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The convergence Newton polygon of a p-adic differential equation II: Continuity and finiteness on Berkovich curves

by

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1. Introduction

In the p-adic setting, linear differential equations exhibit some features that do not appear over the complex numbers. For instance, even if the coefficients of such an equation are entire functions, it may fail to have a solution that converges everywhere. This leads to a non-trivial notion of radius of convergence, which has been actively investigated and found several applications. In this article, we study the variation of the radius of convergence (or more accurately of the slopes of the convergence Newton polygon, which also contains more refined radii) of a module with connection over a curve and prove that it enjoys continuity and finiteness properties.

The precise setting is the following. Let K be a non-archimedean complete valued field of characteristic 0. Let X be a quasi-smooth K-analytic curve, in the sense of Berkovich theory. Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module of finite type endowed with an integrable connection ∇.

If X can be embedded into the affine line $\mathbf{A}_K^{1,\mathrm{an}}$, we have a coordinate on X that we may use to define the radius of convergence. This study has been carried out by the second author in [NP1], to which this article is a follow-up.

Let us now return to the case of a general quasi-smooth curve X. Let x be a K-rational point of X. By the implicit function theorem, in a neighbourhood of the point x the curve is isomorphic to a disc, and we may consider the radius of the biggest disc on which a given horizontal section of $(\mathscr{F},\nabla)$ (i.e. a given solution of the associated differential equation) converges. Considering the radii associated with the various horizontal sections, we define a tuple $\boldsymbol{\mathcal{R}}_S(x)$ that we call the *multiradius of convergence* at x. (It depends on an additional datum S on the curve, to which we shall come back later.) For readers who are familiar with the notion of convergence Newton polygon, let us mention that we here recover the tuple of its slopes, up to a logarithm. Since any point of the curve may be changed into a rational one by a suitable extension of the scalars, we may actually extend the definition of the multiradius of convergence to the whole curve.

It is already interesting to consider the radius of convergence $\mathcal{R}_{S,1}(x)$ at a point x, i.e. the radius of the biggest disc on which $(\mathscr{F},\nabla)$ is trivial, or, equivalently, the minimal radius that appears in the multiradius. In [Ba], Baldassarri proved the continuity of this function.

In the article [NP1], using different methods, the second author thoroughly investigated the multiradius of convergence $\boldsymbol{\mathcal{R}}_S$ in the case where X is an affinoid domain of the affine line. He proved that this function is continuous, piecewise log-linear and satisfies a strong finiteness property: it factorises by the retraction through a finite subgraph of X.

In this paper we prove that those properties extend to arbitrary quasi-smooth K-analytic curves.

THEOREM 3.6. *The map* $\boldsymbol{\mathcal{R}}_S$ *satisfies the following properties*:

(i) *it is continuous*;

(ii) *its restriction to any locally finite graph* Γ *is piecewise log-linear*;

(iii) *on any interval* J, *the log-slopes of its restriction are rational numbers of the form* m/j *with* $m\in\mathbf{Z}$ *and* $j\in[[1,r]]$, *where* r *is the rank of* $\mathscr{F}$ *around* J;

(iv) *there exists a locally finite subgraph* Γ *of* X *and a continuous retraction* $r\colon X\to\Gamma$ *such that the map* $\boldsymbol{\mathcal{R}}_S$ *factorises by* r.

This theorem lays the foundations for a deeper study of p-adic differential equations on curves. In subsequent work we will use it in a crucial way to prove decomposition theorems (see [NP3]) and finiteness results for the de Rham cohomology of modules with connections (see [NP4]). We mention that Baldassarri and Kedlaya have announced similar results (see [K2]).

Let us now give an overview of the contents of the article. We have just explained the rough idea underlying the definition of the multiradius of convergence on a curve X. Obviously, some work remains to be done in order to put together the different local normalisations in a coherent way and give a proper definition of this multiradius. This will be the content of our first section. The first definition was actually given by Baldassarri in [Ba], using a semistable formal model of the curve, which led to some restrictions. Here, we will use Ducros's notion of triangulation, which is a way of cutting a curve into pieces that are isomorphic to virtual discs or annuli. (The existence of triangulations is very closely related to the semistable reduction theorem.) Among other advantages, it lets us deal with arbitrary quasi-smooth curves (regardless of whether they are compact or not, strictly analytic or not, or defined over a field that is trivially valued or not) and enables us to define everything within the realm of analytic spaces. The symbol S that appeared in our notation $\boldsymbol{\mathcal{R}}_S$ for the multiradius above actually referred to the choice of such a triangulation. We will explain how our radius of convergence relates to that of [Ba] and to the usual one in the case of analytic domains of the affine line, as defined, for instance, in [BD] or [NP1].

As indicated above, extension of scalars plays a crucial role in the theory. We devote §2.2 to it and give a precise description of the fibres of the extension maps in the case of curves over algebraically closed fields: they consist of one universal point together with a bunch of virtual open discs that spread out of it. We believe these results to be of independent interest.

Finally, §3 is devoted to the proof of the continuity, log-linearity and finiteness property of the multiradius of convergence $\boldsymbol{\mathcal{R}}_S$. The basic idea is to present the curve X locally as a finite étale cover of an affinoid domain W of the affine line, apply the results of [NP1] to W and pull them back. We will first present the geometric results needed (which essentially come from Ducros's manuscript [D6]) to find a nice presentation of the curve, then we prove a weak version of the finiteness property (local constancy of $\boldsymbol{\mathcal{R}}_S$ outside a locally finite subgraph) and finally we prove the continuity (which will yield the stronger finiteness property) and piecewise log-linearity.

Setting 1.1. For the rest of the article, we fix the following: K is a complete valued field of characteristic 0, p is the characteristic exponent of its residue field $\widetilde{K}$ (either 1 or a prime number), X is a quasi-smooth K-analytic curve,(1) and $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite type endowed with an integrable connection ∇. We fix an algebraic closure $\overline{K}$ of K and denote its completion by K^a.

From §2.3 on, we will assume that the curve X is endowed with a weak triangulation S, and from §3.2 on, we will assume that the field K is algebraically closed.

2. Definitions

In this section, we define the radius of convergence of $(\mathscr{F},\nabla)$ at any point of the curve X. To achieve this task, we will need to understand precisely the geometry of X. Our main tool will be Ducros's notion of triangulation (see [D6, 5.1.13]) that we recall here. The original definition, introduced by Baldassarri in [Ba], in a more restrictive setting, made use of a semistable model of the curve. The two points of view are actually very close (see §2.4.1).

(1) *Quasi-smooth* means that Ω_X is locally free of rank 1, see [D6, 3.1.11]. This corresponds to the notion called "rig-smooth" in the rigid analytic terminology.

Let us point out that the structure of analytic curves has been completely described by Berkovich thanks to the semistable reduction theorem (see [Be1, Chapter 4]). In the book [D6], Ducros recently managed to get the same results working only on the analytic side. As his text is a thorough manuscript with easily quotable results, we have chosen to use it systematically when we need to refer to results on curves, even in the case when they were known before (as, for instance, for bases of neighbourhoods of points).

2.1. Triangulations

First of all, let us fix notation for discs and annuli.

Notation 2.1. Let $\mathbf{A}_K^{1,\mathrm{an}}$ be the affine analytic line with coordinate t. Let M be a complete valued extension of K and let $c\in M$. For $R>0$ we set

$$D_M^+(c,R)=\{x\in\mathbf{A}_M^{1,\mathrm{an}}:|(t-c)(x)|\leqslant R\}$$

and

$$D_M^-(c,R)=\{x\in\mathbf{A}_M^{1,\mathrm{an}}:|(t-c)(x)|<R\}.$$

Denote by $x_{c,R}$ the unique point of the Shilov boundary of $D_M^+(c,R)$.

For R_1 and R_2 such that $0<R_1\leqslant R_2$ we set

$$C_M^+(c,R_1,R_2)=\{x\in\mathbf{A}_M^{1,\mathrm{an}}:R_1\leqslant|(t-c)(x)|\leqslant R_2\}.$$

For R_1 and R_2 such that $0<R_1<R_2$ we set

$$C_M^-(c,R_1,R_2)=\{x\in\mathbf{A}_M^{1,\mathrm{an}}:R_1<|(t-c)(x)|<R_2\}.$$

When it is obvious from the context, we suppress the index M.

Let us now recall that a non-empty connected K-analytic space is a *virtual disc* (resp. *annulus*) if it is isomorphic to a union of discs (resp. annuli whose orientations are preserved by $\mathrm{Gal}(\overline{K}/K)$) over K^a (see [D6, 3.6.32 and 3.6.35]).

The skeleton of a virtual open (resp. closed) annulus C is the set of points Γ_C that have no neighbourhood isomorphic to a virtual open disc. The skeleton Γ_C is homeomorphic to an open (resp. closed) interval and the space C retracts continuously onto it.

Definition 2.2. A subset S of X is a *weak triangulation* of X if

(1) S is locally finite and only contains points of type 2 or 3;

(2) any connected component of $X\setminus S$ is a virtual open disc or annulus.

The union of S and the skeletons of the connected components of $X\setminus S$ that are virtual annuli forms a locally finite graph, which is called the skeleton Γ_S of the weak triangulation S.

Remark 2.3. The fact that the skeleton Γ_S is a locally finite graph is not obvious from the definition. This follows, for instance, from Theorem 3.12 below.

Usually the skeleton is the union of the segments between the points of S but beware that peculiar situations may sometimes appear: for example, the skeleton of an open annulus endowed with the empty weak triangulation is an open segment.

Let us also point out that the skeleton of an open disc endowed with the empty weak triangulation is empty, which sometimes forces us to treat this case separately.

Remark 2.4. Let Z be a connected component of X such that $Z\cap\Gamma_S\neq\varnothing$ (which is always the case except if Z is an open disc such that $Z\cap S=\varnothing$). Then we have a natural continuous topologically proper retraction $Z\to Z\cap\Gamma_S$. It is the identity on $Z\cap\Gamma_S$ and sends a point of $Z\setminus\Gamma_S$ to the unique boundary point of the connected component of $Z\setminus\Gamma_S$ (a virtual open disc by definition) containing it.

Remark 2.5. The definition of a triangulation that is used by Ducros is actually stronger than ours since he requires the connected components of $X\setminus S$ to be relatively compact. For example, the empty weak triangulation of the open disc is excluded.

This property allows him to have a natural continuous retraction $X\to\Gamma_S$, and also to associate with S a nice formal model of the curve X (see [D6, §6.3], for details).

The existence of a triangulation is one of the main results of Ducros's manuscript (see [D6, Théorème 5.1.14]). This result is essentially equivalent to the semistable reduction theorem (see [D2, §4] and [D6, Chapitre 6]).

THEOREM 2.6. (Ducros) *Any quasi-smooth K-analytic curve admits a triangulation, and hence a weak triangulation.*

2.2. Extension of scalars

In this section, we study the effect of extending the scalars on a given weak triangulation.

Notation 2.7. For every complete valued extension L of K, we set $X_L=X\widehat{\otimes}_K L$.

For every complete valued extension L of K, and every complete valued extension M of L, we let $\pi_{M/L}\colon X_M\to X_L$ denote the canonical projection morphism. Further, we set $\pi_M=\pi_{M/K}$.

Starting from the triangulation S of X, it is easy to check that $S_{K^a}=\pi_{K^a}^{-1}(S)$ is a weak triangulation of X_{K^a}.

To go further, we introduce the notion of universal point of X. Roughly speaking, a point is universal if it may be canonically lifted to any base field extension X_L of X. We refer to [Be1, §5.2](2) and [P2] for more information.

Definition 2.8. A point x of X is said to be *universal* if, for any complete valued extension L of K, the tensor norm on the algebra $\mathscr{H}(x)\widehat{\otimes}_K L$ is multiplicative. In this case, x defines a point of X_L that we denote by x_L.

Lemma 2.9. *Let $\varphi: Y\to Z$ be a morphism of K-analytic spaces and let $y\in Y$ be a universal point. Then $\varphi(y)$ is universal.*

Moreover, if L is a complete valued extension of K and $\varphi_L: Y_L\to Z_L$ is the morphism obtained after extension of scalars, then $\varphi(y)_L=\varphi_L(y_L)$.

Proof. For every complete valued extension L of K, we have embeddings

$$\mathscr{H}(\varphi(y))\widehat{\otimes}_K L\hookrightarrow \mathscr{H}(y)\widehat{\otimes}_K L\hookrightarrow \mathscr{H}(y_L).$$

The first map is isometric by [P2, Lemme 3.1], and the second is isometric by the definition of a universal point. We deduce that the tensor norm on $\mathscr{H}(\varphi(y))\widehat{\otimes}_K L$ is multiplicative. Moreover, we get an isometric embedding $\mathscr{H}(\varphi(y)_L)\hookrightarrow\mathscr{H}(y_L)$ which shows that $\varphi_L(y_L)=\varphi(y)_L$. □

In some cases, the universality condition is not restrictive:

Theorem 2.10. ([P2, Corollaire 3.14]) *Over an algebraically closed field, every point is universal.*

This theorem gives a way to lift a given weak triangulation S of X to a weak triangulation S_L of X_L, for any complete valued extension L of K^a.

Notation 2.11. Let L be a complete valued extension of K. We denote by $\mathrm{Gal}^c(L/K)$ the group of isometric automorphisms of L that induce the identity on K.

Lemma 2.12. *Let L be a complete valued extension of K^a. Let $x\in X_{K^a}$ and let $\sigma\in\mathrm{Gal}^c(L/K)$. Then, we have*

$$\sigma(x)_L=\sigma(x_L).$$

Proof. The point $\sigma(x_L)$ lies over $\sigma(x)$ in X_{K^a}, and hence it belongs to

$$\mathcal{M}(\mathscr{H}(\sigma(x))\widehat{\otimes}_{K^a} L).$$

(2) In this reference, where universal points first appeared, they are called “peaked points”.

