\{1, ..., n-1\}$ such that $u|_{\operatorname{int}\tau_i}>0$ and $u|_{\operatorname{int}\tau_j}<0$; and

(3) $u|_{\tau\setminus(\operatorname{int}\tau_0\cup\operatorname{int}\tau_i\cup\operatorname{int}\tau_j)}=0$.

Remark 7.4. Note that, if $u:\tau\to[-1,1]$ is either an opening or a shuffle, $u|_{\partial\tau}=0$.

7.1.1. Pillow cover functions

For each $(n-1)$-simplex σ in E, we set

$$\ell_\sigma = \begin{cases} 2, & \text{if } \nu(\sigma)=-1, \\ 4, & \text{if } \nu(\sigma)=1, \end{cases}$$

and introduce a family of Lipschitz functions

$$\Psi_\sigma: \sigma\times\{1, ..., \ell_\sigma\} \longrightarrow [-1,1],$$

which will form the pillow covers. We consider the two parities separately.

Remark 7.5. For each σ and both parities $\nu(\sigma)$, we may assume that the function Ψ_σ satisfies the additional regularity condition

$$\Psi_\sigma(x,i+1)-\Psi_\sigma(x,i)\geqslant \tfrac{1}{10}\operatorname{dist}(x,\partial\sigma)$$

for $x\in\sigma$ and $i\in\{1,...,\ell_\sigma-1\}$.

We may also assume, from now on, that the mappings Ψ_σ are PL and uniformly Lipschitz, that is, there exists $L\geqslant 1$ (depending only on n) so that every Ψ_σ is L-Lipschitz for every σ in $\partial_\cup\mathbf{\Omega}$ and, in particular, in the cell E.

Case 1. *Functions on negative simplices.*

Suppose $\nu(\sigma)=-1$. We define $u_\sigma\colon\partial\sigma\to[-1,1]$ as follows. Given a face τ of σ, we set $u_\sigma|_\tau$ to be an opening if τ is either an entry or an exit face of σ. If τ is closed, $u_\sigma|_\tau$ is the zero function. Thus we may fix $\Psi_\sigma\colon\sigma\times\{1,2\}\to[-1,1]$ satisfying

(1) $\Psi_\sigma(x,1)=0$ and $\Psi_\sigma(x,2)=u_\sigma(x)$ for all $x\in\partial\sigma$; and

(2) $\Psi_\sigma(x,1)<0<\Psi_\sigma(x,2)$ for all $x\in\operatorname{int}\sigma$.

Case 2. *Functions on positive simplices.*

For $\nu(\sigma)=1$, two functions u_σ and v_σ on $\partial\sigma$ will be used in a similar way. Given a face τ of σ, take $u_\sigma|_\tau$ to be an opening if τ is either an entry or an exit face of σ, and $u_\sigma|_\tau=0$, otherwise. As for v_σ, define $v_\sigma|_\tau=0$ for every face τ of σ which is not an exit face, and take v_σ to be a shuffle on the exit face of σ, if such face exists. Note that u_σ and v_σ have (essentially) pairwise disjoint supports.

We may now fix a function $\Psi_\sigma\colon\sigma\times\{1,...,4\}\to[-1,1]$ so that, for $x\in\partial\sigma$,

(1) $\Psi_\sigma(x,1)=\Psi_\sigma(x,2)=0$ and $\Psi_\sigma(x,3)=\Psi_\sigma(x,4)=u_\sigma(x)$ if $v_\sigma(x)=0$;

(2) $\Psi_\sigma(x,1)=\Psi_\sigma(x,2)=\Psi_\sigma(x,3)=v_\sigma(x)$ and $\Psi_\sigma(x,4)=0$ if $v_\sigma(x)<0$;

(3) $\Psi_\sigma(x,1)=0$ and $\Psi_\sigma(x,2)=\Psi_\sigma(x,3)=\Psi_\sigma(x,4)=v_\sigma(x)$ if $v_\sigma(x)>0$;

while for $x\in\operatorname{int}\sigma$,

(4) $\Psi_\sigma(x,1)<\Psi_\sigma(x,2)<\Psi_\sigma(x,3)<\Psi_\sigma(x,4)$; and

(5) $\Psi_\sigma(x,1)<0<\Psi_\sigma(x,4)$.

7.1.2. Sheets and a pillow cover

The singular $(n-1)$-simplices

$$\hat\sigma_i=\{(x,\Psi_\sigma(x,i)):x\in\sigma\},\tag{7.1}$$

where $i\in\{1,...,\ell_\sigma\}$, constitute the *sheets* of σ (as in [15]), and the union of sheets

$$\hat\sigma=\bigcup_{i=1}^{\ell_\sigma}\hat\sigma_i\tag{7.2}$$

forms a *pillow cover* on σ. Note that a pillow cover of σ consists of either two or four singular $(n-1)$-simplices depending on the parity $\nu(\sigma)$ of σ.

Remark 7.6. Observe that $\{\hat{\sigma}_1, ... \hat{\sigma}_{\ell_\sigma}\}$ is a (singular) triangulation of $\hat{\sigma}$ by singular $(n-1)$-simplices. This triangulation, however, does not induce a simplicial complex, since pairwise intersections of these simplices are generally not unions of sides. For example, $\hat{\sigma}_1 \cap \hat{\sigma}_{\ell_\sigma}$ is not a union of faces of $\hat{\sigma}_1$.

7.1.3. Pillows

We consider next, in more detail, the complementary domains of $\hat{\sigma}$ in $\sigma\times\mathbb{R}$. Let

$$P_\sigma = \{(x,t)\in\sigma\times\mathbb{R} : \Psi_\sigma(x,1)\leqslant t\leqslant \Psi_\sigma(x,\ell_\sigma)\}.$$

We call P_σ a *pillow.* Let also

$$U_\sigma = \{(x,t)\in\sigma\times\mathbb{R} : t\geqslant \Psi_\sigma(x,\ell_\sigma)\}$$

and

$$L_\sigma = \{(x,t)\in\sigma\times\mathbb{R} : t\leqslant \Psi_\sigma(x,1))\}.$$

Regardless of the parity of σ, U_σ and L_σ are bilipschitz equivalent to $\sigma\times[0,\infty)$ and $\sigma\times(-\infty,0]$, respectively. For example, for U_σ, there is the bilipschitz map

$$(x,t)\longmapsto \begin{cases} (x, 2(t-\Psi_\sigma(x,\ell_\sigma)), & \text{if } \Psi_\sigma(x,\ell_\sigma)\leqslant t\leqslant 2\Psi_\sigma(x,\ell_\sigma),\\ (x,t), & \text{if } t\geqslant 2\Psi_\sigma(x,\ell_\sigma),\end{cases}$$

and similarly for L_σ the map

$$(x,t)\longmapsto \begin{cases} (x, 2(t-\Psi_\sigma(x,1)), & \text{if } \Psi_\sigma(x,1)\geqslant t\geqslant 2\Psi_\sigma(x,1),\\ (x,t), & \text{if } t\leqslant 2\Psi_\sigma(x,1).\end{cases}$$

Since $|\Psi_\sigma|\leqslant 1$, these homeomorphisms are the identity outside $\sigma\times[-2,2]$, and the bilipschitz constants of these homeomorphisms depend only on n and the Lipschitz constants of Ψ_σ. Similarly, P_σ is bilipschitz equivalent to an n-cell independent of the parity of σ.

Pillows on a negative simplex

When $\nu(\sigma)=-1$, we observe that ∂P_σ is an essentially disjoint union of $\hat{\sigma}=\hat{\sigma}_1\cup\hat{\sigma}_2$ together with a union of $(n-1)$-cells in $\partial\sigma\times\mathbb{R}$.

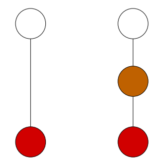

Figure 42. Adjacency graphs $\Gamma((\sigma\times\mathbb{R})\setminus\sigma)$ and $\Gamma((\sigma\times\mathbb{R})\setminus\hat\sigma)$ for σ with negative parity.

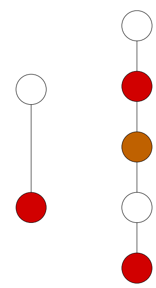

Figure 43. Adjacency graphs $\Gamma((\sigma\times\mathbb{R})\setminus\sigma)$ and $\Gamma((\sigma\times\mathbb{R})\setminus\hat\sigma)$ for σ with positive parity. The merge of the domains P_σ^U and U_σ (as well as P_σ^L and L_σ) in Lemma 7.8 is anticipated by the choice of colors.

Pillows on a positive simplex

When $\nu(\sigma)=1$, the complementary domains have more complicated structure. Now $P_\sigma\setminus\hat\sigma$ has three components with closures P_σ^U, P_σ^M, and P_σ^L, respectively,

$$\begin{aligned}P_\sigma^U &= \{(x,t) : \Psi_\sigma(x,1) \leqslant t \leqslant \Psi_\sigma(x,2)\},\\ P_\sigma^M &= \{(x,t) : \Psi_\sigma(x,2) \leqslant t \leqslant \Psi_\sigma(x,3)\},\\ P_\sigma^L &= \{(x,t) : \Psi_\sigma(x,3) \leqslant t \leqslant \Psi_\sigma(x,4)\}.\end{aligned}$$

Although the letters 'U', 'M', and 'L' refer to 'upper', 'middle', and 'lower', the domains are not named by their position along the t-axis; these names anticipate Lemma 7.8 below. We have

$$\hat\sigma\cap\partial P_\sigma^U = \hat\sigma_1\cup\hat\sigma_2, \quad \hat\sigma\cap\partial P_\sigma^M = \hat\sigma_2\cup\hat\sigma_3, \quad \text{and} \quad \hat\sigma\cap\partial P_\sigma^L = \hat\sigma_3\cup\hat\sigma_4;$$

see Figure 43.

7.2. Pillow covers of adjacent simplices

Recall that $E\subset\partial_\cup\mathbf{\Omega}$ is a planar $(n-1)$-cell, and, to simplify the notation, we have assumed that $E\subset\mathbb{R}^{n-1}\times\{0\}\subset\mathbb{R}^n$.

Let σ be an $(n-1)$-simplex, E be as before, and suppose that σ' is another $(n-1)$-simplex in E sharing an $(n-2)$-simplex with σ. By changing the rôles of σ and $\hat{\sigma}$ if necessary, we may assume that $\nu(\sigma')=-\nu(\sigma)=-1$.

Definition 7.7. The pillow covers $\hat{\sigma}$ and $\hat{\sigma}'$ of σ and σ', respectively, are *compatible* if $\Psi_\sigma(\cdot,2)=\Psi_{\sigma'}(\cdot,1)$ and $\Psi_\sigma(\cdot,3)=\Psi_{\sigma'}(\cdot,2)$ on τ, where τ is the common face of σ and σ'.

From now on we assume that $\hat{\sigma}$ and $\hat{\sigma}'$ are compatible pillow covers. The following lemma recapitulates Rickman's idea on using two types of pillow covers to permute the local rôles of the three domains.