By the definition of a universal point, we have the inequality $\sigma(x_L)\leqslant\sigma(x)_L$ as semi-norms on $\mathscr{H}(\sigma(x))\widehat{\otimes}_{K^a}L$.

The converse inequality follows from the fact that x_L is universal and thus, as semi-norms on $\mathscr{H}(x)\widehat{\otimes}_{K^a}L$, we have $\sigma^{-1}(\sigma(x)_L)\leqslant x_L$. □

Definition 2.13. Let L be a complete valued extension of K and let L^a be the completion of an algebraic closure of L. We see L^α as an extension of K^a. Set

$$S_{L^a}=\{x_{L^a}:x\in S_{K^a}\},\quad \Gamma_{S_{L^a}}=\{x_{L^a}:x\in\Gamma_{S_{K^a}}\},$$
$$S_L=\pi_{L^a/L}(S_{L^a}),\qquad \Gamma_{S_L}=\pi_{L^a/L}(\Gamma_{S_{L^a}}).$$

By Lemma 2.12, the sets S_{L^a} and $\Gamma_{S_{L^a}}$ are invariant under the action of $\mathrm{Gal}^c(L^a/K)$, and thus the sets of the previous definition are well defined and independent of any choice. We write down the following property for future reference.

LEMMA 2.14. *Let L be a complete valued extension of K. The sets S_L and Γ_{S_L} are invariant under the action of* $\mathrm{Gal}^c(L/K)$. □

In extending the weak triangulation of X to the field L, the key point is the following result.

THEOREM 2.15. *Let x be a point of X_{K^a} and let L be a complete valued extension of K^a.*

(1) *If x is of type $i\in\{1,2\}$, then x_L is also of type i. If x is of type $j\in\{3,4\}$, then x_L is either of type j or 2.*

(2) *We have that the fibre $\pi^{-1}_{L/K^a}(x)$ is connected and the connected components of $\pi^{-1}_{L/K^a}(x)\setminus\{x_L\}$ are virtual open discs with boundary $\{x_L\}$. Moreover, the components are open in X_L.*

Before proving the theorem, we will state two consequences.

COROLLARY 2.16. *For any complete valued extension L of K, the set S_L is a weak triangulation of X_L whose skeleton is Γ_{S_L}.*

Proof. It is enough to prove that S_{L^a} is a weak triangulation of X_{L^a} whose skeleton is $\Gamma_{S_{L^a}}$. By Theorem 2.15 (1), the set S_{L^a} contains only points of type 2 or 3. The projection S_{K^a} of S_{L^a} on X_{K^a} is locally finite and there is exactly one point of S_{L^a} above any point of S_{K^a}. We deduce that S_{L^a} is locally finite.

Let $x\in S_{K^a}$ and let C be a connected component of $\pi^{-1}_{L^a/K^a}(x)\setminus\{x_{L^a}\}$. By Theorem 2.15 (2), C is a disc that is open in X_{L^a}. Moreover, since $\pi^{-1}_{L^a/K^a}(x)$ is closed in X_{L^a}, C is also closed in $X_{L^a}\setminus S_{L^a}$. We deduce that the disc C is a connected component of $X_{L^a}\setminus S_{L^a}$.

The complement in $X_{L^a}\setminus S_{L^a}$ of the union of all the connected components of the preceding form is the complement of $\pi^{-1}_{L^a/K^a}(S_{K^a})$ in X_{L^a}, i.e. the union of all sets of the form $\pi^{-1}_{L^a/K^a}(O)$, where O is a connected component of $X_{K^a}\setminus S_{K^a}$. We deduce that the latter are the remaining connected components of $X_{L^a}\setminus S_{L^a}$.

Let O be a connected component of $X_{K^a}\setminus S_{K^a}$. If O is an open disc, then $\pi^{-1}_{L^a/K^a}(O)$ is an open disc too. If O is an open annulus, then $\pi^{-1}_{L^a/K^a}(O)$ is an open annulus too and a direct computation shows that its skeleton is the set $\{x_{L^a}:x\in\Gamma_O\}$, where Γ_O denotes the skeleton of O. The result follows. □

We will now be more precise concerning the open discs that appear in the fibres of the extension of scalars and show that they are actually isomorphic. In order to do so, we will need results about maximally complete valued fields, for which we refer to [Po].

THEOREM 2.17. *Let M be a non-archimedean valued field that is algebraically closed and maximally complete. Let L be a subfield of M. Let L' be a valued extension of L such that there exist a morphism $f:\tilde{L}'\hookrightarrow\widetilde{M}$ between the residue fields, and an injective morphism $g:|L'^*|\hookrightarrow|M^*|$ between the value groups that make the following diagrams commute*:

$$\begin{array}{ccc} & & \tilde{L}' \\ & \nearrow & \downarrow f \\ \tilde{L} & \longrightarrow & \widetilde{M} \end{array} \quad \text{and} \quad \begin{array}{ccc} & & |L'^*| \\ & \nearrow & \downarrow g \\ |L^*| & \longrightarrow & |M^*|. \end{array}$$

Then, there exists an isometric embedding of valued fields $h:L'\hookrightarrow M$ that induces f and g and makes the following diagram commute:

$$\begin{array}{ccc} & & L' \\ & \nearrow & \downarrow h \\ L & \longrightarrow & M. \end{array}$$

Proof. By [Po, Corollary 5], the field M may be embedded in a Mal′tsev–Neumann field M' with the same value group and residue field. Since M is maximally complete, the immediate extension M'/M is an isomorphism. We deduce that M is a Mal′tsev–Neumann field. (Conversely, any Mal′tsev–Neumann field with divisible value group and algebraically closed residue field is algebraically closed and maximally complete by [Po, Theorem 1 and Corollary 4].) The result now follows from [Po, Theorem 2]. □

Corollary 2.18. *Let L be a valued extension of K such that the residual extension $\tilde{L}/\widetilde{K}$ has finite transcendence degree. Let M be a valued extension of K that is algebraically closed and maximally complete. Let $a, b\colon L \hookrightarrow M$ be two K-embeddings of L into M. Then there exists an isometric K-automorphism σ of M such that $b = \sigma \circ a$.*

Proof. By [Bou, §V.14, n. 4, Corollaire 2], there exists an automorphism s of the residue field $\widetilde{M}$ such that $\tilde{b} = \tilde{s} \circ \tilde{a}$. By Theorem 2.17, there exists an isometric endomorphism σ of M that induces s on $\widetilde{M}$ and the identity on $|M^*|$ such that the following diagram commutes:

$$\begin{array}{ccc} & & M \\ & \overset{a}{\nearrow} & \downarrow{\scriptstyle \sigma} \\ L & \xrightarrow{\;b\;} & M. \end{array}$$

In particular, the extension defined by σ is immediate. As M is maximally complete, σ is an isomorphism. □

Remark 2.19. The corollary does not hold if M is not assumed to be maximally complete as proved in [MR]. This answers a question from [DR].

Corollary 2.20. *Let M be an algebraically closed and maximally complete valued extension of K. Let x be a point of X. The Galois group $\mathrm{Gal}^c(M/K)$ acts transitively both on the set of M-rational points of $\pi_{M/K}^{-1}(x)$ and on the set of connected components of $\pi_{M/K}^{-1}(x) \setminus \{x_M\}$.*

Proof. We may assume that the field K is algebraically closed. Let t and t' be two M-rational points of $\pi_{M/K}^{-1}(x)$. They correspond to two embeddings of valued fields $\mathscr{H}(x) \hookrightarrow M$. The residue field $\widetilde{\mathscr{H}(x)}$ has transcendence degree at most 1 over $\widetilde{K}$, and hence, by Corollary 2.18, the embeddings are conjugated. In other words, there exists $\sigma \in \mathrm{Gal}^c(M/K)$ that sends t to t'. This proves the first result.

Let us now consider two connected components D and D' of $\pi_{M/K}^{-1}(x) \setminus \{x_M\}$. By Theorem 2.15, they are discs. Choose two M-rational points t and t' in D and D', respectively. We have just proven that there exists $\sigma \in \mathrm{Gal}^c(M/K)$ such that $\sigma(t) = t'$.

Lemma 2.12 implies that, since x is fixed by σ, the point x_M is fixed too. We deduce that the connected component D of $\pi_{M/K}^{-1}(x) \setminus \{x_M\}$ is mapped into a connected subset of $\pi_{M/K}^{-1}(x)$ containing t' but not x_M. Thus D is mapped into D'. Similarly, we prove that $\sigma^{-1}(D') \subseteq D$. It follows that σ induces an isomorphism between D and D'. □

The rest of the section is dedicated to the proof of Theorem 2.15. We advise the reader who is mainly interested in the applications to differential equations to skip this part and carry on with §2.3.

Let L^a be the completion of an algebraic closure of L. The fibre $\pi^{-1}_{L^a/L}(x_L)$ is finite. Since x is universal, it may be canonically lifted to L^a and the fibre is actually reduced to the point x_{L^a}. We deduce that, for every connected component C of $\pi^{-1}_{L/K^a}(x)\setminus\{x_L\}$, the base change $C\widehat{\otimes}_L L^a$ is a union of connected components of $\pi^{-1}_{L^a/K}(x)\setminus\{x_{L^a}\}$. Recall that the projection map $X_{K^a}\to X$ is the quotient by the action of $\mathrm{Gal}(\overline{K}/K)$. In particular, it is open. Thus, in order to prove that every connected component of $\pi^{-1}_{L/K^a}(x)\setminus\{x_L\}$ is a virtual open disc with boundary $\{x_L\}$ that is open in X_L, it is enough to show that every connected component of $\pi^{-1}_{L^a/K^a}(x)\setminus\{x_{L^a}\}$ is an open disc with boundary $\{x_{L^a}\}$ that is open in X_{L^a}. Thus, from now on, we will assume that L is algebraically closed.

The proof of the theorem will be split into several steps. We first consider the case of points in the affine line.

Lemma 2.21. *Theorem 2.15 holds if x belongs to the affine line.*

Proof. If the point x is of type 1, there is only one point above it and the result is obvious.

Assume that the point x is of type 2 or 3. Then it is the unique point of the Shilov boundary of some closed disc $D^+_{K^a}(c,R)$, with $c\in K^a$ and $R>0$. In the notation above, we have $x=x_{c,R}$.

Here, all the computations can be made explicitly and we check that the canonical lifting x_L of the point x is the point $x_{c,R}$ in $\mathbf{A}^{1,\mathrm{an}}_L$, i.e. x_L is the unique point of the Shilov boundary of the disc $D^+_L(c,R)$.

The point x_L is of type 2 or 3, and hence its complement in $D^+_L(c,R)$ is a disjoint union of open discs of radius R. Let D be such a disc, and assume that it is not contained in the fibre $\pi^{-1}_{L/K^a}(x)$. Then D meets the preimage of a disc of the form $D'=D^-_{K^a}(d,R)$ with $d\in K^a$, i.e. the disc $D^-_L(d,R)$. Two open discs inside an affine line that meet and have the same radius are equal, and hence $D=D^-_L(d,R)$ and $D\cap\pi^{-1}_{L/K^a}(x)=\varnothing$. We conclude that $\pi^{-1}_{L/K^a}(x)\setminus\{x_L\}$ is the disjoint union of the open discs of radius R that are contained in it.

Let us finally assume that x is of type 4. Then, there exists a sequence of K^a-rational points $\{c_n\}_{n\geqslant 0}$ and a sequence of positive real numbers $\{R_n\}_{n\geqslant 0}$ such that the sequence of discs $\{D^+_{K^a}(c_n,R_n)\}_{n\geqslant 0}$ is decreasing with intersection $\{x\}$. We deduce that

$$\pi^{-1}_{L/K^a}(x)=\bigcap_{n\geqslant 0} D^+_L(c_n,R_n).$$

There are two cases. Let us first assume that $\bigcap_{n\geqslant 0} D^+_L(c_n,R_n)$ contains no L-rational point. Then $\pi^{-1}_{L/K^a}(x)$ is a singleton $\{x_L\}$ and x_L is again a point of type 4.

Now, assume that $\bigcap_{n\geqslant 0} D^+_L(c_n,R_n)$ contains an L-rational point c. Then $\pi^{-1}_{L/K^a}(x)$

is the disc $D_L^+(c,R)$, with $R{=}\inf_{n\geqslant 0} R_n{>}0$. In this case, the point x_L is the unique point $x_{c,R}$ of the Shilov boundary of $D_L^+(c,R)$.

In either case, the result of Theorem 2.15 is obvious. □

We will now turn to the case of points on arbitrary curves. Let us begin with the first part of Theorem 2.15. The following classical result, whose proof is based on Krasner's lemma (see, for instance, [D4, 0.21]), will be useful.

LEMMA 2.22. *There exists an affinoid neighbourhood V of x in X and a smooth affine algebraic curve $\mathscr{X}$ over K such that V identifies to an affinoid domain of $\mathscr{X}^{an}$.* □

LEMMA 2.23. *If x is of type $i{\in}\{1,2\}$, then x_L is also of type i. If x is of type $j{\in}\{3,4\}$, then x_L is either of type j or 2.*

Proof. We have seen that the result holds for points in the affine line and we will reduce to this case. By Lemma 2.22, we may assume that X is the analytification of an algebraic curve. Then there exists an affinoid domain X' of X containing x and a finite morphism φ from X' to an affinoid domain Y' of the affine line. Let us consider the commutative diagram

$$\begin{array}{ccc} X'_L & \xrightarrow{\pi_{L/K^a}} & X' \\ \downarrow{\scriptstyle \varphi_L} & & \downarrow{\scriptstyle \varphi} \\ Y'_L & \xrightarrow{\pi_{L/K^a}} & Y'. \end{array}$$

By Lemma 2.9, we have $\varphi_L(x_L){=}\varphi(x)_L$. The result follows since the finite morphisms φ and φ_L preserve types. □

The following lemma will be useful here and later.

LEMMA 2.24. *Let $\varphi\colon Y\to Z$ be a finite morphism between quasi-smooth K-analytic curves. Let y be a point of Y such that the local ring $\mathscr{O}_{Y,y}$ is a field. Then the local ring $\mathscr{O}_{Z,\varphi(y)}$ is a field too and we have*

$$[\mathscr{O}_{Y,y}\colon \mathscr{O}_{Z,\varphi(y)}] = [\mathscr{H}(y)\colon \mathscr{H}(\varphi(y))].$$

Proof. Since $\mathscr{O}_{Y,y}$ is a field, the point y is not rigid, and hence the point $\varphi(y)$ is not rigid either. Thus, $\mathscr{O}_{Z,\varphi(y)}$ is a field.

By [Be2, Theorem 2.3.3] or [P1, Théorème 4.2], the fields $\mathscr{O}_{Y,y}$ and $\mathscr{O}_{Z,\varphi(y)}$ are Henselian. Since K has characteristic 0, the extension $\mathscr{O}_{Y,y}/\mathscr{O}_{Z,\varphi(y)}$ is separable and the equality follows from the beginning of the proof of [Be2, Proposition 2.4.1]. □

Remark 2.25. When the extension $\mathscr{O}_{Y,y}/\mathscr{O}_{Z,\varphi(y)}$ is not separable, the equality

$$[\mathscr{O}_{Y,y}\colon \mathscr{O}_{Z,\varphi(y)}] = [\mathscr{H}(y)\colon \mathscr{H}(\varphi(y))]$$

may fail. We refer the reader to [D5, 5.3.4.2] for a counterexample due to Temkin. He shows that if K is a non-trivially valued and non-algebraically closed field of characteristic p, then there exists a point $z\in \mathbf{A}^{1,\mathrm{an}}_K$ such that $\mathscr{H}(z)=K^a$. Let us consider the endomorphism φ of $\mathbf{A}^{1,\mathrm{an}}_K$ defined by $T\mapsto T^p$. The point z has precisely one preimage y and we have $[\mathscr{O}_y\colon \mathscr{O}_z]=p$, whereas $[\mathscr{H}(y)\colon \mathscr{H}(z)]=1$.

PROPOSITION 2.26. *Every connected component of* $\pi^{-1}_{L/K^a}(x)\setminus\{x_L\}$ *is an open disc with boundary* $\{x_L\}$. *In particular*, $\pi^{-1}_{L/K^a}(x)$ *is connected.*

Proof. If x is of type 1 or 4, then, by [D6, Théorème 4.3.5], it admits a neighbourhood that is isomorphic to a disc and we may apply Lemma 2.21. If x is of type 3, then, by [D6, Théorème 4.3.5], it admits a neighbourhood that is isomorphic to an annulus and we may use Lemma 2.21 again.

Let us now assume that x is of type 2. Since the result only depends on the field $\mathscr{H}(x)$ we may, by Lemma 2.22, assume that X is the analytification of a smooth algebraic curve over K^a. In particular, X has no boundary.

Moreover, we deduce that the residue field $\widetilde{\mathscr{H}(x)}$ is the function field of an irreducible and reduced algebraic curve $\mathscr{C}_x$ over the algebraically closed field $\widetilde{K^a}$. By generic smoothness, there exists a non-empty Zariski open subset of $\mathscr{C}_x$ that admits an étale map to the affine line. Thus there exists $\tilde{\alpha}\in\widetilde{\mathscr{H}(x)}$ such that the extension $\widetilde{\mathscr{H}(x)}/\widetilde{K^a}(\tilde{\alpha})$ is finite and separable. Let us choose $\alpha\in\mathscr{O}_x$ that lifts $\tilde{\alpha}$. Then α induces a morphism φ from a neighbourhood of x in X to $Y=\mathbf{A}^{1,\mathrm{an}}_{K^a}$. By [Be2, Proposition 3.1.4] we may, up to restricting X and Y, assume that φ is finite and that $\varphi^{-1}(\varphi(x))=\{x\}$. Since X and Y are quasi-smooth, the morphism φ is flat. By construction, the extension $\widetilde{\mathscr{H}(x)}/\widetilde{\mathscr{H}(\varphi(x))}$ is finite and separable. Denote its degree by d and set $y=\varphi(x)$.

Since K^a is algebraically closed and x is of type 2, we have

$$|\mathscr{H}(x)^*| = |\mathscr{H}(y)^*| = |K^{a*}|.$$

By [T, Corollary 6.3.6] or [D6, Théorème 4.3.14], the field $\mathscr{H}(y)$ is stable, and thus

$$[\mathscr{H}(x):\mathscr{H}(y)] = \big[\widetilde{\mathscr{H}(x)}:\widetilde{\mathscr{H}(y)}\big]\,[|\mathscr{H}(x)^*|:|\mathscr{H}(y)^*|] = \big[\widetilde{\mathscr{H}(x)}\colon\widetilde{\mathscr{H}(y)}\big] = d.$$

Moreover, the local rings $\mathscr{O}_x$ and $\mathscr{O}_y$ are fields and, by Lemma 2.24, we also have $[\mathscr{O}_x\colon\mathscr{O}_y]=d$. In particular, the morphism φ has degree d.

Let $\tilde{\beta}\in\widetilde{\mathscr{H}(x)}$ be such that $\widetilde{\mathscr{H}(y)}[\tilde{\beta}]=\widetilde{\mathscr{H}(x)}$. Choose $\beta\in\mathscr{O}_x$ that lifts $\tilde{\beta}$. It is easy to check that we also have $\mathscr{O}_y[\beta]=\mathscr{O}_x$. Let $p(T)\in\mathscr{O}_y[T]$ be the monic minimal polynomial of β over $\mathscr{O}_y$. Then p has degree d and its coefficients lie in $\mathscr{H}(y)^\circ$. In particular, the image of $p'(T)$ by the isomorphism $\mathscr{H}(y)[T]/(p(T))\xrightarrow{\sim}\mathscr{H}(x)$ lies in $\mathscr{H}(x)^\circ$. Since $\tilde{\beta}$ is separable, the reduction of $p'(T)$ is non-zero, and hence the image of $p'(T)$ in $\mathscr{H}(x)$ has absolute value 1.

Let $Y'=\mathcal{M}(\mathscr{B})$ be an affinoid domain of Y that contains y such that the coefficients of $p(T)$ belong to $\mathscr{B}$. Set $\mathscr{A}=\mathscr{B}[T]/(p(T))$ and $X'=\mathcal{M}(\mathscr{A})$. Let $\psi\colon X'\to Y'$ be the natural morphism. The preimage of y by ψ coincides with the spectrum of

$$\mathscr{H}(y)[T]/(p(T))\simeq\mathscr{H}(x),$$

and hence it contains a single point and the completed residue field of this point is isomorphic to $\mathscr{H}(x)$. We still denote this point by x.

Let us now consider the commutative diagram

$$\begin{array}{ccc} X'_L & \xrightarrow{\pi_{L/K^a}} & X' \\ \downarrow{\scriptstyle \psi_L} & & \downarrow{\scriptstyle \psi} \\ Y'_L & \xrightarrow{\pi_{L/K^a}} & Y'. \end{array}$$

Since y is a point of type 2 of the affine line, the residue field $\widetilde{\mathscr{H}(y)}$ is purely of transcendence degree 1, and hence isomorphic to $\widetilde{K^a}(u)$, where u is an indeterminate. By assumption, the reduction $\tilde{p}(T)$ of $p(T)$ is irreducible over $\widetilde{\mathscr{H}(y)}$. Since $\widetilde{K^a}$ is algebraically closed, by [G, Proposition 4.3.9], the ring $(\widetilde{K^a}(u)[T]/(\tilde{p}(T)))\otimes_{\widetilde{K^a}}\tilde{L}=\tilde{L}(u)[T]/(\tilde{p}(T))$ is a domain. Thus the polynomial $\tilde{p}(T)$ is still irreducible over $\tilde{L}(u)\simeq\widetilde{\mathscr{H}(y_L)}$. In particular, $p(T)$ is irreducible over $\mathscr{H}(y_L)$ and there is only one point in X'_L above y_L, which can only be x_L, by Lemma 2.9.

We have just shown that $\varphi_L^{-1}(y_L)=\{x_L\}$, and hence it is enough to prove that the preimage of every connected component of $\pi_{L/K^a}^{-1}(y)\setminus\{y_L\}$ is an open disc whose boundary contains x_L.

By Lemma 2.21, the connected components of $\pi_{L/K^a}^{-1}(y)\setminus\{y_L\}$ are open discs. Let D be one of them and choose a coordinate S on it. Let $z\in D(L)$ and let $p_z(T)$ be the image of $p(T)$ in $\mathscr{H}(z)[T]\simeq L[T]$. There are exactly d points $z_1,...,z_d$ in X'_L above z, and these points correspond to the zeroes of $p_z(T)$ in L. Since every z_i lies over x, the image of $p'_z(T)$ in $\mathscr{H}(z_i)$ has absolute value 1. We deduce that the reduction $\tilde{p}_z(T)$ is separable over $\widetilde{\mathscr{H}(z)}$, and hence the reductions $\tilde{z}_1,...,\tilde{z}_d\in\tilde{L}$ are distinct. Moreover, for every $i\in[[1,d]]$, we have $\tilde{p}'_z(\tilde{z}_i)\neq 0$.

We may consider $p(T)$ as a polynomial with coefficients in $\mathscr{O}(D)$, or even in $\mathscr{O}(D)^\circ = L^\circ[[S]]$. By Henselianity, the zeroes $\tilde{z}_1, ..., \tilde{z}_d \in \tilde{L}$ lift to d elements of $L^\circ[[S]]$, and hence gives rise to d sections of φ_L over D. Of course, these sections correspond to the zeroes of $p(T)$ in $\mathscr{O}(D)$.

For every point z' of D of type 2, 3 or 4, the ring $\mathscr{O}(D)$ embeds into $\mathscr{H}(z')$. Thus the d sections are distinct above z'. For every point z' of D of type 1, by redoing the previous argument with z' instead of z, we prove that the sections are also distinct above z'. We deduce that the sections are disjoint everywhere, and hence $\varphi_L^{-1}(D)$ is a disjoint union of d connected components $C_1, ..., C_d$.

Let $i \in [[1, d]]$. The map φ_L induces a map $C_i \to D$ that is a finite morphism of degree 1, and hence is an isomorphism. This proves the first part of the result. Moreover, X'_L is compact and C_i is not, and so the boundary B_i of C_i is non-empty. The boundary of D in Y'_L is $\{y_L\}$. For every neighbourhood U of y_L in Y'_L, B_i belongs to $\psi^{-1}(U)$. We deduce that $\psi(B_i) = \{y_L\}$, and so $B_i = \{x_L\}$, which concludes the proof. □

We still have to prove that the connected components of $\pi^{-1}_{L/K^a}(x) \setminus \{x_L\}$, which are isomorphic to open discs, are open in X_L. This is a general fact that is related to the analytic version of Zariski's main theorem (see [Be2, Proposition 3.1.4] for morphisms with no boundary, which is enough for our needs, or [D1, Théorème 3.2] in general).

LEMMA 2.27. *Let $\varphi: Y \to Z$ be a morphism between quasi-smooth K-analytic curves. Assume that Y has no boundary and that φ is injective. Then φ is open.*

Proof. Let $y \in Y$. By [Be2, Proposition 3.1.4], there exist affinoid neighbourhoods V and W of y and $\varphi(y)$, respectively, such that φ induces a finite morphism $\psi: V \to W$. Since ψ is a finite map between quasi-smooth curves, it is flat, and hence open by [Be2, Proposition 3.2.7]. □

Remark 2.28. In a previous version of this article, a different strategy was used to prove Proposition 2.26. It involved reduction techniques that were very close to the ones used in the proof of Theorem 2.10 in [P2]. In the case of points of type 2, we were able to prove that there exists an affinoid domain V of X_L containing $\pi^{-1}_{L/K^a}(x)$ such that the reduction map $V \to \widetilde{V}$ sends the point x_L to a generic point, whereas every other point of $\pi^{-1}_{L/K^a}(x)$ is sent to a smooth point. We could then conclude by a result of Bosch that ensures that the preimage of a smooth point by the reduction map is an open disc (see [Bos, Satz 6.3]). Unfortunately, since the last result is only available in the strictly affinoid case and over non-trivially valued fields, we had to distinguish several cases, which made the paper longer and more technical. The current proof of Proposition 2.26 was suggested by a referee. Let us finally mention that Lemma 2.27 can also be proven by use of the above reduction techniques, using the simple fact that the preimage of a closed point is open.

Remark 2.29. Theorem 2.15 actually holds regardless of the characteristic of the field. As it is written, our proof works only when K has characteristic 0 (contrary to the one described in Remark 2.28) due to Lemma 2.24. If K is not trivially valued, this can be easily fixed by slightly moving the element $\beta\in\mathscr{O}_x$ that we choose in the proof of Proposition 2.26 (and using Krasner's lemma) in order to assume that $\mathscr{O}_x/\mathscr{O}_y$ is separable.