LEMMA 7.8. *Let $\hat{\sigma}$ and $\hat{\sigma}'$ be compatible pillow covers of σ and σ', respectively. Then*

$$((\sigma\cup\sigma')\times\mathbb{R})\setminus(\hat{\sigma}\cup\hat{\sigma}')$$

has three components Ω^U, Ω^M, and Ω^L satisfying

$$\overline{\Omega^U}=U_\sigma\cup P_\sigma^U\cup U_{\sigma'},\quad \overline{\Omega^M}=P_\sigma^M\cup P_{\sigma'},\quad \textit{and}\quad \overline{\Omega^L}=L_\sigma\cup P_\sigma^L\cup L_{\sigma'}.$$

Proof. It suffices to observe that the closures of P_σ^U and $U_{\sigma'}$ meet in the $(n-1)$-cell

$$\{(x,t)\in\tau\times\mathbb{R}:\Phi_\sigma(x,3)\leqslant t\leqslant\Phi_\sigma(x,4)\}.$$

Similarly, $P_\sigma^L\cap L_{\sigma'}$ is an $(n-1)$-cell. □

Using the notation of Lemma 7.8, we now make a few observations on the natural triangulation of $\hat{\sigma}\cup\hat{\sigma}'$ into sheets and domains Ω^U, Ω^M, and Ω^L.

For $\hat{\sigma}'$, the pairwise intersections of the domains Ω^L, Ω^M, and Ω^U with $\hat{\sigma}'\times\mathbb{R}$ are (up to a closure) $L_{\sigma'}$, $P_{\sigma'}$, and $U_{\sigma'}$. Thus

$$\partial\Omega^L\cap\hat{\sigma}'=\hat{\sigma}'_1,\quad \partial\Omega^M\cap\hat{\sigma}'=\hat{\sigma}'_1\cup\hat{\sigma}'_2,\quad\text{and}\quad \partial\Omega^U\cap\hat{\sigma}'=\hat{\sigma}'_2.$$

The situation is slightly more complicated with $\hat{\sigma}$. Note first that $\Omega^M\cap(\sigma\times\mathbb{R})$ is P_σ^M up to closure. Thus

$$\partial\Omega^M\cap\hat{\sigma}=\hat{\sigma}_2\cup\hat{\sigma}_3,$$

and we have

$$\hat{\sigma}_2=\Omega^U\cap\Omega^M\cap(\sigma\times\mathbb{R})\quad\text{and}\quad \hat{\sigma}_3=\Omega^L\cap\Omega^M\cap(\sigma\times\mathbb{R}).$$

Moreover,

$$\overline{\Omega^L\cap(\sigma\times\mathbb{R})}=\overline{L_\sigma\cup P_\sigma^L}\quad\text{and}\quad \overline{\Omega^U\cap(\sigma\times\mathbb{R})}=\overline{U_\sigma\cup P_\sigma^U},$$

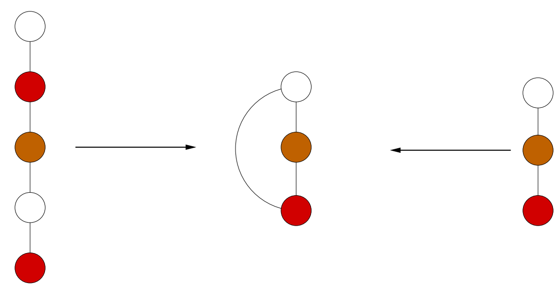

Figure 44. Adjacency graphs $\Gamma((\sigma\times\mathbb{R})\setminus\hat\sigma)$, $\Gamma(((\sigma\cup\sigma')\times\mathbb{R})\setminus(\hat\sigma\cup\hat\sigma'))$ and $\Gamma((\sigma'\times\mathbb{R})\setminus\hat\sigma')$ with maps of graphs induced by inclusions.

and

$$\partial\Omega^L\cap\hat\sigma=\partial L_\sigma\cup\partial P^L_\sigma=\hat\sigma_1\cup\hat\sigma_3\cup\hat\sigma_4 \quad\text{and}\quad \partial\Omega^U\cap\hat\sigma=\partial U_\sigma\cup\partial P^U_\sigma=\hat\sigma_4\cup\hat\sigma_1\cup\hat\sigma_2.$$

This 'shuffle' will allow our domains $\{\Omega_\ell\}_{\ell=1}^3$ to connect near $\partial_\cup\mathbf{\Omega}$. The proof of the following lemma is left to the interested reader; the situation is captured by the suggestive figure in [15, Figure 7.2] and Figure 44.

Lemma 7.9. *With the notation above, we have*

$$\hat\sigma_1\cup\hat\sigma_4=\Omega^U\cap\Omega^L\cap(\sigma\times\mathbb{R}),$$
$$\hat\sigma_2=\Omega^L\cap\Omega^M\cap(\sigma\times\mathbb{R}),$$
$$\hat\sigma_3=\Omega^M\cap\Omega^U\cap(\sigma\times\mathbb{R}).$$

Furthermore,

$$\hat\sigma_1'=\Omega^L\cap\Omega^M\cap(\sigma'\times\mathbb{R}) \quad\textit{and}\quad \hat\sigma_2'=\Omega^M\cap\Omega^U\cap(\sigma'\times\mathbb{R}).$$

Our discussion shows that the domains Ω^U, Ω^M, and Ω^L are bilipschitz equivalent to either $(\sigma\cup\sigma')\times(0,\infty)$, $(\sigma\cup\sigma')\times(-\infty,0)$, or to $\mathbb{B}^n$. We formalize this observation as follows.

Lemma 7.10. *Let $\hat\sigma$ and $\hat\sigma'$ be compatible Lipschitz pillow covers on σ and σ', respectively. Then*

(1) *there exist bilipschitz homeomorphisms*

$$h^U_{\sigma,\sigma'}\colon(\sigma\cup\sigma')\times(0,\infty)\longrightarrow(\Omega^U,d_{\Omega^U}) \quad\textit{and}\quad h^L_{\sigma,\sigma'}\colon(\sigma\cup\sigma')\times(-\infty,0)\longrightarrow(\Omega^L,d_{\Omega^L}),$$

whose supports are contained in $\sigma\cup\sigma'\times\left[-\frac12,\frac12\right]$, *such that* $h^U_{\sigma,\sigma'}$ *and* $h^L_{\sigma,\sigma'}$ *extend to* BLD-*maps* $(\sigma\cup\sigma')\times[0,\infty)\to\overline{\Omega^U}$ *and* $(\sigma\cup\sigma')\times(-\infty,0]\to\overline{\Omega^L}$, *respectively*; *and*

(2) *the closure of Ω^M is a bilipschitz n-cell.*

The bilipschitz (*and* BLD) *constants are quantitative in the sense that they depend only on* n, *the Lipschitz constants of* Ψ_σ *and* $\Psi_{\sigma'}$ *and the minimal bilipschitz constants of the homeomorphisms* $\sigma\to\mathbb{B}^{n-1}$ *and* $\sigma'\to\mathbb{B}^{n-1}$.

7.3. Maps on pairs of sheets

The pillow construction on the union $\sigma\cup\sigma'$ of two adjacent simplices σ and σ' gives rise to maps $\hat\sigma\cup\hat\sigma'\to\widehat{\mathbb{S}}^{n-1}$, where $\widehat{\mathbb{S}}^{n-1}=\mathbb{S}^{n-1}\cup\mathbb{B}^{n-1}\subset\mathbb{R}^n$. We now discuss these local maps in more detail.

We write $\mathbb{S}^{n-1}=\mathbb{S}^{n-1}_+\cup\mathbb{S}^{n-1}_-$, where $\mathbb{S}^{n-1}_+$ and $\mathbb{S}^{n-1}_-$ are the upper and lower hemispheres of $\mathbb{S}^{n-1}$, i.e. $\mathbb{S}^{n-1}_+\cap\mathbb{S}^{n-1}_-=\partial\mathbb{B}^{n-1}$. Then $\mathbb{R}^n\setminus\widehat{\mathbb{S}}^{n-1}$ has three components denoted D^U, D^L, and D^M so that $\partial D^U=\mathbb{S}^{n-1}_+\cup\mathbb{B}^{n-1}$, $\partial D^L=\mathbb{S}^{n-1}_-\cup\mathbb{B}^{n-1}$, and $\partial D^M=\mathbb{S}^{n-1}$. We fix n points $\{y_0,...,y_{n-1}\}$ on $\partial\mathbb{B}^{n-1}$ and view $\widehat{\mathbb{S}}^{n-1}$ as a CW-complex having three $(n-1)$-cells $\mathbb{S}^{n-1}_+$, $\mathbb{S}^{n-1}_-$, and $\mathbb{B}^{n-1}$ and vertices $\{y_0,...,y_{n-1}\}$.

Let σ and σ' be adjacent $(n-1)$-simplices in E and let $\hat\sigma$ and $\hat\sigma'$ be compatible Lipschitz pillows on σ and σ', respectively. By changing the rôles of σ and σ' if necessary, we may assume that $\nu(\sigma)=-\nu(\sigma')=1$. Let $\vartheta\colon\sigma^{(0)}\cup\sigma'^{(0)}\to\{0,...,n-1\}$ be the labeling function of $\mathbf{\Omega}$ restricted to $\sigma\cup\sigma'$.

Although the singular simplices $\Delta=\{\hat\sigma_1,...,\hat\sigma_4,\hat\sigma'_1,\hat\sigma'_2\}$ again do not define a simplicial complex, there exists a continuous map $f\colon\hat\sigma\cup\hat\sigma'\to\widehat{\mathbb{S}}^n$ satisfying

(S1) f maps each singular simplex in Δ to one of the simplices $\mathbb{S}^{n-1}_+$, $\mathbb{S}^{n-1}_-$, and $\mathbb{B}^{n-1}$ in a bilipschitz manner;

(S2) $f(v)=y_{\vartheta(v)}$ for all $v\in\sigma^{(0)}\cup(\sigma')^{(0)}$; and

(S3) if $\{X,Y\}\subset\{U,L,M\}$ is a pair then $f(\Omega^X\cap\Omega^Y)=D^X\cap D^Y$.

Since f is bilipschitz on singular simplices, it is discrete and

$$\frac{1}{\mathcal{L}}\ell(\gamma)\leqslant\ell(f\circ\gamma)\leqslant\mathcal{L}\ell(\gamma)$$

for all paths γ in $\sigma\cup\sigma'$, where $\mathcal{L}$ is the maximum of the bilipschitz constants of f restricted to simplices in Δ. Furthermore, in the sense of the following lemma, f is a branched cover in the interior of $\hat\sigma\cup\hat\sigma'$.

LEMMA 7.11. *Let* $O=(\hat\sigma\cup\hat\sigma')\cap\operatorname{int}(\sigma\cup\sigma')\times\mathbb{R}$. *Then* $f|_O\colon O\to\widehat{\mathbb{S}}^n$ *is a branched cover and the branch set of* $f|_O$ *is the set*

$$O\cap\{y\in\sigma\cap\sigma':\Psi_\sigma(y,1)=\Psi_\sigma(y,4)\}\subset\mathbb{R}^n.$$

In particular, $f|_O$ *is an open map.*

Proof. Let τ be the common face of σ and σ'. Let $S=\hat\sigma\cup\hat\sigma'$ and

$$G=\bigcup_{i=1}^{4}\operatorname{int}\hat\sigma_i\cup\bigcup_{j=1}^{2}\operatorname{int}\hat\sigma'_j.$$

Then

$$S=G\cup(S\cap(\tau\times\mathbb{R}))\cup(S\cap\partial(\sigma\cup\sigma')\times\mathbb{R}).$$

Clearly $G\subset O$ and $f|_G\colon G\to\widehat{\mathbb{S}}^n$ is a local homeomorphism. Suppose now that $x=(y,t)\in O\cap(\tau\times\mathbb{R})$. Then $f(x)\in\mathbb{S}^n\cap\mathbb{B}^n$.