If K is trivially valued, a direct proof is possible. Indeed, if K is algebraically closed, which we may assume, any type-2 point x appears inside the analytification X^{an} of a smooth connected projective algebraic curve X. By [Be1, 1.4.2], the point x is then the only point of type 2 in X^{an} and its complement is a disjoint union of open discs. Since analytification commutes with extension of scalars, the result easily follows, by arguments that are similar to those used in the proof of Lemma 2.21.

Remark 2.30. When working on this paper, Ducros told us that he also had a proof of Theorem 2.15 (in any characteristic) which is based on computations of étale cohomology groups. At that time it was not available in written form, but the reader may now find it in [D6, §5.3].

2.3. Radius of convergence

For the rest of the article, we assume that X is endowed with a weak triangulation S.

Definition 2.31. Let $x\in X$ and let L be a complete valued extension of K such that X_L contains an L-rational point t_x over x. We denote by $D(t_x,S_L)$ the biggest open disc centred at t_x that is contained in $X_L\setminus S_L$, i.e. the connected component of $X_L\setminus\Gamma_{S_L}$ that contains t_x.

Remark 2.32. Assume that $x\notin\Gamma_S$. In this case, the connected component C of $X\setminus\Gamma_S$ that contains x is a virtual disc and $D(t_x,S_L)$ is the connected component of C_L that contains t_x.

Assume that $x\in\Gamma_S$. In this case, $D(t_x,S_L)$ is the biggest open disc centred in t_x that is contained in $\pi_L^{-1}(x)$.

In particular, the definition of the disc $D(t_x,S_L)$ depends only on the skeleton Γ_S and not on the weak triangulation S itself.

The following lemma is an easy consequence of the definitions.

LEMMA 2.33. *Let M/L be an extension of complete valued fields over K. Let t_x be an L-rational point of X_L. The point t_x naturally gives rise to an M-rational point $t_{x,M}$ of X_M. Then we have a natural isomorphism*

$$D(t_x, S_L)\widehat{\otimes}_L M \xrightarrow{\sim} D(t_{x,M}, S_M). \qquad \square$$

The next result follows from Corollary 2.18.

LEMMA 2.34. *Let M be an algebraically closed and maximally complete valued extension of K. Let t_x and t'_x be two points in X_M that project onto the same point on X. Then there exists $\sigma \in \mathrm{Gal}^c(M/K)$ that sends t_x to t'_x and induces an isomorphism $D(t_x, S_M) \xrightarrow{\sim} D(t'_x, S_M)$.*

Proof. By Corollary 2.18, there exists $\sigma \in \mathrm{Gal}^c(M/K)$ that sends t_x to t'_x. By Lemma 2.14, the skeleton S_M is invariant under $\mathrm{Gal}^c(M/K)$. Hence, the isomorphism ψ_σ of X_M induced by σ sends the disc $D(t_x, S_M)$ to a disc that contains t'_x and does not meet S_M. We deduce that $\psi_\sigma(D(t_x, S_M)) \subset D(t'_x, S_M)$. Using the same argument with σ replaced by σ^{-1}, one shows the reverse inclusion. $\square$

In the introduction we explained that the radius of convergence was to appear as the radius of some disc. Unfortunately, the radius of a disc is not invariant under isomorphisms. This leads us to define the radius of convergence as a relative radius inside a fixed bigger disc. The lemma that follows will help to show that it is well defined.

LEMMA 2.35. *Let $R_1, R_2 > 0$. Up to a translation of the coordinate t, any isomorphism $\alpha: D^-_K(0, R_1) \xrightarrow{\sim} D^-_K(0, R_2)$ is given by a power series of the form*

$$f(t) = \sum_{i \geqslant 1} a_i t^i \in K[[t]],$$

where $|a_1| = R_2/R_1$ and, for every $i \geqslant 2$, $|a_i| \leqslant R_2/R_1$. In particular, α multiplies distances by the constant factor R_2/R_1, i.e. for every complete valued extension L of K and every x and y in $D^-_K(0, R_1)(L)$, we have

$$|\alpha(x) - \alpha(y)| = \frac{R_2}{R_1}|x - y|.$$

As a consequence, such an isomorphism may only exist when $R_2/R_1 \in |K^*|$. $\square$

We may now adapt the usual definition of radius of convergence (see [NP1, §3.2] as well as [K1, Notation 11.3.1 and Definition 11.9.1]).

Definition 2.36. Let $\mathscr{F}$ be a locally free $\mathscr{O}_X$-module of finite type with an integrable connection ∇. Let x be a point in X and L be a complete valued extension of K such that X_L contains an L-rational point t_x over x. Let us consider the pull-back $(\mathscr{F}^x,\nabla^x)$ of $(\mathscr{F},\nabla)$ on $D(t_x,S_L)\simeq D_L^-(0,R)$. Let $r{=}\mathrm{rk}(\mathscr{F}^x)$. For $i\in[[1,r]]$, we denote by $\mathcal{R}'_{S,i}(x,(\mathscr{F},\nabla))$ the radius of the biggest open subdisc of $D_L^-(0,R)$ centred at 0 and on which the connection $(\mathscr{F}^x,\nabla^x)$ admits at least $r-i+1$ horizontal sections that are linearly independent over L. Let us define the ith radius of convergence of $(\mathscr{F},\nabla)$ at x by $\mathcal{R}_{S,i}(x,(\mathscr{F},\nabla)){=}\mathcal{R}'_{S,i}(x,(\mathscr{F},\nabla))/R$ and the multiradius of convergence of $(\mathscr{F},\nabla)$ at x by

$$\boldsymbol{\mathcal{R}}_S(x,(\mathscr{F},\nabla))=(\mathcal{R}_{S,1}(x,(\mathscr{F},\nabla)),...,\mathcal{R}_{S,r}(x,(\mathscr{F},\nabla)))\in(0,1]^r.$$

We will frequently suppress ∇ from the notation when it will be clear from the context.

Remark 2.37. With the previously used notation, one may also consider the radius of the biggest open subdisc of $D_L^-(0,R)$ centred at 0 on which the connection $(\mathscr{F}^x,\nabla^x)$ is trivial as, for instance, in [Ba, Definition 3.1.7]. This way, one recovers the radius $\mathcal{R}_{S,1}$.

Remark 2.38. By Remark 2.32, the radii depend only on the skeleton Γ_S and not on the weak triangulation S itself.

Definition 2.36 is independent of the choices made and invariant by extensions of the ground field K, due to the preceding lemmas (first prove the independence of the isomorphism $D(t_x,S_L)\simeq D_L^-(0,R)$ and in particular of R, then the invariance under base-change for rational points and finally reduce to the case where L is algebraically closed and maximally complete). We state the following for future reference.

LEMMA 2.39. *Let L be a complete valued extension of K. For any $x\in X_L$, we have*

$$\boldsymbol{\mathcal{R}}_{S_L}(x,\pi_L^*(\mathscr{F},\nabla))=\boldsymbol{\mathcal{R}}_S(\pi_L(x),(\mathscr{F},\nabla)).\qquad\square$$

Let us now explain how the function behaves with respect to changing triangulations. Let S' be a weak triangulation of X such that $\Gamma_{S'}$ contains Γ_S. Let $x\in X$ and let L be a complete valued extension of K such that X_L contains an L-rational point t_x over x. By Remark 2.32, the disc $D(t_x,S'_L)$ is contained in $D(t_x,S_L)\simeq D_L^-(0,R)$. Let R' be its radius as a subdisc of $D_L^-(0,R)$ and set $\varrho_{S',S}(x){=}R'/R\in(0,1]$. Remark that $\varrho_{S',S}$ is constant and equal to 1 on Γ_S. It is now easy to check that, for any $i\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$, we have

$$\mathcal{R}_{S',i}(x,(\mathscr{F},\nabla))=\min\left\{\frac{\mathcal{R}_{S,i}(x,(\mathscr{F},\nabla))}{\varrho_{S',S}(x)},1\right\}.\qquad(2.1)$$

2.4. Comparison with other definitions

We compare the radius of convergence that we introduced in Definition 2.36 to other radii that appear in the literature.

2.4.1. Baldassarri's definition using semistable models

The first definition of radius of convergence on a curve was given by Baldassarri in [Ba]. It was our main source of inspiration and our definition is very close to his.

Assume that the absolute value of K is non-trivial and that the curve X is strictly K-affinoid. In this case, it is known that there exists a finite separable extension L/K such that the curve X_L admits a semistable formal model $\mathfrak{X}$ over L°.

There actually exists a strong relation between the semistable models of X_L and its triangulations (see [D6, §6.4] for a detailed account). Indeed, for any generic point $\tilde{\xi}$ of the special fibre $\mathfrak{X}_s$ of $\mathfrak{X}$, let ξ be the unique point of the generic fibre $\mathfrak{X}_\eta = X_L$ whose reduction is equal to $\tilde{\xi}$. Gathering all points ξ, we construct a finite set $S(\mathfrak{X})$ that is a triangulation of X_L. Let us remark that our notation unfortunately disagrees with Baldassarri's: in [Ba], $S(\mathfrak{X})$ denotes the skeleton ($\Gamma_{S(\mathfrak{X})}$ in our notation) and not the triangulation.

Let $x \in X_L(L)$. In [Ba], Baldassarri considers the biggest disc that does not meet the skeleton $\Gamma_{S(\mathfrak{X})}$ of $\mathfrak{X}$ (see [Ba, Definition 1.6.6]), which is nothing but our disc $D(x, S(\mathfrak{X}))$. Since every point of $S(\mathfrak{X})$ is of type 2, this disc is actually isomorphic to the open unit disc $D^-_L(0,1)$. Baldassarri then defines the radius of convergence $\mathcal{R}_{\mathfrak{X}}(x, \pi_L^*(\mathscr{F}, \nabla))$ of $\pi_L^*(\mathscr{F}, \nabla)$ at x as the radius r of the biggest open subdisc of $D^-_L(0,1)$ centred at 0 and on which $(\mathscr{F}, \nabla)$ is trivial (see [Ba, Definition 3.1.8]). This is compatible with our definition that uses a relative radius (see Lemma 2.35 and Remark 2.37). Finally, we have proved that, for any $x \in X_L(L)$, we have

$$\mathcal{R}_{\mathfrak{X}}(x, \pi_L^*(\mathscr{F}, \nabla)) = \mathcal{R}_{S_L,1}(x, \pi_L^*(\mathscr{F}, \nabla)).$$

Baldassarri extends his definition to other points of the curve by extending the scalars so as to make them rational (see [Ba, Definition 3.1.11]). One may check that, for any complete valued extension M of L, $\mathfrak{X} \widehat{\otimes}_{L^\circ} M^\circ$ is a semistable model of X_M and that $S(\mathfrak{X} \widehat{\otimes}_{L^\circ} M^\circ) = S(\mathfrak{X})_M$. Hence our definition coincides with Baldassarri's everywhere.

Let us also point out that, conversely, for any triangulation S of X that only contains points of type 2, there exists a finite separable extension L/K and a semistable formal model $\mathfrak{X}$ of X_L such that $S(\mathfrak{X}) = S_L$. Hence, under the hypotheses of this section and if we restrict to triangulations that only contain points of type 2, our definition is essentially equivalent to Baldassarri's. (Remark that, in the non-strict case, triangulations must be allowed to contain points of type 3. We, as well as Ducros, allow this in general.)

Finally, let us mention that Baldassarri actually considers the slightly more general situation $X=\overline{X}\setminus\{z_1,...,z_r\}$, where $\overline{X}$ is a compact curve as above and $z_1,...,z_r$ are K-rational points. In this case, he constructs the skeleton of X by branching on the skeleton of $\overline{X}$ a half-line ℓ_i that extends in the direction of z_i, for each i. The definition of radius of convergence may then be adapted.

Let us mention that this more general situation is already covered in our setting, since we did not require the curves to be compact. To find the same skeleton, it is enough to begin with the triangulation of $\overline{X}$ and add, for each i, a sequence of points that lie on ℓ_i and tend to z_i.

2.4.2. The definition for analytic domains of the affine line

Assume that X is an analytic domain of the affine line $\mathbf{A}^{1,\mathrm{an}}_K$. The choice of a coordinate t on $\mathbf{A}^{1,\mathrm{an}}_K$ provides a global coordinate on X and it seems natural to use it in order to measure the radii of convergence. This normalisation has been used by Baldassarri and Di Vizio in [BD] (for the first radius) and by the second author in [NP1]. We will call the radii we define in this setting *embedded*.

From now on, we will assume that X is not the affine line. Let us first give a definition of radii that does not refer to any triangulation.

Definition 2.40. Let x be a point of X and L be a complete valued extension of K such that X_L contains an L-rational point t_x over x. Let $D(t_x,X_L)$ be the biggest open disc centred at t_x that is contained in X_L (such a disc exists since $X\neq\mathbf{A}^{1,\mathrm{an}}_K$).

Let us consider the pull-back $(\mathscr{F}^x,\nabla^x)$ of $(\mathscr{F},\nabla)$ on $D(t_x,X_L)$ and let $r=\mathrm{rk}(\mathscr{F}^x)$. For $i\in[[1,r]]$ we let $\mathcal{R}^{\mathrm{emb}}_i(x,(\mathscr{F},\nabla))$ denote the radius of the biggest open subdisc of $D(t_x,X_L)$ that is centred at t_x, measured using the coordinate t on $\mathbf{A}^{1,\mathrm{an}}_L$, and on which the connection $(\mathscr{F}^x,\nabla^x)$ admits at least $r-i+1$ horizontal sections that are linearly independent over L.