There are four cases to consider. Suppose first that y has a neighborhood O' in τ such that $\Psi_\sigma(y',1)=\Psi_\sigma(y',2)$ for $y'\in O'$. Then also $\Psi_\sigma(y',1)=\Psi_{\sigma'}(y',1)$ and $\Psi_\sigma(y',3)=\Psi_\sigma(y',4)=\Psi_{\sigma'}(y',2)$ for $y'\in O'$ by compatibility, and so either $t=\Psi_\sigma(y,1)=\Psi_{\sigma'}(y,1)$ or $t=\Psi_\sigma(y,3)=\Psi_{\sigma'}(y,2)$. In either case, there are exactly three simplices T_U, T_L, and T_M among the simplices $\{\hat\sigma_1,...,\hat\sigma_4,\hat\sigma'_1,\hat\sigma'_2\}$ with $x\in T_U\cap T_L\cap T_M$ and $f(T_U)=\partial D^U$, $f(T_L)=\partial D^L$, and $f(T_M)=\partial D^M$. When y has a neighborhood O' with $\Psi_\sigma(y',1)=\Psi_\sigma(y',3)$ or $\Psi_\sigma(y',2)=\Psi_\sigma(y',4)$ for $y'\in O'$, the argument is similar. In all these cases, f is a homeomorphism in a neighborhood of x.

In the remaining case, $x\in O\cap(\tau\times\mathbb{R})$ and $\Psi_\sigma(y,1)=\Psi_\sigma(y,4)$. Then x belongs to all six singular simplices, and f is a branched double cover near x. □

7.4. Pillow covers of cells

Suppose again that E is a planar $(n-1)$-cell, i.e. E is contained in an $(n-1)$-plane P. We may take $P=\mathbb{R}^{n-1}\times\{0\}$ as in the beginning of §7.

Having $\nu=\nu_{E,\mathbf{\Omega}}$ at our disposal, we fix a maximal tree $\widehat{\mathcal{E}}\subset\Gamma(E^{(n-1)})$ and obtain, for every $\sigma\in E^{(n-1)}$, a pillow $\hat\sigma$ compatible with the simplices adjacent to σ in E. The set

$$\widehat{E}=\bigcup_{\sigma\in E^{(n-1)}}\hat\sigma$$

is a *pillow cover on E*. By our convention, all pillow covers $\hat\sigma$ for $\sigma\in E^{(n-1)}$ are $\mathcal{L}$-Lipschitz for $\mathcal{L}\geqslant1$ depending only on n, so that $\widehat{E}$ is an *$\mathcal{L}$-Lipschitz pillow cover*.

Lemmas 7.8 and 7.10 on metric properties of the pillow cover construction for pairs of simplices have counterparts for an $(n-1)$-cell contained in a hyperplane. The proofs are verbatim so we merely state the results.

LEMMA 7.12. *Let E be a cubical $(n-1)$-cell in $\mathbb{R}^{n-1}$ and $\widehat{E}\subset E\times\left[-\frac12,\frac12\right]$ be an $\mathcal{L}$-Lipschitz pillow on E. Then*

$$E\times[-1,1]\setminus\widehat{E}$$

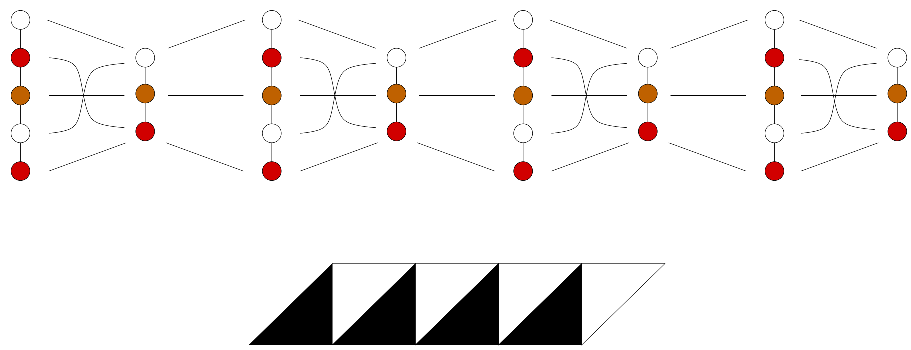

Figure 45. The adjacency of the domains Ω^U, Ω^M, and Ω^L over a 2-cell in $\mathbb{R}^2\times\{0\}\subset\mathbb{R}^3$.

has three components Ω^U, Ω^M, and Ω^L, each bilipschitz equivalent to $\mathbb{B}^n$ in their inner metric, so that $\Omega^U\supset E\times\{1\}$ and $\Omega^L\supset E\times\{-1\}$. The bilipschitz constant is quantitative and depends only on n and $\mathcal{L}$.

Lemma 7.13. *Let E be a cubical $(n-1)$-cell in $\mathbb{R}^{n-1}$ and $\widehat{E}\subset E\times\left[-\frac{1}{2},\frac{1}{2}\right]$ be an $\mathcal{L}$-Lipschitz pillow on E. Then*

(1) *there exists a bilipschitz homeomorphism $h^U_E\colon E\times(0,1)\to(\Omega^U,d_{\Omega^U})$ having a* BLD-*extension $\bar{h}^U_E\colon E\times[0,1]\to\overline{\Omega^U}$ so that $\bar{h}^U_E$ is the identity on $E\times\{1\}\cup\partial E\times[0,1]$; and*

(2) *there exists a bilipschitz homeomorphism $h^L_E\colon E\times(-1,0)\to(\Omega^L,d_{\Omega^L})$ having a* BLD-*extension $\bar{h}^L_E\colon E\times[-1,0]\to\overline{\Omega^L}$ so that $\bar{h}^L_E$ is the identity on $E\times\{-1\}\cup\partial E\times[-1,0]$. The statement is quantitative in the sense that the bilipschitz constant depends only on n and $\mathcal{L}$.*

In order to define maps $\widehat{E}\to\widehat{\mathbb{S}}^n$, we fix points $\{y_0,\dots,y_{n-1}\}\subset\mathbb{S}^{n-1}\cap\mathbb{B}^{n-1}$, as in §7.3. The following lemma is a counterpart of the construction in §7.3.

Lemma 7.14. *Let E be a cubical planar n-cell in $\mathbb{R}^n$ and $\widehat{E}\subset E\times\left[-\frac{1}{2},\frac{1}{2}\right]$ be a pillow on E. Then there exists a map $f_E\colon\widehat{E}\to\widehat{\mathbb{S}}^n$, which is a branched cover on* $\operatorname{int}\widehat{E}=\widehat{E}\cap(\operatorname{int}E\times\mathbb{R})$, *so that $f_E|_{\hat\sigma\cup\hat\sigma'}$ satisfies* (S1)–(S3) *from* §7.3 *for every pair of adjacent simplices σ and σ' in $E^{(n-1)}$. The* BLD-*constant of $f_E|_{\operatorname{int}\widehat{E}}$ is quantitative in the sense that it depends only on n, $\mathcal{L}$, and the points $\{y_0,\dots,y_{n-1}\}$.*

Proof. The mapping f_E is readily obtained as in the discussion in §7.3, so it suffices to discuss the uniformity of the BLD-constant of $f_E|_{\operatorname{int}\widehat{E}}$. Since E is given a standard simplicial structure, all simplices σ in $E^{(n-1)}$ are congruent. For every $\sigma\in E^{(n-1)}$, faces of σ are of one of the following three different types: *entry*, *exit*, and *closed* faces. By fixing opening and shuffle functions invariant under congruences, we may assume that pillows over simplices, with the same combinatorics, are congruent. More precisely, there exist simplices $\sigma_1,\dots,\sigma_r$ in $E^{(n-1)}$ and compatible pillows so that, for every $\sigma\in E^{(n-1)}$, there exists an isometry I_σ of $\mathbb{R}^n$, preserving $\mathbb{R}^{n-1}\times[0,\infty)$, and $1\leqslant i_\sigma\leqslant r$ so that $I_\sigma(\sigma)=\sigma_{i_\sigma}$ and $I_\sigma(\hat\sigma)=\hat\sigma_{i_\sigma}$.

Thus we fix a finite collection of Lipschitz maps $f_i\colon\hat\sigma_i\to\widehat{\mathbb{S}}^{n-1}$ and use the isometries I_σ to obtain a map $f_E\colon\widehat{E}\to\widehat{\mathbb{S}}^{n-1}$. The BLD-constant of $f_E|_{\operatorname{int}\widehat{E}}$ then clearly depends only on the Lipschitz constants of this finite collection $f_1,\dots,f_r$, depending only on n, $\mathcal{L}$, and the choice of points $\{y_0,\dots,y_{n-1}\}$. □

Remark 7.15. The standard simplicial structure of E is not essential for the proof of Lemma 7.14. In fact, given any simplicial complex P in $\mathbb{R}^n$ with $|P|=E$, it is easy to observe that there exists a pillow $\widehat{E}$ on E consisting of compatible pillows $\hat\sigma$ for $\sigma\in P^{(n-1)}$, and a map $f_{E,P}\colon\widehat{E}\to\widehat{\mathbb{S}}^{n-1}$ satisfying the properties of Lemma 7.14 with the only exception that the BLD-constant of $f_{E,P}|_{\operatorname{int}\widehat{E}}$ now depends also on the bilipschitz constants of affine parametrizations $[0,e_1,\dots,e_{n-1}]\to\sigma$ for $\sigma\in P^{(n-1)}$. Although, this observation is essential in what follows, we leave the simple modification of the proof of Lemma 7.14 to the interested reader.

Suppose now that E is a cubical $(n-1)$-cell in $\mathbb{R}^n$. Since E is a PL $(n-1)$-cell, there exists a PL-homeomorphism $E\to E'$, where E' is an $(n-1)$-cell in $\mathbb{R}^{n-1}$. More precisely, there exists a simplicial complex P so that $|P|=E$ and a simplicial homeomorphism $\varphi\colon E\to E'$ with respect to P.

Let E be a cubical $(n-1)$-cell E in $\mathbb{R}^n$ and let $\mathcal{Q}(E)$ be the collection of all unit n-cubes Q in $\mathbb{R}^n$ with $Q\cap\operatorname{int}E\neq\varnothing$, and let $|\mathcal{Q}(E)|$ be the union of these cubes. Set

$$\mathcal{N}(E)=B_\infty\big(E,\tfrac13\big)\cap|\mathcal{Q}(E)|.$$

In particular, we have

$$\mathcal{N}(E')=E'\times\big[-\tfrac13,\tfrac13\big]$$

for a planar $(n-1)$-cell E' in $\mathbb{R}^n$, and the pair $(\mathcal{N}(E),E)$ is PL-homeomorphic to a proper cell pair $(\overline{B}^n,\overline{B}^{n-1})$; see [18, Chapter 4].