As before, one checks that the definition of $\mathcal{R}^{\mathrm{emb}}_i(x,(\mathscr{F},\nabla))$ only depends on the point x and not on L or t_x. This radius is denoted by $\mathcal{R}^{\mathscr{F}}_i(x)$ in [NP1, §3.2]. There, the second author works over an affinoid domain V of the affine line.

Although possibly superfluous in this context, it is also possible to state a definition that depends on the weak triangulation S of X.

Definition 2.41. Let x be a point of X and let L be a complete valued extension of K such that X_L contains an L-rational point t_x over x. As in Definition 2.31, consider

$D(t_x, S_L)$, the biggest open disc centred at t_x that is contained in $X_L \setminus S_L$. We denote by $\varrho_S(x)$ the radius of $D(t_x, S_L)$, measured using the coordinate t on $\mathbf{A}_L^{1,\mathrm{an}}$.

Let us consider the pull-back $(\mathscr{F}^x, \nabla^x)$ of $(\mathscr{F}, \nabla)$ on $D(t_x, S_L)$ and let $r{=}\mathrm{rk}(\mathscr{F}_x)$. For $i \in [[1, r]]$ we let $\mathcal{R}_{S,i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))$ denote the radius of the biggest open subdisc of $D(t_x, S_L)$ centred at t_x, measured using the coordinate t on $\mathbf{A}_L^{1,\mathrm{an}}$, on which the connection $(\mathscr{F}^x, \nabla^x)$ admits at least $r-i+1$ horizontal sections that are linearly independent over L.

Once again, the definitions of $\varrho_S(x)$ and $\mathcal{R}_{S,i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))$ are independent of the choices of L and t_x.

The radii we have just defined may easily be linked to the one we introduced in Definition 2.36. The second case is the simplest one: for any $i \in [[1, \mathrm{rk}(\mathscr{F}_x)]]$, we have

$$\mathcal{R}_{S,i}(x, (\mathscr{F}, \nabla)) = \frac{\mathcal{R}_{S,i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))}{\varrho_S(x)}. \tag{2.2}$$

Since X is not the affine line, we have the following result, whose proof we leave to the reader.

Lemma 2.42. *The set of skeletons of the weak triangulations of X admits a smallest element $\Gamma_0(X)$. It coincides with the analytic skeleton of X, i.e. the set of points that have no neighbourhood isomorphic to a virtual open disc.* □

Let S_0 be a weak triangulation of X whose skeleton is Γ_0. By Remark 2.38, the radii computed with respect to a weak triangulation only depend on the skeleton of the triangulation. In particular, the radii computed with respect to S_0 are well defined. It is now easy to check that, for every $i \in [[1, \mathrm{rk}(\mathscr{F}_x)]]$, we have

$$\mathcal{R}_{S_0,i}(x, (\mathscr{F}, \nabla)) = \frac{\mathcal{R}_{S_0,i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))}{\varrho_{S_0}(x)} = \frac{\mathcal{R}_{i}^{\mathrm{emb}}(x, (\mathscr{F}, \nabla))}{\varrho_{S_0}(x)}. \tag{2.3}$$

3. The result

In §2 we defined the radius of convergence of $(\mathscr{F}, \nabla)$ at any point of the curve X. We now investigate its properties.

3.1. Statement

To precisely state the result we are interested in, we first need to define the notion of locally finite subgraph of X.

From the existence of weak triangulations, one deduces that every point of X has a neighbourhood that is uniquely arcwise connected. In particular, the curve X may be covered by uniquely arcwise connected analytic domains. On such a subset, it makes sense to speak of the segment $[x,y]$ joining two given points x and y, and hence one has a notion of convex subsets (see also [BR, §2.5]).

Definition 3.1. A subset Γ of X is said to be a *finite* (resp. *locally finite*) *subgraph* of X if there exists a finite (resp. locally finite) family $\mathscr{V}$ of affinoid domains of X that covers Γ and such that, for every element V of $\mathscr{V}$, we have

(1) V is uniquely arcwise connected;

(2) $\Gamma \cap V$ is the convex hull of a finite number of points.

We now want to define a notion of log-linearity. To do so, we first need to explain how to measure distances.

Definition 3.2. Let C be a closed virtual annulus over K. Its preimage over K^a is a finite union of closed annuli. If $C^+_{K^a}(c, R_1, R_2)$ is one of them, we set

$$\mathrm{Mod}(C) = \frac{R_2}{R_1}.$$

It is independent of the choices.

Notation 3.3. Let I be a closed interval inside a virtual disc or annulus and assume that I contains only points of type 2 or 3. Then I is the skeleton of a virtual closed subannulus $I^\sharp$ and we set

$$\ell(I) = \log \mathrm{Mod}(I^\sharp).$$

Remark 3.4. Pushing these ideas further, one can show that it is possible to define a canonical length for any closed interval inside a curve that contains only points of type 2 or 3 (see [D6, Proposition 4.5.7]). The definition may actually be extended to the whole curve (see [D6, Corollaire 4.5.8]).

Definition 3.5. Let $X_{[2,3]}$ be the set of points of X that are of type 2 or 3. Let J be an open interval inside $X_{[2,3]}$ and identify it with a real interval. A map $f\colon J \to \mathbf{R}^*_+$ is said to be *log-linear* if there exists $\gamma \in \mathbf{R}$ such that, for every $a, b \in J$, with $a < b$, we have

$$\log f(b) - \log f(a) = \gamma\, \ell([a,b]).$$

Let Γ be a locally finite subgraph of X. A map $f\colon \Gamma \to \mathbf{R}^*_+$ is said to be *piecewise log-linear* if Γ may be covered by a locally finite family $\mathscr{J}$ of closed intervals such that, for every $J \in \mathscr{J}$, the restriction of f to $\mathring{J}$ is log-linear.

We may now state the theorem we want to prove.

THEOREM 3.6. *The map* $\boldsymbol{\mathcal{R}}_S$ *satisfies the following properties*:

(i) *it is continuous*;

(ii) *its restriction to any locally finite graph* Γ *is piecewise log-linear*;

(iii) *on any interval* J, *the log-slopes of its restriction are rational numbers of the form* m/j *with* $m\in\mathbf{Z}$ *and* $j\in[[1,r]]$, *where* r *is the rank of* $\mathscr{F}$ *around* J;

(iv) *there exists a locally finite subgraph* Γ *of* X *and a continuous retraction* $r\colon X\to\Gamma$ *such that the map* $\boldsymbol{\mathcal{R}}_S$ *factorises by* r.

The continuity of the radius of convergence $\mathcal{R}_{S,1}$ has been proven by Baldassarri and Di Vizio in the case of affinoid domains of the affine line (see [BD]) and by Baldassarri in general (see [Ba]). Baldassarri's setting is actually slightly less general than ours, but his result extends easily.

For the multiradius of convergence on affinoid domains of the affine line, the result is due to the second author (see [NP1, Theorem 3.9]).

THEOREM 3.7. (Pulita) *Theorem 3.6 holds when* X *is an affinoid domain of the affine line.*

Let us write down a corollary of Theorem 3.7 about open discs that will be useful later.

COROLLARY 3.8. *Assume that* X *is an open disc endowed with the empty triangulation. Let* x *be a point of* X *and let* $[x,y]$ *be a segment with initial point* x. *The restriction of the map* $\boldsymbol{\mathcal{R}}_S$ *to the segment* $[x,y]$ *is continuous at the point* x *and log-linear in the neighbourhood of* x *with slope of the form* m/j *with* $m\in\mathbf{Z}$ *and* $j\in[[1,\mathrm{rk}(\mathscr{F})]]$.

Proof. We identify X with the disc $D^-(0,R)$ for some $R>0$. Let $i\in[[1,\mathrm{rk}(\mathscr{F})]]$. Pick $r\in(0,1)$ such that $[x,y]\subset D^-(0,rR)$.

Let us first assume that $\mathcal{R}_{\varnothing,i}(\cdot,\mathscr{F})$ is smaller than or equal to r on $[x,y]$, and let us endow X with the triangulation $S=\{x_{0,rR}\}$. By formula (2.1), for every $z\in[x,y]$, we have

$$\mathcal{R}_{S,i}(z,\mathscr{F})=r\mathcal{R}_{\varnothing,i}(z,\mathscr{F}).$$

Moreover, for every $z\in D^+(0,rR)$, we have $\mathcal{R}_{S,i}(z,\mathscr{F})=r\mathcal{R}_{S,i}(z,\mathscr{F}|_{D^+(0,rR)})$. The result now follows from Theorem 3.7.

We may now assume that there exists $z\in[x,y]$ and $r'\in(r,1)$ such that $\mathcal{R}_{\varnothing,i}(z,\mathscr{F})=r'$. This means that the module $\mathscr{F}$ has a trivial submodule of corank $i-1$ on $D^-(0,r'R)$, and hence $\mathcal{R}_{\varnothing,i}(\cdot,\mathscr{F})=r'$ on $D^-(0,r'R)$ and the result follows trivially. $\square$

Remark 3.9. Let us mention that if one is only interested in the first radius of convergence $\mathcal{R}_{S,1}$ on boundary-free curves (or with overconvergent connections), it is possible to get the result of Theorem 3.6 in a much shorter way via potential theory (see [PP]).

Remark 3.10. The result of Theorem 3.6 can be strengthened. For instance, one can require that the graph Γ contains no point of type 4. This is equivalent to the fact that the radii are constant in the neighbourhood of every point of type 4, a result that is due to Kedlaya (see [K2, Lemma 4.5.14]).

Remark 3.11. Let J be an interval inside X. One could expect that the restriction of the map $\boldsymbol{\mathcal{R}}_S$ to J is log-linear around every point of type 3 that does not belong to S. We know how to prove this for $\mathcal{R}_{S,1}$ or when no radius is solvable but the general case seems trickier. This is related to the general question of super-harmonicity for the partial heights of the convergence Newton polygon (i.e. the sum of the logarithms of the first j slopes for varying j), which is an open problem (see [NP4, Remark 2.4.10] for the description of a hypothetical situation where it would fail).

Nevertheless, we are able to prove that if Γ_S has no end-points of type 3, then the smallest graph Γ that contains Γ_S and satisfies the properties of Theorem 3.6 (for the multi-radius $\boldsymbol{\mathcal{R}}_S$ and not some individual $\mathcal{R}_{S,i}$) exists and has no end-points of type 3 either. We refer to [NP3] for more on this issue.

Let us now turn to the proof of Theorem 3.6. By Lemma 2.39, it is enough to prove the result after a scalar extension. *From now on, we will assume that K is algebraically closed.*

3.2. A geometric result

The overall strategy of our proof is the following. By the definition of weak triangulation, the set $X \setminus S$ is a union of open discs and annuli, and for those we may use the results of the second author (see Theorem 3.7). We still need to investigate what happens around the points of the triangulation S. To carry out this task, we will write the curve X, locally around these points, as a finite étale cover of a subset of the affine line, consider the push-forward of $(\mathscr{F}, \nabla)$, which is well understood due to Theorem 3.7 again, and relate its radii of convergence to the original radii.

We will need to find étale morphisms from open subsets of X to the affine line that satisfy nice properties. The main result we use has been adapted from the proof of [D6, Théorème 4.4.15]. For the convenience of the reader, we have decided to sketch the proof of the whole result.

In what follows, we use Ducros's notion of "branch" (see [D6, §1.7]), which roughly corresponds to that of a direction out of a point. More precisely, for every open subtree V of X containing a point x, there is a bijection between the set of branches out of x and the set of connected components of $V\setminus\{x\}$. Such a connected component is called a *section* of the corresponding branch. Every branch out of a point of type 2 or 3 admits a section that is isomorphic to an open annulus.

There are well-defined notions of direct and inverse images of branches that correspond to the intuitive ones. Let $\varphi\colon X\to Y$ be a morphism of curves and let $x\in X$ be a point such that $\varphi^{-1}(\varphi(x))$ is finite. Then the image of a branch out of x is a branch out of $\varphi(x)$ and the preimage of a branch out of $\varphi(x)$ is a union of branches out of x.

Let x be a point of X of type 2 and consider the complete valued field $\mathscr{H}(x)$ associated with it. Since $\widetilde{K}$ is algebraically closed, the residue field $\widetilde{\mathscr{H}(x)}$ is the function field of a unique projective smooth connected curve $\mathscr{C}$ over $\widetilde{K}$, called the residual curve (see [D6, 3.3.5.2]). If x lies in the interior of X, then the closed points of $\mathscr{C}$ correspond bijectively to the branches out of the point x (see [D6, 4.2.11.1]).

We say that a property holds for almost every element of a set E if it holds for all but finitely many of them.

THEOREM 3.12. *Let x be a point of X of type 2. Let $b_1, ..., b_t$ and c be distinct branches out of x. Let N be a positive integer. There exists an affinoid neighbourhood Z of x in X, a quasi-smooth affinoid curve Y, an affinoid domain W of $\mathbf{P}^{1,an}_K$ and a finite étale map $\psi\colon Y\to W$ such that the following hold*:

(1) *Z is isomorphic to an affinoid domain of Y and x lies in the interior of Y*;

(2) *the degree of ψ is prime to N*;

(3) $\psi^{-1}(\psi(x))=\{x\}$;

(4) *almost every connected component of $Y\setminus\{x\}$ is an open unit disc with boundary $\{x\}$*;

(5) *almost every connected component of $W\setminus\{\psi(x)\}$ is an open unit disc with boundary $\{\psi(x)\}$*;

(6) *for almost every connected component V of $Y\setminus\{x\}$, we have that the induced morphism $V\to\psi(V)$ is an isomorphism*;

(7) *for every $i\in[[1,t]]$, the morphism ψ induces an isomorphism between a section of b_i and a section of $\psi(b_i)$ and we have $\psi^{-1}(\psi(b_i))\subseteq Z$*;

(8) $\psi^{-1}(\psi(c))=\{c\}$.