We apply these observations to small $(n-1)$-cells in $\mathbb{R}^n$, and summarize the needed properties in the following lemma, omitting details. Note that the uniform bound of the bilipschitz constant follows directly from the finiteness of the congruence classes of $(n-1)$-cells in the statement.

LEMMA 7.16. *Let E be a cubical $(n-1)$-cell in a cube $Q\subset\mathbb{R}^n$ of side-length* 3. *Then there exist $\mathcal{L}\geqslant1$ depending only on n, a planar cubical $(n-1)$-cell E', and an $\mathcal{L}$-bilipschitz* PL*-homeomorphism $\varphi_E\colon\mathcal{N}(E)\to\mathcal{N}(E')$ so that $\varphi_E(E)=E'$. Moreover, there is a simplicial complex P such that $|P|=E$ and φ_E is piecewise affine with respect to P.*

Using Lemma 7.16, we may define pillow covers for small $(n-1)$-cells in $\mathbb{R}^n$. Let E be a cubical $(n-1)$-cell contained in a cube of side-length 3. Suppose E' is a planar $(n-1)$-cell and $\varphi_E: \mathcal{N}(E) \to \mathcal{N}(E')$ is a PL-homeomorphism as in Lemma 7.16. Then $\varphi_E(E^{(n-1)})$ is a triangulation of E'. Although $\varphi_E(E^{(n-1)})$ is not the standard triangulation of E', we obtain a pillow $\widehat{E}'$ on E' in $\mathcal{N}(E')$ with respect to this triangulation, and call $\widehat{E} = \varphi_E^{-1}(\widehat{E}')$ a *pillow cover* of E.

Given an $(n-1)$-simplex σ in E, we also say that $\hat{\sigma} = \varphi^{-1}(\widehat{E} \cap (\varphi(\sigma) \times [-1,1]))$ is the *pillow over σ in $\widehat{E}$*. By the finiteness of congruence classes, we conclude that the results in the beginning of this section hold also for these pillow covers almost verbatim.

7.5. Proof of Proposition 7.1

Let $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$ be a rough Rickman partition of $\mathbb{R}^n$ having the tripod property. Thus $\partial_\cup \mathbf{\Omega}$ has an essential partition into cubical $(n-1)$-cells $\Delta = \{E_\ell\}_{\ell \geqslant 0}$.

Given adjacent E_ℓ and $E_{\ell'}$ in Δ belonging to different $\mathbf{\Omega}$-equivalence classes (recall Definition 4.2), there exists, by property $(\Delta 2)$ of Definition 4.4, a unique $E_{\ell''}$ in Δ so that the cells E_ℓ, $E_{\ell'}$, and $E_{\ell''}$ are mutually adjacent, contained in the same cube of side-length 3, and belong to different $\mathbf{\Omega}$-equivalence classes. If we write $E_\ell \sim E_{\ell'}$, the relation $\sim$ defines an equivalence relation in Δ which subdivides Δ into equivalence classes containing exactly three elements.

Let

$$\mathcal{N}(\partial_\cup \mathbf{\Omega}) = B_\infty\left(\partial_\cup \mathbf{\Omega}, \tfrac{1}{3}\right)$$

be the $\frac{1}{3}$-neighborhood of $\partial_\cup \mathbf{\Omega}$ in $\mathbb{R}^n$, and for each ℓ define

$$\mathcal{N}_\ell = \{x \in \mathcal{N}(\partial_\cup \mathbf{\Omega}) : \operatorname{dist}(x, E_\ell) = \operatorname{dist}(x, \partial_\cup \mathbf{\Omega})\}.$$

Then $\{\mathcal{N}_\ell\}_{\ell \geqslant 0}$ is an essential partition of $\mathcal{N}(\partial_\cup \mathbf{\Omega})$. Moreover, $\mathcal{N}_\ell$ is PL-homeomorphic to $\mathcal{N}(E_\ell)$ for every ℓ. As there are only a finite number of congruence classes of $\mathcal{N}_\ell$ and $\mathcal{N}(E_\ell)$, we have that $\mathcal{N}_\ell$ is bilipschitz to $\mathcal{N}(E_\ell)$, the constant depending only on n.

Suppose E_{ℓ_0}, E_{ℓ_1}, and E_{ℓ_2} are equivalent $(n-1)$-cells in Δ. We create pillows $\widehat{E}_{\ell_0}$, $\widehat{E}_{\ell_1}$, and $\widehat{E}_{\ell_2}$ simultaneously. Let $E_{[\ell]} = E_{\ell_0} \cup E_{\ell_1} \cup E_{\ell_2}$ and $\mathcal{N}_{[\ell]} = \mathcal{N}_{\ell_0} \cup \mathcal{N}_{\ell_1} \cup \mathcal{N}_{\ell_2}$. We fix, for $m = 0, 1, 2$, indices $\{i_m, j_m, k_m\} = \{1, 2, 3\}$ such that $E_{\ell_m} \cap \Omega_{k_m}$ is an $(n-2)$-cell and $E_{\ell_m} \subset \Omega_{i_m} \cap \Omega_{j_m}$.

Let

$$Y = (\mathbb{R}^{n-1} \times \{0\}) \cup (\{0\} \times \mathbb{R}^{n-2} \times [0, \infty)) \subset \mathbb{R}^n.$$

Since $E_{[\ell]} = E_{\ell_0} \cup E_{\ell_1} \cup E_{\ell_2}$ is a union of equivalent elements in Δ, we may fix essentially disjoint $(n-1)$-cells E'_{ℓ_0}, $E'_{\ell,1}$, and $E'_{\ell,2}$ in Y such that there exists a map

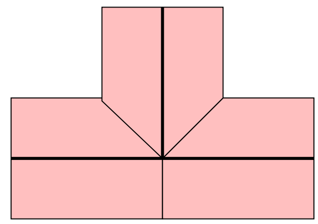

Figure 46. The cells E'_ℓ, $E'_{\ell'}$, and $E'_{\ell''}$ meeting at $\{0\}\times\mathbb{R}^{n-2}\times\{0\}$ and the partition of $\mathcal{N}(E'_{[\ell]})$.

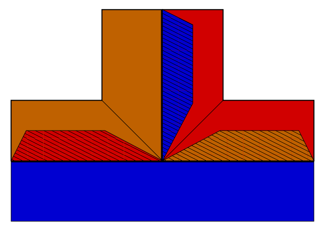

Figure 47. Three components D'_m waiting to be connected to the corresponding components U_m.

$\phi_{[\ell]}\colon E'_{[\ell]}\to E_{[\ell]}$, where $E'_{[\ell]}=E'_{\ell_0}\cup E'_{\ell_1}\cup E'_{\ell_2}$, which is a PL-homeomorphism

$$E'_{\ell_0}\cap E'_{\ell_1}\cap E'_{\ell_2}\to E'_{\ell_0}\cap E'_{\ell_1}\cap E'_{\ell_2} \quad\text{and}\quad E'_{\ell_k}\to E_{\ell_k} \text{ for each } k.$$

The map $\phi_{[\ell]}$ extends to a PL-map $\phi_{[\ell]}\colon\mathcal{N}(E'_{[\ell]})\to\mathcal{N}_{[\ell]}$ which is a homeomorphism from the interior of $\mathcal{N}(E'_{[\ell]})$ to the interior of $\mathcal{N}_{[\ell]}$, where $\mathcal{N}(E'_{[\ell]})=\bigcup_{m=0}^2\mathcal{N}(E'_{\ell_m})$. The connected components of $\mathcal{N}(E'_{[\ell]})\setminus Y$ are $U_m=\psi_{[\ell]}(\operatorname{int}\Omega_m\cap\mathcal{N}_{[\ell]})$ for $m=0,1,2$.

Again, by finiteness of the congruence classes, $\phi'_{[\ell]}=\phi_{[\ell]}|_{\operatorname{int}E'_{[\ell]}}\colon\operatorname{int}E'_{[\ell]}\to\operatorname{int}E_{[\ell]}$ is bilipschitz (in the inner metric) with constant depending only on n. Each map $\phi'_{[\ell]}$ induces a triangulation on $E'_{[\ell]}$, and we denote by ν the parity function $\sigma\mapsto\nu_{\mathbf{\Omega}}(\phi_{[\ell]}\circ\sigma)$ defined on the $(n-1)$-simplices in the induced triangulation of $E'_{[\ell]}$.

In terms of this function ν on $E'_{[\ell]}$, we fix, for every $m=0,1,2$, a Lipschitz pillow $\widehat{E}'_{\ell_m}\subset B_\infty\big(E'_{\ell_m},\frac{1}{3}\big)$. By Lemma 7.12, $\mathcal{N}(E'_{\ell_m})\setminus\widehat{E}'_{\ell_m}$ has three components and there exists a unique component $D'_m\subset\mathcal{N}(E'_{\ell_m})\setminus\widehat{E}'_{\ell_m}$ which does not meet $\partial\mathcal{N}(E'_{\ell_m})$ essentially; that is, the intersection $D'_m\cap\mathcal{N}(E'_{\ell_m})$ does not contain $(n-1)$-simplices.

We observe that the set $\bigcup_{m=0}^2\widehat{E}'_{\ell_m}$ has six complementary components in $\mathcal{N}(E'_{[\ell]})$; see Figure 47. We now modify the pillows $\widehat{E}'_{\ell_m}$; informally, by connecting each D'_m to U_m, there will only be three complementary components.

For $m=0,1,2$, let $\sigma_{\ell_m}\subset E'_{\ell_m}$ be simplices meeting on a common face $\tau\subset\sigma_{\ell_0}\cap\sigma_{\ell_1}\cap\sigma_{\ell_2}$. By Lemma 6.1, all simplices σ_{ℓ_m} have the same ν-parity. For notational simplicity, we consider only the case $\nu(\sigma_{\ell_m})=-1$; the case $\nu(\sigma_{\ell_m})=1$ is similar and is left to the reader.

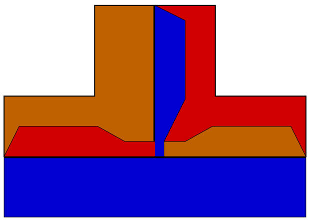

Figure 48. Domains after modification; a side view.

We fix three subsimplices τ_0, τ_1, and τ_2 of τ by subdiving $\phi(\tau)$ into congruent subsimplices of side-length $\frac{1}{3}$ and then fixing three of the preimages of these subsimplices in τ.

Since $\nu(\sigma_{\ell_0})=\nu(\sigma_{\ell_1})=-1$, the sheets $\hat{\sigma}_{\ell_0}$ and $\hat{\sigma}_{\ell_1}$ of σ_0 and σ_1, respectively, are determined by the functions $\Psi_{\sigma_{\ell_0}}$ and $\Psi_{\sigma_{\ell_1}}$. We modify these functions so that

$$\Psi_{\sigma_{\ell_r}}(\operatorname{int}\tau_r\times\{2\})\subset\left(0,\tfrac{1}{3}\right)$$

for $r=0,1$, and denote the new sheets obtained in this manner by $\tilde{\sigma}_{\ell_r}$ for $r=0,1$. We denote also by $\widetilde{D}'_r$ the component of $\mathcal{N}(E'_{\ell_r})\setminus\tilde{\sigma}_{\ell_r}$ which does not meet $\partial\mathcal{N}(E'_{\ell_r})$ essentially.

For $r=0,1$, let $\widetilde{U}_{k_r}$ be the components of $\mathcal{N}(E'_{\ell_r})\setminus\tilde{\sigma}_{\ell_r}$ contained in U_{k_r}. It is now easy to observe that $\widetilde{D}'_r\subset\widetilde{U}_{k_r}$ is connected. Indeed, the $(n-2)$-cell

$$D_r=\{(x,t)\in\tau_r\times\mathbb{R}:\Psi_{\sigma_{\ell_r}}(x,1)\leqslant t\leqslant\Psi_{\sigma_{\ell_r}}(x,2)\}$$

for $r=0,1$, lies on the boundary of $\widetilde{D}'_r$ and is contained in $\widetilde{U}_{k_r}$. Furthermore, we have that the interior of $\operatorname{cl}(\widetilde{D}'_r\cup U_{k_r})$ is bilipschitz to $\mathbb{B}^n$ in the inner metric.

Without changing notation, we modify the sheet $\hat{\sigma}_{\ell_2}$ accordingly in order to preserve compatibility with other sheets after this change on $\tau_0\cup\tau_1$. The sheet modification is now applied to $\hat{\sigma}_{\ell_2}$ to obtain a new compatible sheet $\tilde{\sigma}_{\ell_2}$ so that the component D'_2 of $B_\infty\left(E'_2,\frac{1}{3}\right)\setminus\tilde{\sigma}_{\ell_2}$ is connected to U_{k_2}. We leave the details of this step to the interested reader.

We make some observations on the construction of the modified sheets $\tilde{\sigma}_{\ell_m}$ for $m=0,1,2$. First note that, although $\tilde{\sigma}_{\ell_m}$ is not homeomorphic to $\hat{\sigma}_{\ell_m}$ there exist maps $h_{\ell_m}:\tilde{\sigma}_{\ell_m}\to\hat{\sigma}_{\ell_m}$ such that h_{ℓ_m} is a homeomorphism in the interior of $\tilde{\sigma}_{\ell_m}$ and

$$h_{\ell_m}|_{\tilde{\sigma}_{\ell_m}\cap\hat{\sigma}_{\ell_m}}=\mathrm{id}.$$

In particular, $\tilde{\sigma}_{\ell_m}$ has the same number of singular simplices as does $\hat{\sigma}_{\ell_m}$ and the map h_{ℓ_m} restricts to a map between singular simplices.

Second, let

$$\widetilde{E}'_{[\ell]} = \tilde{\sigma}_{\ell_0} \cup \tilde{\sigma}_{\ell_1} \cup \tilde{\sigma}_{\ell_2}.$$

Then

$$\mathcal{N}(E'_{[\ell]}) \setminus \widetilde{E}'$$

has three connected components $\widetilde{U}_1$, $\widetilde{U}_2$, and $\widetilde{U}_3$ with the property

$$\partial \widetilde{U}_r \cap \partial U_r = B_\infty\big(Y, \tfrac{1}{3}\big) \cap \partial U_r,$$

and for every $r = 1, 2, 3$, there exists a bilipschitz homeomorphism

$$(\widetilde{U}_r, d_{\widetilde{U}_r}) \longrightarrow (U_r, d_{U_r}),$$

which is the identity on $\partial \widetilde{U}_r \cap \partial U_r$.

Let

$$\widetilde{E}_{[\ell]} = \phi_{[\ell]}(\widetilde{E}'_{[\ell]}).$$

Due to the convention on closed edges on the boundary of $\partial(\bigcup_{r=0}^2 E_{\ell_r})$, we have that

$$\widetilde{E}_{[\ell]} \cap \widetilde{E}_{[\ell']} = E_{[\ell]} \cap E_{[\ell']}$$

for all ℓ and ℓ'.

Let

$$X = \bigcup_{[\ell]} \widetilde{E}_{[\ell]}$$

denote the union over the equivalence classes $[\ell]$ of indices. Then $\mathbb{R}^n \setminus X$ has components $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$, and $\widetilde{\Omega}_3$. Using the congruence classes of the pillows $\widehat{E}_{[\ell]}$, we may assume that the pillows $\widetilde{E}_{[\ell]}$ are uniformly Lipschitz. Then the components $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$, and $\widetilde{\Omega}_3$ are bilipschitz equivalent to the components Ω_1, Ω_2, and Ω_3 of our original Rickman partition, respectively, in their inner metric. Furthermore, these bilipschitz homeomorphisms $(\Omega_m, d_{\Omega_m}) \to (\widetilde{\Omega}_m, d_{\widetilde{\Omega}_m})$, $m = 1, 2, 3$, extend to BLD-maps $\mathrm{cl}(\Omega_m) \to \mathrm{cl}(\widetilde{\Omega}_m)$. If we set $\widetilde{\boldsymbol{\Omega}} = (\widetilde{\Omega}_1, \widetilde{\Omega}_2, \widetilde{\Omega}_3)$, then $X = \partial_\cup \widetilde{\boldsymbol{\Omega}}$.

Finally, we obtain a BLD-map $f\colon \partial_\cup \widetilde{\boldsymbol{\Omega}} \to \widehat{\mathbb{S}}^{n-1}$. Relabel the components of $\mathbb{R}^n \setminus \widehat{\mathbb{S}}^n$ by D_1, D_2, and D_3 so that $D_1 = D^U$, $D_2 = D^L$, and $D_3 = D^M$.

By Remark 7.15, we may fix a map $g_{[\ell]}\colon \widetilde{E}'_\ell \to \widehat{\mathbb{S}}^{n-1}$ as in Lemma 7.14. By Lipschitz uniformity of the pillows $\widetilde{E}'_{[\ell]}$, we may assume that $g_{[\ell]}|_{\mathrm{int}\, \widetilde{E}'_{[\ell]}}$ is BLD with BLD-constant depending only on n.

Let $f_{[\ell]}\colon \widetilde{E}_{[\ell]} \to \widehat{\mathbb{S}}^{n-1}$ be the unique map satisfying $f_{[\ell]} \circ \phi_{[\ell]} = g_{[\ell]}$.

Given adjacent pillows $\widetilde{E}_{[\ell]}$ and $\widetilde{E}_{[\ell']}$, the mappings $f_{[\ell]}$ and $f_{[\ell']}$ are both defined on $\widetilde{E}_{[\ell]} \cap \widetilde{E}_{[\ell']}$. By uniformity of the BLD-constants, we may modify one of the mappings

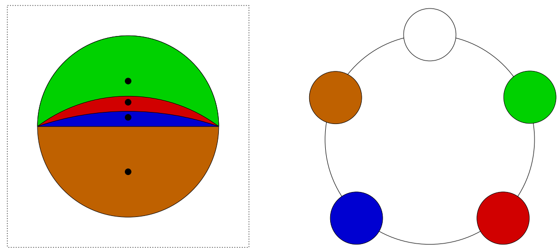

Figure 49. The adjacency graph for the cells in Figure 1; $p=4$.

$f_{[\ell]}$ and $f_{[\ell']}$ slightly to obtain a new collection of uniformly BLD-mappings so that the mappings $f_{[\ell]}$ and $f_{[\ell']}$ agree on $\widetilde{E}_{[\ell]}\cap\widetilde{E}_{[\ell']}$ for every $\ell\neq\ell'$. The map f, defined by $f|_{\widetilde{E}_{[\ell]}}=f_{[\ell]}$, is BLD.

This concludes the proof of Proposition 7.1.

8. Finishing touch

In this section we prove Theorem 1.4 and Proposition 1.5. The proofs are slight generalizations of Theorem 5.1 and Proposition 7.1. The proof of Proposition 1.5 is a straightforward modification, so we merely indicate the differences. For Theorem 1.4, we introduce a particular class of rough Rickman partitions, called *skewed Rickman partitions*, and show that the method to obtain a rough Rickman partition in the proof of Theorem 5.1 may be modified to obtain skewed Rickman partitions.

8.1. Skewed Rickman partitions

For general $p>2$, choose points $\{y_0, ..., y_p\}$ in $\mathbb{S}^n$ as in the introduction, that is, $y_0=e_{n+1}$ and $y_r=(0,t_r)\in\mathbb{R}^n$, where $-\frac{1}{2}=t_1<0<t_2<...<t_p=\frac{1}{2}$. Take n-cells $E_0, ..., E_p$ as in the introduction, i.e. $E_0=\mathrm{cl}(\mathbb{S}^n\setminus\mathbb{B}^n)$, $E_1\cup...\cup E_p=\mathbb{B}^n$, $y_r\in\mathrm{int}\,E_r$, so that $D_r=E_r\cap E_{r+1}$ is an $(n-1)$-cell for $r=0, ..., p$ (mod $p+1$). Note that ∂E_r is an $(n-1)$-sphere consisting of $(n-1)$-cells $D_r\cup D_{r-1}$ (mod $p+1$).

Let

$$\widehat{\mathbb{S}}^{n-1}_p=\bigcup_{r=0}^{p}\partial E_r.$$

We emphasize that $E_i\cap E_j=\mathbb{S}^{n-2}$ for $|i-j|>1$. Let $\mathcal{E}_p=(E_0, E_1, ..., E_p)$. Then $\mathcal{E}_p$ is an essential partition of $\mathbb{S}^n$, $\partial_\cup\mathcal{E}_p=\widehat{\mathbb{S}}^{n-1}_p$, $\partial_\cap\mathcal{E}_p=\mathbb{S}^{n-2}$ and the adjacency graph $\Gamma(\mathcal{E}_p)$ is cyclic.

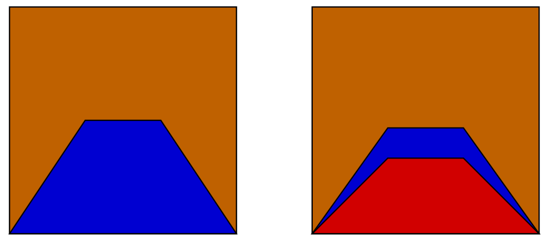

Figure 50. Schematic figure on n-cells A_j in A for $p=3$ and $p=4$.

Let q be a k-cube. A PL-embedding $\varphi\colon q\to\mathbb{R}^n$ is a PL *k-cube* and a complex composed of PL-cubes is a *skew complex* if the PL k-cubes are uniformly bilipschitz equivalent. A Rickman partition $\mathbf{\Omega}$ is *skew* if $\partial_\cup\mathbf{\Omega}$ is a skew complex.

The tripod property (Definition 4.4) admits a straightforward generalization to skew complexes $\mathbf{\Omega}=(\Omega_0,...,\Omega_p)$. Indeed, instead of having three elements in an equivalence class, we require that $\partial_\cup\mathbf{\Omega}$ have an essential partition Δ into (skew) $(n-1)$-cells, and we require that Δ in turn admit a partition into groups of $p+1$ elements, each $(n-1)$-cell between different elements of $\mathbf{\Omega}$, and all having a common intersection containing an $(n-2)$-cell. In this case we say that the skew complex satisfies a *generalized tripod property.*

We show that $\mathbb{R}^n$ admits a skew Rickman partition $\mathbf{\Omega}=(\Omega_0,...,\Omega_p)$ for each $p>2$.