Proof. By Lemma 2.22, there exists an affinoid neighbourhood Z of x in X and a smooth affine algebraic curve $\mathscr{X}$ over K such that V identifies to an affinoid domain of $\mathscr{X}^{\rm an}$. We may assume that $\mathscr{X}$ is projective and connected. Let $\mathscr{C}$ be the residual curve at the point x. Let g be a rational function on $\mathscr{C}$ that induces a generically étale morphism $\mathscr{C}\to\mathbf{P}^1_{\tilde{K}}$. Let f be a rational function on $\mathscr{X}$ such that $|f(x)|=1$ and $\tilde{f}=g$. Let $f^{\mathrm{an}}\colon\mathscr{X}^{\mathrm{an}}\to\mathbf{P}^{1,\mathrm{an}}_K$ be the associated morphism.

Let us first remark that, for almost every connected component V of $\mathscr{X}^{\mathrm{an}}\setminus\{x\}$, V meets a unique branch out of x and $f^{\mathrm{an}}(V)$ is a connected component of $\mathbf{P}^{1,\mathrm{an}}_K\setminus\{f^{\mathrm{an}}(x)\}$.

Since the map induced by g is generically étale, it is, for almost every connected component V of $\mathscr{X}^{\mathrm{an}}\setminus\{x\}$, unramified at the closed point of $\mathscr{C}$ corresponding to the branch associated with V. From this we deduce that the morphism $V\to f^{\mathrm{an}}(V)$ induced by f^{an} has degree 1 (see [D6, Théorème 4.3.13]), and hence it is an isomorphism.

Finally, choose an affinoid neighbourhood W of $f^{\mathrm{an}}(x)$ in $\mathbf{P}^{1,\mathrm{an}}_K$ such that the different points of $(f^{\mathrm{an}})^{-1}(f^{\mathrm{an}}(x))$ belong to different connected components of $(f^{\mathrm{an}})^{-1}(W)$. Let Y be the connected component containing x, and $\psi\colon Y\to W$ be the induced morphism. Properties (3) and (6) are satisfied by construction. Since $\psi(x)$ is a point of type 2, property (5) is clear. Property (4) follows.

For any $i\in[[1,t]]$, let $\tilde{b}_i$ be the closed point of the residue curve $\mathscr{C}$ associated with the branch b_i. Let $\tilde{c}$ be the closed point associated with the branch c. Finally, let $\tilde{a}_1,...,\tilde{a}_u$ be the closed points associated with the branches of $\mathscr{X}$ out of x that do not belong to X. In the lemma below, we will prove that there exists a rational function g as above whose degree d is prime to N, such that g has a simple zero at every $\tilde{b}_i$, a unique pole at $\tilde{c}$ and takes the value 1 at every $\tilde{a}_j$. Property (8) then follows from the link between branches out of a point and closed points of the residue curve. The first part of property (7) follows from [D6, Théorème 4.3.13], as above. The second follows from the fact that $g^{-1}(g(\tilde{b}_i))=g^{-1}(0)$ contains none of the points $\tilde{a}_j$. Moreover, if we assume that the set of branches b_i is non-empty, which we can always do, then the map g has a simple zero, which forces it to be generically étale.

Regarding the degree of ψ, we note that $\big[\widetilde{\mathscr{H}(x)}\colon\widetilde{\mathscr{H}(\psi(x))}\big]=\deg(g)=d$. Since K is algebraically closed and x is of type 2, we have $|\mathscr{H}(x)^*|=|\mathscr{H}(\psi(x))^*|=|K^*|$. By [T, Corollary 6.3.6] or [D6, Théorème 4.3.14], the field $\mathscr{H}(\psi(x))$ is stable, and hence

$$[\mathscr{H}(x):\mathscr{H}(\psi(x))]=\big[\widetilde{\mathscr{H}(x)}:\widetilde{\mathscr{H}(\psi(x))}\big]\,[|\mathscr{H}(x)^*|:|\mathscr{H}(\psi(x))^*|]=d.$$

By Lemma 2.24, the degree of ψ at x is d. Since $\psi^{-1}(\psi(x))=\{x\}$, the degree of ψ itself is also d. □

LEMMA 3.13. *Let C be a projective smooth connected curve over a field k, let $x_1,...,x_t,y,z_1,...,z_u$ be distinct closed points of C, and let $N,n_1,...,n_t$ be positive integers. There exists a rational function g on C with degree prime to N such that*

(i) *for every $i\in[[1,t]]$, g has a zero of order n_i at x_i;*

(ii) *for every* $j\in[[1,u]]$, g *takes the value* 1 *at* z_j;

(iii) g *has a unique pole, which lies at* y.

Proof. Let d be a positive integer. Consider the divisors

$$D=(d-1)(y)-(n_1+1)(x_1)-...-(n_t+1)(x_t)-(z_1)-...-(z_u)$$

and

$$D'=d(y)-n_1(x_1)-...-n_t(x_t)$$

on C. The cokernel of the natural injection $\mathscr{O}(D)\to\mathscr{O}(D')$ is a skyscraper sheaf $\mathscr{G}$ supported on $\{x_1,...,x_t,y,z_1,...,z_u\}$. Let s be the global section of $\mathscr{G}$ that takes the value 1 at every point of the support.

Choose d big enough so as to have $H^1(C,\mathscr{O}(D))=0$. We may also assume that d is prime to N. Then we have the exact sequence

$$0\longrightarrow H^0(C,\mathscr{O}(D))\longrightarrow H^0(C,\mathscr{O}(D'))\longrightarrow H^0(C,\mathscr{G})\longrightarrow 0.$$

Let s' be an element of $H^0(C,\mathscr{O}(D'))$ that lifts s. The associated rational function satisfies the required properties. □

3.3. Proof of the finiteness property

In this section, we will prove that the map $\boldsymbol{\mathcal{R}}_S(\cdot,(\mathscr{F},\nabla))$ is locally constant outside a locally finite subgraph Γ of X.

By the definition of a triangulation, $X\setminus S$ is a union of open discs and annuli, each of which may be handled by Theorem 3.7. We still need to investigate the behaviour of the multiradius around the points of the triangulation. Let us first remark that, as far as the finiteness property is concerned, it is harmless to change triangulations.

Lemma 3.14. *Let* S *and* S' *be two weak triangulations of* X. *There exists a locally finite subgraph* Γ *of* X *outside which the map* $\boldsymbol{\mathcal{R}}_S(\cdot,\mathscr{F})$ *is locally constant if and only if there exists a locally finite subgraph* Γ' *of* X *outside which the map* $\boldsymbol{\mathcal{R}}_{S'}(\cdot,\mathscr{F})$ *is locally constant.*

Proof. It is possible to construct a triangulation S'' that contains both S and S'. Hence we may assume that $S\subset S'$. In this case, formula (2.1) shows that the property for S implies the property for S'.

Let us assume that there exists a locally finite subgraph Γ' of X outside which the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,\mathscr{F})$ is locally constant. We may assume that Γ' contains $\Gamma_{S'}$. Let U be a connected component of $X\setminus\Gamma'$. It is enough to prove that the map $\boldsymbol{\mathcal{R}}_S(\cdot,\mathscr{F})$ is constant on U. Let V be the connected component of $X\setminus\Gamma_S$ that contains U. Both U and V are discs and the distance function $\varrho_{S,S'}$ (see the paragraph preceding formula (2.1)) is constant on U. Let ϱ be its value and let $r=\mathrm{rk}(\mathscr{F}|_U)$. For any $i\in[[1,r]]$ and any $x\in U$, we have

$$\mathcal{R}_{S',i}(x,\mathscr{F})=\min\left\{\frac{\mathcal{R}_{S,i}(x,\mathscr{F})}{\varrho},1\right\}.$$

We now have two cases. Fix $i\in[[1,r]]$. If there exists $x\in U$ such that $\mathcal{R}_{S,i}(x,\mathscr{F})$ is at least ϱ, then $\mathcal{R}_{S,i}(\cdot,\mathscr{F})$ is constant on U (which is contained in an open disc of relative radius ϱ). Otherwise, the maps $\mathcal{R}_{S,i}(\cdot,\mathscr{F})$ and $\varrho\mathcal{R}_{S',i}(\cdot,\mathscr{F})$ coincide on U, and hence both are constant. □

In our study, we will need to restrict the connection to some subspaces. Unfortunately, the multiradius of convergence may vary under this operation. In the following lemma, we gather a few easy cases where the resulting multiradius may be controlled. Recall that we denote by $x_{c,R}$ the unique point of the Shilov boundary of $D^+(c,R)$.

LEMMA 3.15. *Let x be a point of Γ_S. Let C be an open disc or annulus inside $X\setminus\Gamma_S$ such that $\overline{C}\cap\Gamma_S=\{x\}$.*

(a) *Assume that C is an open disc. Then, for any $y\in C$, we have*

$$\boldsymbol{\mathcal{R}}_{\varnothing}(y,\mathscr{F}|_C)=\boldsymbol{\mathcal{R}}_S(y,\mathscr{F}).$$

(b) *Assume that C is an open annulus. Identify C with an annulus $C^-(0,R_1,R_2)$, with coordinate t, in such a way that $\lim_{R\to R_2}x_{0,R}=x$. Then, for any $y\in C$ and any $i\in[[1,\mathrm{rk}(\mathscr{F}|_C)]]$, we have*

$$\mathcal{R}_{\varnothing,i}(y,\mathscr{F}|_C)=\min\left\{\frac{R_2}{|t(y)|}\mathcal{R}_{S,i}(y,\mathscr{F}),1\right\}.$$

Moreover, if $\mathcal{R}_{\varnothing,i}(x_{0,R},\mathscr{F}|_C)=1$ for all R close enough to R_2, then either $\mathcal{R}_{S,i}(\cdot,\mathscr{F})=1$ on C, or $\mathcal{R}_{S,i}(x_{0,R},\mathscr{F})=R/R_2$ for all R close enough to R_2.

Proof. Assume we are in case (a). The set C is an open disc and the point x lies at its boundary. As a consequence, for any complete valued extension L of K and any L-rational point $\tilde{y}$ of X_L, the disc $D(\tilde{y},S_L)$ is equal to C_L. In particular, the multiradius of convergence of $\mathscr{F}$ on C only depends on the restriction of $\mathscr{F}$ to C and the result follows.

Assume we are in case (b). The connected component of $X\setminus\Gamma_S$ that contains C is an open disc D. We may identify it with $D^-(0,R_2)$, with coordinate t, in a way that is compatible with the identification of C and $C^-(0,R_1,R_2)$. By case (a), restricting the connection to D, endowed with the empty weak triangulation, leaves the radii unchanged.

Consider the weak triangulation $T=\{x_{0,R_1}\}$ of D. Its skeleton is

$$\Gamma_T=\{x_{0,R}\,|\,R_1\leqslant R<R_2\}.$$

Since $\Gamma_C=C\cap\Gamma_T$ and the radii only depend on the skeleton, for every $y\in C$ we have

$$\boldsymbol{\mathcal{R}}_\varnothing(y,\mathscr{F}|_C)=\boldsymbol{\mathcal{R}}_T(y,\mathscr{F}|_D).$$

We may now apply formula (2.1) to compute the right-hand side and the result follows.

Let us now prove the final statement. By case (a), we have $\mathcal{R}_{S,i}(\cdot,\mathscr{F})=\mathcal{R}_{\varnothing,i}(\cdot,\mathscr{F}|_D)$ on D. Let us assume that there exists $R'\in(R_1,R_2)$ such that $\mathcal{R}_{\varnothing,i}(x_{0,R},\mathscr{F}|_C)=1$ for all $R\in(R',R_2)$. This implies that $\mathcal{R}_{\varnothing,i}(x_{0,R},\mathscr{F}|_D)\geqslant R/R_2$ for all $R\in(R',R_2)$. If the latter is an equality everywhere, we are done.

Otherwise, there exists $R''\in(R',R_2)$ such that $\mathcal{R}_{\varnothing,i}(x_{0,R''},\mathscr{F}|_D)>R''/R_2$. We may then write this radius in the form r/R_2, with $r\in(R'',R_2]$. This means that the module $\mathscr{F}$ has a trivial submodule of corank $i-1$ on $D^-(0,r)$, and hence $\mathcal{R}_{\varnothing,i}(\cdot,\mathscr{F}|_D)=r/R_2$ on $D^-(0,r)$. If $r=R_2$, we deduce that $\mathcal{R}_{\varnothing,i}(\cdot,\mathscr{F}|_D)=1$ on D. If $r<R_2$, then we have $\mathcal{R}_{\varnothing,i}(x_{0,R},\mathscr{F}|_D)=R/R_2$ for every $R\in[r,R_2)$. Indeed, if it were not the case, then we would be able to repeat the previous argument with some $R'''\in(r,R_2)$ and show that $\mathcal{R}_{\varnothing,i}(\cdot,\mathscr{F}|_D)=s/R_2$, with $s\in(r,R_2]$, in a neighbourhood of 0 in D. This is a contradiction. □

LEMMA 3.16. *Let C be an open annulus inside $X\setminus S$ such that $\Gamma_C\cap\Gamma_S\neq\varnothing$. Then, we have $\Gamma_C=C\cap\Gamma_S$ and, for any $y\in C$,*

$$\boldsymbol{\mathcal{R}}_\varnothing(y,\mathscr{F}|_C)=\boldsymbol{\mathcal{R}}_S(y,\mathscr{F}).$$

Proof. Let C' be the connected component of $X\setminus S$ that contains C. As $\Gamma_C\cap\Gamma_S\neq\varnothing$, C' is an annulus and we have $\Gamma_{C'}=C'\cap\Gamma_S$. An inclusion of annuli whose skeletons meet induces an inclusion of their skeletons and we deduce that $\Gamma_C=C\cap\Gamma_S$. The result is now proved as case (a) of Lemma 3.15 using Remark 2.32. □

Using Theorem 3.12, we now prove a kind of generic finiteness of the multiradius around a point of the triangulation that is of type 2.