Proposition 8.1. *Given $n\geqslant 3$ and $p>2$ there exists a skew Rickman partition $\mathbf{\Omega}=(\Omega_0,...,\Omega_p)$ supporting the (generalized) tripod property.*

8.1.1. Skew structures on atoms and molecules

An essential partition $\mathcal{S}$ of an n-cell C is *skew* if the elements of $\mathcal{S}$ are skew n-cells. Before proceeding further, we discuss skew partitions for (flat) atoms and (non-flat) molecules.

Let A be an r-fine $\mathbb{R}^{n-1}$-based atom in $\mathbb{R}^n$; let $F=A\cap\mathbb{R}^{n-1}$ and $C=\partial A-F$, where F refers to 'floor' and C to 'ceiling'. Note that F and C are $(n-1)$-cells. We partition A into skew atoms $A_1,...,A_{p-1}$ as follows.

Let $L_1=F$, $L_p=C$, and define, for $j=2,...,p-1$, $L_j=\{(x,\delta_{B,j}(x))\in A:x\in F\}$, where $\delta_{B,j}\colon F\to\left[0,\frac{1}{3}r\right]$ is the function

$$\delta_{B,j}(x)=\frac{j}{p}\max\left\{\frac{r}{3},\operatorname{dist}(x,F\cap C)\right\}\quad\text{for } x\in F.$$

For every $j=1,...,p-1$, $L_j\cup L_{j+1}$ bounds a unique n-cell A_j with boundary $L_j\cup L_{j+1}$.

Now the essential partition

$$\mathcal{S}(A)=(A_1,...,A_{p-1})\tag{8.1}$$

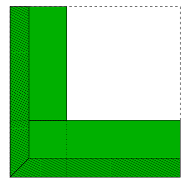

Figure 51. The join of two skew non-planar building blocks.

is a skew partition of A. Note that $F\subset\partial A_1$, $C\subset\partial A_{p-1}$, and $A_j\cap A_{j+1}$ is the $(n-1)$-cell L_{j+1} for all $j=1,...,p-1$.

We leave the details of the following lemma to the interested reader.

LEMMA 8.2. *Let A be an r-fine $\mathbb{R}^{n-1}$-based atom in $\mathbb{R}^n$ and $\mathcal{S}(A)=(A_1,...,A_{p-1})$ be a skew partition of A in (8.1) for $p>2$. Then there exist L-bilipschitz homeomorphisms $\varphi_j: A\to A_j$ for $j=1,...,p-1$, where $L=L(n,p)$, such that on $F=A\cap\mathbb{R}^{n-1}$ and $C=\partial A-F$ we have*

(i) $\varphi_1|_F=\mathrm{id}$ *and* $\varphi_{p-1}|_C=\mathrm{id}$, $\varphi_j|_{F\cap C}=\mathrm{id}$ *for each* j; *and*

(ii) $\varphi_j(F)=L_j$ *and* $\varphi_j(C)=L_{j+1}$ *for each* j.

Skew partitions of atoms merge to produce skew partitions of molecules.

LEMMA 8.3. *There exists $L=L(n,p)$ with the following properties. Let M be a molecule consisting of building blocks on the boundary of an n-cube Q so that pairwise unions of adjacent building blocks of different scales are planar. Then there exist an essential skew partition $\mathcal{S}(M)=(M_1,...,M_{p-1})$ of M into n-cells and L-bilipschitz homeomorphisms $\psi_j: M\to M_j$, $j=1,...,p-1$, for which*

(a) $\partial M\cap\partial Q\subset\partial M_1$ *and* $\partial M-\partial Q\subset\partial M_{p-1}$;

(b) $\psi_i(M)\cap\psi_j(M)$ *is an* $(n-1)$-*cell if* $j=i+1$; *and*

(c) $\psi_i(M)\cap\psi_j(M)=\partial M\cap\partial Q$ *for* $|i-j|>1$.

Proof. It suffices to consider two cases: (i) a non-planar atom in $\Gamma(M)$, and (ii) two adjacent atoms in $\Gamma(M)$.

Suppose first that A is a non-planar atom in $\Gamma(M)$. Then A consists of planar parts, all meeting in pairs of building blocks. Thus the general case follows from the special case of building blocks B and B' based on different faces of an n-cube, say Q' (see Figure 51). There exists a cube q of side-length r in $B\cup B'$ contained in one of the building blocks, say B, so that $q\cap B'=B\cap B'$. Since $A'=B'\cup q$ is an atom, we find skew atoms A'_j and B_j for $j=1,...,p-1$, in A' and B, respectively, so that $A'_j\cup B_j$ is an n-cell for each j and A'_1 and B_1 meet $\partial Q'$. Since $A'_j\cup B_j$ are n-cells for $j=1,...,p-1$, it is now easy to define non-planar n-cells $A_1,...,A_{p-1}$ forming an essential partition of A.

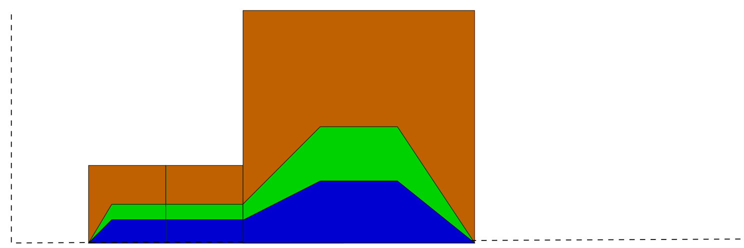

Figure 52. A skew partitioned molecule; $p=4$.

Suppose now that A is an r-fine atom adjacent to a $\frac{1}{3}r$-fine atom A'. Again, there exist building blocks $B\subset A$ and $B'\subset A'$ such that $A'\cap A=B\cap B'$. We may assume that $B\cup B'$ is $\mathbb{R}^{n-1}$-based. Let $\mathcal{S}(B)=(B_1, ..., B_{p-1})$ and $\mathcal{S}(B')=(B'_1, ..., B'_{p-1})$ be skew partitions of B and B'. Let $\varphi_j: B\to B_j$ and $\varphi'_j: B'\to B'_j$ be homeomorphisms as in Lemma 8.2. It is now easy to modify these homeomorphisms on $A\cap A'$ to obtain homeomorphisms $\widetilde{\varphi}_j$ and $\widetilde{\varphi}'_j$ for $j=1, ..., p-1$, so that each $\widetilde{\varphi}_j(B)\cup\widetilde{\varphi}_j(B')$ is an n-cell. Since the modification is local, we may also assume that the mappings $\widetilde{\varphi}_j$ and $\widetilde{\varphi}'_j$ are uniformly bilipschitz with constant depending only on n. We leave the further details to the interested reader; see Figure 52. □

8.1.2. Coarsification of skew partitions

In the proof of Proposition 8.1 we use generalizations of primary and secondary modifications introduced in §5. The rearrangements are given using skew partitions having properties similar to cubical partitions. We now introduce the necessary terminology.

In this section, let C be a cubical n-cell and $\mathcal{S}=(S_1, ..., S_{p-1})$ be a skew essential partition of C into n-cells.

Let $\alpha\in\mathbb{Z}_+$ and let $\mathcal{Q}_\alpha(C)$ be a subdivision of C into pairwise disjoint n-cubes of side-length $3^{-\alpha}$. We assign to each $q\in\mathcal{Q}_\alpha(C)$ an index $i_q\in\{1, ... p-1\}$ with i_q being the minimal index for which $q\cap S_i$ has non-empty interior. Let $\mathcal{M}_\alpha(\mathcal{S})$ be the set of cubes q in $\mathcal{Q}_\alpha(C)$ for which $\mathrm{int}(q\cap S_i)\neq\varnothing$ for more than two indices $i\in\{1, ..., p-1\}$, and let

$$E_{\alpha,i}(\mathcal{S}) = |\{q\in\mathcal{Q}_\alpha(C)\setminus\mathcal{M}_\alpha(\mathcal{S}) : i_q = i\}|.$$

Remark 8.4. Clearly, we have that $(E_{\alpha,1}(\mathcal{S}), ..., E_{\alpha,p-1}(\mathcal{S}))$ is an essential partition of $C-|\mathcal{M}_\alpha(\mathcal{S})|$. Although the cubical sets $E_{\alpha,i}(\mathcal{S})$ need not be n-cells for all $\alpha\in\mathbb{Z}_+$, since $\mathcal{S}$ is a skew partition, there exists $\alpha_0\in\mathbb{Z}_+$ for which $(E_{\alpha,1}(\mathcal{S}), ..., E_{\alpha,p-1}(\mathcal{S}))$ is an essential partition of $C-|\mathcal{M}_\alpha(\mathcal{S})|$ into n-cells for $\alpha\geqslant\alpha_0$.

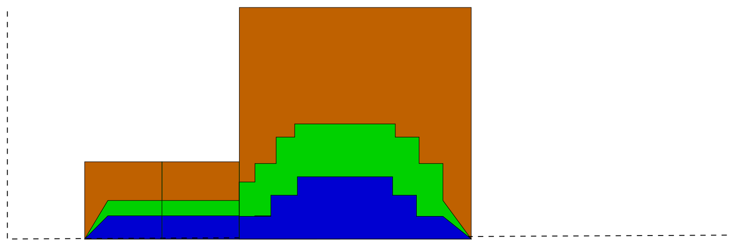

Figure 53. A coarsification of the skew partition in Figure 52.

Definition 8.5. Let $\alpha\in\mathbb{Z}_+$ and $\mathcal{S}=(S_1,...,S_{p-1})$ be a skew partition of an n-cell C. The essential partition $\widehat{\mathcal{S}}_\alpha=(\widehat{S}_1,...,\widehat{S}_{p-1})$ of C, where

$$\widehat{S}_i=E_{\alpha,i}(\mathcal{S})\cup(|\mathcal{M}_\alpha(\mathcal{S})|\cap S_i), \tag{8.2}$$

is an *ε-coarsification of $\mathcal{S}$ for $\varepsilon>0$* if, for each $i=1,...,p-1$, $\widehat{S}_i$ is an n-cell, $E_{\alpha,i}(\mathcal{S})\neq\varnothing$, $\widehat{S}_i\cap\widehat{S}_{i-1}$ is an $(n-1)$-cell, and $\operatorname{dist}_{\mathcal{H}}(S_i,\widehat{S}_i)<\varepsilon$.

In the proof of Proposition 8.1, we modify the earlier $\mathcal{C}$-modifications to produce skew partitions. Heuristically, in a generalized $\mathcal{C}$-modification, we rearrange the domain $\widehat{S}_i$ of a skew partition $\widehat{\mathcal{S}}_\alpha=(\widehat{S}_1,...,\widehat{S}_{p-1})$ of a cube using atoms along common boundaries $\widehat{S}_{i-1}\cap\widehat{S}_i$. To obtain a repartition of a cube satisfying a collapsibility condition analogous to λ-collapsibility, we must restrict the combinatorial length of the atoms created by this generalized $\mathcal{C}$-modification. With this aim in mind, we now introduce an additional condition for coarsified skew partions.

To motivate this condition, consider a $\mathcal{C}$-cube Q of color i in a rough Rickman partition $\widetilde{\mathbf{\Omega}}=(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)$ of $\mathbb{R}^n$. Then $Q\cap\widetilde{\Omega}_j$ is contained in at most $2n-2$ faces of Q for $j\neq i$; see e.g. Figure 34 for $n=3$. Thus $Q\cap\widetilde{\Omega}_j$ would meet at most $3^{\alpha(n-1)}(2n-2)$ cubes in $\mathcal{Q}_\alpha(Q)$.

Now, let $\mathcal{S}=(S_1,...,S_{p-1})$ be a skew partition of an n-cube Q and $\widehat{\mathcal{S}}_\alpha=(\widehat{S}_1,...,\widehat{S}_{p-1})$ be an ε-coarsification of $\mathcal{S}$ for some $\alpha\in\mathbb{Z}_+$, and $\varepsilon\in(0,1)$. For $|i-j|=1$, let

$$P_{ij}(\widehat{\mathcal{S}}_\alpha)=\{q\in\mathcal{Q}_\alpha(Q):q\subset\widehat{S}_i \text{ and } q\cap\widehat{S}_j \text{ contains a face of } q\}.$$

Definition 8.6. The coarsification $\widehat{\mathcal{S}}_\alpha$ of $\mathcal{S}$ is *small* if, for each $|i-j|=1$, there exists a tree

$$\Gamma\subset\Gamma(P_{i,j}(\widehat{\mathcal{S}}_\alpha)\cup P_{j,i}(\widehat{\mathcal{S}}_\alpha)) \tag{8.3}$$

containing $P_{ij}(\widehat{\mathcal{S}}_\alpha)$ in its vertex set with $\#\Gamma<3^{\alpha(n-1)}(2n-2)$.

Remark 8.7. It is straightforward to check that for the skew partition $\mathcal{S}(M)$ of Lemma 8.3 the coarsified partitions $\widehat{\mathcal{S}}_\alpha(M)\cap Q$ are small for all cubes $Q\in\Gamma^{\mathrm{int}}(M)$.

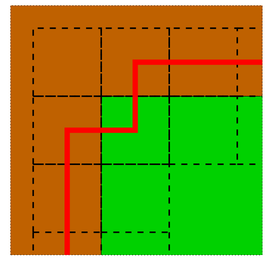

Figure 54. Schematic figure of the tree Γ; a detail.

Before confronting the proof of Proposition 8.1, we introduce a combinatorial notion related to skew partitions. We say that an essential partition $\mathcal{S}=(S_1, ..., S_{p-1})$ of a cube is *linear* if the adjacency graph $\Gamma(\mathcal{S})$ is an arc, that is, each vertex in $\mathcal{S}$ has valence at most 2. Furthermore, we may also assume from now on that elements in $\mathcal{S}$ are indexed so that S_1 and S_{p-1} have valence 1 in $\Gamma(\mathcal{S})$ and the neighbors of S_i are S_{i-1} and S_{i+1} for each $i=2, ..., p-2$.

Remark 8.8. The skew partition $\mathcal{S}(M)$ in Lemma 8.3 is linear and $\mathcal{S}(M)\cap Q$ is linear for each $Q\in\Gamma^{\text{int}}(M)$. Note also that, for $\alpha\in\mathbb{Z}_+$ large enough, coarsifications $\widehat{\mathcal{S}}_\alpha(M)$ of $\mathcal{S}(M)$ are also linear.

8.1.3. Proof of Proposition 8.1

We construct the essential partition $\mathbf{\Omega}$ with the same scheme as in §5 but now with skew partitions and coarsification methods. Apart from coarsification, this approach is similar to Rickman's in [15, §8.1]. Since the methods are based on those of §5 with the modifications already introduced in §7, we merely sketch the argument.

Mimicking the proof of Theorem 5.1, we construct a sequence $\{\mathcal{S}_m\}_{m\geqslant 0}$ of essential partitions of n-cells $3^m([0,3]^{n-1}\times[-3,3])$ analogous to the sequence $\{\mathbf{\Omega}_m\}_{m\geqslant 0}$. Recall that

$$\mathbf{\Omega}_0=(\Omega_{0,1},\Omega_{0,2},\Omega_{0,3})=([0,3]^n,[0,3]^{n-1}\times[-3,0],[3,6]\times[0,3]^{n-1})$$

and $\mathbf{\Omega}_1=(3\Omega_{0,1}-(A_2\cup A_3), 3\Omega_{0,2}\cup A_2, 3\Omega_{0,3}\cup A_3)$, where A_2 and A_3 are atoms.

It is not necessary to define $\mathcal{S}_0$, and we set directly

$$\mathcal{S}_1=([0,9]^n-A_3,[0,9]^{n-1}\times[-9,0],S_{1,2},...,S_{1,p}),$$

where $(S_{1,2}, ..., S_{1,p})$ is the skew partition $\mathcal{S}(3A_3)$ into $p-1$ n-cells as in (8.1).

Construction of S_2; *first generalized modifications.*

We construct $\mathcal{S}_2$ from $\mathcal{S}_1$ by independent generalized $\mathcal{D}$-modifications; note that $\mathbf{\Omega}_2$ is obtained from $\mathbf{\Omega}_1$ by a secondary $\mathcal{C}$-modification, as observed in Remark 5.22. In this particular case it suffices to observe that, in the construction of $\mathbf{\Omega}_2$, we extend the atom $3A_3$ to a molecule M by attaching 1-atoms. Thus, to obtain $\mathcal{S}_2$ from $\mathcal{S}_1$, it is enough to extend the skew partition $3(S_{1,2}, ..., S_{1,p})$ to a skew partition of M; cf. Lemma 8.3. This extension of the skew partition $3\mathcal{S}_1$ into each 1-fine atom is the *generalized $\mathcal{D}$-modification*. Thus

$$\mathcal{S}_2 = (S_{2,0}, ..., S_{2,p}) = ([0,27]^n - M, [0,27]^{n-1} \times [-27,0], S_{2,2}, ..., S_{2,p}),$$

where $(S_{2,2}, ..., S_{2,p})$ is a skew partition of M.

In later steps, we also use similar generalizations of secondary modifications. Note that we use these generalizations alongside with (original) $\mathcal{D}$-modifications and secondary modifications.

Construction of $\mathcal{S}_3$; generalized $\mathcal{C}$-modifications.

To obtain $\mathcal{S}_3 = (S_{3,0}, ..., S_{3,p})$ from $\mathcal{S}_2$, we use generalized $\mathcal{D}$-modifications and generalized $\mathcal{C}$-modifications in rescaled $\mathcal{C}$-cubes. Note that, for $Q \in \Gamma^{\mathrm{int}}(3A_3)$, the essential partition $Q \cap \mathcal{S}_2$ is a skew partition of Q into $p-1$ skew n-cells meeting the remaining two elements of $\mathcal{S}_2$ analogously as in the situation with a $\mathcal{C}$-cube; recall that $3A_3 \subset \Omega_{2,3}$ was adjacent to the domains $\Omega_{2,1}$ and $\Omega_{2,2}$ in $\mathbf{\Omega}_2$ (see §5.1.3). We therefore call Q a *generalized $\mathcal{C}$-cube*.

Using notation related to Q and the skew partition $\mathcal{S}_2$, we now describe the generalized $\mathcal{C}$-modification in $3Q$. For this modification, we consider cubes in two scales $3^{-\beta}$ and $3^{-\alpha}$ for $\alpha > \beta \geqslant p$. Thus we divert here from the convention that side-lengths of cubes are at least 1.

First, let $\beta \geqslant p$ be an integer, to be determined later, for which we may choose, for $i = 2, ..., p$, a cube $q_i \in \mathcal{M}_\beta(3(Q \cap \mathcal{S}_2))$ so that $\mathrm{dist}_\infty(q_i, q_j) \geqslant 3^{-\beta}$ for $i \neq j$. Note that each cube in $\mathcal{M}_\beta(3(Q \cap \mathcal{S}_2))$ is adjacent to $3S_{2,0}$ and $3S_{2,1}$.

Second, let $\alpha > \beta$, to be determined later, so that $\widehat{\mathcal{S}}_\alpha = (\widehat{S}_1, ..., \widehat{S}_{p-1})$, where

$$\widehat{S}_i = E_{\alpha,i}(3(Q \cap \mathcal{S}_2)) \cup (|\mathcal{M}_\alpha(3(Q \cap \mathcal{S}_2))| \cap S_i),$$

is a 1-coarsification of $\mathcal{S} = 3(Q \cap \mathcal{S}_2) = (S_1, ..., S_{p-1})$ as in (8.2). By increasing α, if necessary, there exists for each $i = 1, ..., p-1$ adjacent cubes $q_i', q_i'' \in \mathcal{Q}_\alpha(Q)$ so that $q_i' \subset q_i$ and $q_i'' \in P_{i,i-1}(\widehat{\mathcal{S}}_\alpha)$; when $i = 1$, we assume that q_1'' meets $\partial(3Q)$.

We now modify the cells $3\widehat{S}_2, ..., 3\widehat{S}_{p-1}$ in $3Q$ as follows; the modification of $3\widehat{S}_1$ is similar and postponed to the end of the process.

For each $i = 2, ..., p-1$, let Γ_i be a maximal tree as in (8.3). Let a_i' be the associated $3^{-\alpha-2}$-fine atom, and let $a_i = a_i' \cup q_i'$; then this allows a_i to enter both $\widehat{S}_i$ and

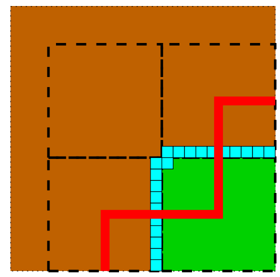

Figure 55. A schematic figure of an atom a_i associated with a tree Γ_i; a detail.

$\widehat{S}_{i-1}$, see Figure 55. Fix also a small skew partition $(a_{i,1}, ..., a_{i,p-1})$ of a_i so that $(\widehat{S}_{i-1}, a_{i,1}, ..., a_{i,p-1}, \widehat{S}_i - a_i)$ is a skew partition of $\widehat{S}_{i-1} \cup \widehat{S}_i$ with cyclic adjacency graph; cf. Lemma 8.3.

To connect the cells $a_{i,k}$ to the cells $\widehat{S}_j$ for $j \neq i$ and $k \in \{1, ..., p-1\}$, note that for each $i=2, ..., p-1$ there exists a unique graph isomorphism

$$\theta_i: \Gamma(\widehat{\mathcal{S}}_\alpha) \longrightarrow \Gamma(\widehat{S}_{i-1} - a_i, a_{i,1}, ..., a_{i,p-1}, \widehat{S}_i - a_i)$$

satisfying $\theta_i(\widehat{S}_{i-1}) = \widehat{S}_{i-1} - a_i$ and $\theta_i(\widehat{S}_i) = \widehat{S}_i - a_i$.

Fix now, on each cube q_i, a small skew partition $(q_{i,1}, ..., q_{i,p-1})$ so that $q_{i,j} \cup (\widehat{S}_j - q_i)$ and $q_{i,j} \cup \theta_i(a_j)$ are skew n-cells.

Then, by attaching the cells $q_{i,j} \cup \theta_i(a_j)$ to the cells $\widehat{S}_j$ for $2 \leqslant j \leqslant p-1$, we obtain cells Q_j for which the system $(\widehat{S}_1, Q_2, ... Q_{p-1})$ produces the desired skew partition of $3Q$ after we make an analogous extension of both $\widehat{S}_1$ and Q_j along $\partial(3Q)$. We leave this last detail to the interested reader.