PROPOSITION 3.17. *Let x be a point of S of type 2. There exists an affinoid domain Y_x of X such that $Y_x\cap S=\{x\}$, every connected component of $Y_x\setminus\{x\}$ is an open disc, and for every $i\in[[1,r]]$ the map $\mathcal{R}_{S,i}(\cdot,\mathscr{F})|_{Y_x}$ is locally constant outside a finite subgraph Γ_x of Y_x that contains x.*

Proof. Let us consider a finite étale map $\psi: Y_x \to W_x$ as in Theorem 3.12 (without any branches b_i or c, and with $N=1$). Note that almost every branch out of x belongs to X. It is possible to restrict W_x and Y_x by removing a finite number of connected components of $W_x \setminus \{\psi(x)\}$ and $Y_x \setminus \{x\}$ in order to assume that Y_x is an affinoid domain of X, that $S \cap Y_x = \{x\}$ and that properties (4)–(6) of Theorem 3.12 hold for every, and not only almost every, connected component that appears in their statements. Beware that Y_x will no longer be a neighbourhood of x.

Since ψ is finite étale, we may consider the push-forward $\psi_*(\mathscr{F}, \nabla)|_{Y_x}$ of $(\mathscr{F}, \nabla)|_{Y_x}$ to W_x. By property (5), the subset $T=\{\psi(x)\}$ of W_x is a weak triangulation of W_x.

Let V be a connected component of $Y_x \setminus \{x\}$. By property (6), its image $\psi(V)$ is a connected component of $W_x \setminus \{\psi(x)\}$, and hence it is an open disc. Let $V' = \psi^{-1}(\psi(V))$ and let $V_1 = V, V_2, ..., V_d$ be its connected components. By (4) and (6), for every $i \in [\![1, d]\!]$, V_i is a disc and the induced morphism $\psi_i: V_i \to \psi(V)$ is an isomorphism. By case (a) of Lemma 3.15, for every y in V' we have

$$\boldsymbol{\mathcal{R}}_S(y, \mathscr{F}) = \boldsymbol{\mathcal{R}}_\varnothing(y, \mathscr{F}|_{V'})$$

and, for every z in $\psi(V)$,

$$\boldsymbol{\mathcal{R}}_T(z, \psi_*\mathscr{F}|_{Y_x}) = \boldsymbol{\mathcal{R}}_\varnothing(z, (\psi_*\mathscr{F}|_{Y_x})|_{\psi(V)}) = \boldsymbol{\mathcal{R}}_\varnothing(z, \psi'_*(\mathscr{F}|_{V'})),$$

where we let $\psi': V' \to \psi(V)$ be the induced morphism.

Since ψ' is a trivial cover of degree d, the situation is simple. Actually, the module $\psi'_*(\mathscr{F}|_{V'})$ over $\psi(V)$ splits as $\bigoplus_{i=1}^d \psi_{i*}(\mathscr{F}|_{V_i})$. By [NP1, Proposition 3.5], we deduce that, for every $y \in V$, every component of the multiradius of convergence $\boldsymbol{\mathcal{R}}_S(y, \mathscr{F})$ appears in $\boldsymbol{\mathcal{R}}_T(\psi(y), \psi_*\mathscr{F}|_{Y_x})$.

By Theorem 3.7, the map $\boldsymbol{\mathcal{R}}_T(\cdot, \psi_*\mathscr{F}|_{Y_x})$ is locally constant outside a finite subgraph Γ_x of W_x that contains x. This finite graph only meets a finite number of connected components $U_1, ..., U_n$ of $W_x \setminus \{x\}$. For every $i \in [\![1, n]\!]$, $\Gamma_x \cap (U_i \cup \{\psi(x)\})$ is a finite graph containing x. By property (3), we have $\psi^{-1}(\psi(x)) = \{x\}$ and, by property (6), the induced morphism $\psi^{-1}(U_i) \to U_i$ is a trivial cover. We deduce that $\psi^{-1}(\Gamma_x \cap (U_i \cup \{x\}))$ is a finite graph, and hence so is $\psi^{-1}(\Gamma_x)$. We will prove that the map $\boldsymbol{\mathcal{R}}_S(\cdot, \mathscr{F})$ is locally constant on its complement.

Let U be a connected open subset of $Y_x \setminus \psi^{-1}(\Gamma_x)$. Then U is contained in a connected component D of $Y_x \setminus \{x\}$. By property (4), D is an open disc that contains no point of S and the point x lies at its boundary. By Lemma 3.15 (a) and Corollary 3.8, the map $\boldsymbol{\mathcal{R}}_S(\cdot, \mathscr{F})$ is continuous on D, and hence also on U.

For every point $y \in U$, the components of the r-tuple $\boldsymbol{\mathcal{R}}_S(y, \mathscr{F})$ are equal to some of the components of the dr-tuple $\boldsymbol{\mathcal{R}}_T(\psi(y), \psi_*\mathscr{F})$. Since $\boldsymbol{\mathcal{R}}_T(\cdot, \psi_*\mathscr{F})$ is constant on $\psi(U)$, there are only finitely many possible such values. We deduce that the restriction of the map $\boldsymbol{\mathcal{R}}_S(\cdot,\mathscr{F})$ to U is continuous with values in a finite set, and hence is constant. □

Remark 3.18. Every connected component of $Y_x\setminus\{x\}$ is a connected component of $X\setminus S$.

Let x be a point of S of type 2. The affinoid domain Y_x of the previous corollary contains entirely almost every branch out of x. We still need to prove the finiteness property on the remaining branches. This is the object of the following proposition (where we also handle points of type 3).

PROPOSITION 3.19. *Let x be a point of S and b be a branch out of it. There exists an open annulus $C_{x,b}$ that is a section of b and satisfies $\overline{C}_{x,b}\cap S=\{x\}$. There also exists a finite subgraph $\Gamma_{x,b}$ of $\overline{C}_{x,b}$ such that the map $\boldsymbol{\mathcal{R}}_\varnothing(\cdot,\mathscr{F}|_{C_{x,b}})$ is locally constant outside $\Gamma_{x,b}\cap C_{x,b}$.*

Proof. By [D6, Théorème 4.3.5], any point of type 3 has a neighbourhood that is isomorphic to a closed annulus. Hence, around such a point, we may conclude by Theorem 3.7.

Let us now assume that x is a point of type 2. The proof will closely follow that of Proposition 3.17. Let us consider a finite étale map $\psi: Y\to W$ as in Theorem 3.12 with $b_1=b$ (with no branches b_i or c, and with $N=1$). Let $C_{x,b}$ be an open annulus that is a section of b, that satisfies $\overline{C}_{x,b}\cap S=\{x\}$ and $\psi^{-1}(\psi(C_{x,b}))\subseteq X$, and is such that the induced map $\psi_0: C_{x,b}\to\psi(C_{x,b})$ is an isomorphism. We may restrict W and Y by removing a finite number of connected components of $W\setminus\{\psi(x)\}$ and $Y\setminus\{x\}$ in order to assume that Y is an affinoid domain of X containing $\psi^{-1}(\psi(C_{x,b}))\cup\{x\}$ (but not necessarily a neighbourhood of x anymore). Let us consider the push-forward $\psi_*(\mathscr{F},\nabla)$ of the connection $(\mathscr{F},\nabla)$ to W. We endow W with a weak triangulation T such that

$$T\cap\psi(\overline{C}_{x,b})=\partial\psi(C_{x,b})=\psi(\partial C_{x,b}).$$

Arguing as in the proof of Proposition 3.17, we show that, for every $z\in\psi(C_{x,b})$, we have

$$\boldsymbol{\mathcal{R}}_T(z,\psi_*\mathscr{F})=\boldsymbol{\mathcal{R}}_\varnothing(z,\psi'_*(\mathscr{F}|_{P_{x,b}})),$$

where $\psi': P_{x,b}=\psi^{-1}(\psi(C_{x,b}))\to\psi(C_{x,b})$ is the induced morphism.

The subset $P_{x,b}$ has several connected components, one of which is $C_{x,b}$. We deduce that $(\psi_0)_*(\mathscr{F}|_{C_{x,b}})$ is a direct factor of $\psi'_*(\mathscr{F}|_{P_{x,b}})$. Since ψ_0 is an isomorphism, we have that, for every $y\in C_{x,b}$, every component of the multiradius $\boldsymbol{\mathcal{R}}_\varnothing(y,\mathscr{F}|_{C_{x,b}})$ appears in $\boldsymbol{\mathcal{R}}_T(\psi(y),\psi_*\mathscr{F})$. By Theorem 3.7, the map $\boldsymbol{\mathcal{R}}_T(\cdot,\psi_*\mathscr{F})$ is locally constant outside a finite subgraph Γ of W. Using an argument of continuity as in the last paragraph of the proof of Proposition 3.17, we deduce that the map $\boldsymbol{\mathcal{R}}_\varnothing(\cdot,\mathscr{F}|_{C_{x,b}})$ is locally constant outside $\psi^{-1}(\Gamma)$. Note that $\psi^{-1}(\Gamma)$ is a finite subgraph of $\overline{C}_{x,b}$. □

Remark 3.20. In the situation of Proposition 3.19, let C be an open annulus that is a section of b. The coefficients of the matrix of the connection on C converge in a neighbourhood of $\overline{C}$ in X. If X were an affinoid domain of the affine line, we would deduce that these coefficients are analytic elements on C (in the sense of [K1, §8.5]) and then conclude by [NP1, Corollary 3.10 (i)]. Unfortunately, in the general case, such functions do not give rise to analytic elements.

We may now conclude the proof of the finiteness property. By Lemma 3.14, we may enlarge the weak triangulation S into a triangulation in the sense of Ducros, which means that every connected component of $X\setminus S$ is relatively compact.

Let $S_{[2]}$ (resp. $S_{[3]}$) be the set of points of S that are of type 2 (resp. 3). With every $x\in S_{[2]}$, we associate an affinoid domain Y_x of X containing x by Proposition 3.17. Let $b_{x,1},...,b_{x,n(x)}$ be the branches out of x that do not belong to Y_x. For every $x\in S_{[3]}$, we set $Y_x=\{x\}$ and denote by $b_{x,1},...,b_{x,n(x)}$ the branches out of x (and hence $n(x)\leqslant 2$).

With every $x\in S$ and every $i\in[[1,n(x)]]$, we associate an open annulus $C_{x,b_{x,i}}$ by Proposition 3.19. We may shrink the annuli $C_{x,b_{x,i}}$ in order to assume that they do not overlap. We now enlarge the triangulation S of X into a triangulation S' such that every annulus $C_{x,b_{x,i}}$ is a connected component of $X\setminus S'$. By Lemma 3.14 again, this does not affect the result we want to prove.

For every $x\in S_{[2]}$, we have constructed a finite subgraph Γ_x of Y_x outside which the map $\boldsymbol{\mathcal{R}}_S(\cdot,\mathscr{F})|_{Y_x}$, and hence also the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,\mathscr{F})|_{Y_x}$, is locally constant. Set

$$\Gamma_Y=\bigcup_{x\in S_{[2]}}\Gamma_x.$$

Every point of X has a neighbourhood that meets at most one of the Y_x, and hence Γ_Y is locally finite.

For every $x\in S$ and every $i\in[[1,n(x)]]$, we have constructed a finite subgraph $\Gamma_{x,b_{x,i}}$ of $\overline{C}_{x,b_{x,i}}$ such that the map $\boldsymbol{\mathcal{R}}_\varnothing(\cdot,\mathscr{F}|_{C_{x,b_{x,i}}})=\boldsymbol{\mathcal{R}}_{S'}(\cdot,\mathscr{F})|_{C_{x,b_{x,i}}}$ (see Lemma 3.16) is locally constant outside $\Gamma_{x,b_{x,i}}\cap C_{x,b_{x,i}}$. Set

$$\Gamma_C=\bigcup_{\substack{x\in S\\ 1\leqslant i\leqslant n(x)}}\Gamma_{x,b_{x,i}}.$$

Every point of X has a neighbourhood that meets only a finite number of the $C_{x,b_{x,i}}$ (and actually at most one for points in $X\setminus S$), and hence Γ_C is locally finite.

Let $\mathscr{E}$ be the set of connected components of $X\setminus\bigcup_{x\in S} Y_x$, and let $E\in\mathscr{E}$. By Remark 3.18, E is a connected component of $X\setminus S$. Assume that E is a disc. Since S is a triangulation (and not only a weak triangulation), there exists a point $x\in S$ that lies at the boundary of E. Let b be the branch out of x that is defined by E. Then b does not belong to any $Y_{x'}$ with $x'\in S_{[2]}$, and hence we can consider the annulus $C_{x,b}$. By assumption, the boundary point $z_{x,b}$ of $C_{x,b}$ inside E belongs to S'. The complement Z_E of $C_{x,b}$ in E is a closed disc with boundary point $z_{x,b}$. In particular, $\{z_{x,b}\}$ is a triangulation of Z_E. By Theorem 3.7 there exists a finite subgraph Γ_E of Z_E such that the restriction of the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,\mathscr{F})$ to Z_E is locally constant outside Γ_E.

If E is an annulus, we argue in a similar way to define a closed subannulus Z_E of E whose complement is the union of two $C_{x,b}$'s and a finite subgraph Γ_E of Z_E such that the restriction of the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,\mathscr{F})$ to Z_E is locally constant outside Γ_E. Set

$$\Gamma_{\mathscr{E}}=\bigcup_{E\in\mathscr{E}}\Gamma_E.$$

Every point of X has a neighbourhood that meets at most one of the Z_E, and hence $\Gamma_{\mathscr{E}}$ is locally finite.

On the whole, the set $\Gamma=\Gamma_Y\cup\Gamma_C\cup\Gamma_{\mathscr{E}}$ is a locally finite subgraph of X. Moreover, since X is covered by the union of the Y_x, the $C_{x,b}$ and the Z_E, the map $\boldsymbol{\mathcal{R}}_{S'}(\cdot,\mathscr{F})$ is locally constant on $X\setminus\Gamma$.

3.4. Proof of continuity and log-linearity at points of the skeleton

We will now prove the continuity and log-linearity at points of the skeleton in a given direction. Let us begin with the case of points of type 3, which is easier.

LEMMA 3.21. *Let x be a point of Γ_S of type 3 and let $[x,y]$ be a segment with initial point x. For every $i\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$, the restriction of the map $\mathcal{R}_{S,i}(\cdot,\mathscr{F})$ to the segment $[x,y]$ is continuous at the point x and log-linear in the neighbourhood of x with slope of the form m/j with $m\in\mathbf{Z}$ and $j\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$.*

Proof. By [D6, Théorème 4.3.5], the point x has a neighbourhood that is isomorphic to an annulus. If $\Gamma_S\cap[x,y]\neq\varnothing$, then there exist $z\in(x,y]$ and a closed annulus C in X such that $[x,z]=\Gamma_S\cap C$ is the skeleton of C. Since the radii only depend on the skeleton, we have $\mathcal{R}_{S,i}(\cdot,\mathscr{F})|_C=\mathcal{R}_{\{x,z\},i}(\cdot,\mathscr{F}|_C)$ and the results we want now follow from Theorem 3.7.

If $\Gamma_S\cap[x,y]=\varnothing$, then there exists a closed disc D in X, with boundary point x, that contains y and is such that $D\cap S=\{x\}$. We may then conclude as before. □

We will now turn to the case of points of type 2. Let us first state a result that relates the multiradius of convergence after push-forward by an étale map to the original multiradius of convergence. Recall that K is assumed to be algebraically closed.

LEMMA 3.22. *Let Z be a quasi-smooth K-analytic curve. Let $\psi: X \to Z$ be a finite morphism. Let $x \in X$ be a point of type 2 or 3. Assume that $d = [\mathscr{H}(x) : \mathscr{H}(\psi(x))]$ is prime to p.*(3) *Let L be an algebraically closed complete valued extension of $\mathscr{H}(x)$. Then every connected component of $\pi_L^{-1}(\psi(x)) \setminus \{\psi(x)_L\}$ is a disc over which the morphism ψ_L induces a trivial cover of degree d.*

Proof. By Theorem 2.15 (1), the point x_L has type 2 or 3. Moreover, by the definition of x_L (see Definition 2.8), the norm associated with x_L is induced by the tensor norm on $\mathscr{H}(x) \widehat{\otimes}_K L$. Hence

$$\sqrt{|\mathscr{H}(x_L)^*|} = \sqrt{\|\mathscr{H}(x) \widehat{\otimes}_K L \setminus \{0\}\|} = \sqrt{|L^*|},$$

since L contains $\mathscr{H}(x)$. We deduce that the point x_L is of type 2. As a consequence, the point $\psi_L(x_L)$, where $\psi_L: X_L \to Z_L$ is the morphism induced by ψ, is also of type 2.

Recall that, by Lemma 2.9, we have $\psi_L(x_L) = \psi(x)_L$ and that, by Theorem 2.15, every connected component of $\pi_L^{-1}(\psi(x)) \setminus \{\psi(x)_L\}$ is a disc.

Let $\mathscr{C}$ and $\mathscr{D}$ be the residual curves at x_L and $\psi(x)_L$, respectively. Using the stability of the field $\mathscr{H}(\psi(x)_L)$ and arguing as in the proof of Theorem 3.12, we show that the morphism $\tilde{\psi}_L: \mathscr{C} \to \mathscr{D}$ induced by ψ has degree d. Since d is prime to p and $\tilde{L}$ is algebraically closed, $\tilde{\psi}_L$ is generically étale, and hence unramified at almost every closed point. By [D6, Théorème 4.3.13], we deduce that ψ_L has degree 1 on almost every branch out of x_L and the result follows for almost every connected component of $\pi_L^{-1}(\psi(x)) \setminus \{\psi(x)_L\}$ (see the proof of Theorem 3.12 for more details).

It remains to show that the result holds for every connected component of the space $\pi_L^{-1}(\psi(x)) \setminus \{\psi(x)_L\}$. Let C be one of them. If L is maximally complete, then, by Corollary 2.18, there exists $\sigma \in \mathrm{Gal}^c(L/K)$ that maps C onto a connected component over which ψ_L induces a trivial cover of degree d and the result follows.

Otherwise, we embed L into a field M that is algebraically closed and maximally complete. Let e be the number of connected components of $\psi_L^{-1}(C)$. It is enough to prove that $e = d$. Since L is algebraically closed, these connected components are geometrically connected (see [D3, Théorème 7.11]) and $\pi_{M/L}^{-1}(\psi_L^{-1}(C)) = \psi_M^{-1}(\pi_{M/L}^{-1}(C))$ still has e connected components.

Finally, note that $\pi_{M/L}^{-1}(C)$ is a connected component of $\pi_M^{-1}(\psi(x)) \setminus \{\psi(x)_M\}$, and hence ψ_M induces a trivial cover of degree d over it, by the previous arguments. It follows that $e = d$. □

(3) If $\widetilde{K}$ has characteristic 0, then $p = 1$ and this condition is always satisfied.

Lemma 3.23. *Let Z be a quasi-smooth K-analytic curve endowed with a weak triangulation T. Let $\psi\colon X\to Z$ be a finite étale morphism and let $x\in\Gamma_S\cap\psi^{-1}(\Gamma_T)$. Assume that the degree $d=[\mathscr{H}(x)\colon\mathscr{H}(\psi(x))]$ is prime to p. Then, for every $i\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$ and $j\in[[1,d]]$, we have*

$$\mathcal{R}_{T,d(i-1)+j}(\psi(x),\psi_*(\mathscr{F},\nabla))=\mathcal{R}_{S,i}(x,(\mathscr{F},\nabla)).$$

Proof. Since $x\in\Gamma_S$ and $\psi(x)\in\Gamma_T$, the radii of convergence at these points only depend on the restrictions of the connections to these points (see Remark 2.32). Let L be an algebraically closed complete valued extension of $\mathscr{H}(x)$. By the previous lemma, every connected component of $\pi_L^{-1}(\psi(x))\setminus\{\psi(x)_L\}$ is a disc over which the morphism ψ_L induces a trivial cover of degree d.

Let W be such a connected component and $\tilde{y}$ be an L-rational point of it. Letting $\psi_L^{-1}(\tilde{y})=\{\tilde{x}_1,\ldots,\tilde{x}_d\}$ and arguing as in the proof of Proposition 3.17, we show that $\boldsymbol{\mathcal{R}}_{T_L}(\tilde{y},\psi_{L*}\pi_L^*\mathscr{F})$ is obtained by concatenating and reordering the tuples

$$\boldsymbol{\mathcal{R}}_{S_L}(\tilde{x}_1,\pi_L^*\mathscr{F}),\quad \ldots,\quad \boldsymbol{\mathcal{R}}_{S_L}(\tilde{x}_d,\pi_L^*\mathscr{F}).$$

We have $\pi_L(\tilde{y})=\psi(x)$ and, for any $i\in[[1,d]]$, $\pi_L(\tilde{x}_i)=x$. From Lemma 2.39, we deduce that $\boldsymbol{\mathcal{R}}_{T_L}(\tilde{y},\psi_{L*}\pi_L^*\mathscr{F})=\boldsymbol{\mathcal{R}}_T(\psi(x),\psi_*\mathscr{F})$ and $\boldsymbol{\mathcal{R}}_{S_L}(\tilde{x}_i,\pi_L^*\mathscr{F})=\boldsymbol{\mathcal{R}}_{S_L}(x,\mathscr{F})$ for any $i\in[[1,d]]$. The result follows. ☐

Proposition 3.24. *Let x be a point of Γ_S of type 2 and $[x,y]$ be a segment with initial point x. For every $i\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$, the restriction of the map $\mathcal{R}_{S,i}(\cdot,\mathscr{F})$ to the segment $[x,y]$ is continuous at the point x and log-linear in the neighbourhood of x with slope of the form m/j with $m\in\mathbf{Z}$ and $j\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$.*

Proof. Up to shrinking $[x,y]$, we may assume that the open interval (x,y) is the skeleton Γ_C of an open annulus C. Let c be the branch out of x associated with C.

Let us now consider a finite étale map $\psi\colon Y\to W$ that satisfies the conclusions of Theorem 3.12 with c as above and $N=p$ (and no branches b_i). In particular, we have $\psi^{-1}(\psi(c))=\{c\}$ and the degree d of ψ is prime to p. Replacing C by a subannulus, we may assume that $C\subseteq Z$, that $S\cap C=\varnothing$ and that, for every $z\in\Gamma_C$, $\psi^{-1}(\psi(z))=\{z\}$, and hence $[\mathscr{H}(z)\colon\mathscr{H}(\psi(z))]=d$, by Lemma 2.24.

Since $\psi^{-1}(\psi(C))=C$ and $\psi^{-1}(\psi(x))=\{x\}$, we may restrict W and Y by removing a finite number of connected components of $W\setminus\{\psi(x)\}$ and $Y\setminus\{x\}$ in order to assume that Y is an affinoid domain of X containing $C\cup\{x\}$ (but not necessarily a neighbourhood of x anymore). Let us endow W with a triangulation T whose skeleton contains $\psi(\Gamma_C)$ and $\psi(x)$.

By Lemma 3.23, for any $i\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$, we have

$$\mathcal{R}_{S,i}(x,\mathscr{F})=\mathcal{R}_{T,di}(\psi(x),\psi_*\mathscr{F}).$$

We will now distinguish two cases. First assume that $\Gamma_C\cap\Gamma_S\neq\varnothing$. By Lemmas 3.16 and 3.23, for any $z\in\Gamma_C$ and any $i\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$ we have

$$\mathcal{R}_{S,i}(z,\mathscr{F})=\mathcal{R}_{T,di}(\psi(z),\psi_*\mathscr{F}).$$

By Theorem 3.7, the map $\mathcal{R}_{T,di}(\cdot,\psi_*\mathscr{F})$ is continuous on W and piecewise log-linear on every segment inside it with slopes as requested. The result follows.

Let us now assume that $\Gamma_C\cap\Gamma_S=\varnothing$. By Lemma 3.23 and case (b) of Lemma 3.15, whose notation we borrow, for any $i\in[[1,\mathrm{rk}(\mathscr{F}_x)]]$ and any $R\in(R_1,R_2)$, we have

$$\mathcal{R}_{T,di}(\psi(x_{0,R}),\psi_*\mathscr{F})=\mathcal{R}_{\varnothing,i}(x_{0,R},\mathscr{F}|_C)=\min\left\{\frac{R_2}{R}\,\mathcal{R}_{S,i}(x_{0,R},\mathscr{F}),1\right\}.$$

If $\mathcal{R}_{T,di}(\psi(x_{0,R}),\psi_*\mathscr{F})<1$ for all R close enough to R_2, then we have

$$\mathcal{R}_{T,di}(\psi(x_{0,R}),\psi_*\mathscr{F})=\frac{R_2}{R}\mathcal{R}_{S,i}(x_{0,R},\mathscr{F})$$

and we conclude as before. Otherwise, the set of R such that $\mathcal{R}_{T,di}(\psi(x_{0,R}),\psi_*\mathscr{F})=1$ contains R_2 in its closure. There exists $R'\in(R_1,R_2)$ such that $\mathcal{R}_{T,di}(\cdot,\psi_*\mathscr{F})$ is log-linear on $[\psi(x_{0,R'}),\psi(x)]$, which implies that we actually have $\mathcal{R}_{T,di}(\psi(x_{0,R}),\psi_*\mathscr{F})=1$ for all R close enough to R_2. The result then follows from the last statement of Lemma 3.15 using the fact that

$$\mathcal{R}_{S,i}(x,\mathscr{F})=\mathcal{R}_{T,di}(\psi(x),\psi_*\mathscr{F})=1. \qquad\square$$

3.5. Conclusion of the proof

In §3.3, we proved that there exists a locally finite subgraph Γ of X outside which the map $\boldsymbol{\mathcal{R}}_S$ is locally constant. As a consequence, to prove that $\boldsymbol{\mathcal{R}}_S$ is continuous, it is now enough to prove that it is continuous at the initial point x of every closed interval $[x,y]$ inside X. If x belongs to Γ_S, this was done in Lemma 3.21 and Proposition 3.24. Otherwise, x belongs to an open disc and we may use Corollary 3.8. This proves property (i) of Theorem 3.6.

Properties (ii) and (iii), which deal with the piecewise log-linearity and the form of the slopes, are dealt with in the same way.

Finally, remark that it is possible to enlarge Γ into a locally finite subgraph Γ' of X such that $X\setminus\Gamma'$ is a disjoint union of relatively compact open discs. In this case, by Remark 2.4, there exists a natural continuous retraction $X\to\Gamma'$. The map $\boldsymbol{\mathcal{R}}_S$ factorises by it and Theorem 3.6 is now proved.

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