We conclude by noting that, since the atoms a_i have side-length $3^{-\alpha}$, we do not need to rearrange their scaled copies before constructing $\mathcal{S}_{3+(\alpha+2)}$. At that stage, cubes in $\Gamma^{\text{int}}(3^{\alpha+1} a_i)$ are generalized $\mathcal{C}$-cubes. A similar comment applies to the cubes q_i and the construction of $\mathcal{S}_{3+(\beta+2)}$. Note also that it suffices to fix, up to an isometry, one essential partition for a cube of side-length $3^{-\beta}$ for all generalized $\mathcal{C}$-modifications. In particular, we may fix parameters α and β to depend only on n and p.

Construction of $\mathbf{\Omega}$; *inductive process.*

With these generalized primary and secondary rearrangements at our disposal, we proceed as in §5 and obtain an essential partition $\mathcal{S}_m$ from $\mathcal{S}_{m-1}$ for every $m > 3$. Similarly as in the proof of Theorem 1.4 (for $p=2$) we may arrange it so that these essential partitions yield an essential partition $\mathbf{\Omega} = (\Omega_0, ..., \Omega_p)$ of $\mathbb{R}^n$ satisfying the (generalized) tripod property; again Ω_0 and Ω_1 are connected and $\Omega_2, ..., \Omega_p$ have 2^{n-1} components each.

Note that the combinatorial length estimate for small skew partitions yields that Ω_0, Ω_1, and each component of Ω_r for $r\geqslant 2$ are λ-collapsible in a natural generalized sense; in dimension $n=3$ we use again the particular configurations illustrated in §5.3.4 to obtain collapsibility. Analogously as in §5.3.3 and §5.4, we obtain that Ω_0, Ω_1, and each component of Ω_r are bilipschitz to $\mathbb{R}^{n-1}\times[0,\infty)$. Thus $\mathbf{\Omega}$ is a skew Rickman partition satisfying the (generalized) tripod property and we have proved Proposition 8.1.

8.2. Proof of Proposition 1.5

Let $\mathbf{\Omega}=(\Omega_0,...,\Omega_p)$ be a skew Rickman partition as in Proposition 8.1. Then $\partial_\cup\mathbf{\Omega}$ carries a uniformly bilipschitz triangulation into $(n-1)$-simplices together with an associated labeling function.

Due to the cyclic combinatorics of domains in $\mathbf{\Omega}$, that is, since $\Omega_j\cap\Omega_{j+1}$ is locally an $(n-1)$-cell for $j=0,...,p$ (mod $p+1$), we define a parity function $\nu_{\partial_\cup\mathbf{\Omega}}\colon(\partial_\cup\mathbf{\Omega})^{n-1}\to\{\pm1\}$ for $p>2$ analogous to the case $p=2$ in §6.

To construct a pillow cover over the triangulation of $\partial_\cup\mathbf{\Omega}$ it suffices to discuss pillows over pairs of adjacent $(n-1)$-simplices. We merely describe the differences from the case $p=2$; apart from these slight modifications we proceed as in §7.

Let σ and σ' be an adjacent pair of $(n-1)$-simplices on $\partial_\cup\mathbf{\Omega}$ and suppose that $\nu_{\partial_\cup\mathbf{\Omega}}(\sigma)=-1$. We may also assume, to simplify notation, that $\sigma\cup\sigma'\subset\mathbb{R}^{n-1}\times\{0\}$. In this case the sheets $\hat\sigma_1,...,\hat\sigma_p$ on σ are given by the graph of a function $\Psi_\sigma\colon\sigma\times\{1,...,p\}\to\mathbb{R}$ similarly as in §7.1. The sheets $\hat\sigma'_1,...\hat\sigma'_{p+2}$ on σ' are similarly given by the graph of a function $\Psi_{\sigma'}\colon\sigma'\times\{1,...,p+2\}\to\mathbb{R}$. We require that these pillows satisfy compatibility conditions analogous to those of Definition 7.7 in §7.2. Since local modifications of pillows are similar to the case $p=2$, we leave the finer details to the interested reader and discuss in detail only the 'shuffle' of domains.

Suppose for now that we have fixed the functions Φ_σ and $\Phi_{\sigma'}$ providing us with sheets for the simplices σ and σ', respectively. Let $D_0,D_1,...,D_p$ be the components of

$$(\sigma\times\mathbb{R})\setminus\bigcup_{i=1}^{p}\hat\sigma_i$$

so that $\hat\sigma_1\subset\partial D_0$, $\hat\sigma_i\cup\hat\sigma_{i+1}\subset\partial D_i$ for $i=1,...,p-1$, and $\hat\sigma_p\subset\partial D_p$. Let $D'_0,...,D'_{p+2}$ be the components of

$$(\sigma'\times\mathbb{R})\setminus\bigcup_{j=1}^{p+2}\hat\sigma'_j$$

in the same order, that is, $\hat\sigma'_0\subset\partial D'_0$, $\hat\sigma'_j\cup\hat\sigma'_{j+1}\subset\partial D'_j$ for $j=1,...,p+1$, and $\hat\sigma_{p+2}\subset\partial D'_{p+2}$.

Following the method in §7.2, we may assume that, for the functions Φ_σ and $\Phi_{\sigma'}$, the sets $D_0\cup D'_0\cup D'_{p+1}$, $D_i\cup D'_{p+1-i}$ for $i=1,...,p-1$, and $D_p\cup D'_{p+2}\cup D'_1$ are connected components of

$$((\sigma\cup\sigma')\times\mathbb{R})\setminus\left(\bigcup_{i=1}^{p}\hat{\sigma}_i\cup\bigcup_{j=1}^{p+2}\hat{\sigma}_j\right). \tag{8.4}$$

Note that in order to merge the sets D_i and D'_j this way it suffices to subdivide the set $\tau_0\subset\tau=\sigma\cap\sigma'$, defined in §7.1, into $(n-2)$-simplices and to define several openings this way.

This 'shuffle' allows the domains D_p and D'_p to be connected across $\bar{\sigma}\cup\sigma'$ and preserves the global adjacency structure on these domains when passing from $\mathbf{\Omega}$ to $\widetilde{\mathbf{\Omega}}$.

To fix notation, suppose that simplices σ and σ' in $\partial_\cup\mathbf{\Omega}$ are between the domains Ω_ℓ and $\Omega_{\ell+1}$ for $\ell\in\{0,...,p\}$, where we understand that $\ell+1=0$ if $\ell=p$. We may assume that locally near $\sigma\cup\sigma'$, Ω_ℓ is contained in $(\sigma\cup\sigma')\times(-\infty,0]$.

We begin with the negative simplex σ. The adjacency graph $\Gamma(\mathbf{\Omega}\cap(\sigma\times\mathbb{R}))$ near σ consists only of an edge between Ω_ℓ and $\Omega_{\ell+1}$. The adjacency graph of the domains $D_0,...,D_p$, on the other hand, is an arc from D_0 to D_p. By construction of the essential partition $\widetilde{\mathbf{\Omega}}$, the sets $D_0,...,D_p$ are contained in elements of the essential partition $\widetilde{\mathbf{\Omega}}$. Since $\widetilde{\mathbf{\Omega}}$ has the same cyclic adjacency graph as $\mathbf{\Omega}$ and $\Gamma(\widetilde{\mathbf{\Omega}}\cap(\sigma\times\mathbb{R}))$ is an arc of length p, we note that the domains $D_0,...,D_p$ belong to the sets $\widetilde{\Omega}_\ell,\widetilde{\Omega}_{\ell-1},...,\widetilde{\Omega}_1,\widetilde{\Omega}_p,...,\widetilde{\Omega}_{\ell+1}$, in this order.

For the positive simplex σ', we note that, by (8.4) and by the same argument, the domains $D'_0,...,D'_{p+2}$ are contained in the domains $\widetilde{\Omega}_\ell,\widetilde{\Omega}_{\ell+1},...,\widetilde{\Omega}_p,\widetilde{\Omega}_1,...,\widetilde{\Omega}_\ell,\widetilde{\Omega}_{\ell+1}$ in this order.

As a remark, we note that if we merge the graphs $\Gamma(\widetilde{\mathbf{\Omega}}\cap(\sigma\times\mathbb{R}))$ and $\Gamma(\widetilde{\mathbf{\Omega}}\cup(\sigma'\times\mathbb{R}))$ by identifying vertices corresponding to the domains D_0 and D_p with D'_0 and D'_{p+2}, respectively, we obtain a cyclic graph which is a natural double cover of $\Gamma(\widetilde{\mathbf{\Omega}})$.

This remark concludes the construction of the essential partition $\widetilde{\mathbf{\Omega}}$ and the proof of Proposition 1.5.

Corollary 8.9. *The domains* $\operatorname{int}\widetilde{\Omega}_0,...,\operatorname{int}\widetilde{\Omega}_p$, *as well as* $\operatorname{int}\Omega_0,...,\operatorname{int}\Omega_p$, *are uniform domains.*

Proof. Since the domains $\operatorname{int}\Omega'_1$, $\operatorname{int}\Omega'_2$, and $\operatorname{int}\Omega'_3$ are uniform domains by Corollary 5.2, we have that the domains $\operatorname{int}\Omega_0,...,\operatorname{int}\Omega_p$ are uniform domains by bilipschitz invariance of the uniformity condition. As $\operatorname{int}\Omega_k$ is bilipschitz to $(\operatorname{int}\widetilde{\Omega}_k, d_{\operatorname{int}\widetilde{\Omega}_k})$, we have that $\operatorname{int}\widetilde{\Omega}_k$ is a uniform domain for each $k=0,...,p$. □

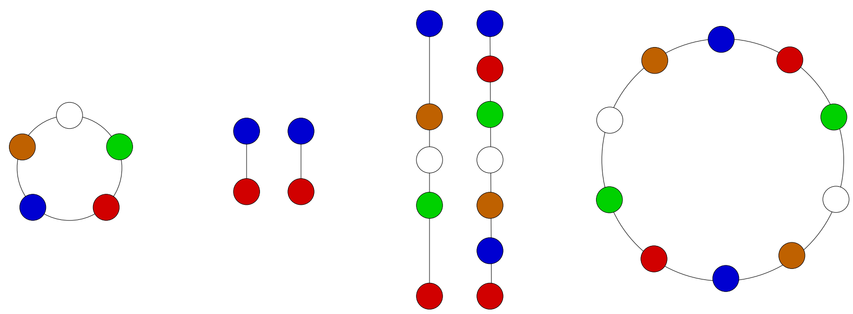

Figure 56. Case $p=5$. From left to right: the cyclic adjacency graph of $\widetilde{\mathbf{\Omega}}$, the adjacency graphs $\Gamma(\mathbf{\Omega}\cap(\sigma\times\mathbb{R}))$ and $\Gamma(\mathbf{\Omega}\cap(\sigma'\times\mathbb{R}))$, the adjacency graphs $\Gamma(\widetilde{\mathbf{\Omega}}\cap(\sigma\times\mathbb{R}))$ and $\Gamma(\widetilde{\mathbf{\Omega}}\cap(\sigma'\times\mathbb{R}))$, and the merge of $\Gamma(\widetilde{\mathbf{\Omega}}\cap(\sigma\times\mathbb{R}))$ and $\Gamma(\widetilde{\mathbf{\Omega}}\cap(\sigma'\times\mathbb{R}))$.

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