A SUBSPACE THEOREM FOR SUBVARIETIES

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Abstract. In this paper, we establish a height inequality, in terms of an (ample) line bundle, for a sum of subschemes located in ℓ-sub-general position in an algebraic variety, which extends the main result of McKinnon and Roth in [14]. The inequality obtained in this paper connects the result of McKinnon and Roth [14] (the case when the subschemes are points) and the results of Corvarja-Zannier [4], Evertse-Ferretti [7], Ru [19], and Ru-Vojta [20] (the case when the subschmes are divisors). Furthermore, our approach gives an alternative short and simpler proof of the main result in [14].

1. Introduction and Statements

In their recent Invent. Math. paper (see [14]), McKinnon and Roth introduced the approximation constant $\alpha_x(L)$ to an algebraic point x on an algebraic variety V with an ample line bundle bundle L. The invariant $\alpha_x(L)$ measures how well x can be approximated by rational points on X with respect to the height function associate to L. They showed that $\alpha_x(L)$ is closely related to the Seshadri constant $\epsilon_x(L)$ measuring the local positivity of L at x. They also showed that the invariant $\alpha_x(L)$ can be computed through another invariant $\beta_x(L)$ in the height inequality they established (see Theorem 5.1 and Theorem 6.1) in [14]. By computing the Seshadri constant $\epsilon_x(L)$ for the case of $V = \mathbb{P}^1$, their result recovers the Roth's theorem, so the height inequality they established can be viewed as the generalization of the Roth's theorem to arbitrary projective varieties.

In this short note, we give such results a short and simpler proof. Furthermore, we extend the results from the points of a projective variety to subschemes. The generalized result in terms of subschemes connects, as well as gives a clearer explanation, the above mentioned result of McKinnon and Roth [14] with the recent Diophantine approximation results in term of the divisors obtained by Corvarja-Zannier [4], Evertse-Ferretti [7] , Ru [19], and Ru-Vojta [20].

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We now state our result. Let V be a projective variety defined over a number field k.

Definition 1.1. *Let Y be a closed subscheme of V and $\mathfrak{I}_Y$ the ideal sheaf of Y. Let $\mathcal{L}$ be a line sheaf on V with $h^0(V,\mathcal{L}^N)\geq 1$ for N big enough. We define*

$$\beta_{\mathcal{L},Y}:=\limsup_{N\to\infty}\frac{\sum_{m=1}^{\infty}h^0(V,\mathcal{L}^N\otimes\mathfrak{I}_Y^m)}{N\cdot h^0(V,\mathcal{L}^N)}.$$

Remark 1.2. (a) Let Y be a closed subscheme of V of codimension at least two and $\pi:\tilde{V}\to V$ be the blow-up along Y, and E be the exceptional divisor. Let L be a line bundle over V with $h^0(V,NL)\geq 1$ for N big enough and $\mathcal{L}$ be the corresponding line sheaf. Then

$$\beta_{L,Y}:=\limsup_{N\to\infty}\frac{\sum_{m=1}^{\infty}h^0(\tilde{V},N\tilde{\pi}^*L-mE)}{N\cdot h^0(V,NL)}=\beta_{\mathcal{L},Y}.$$

(b) Let D be an effective divisor on V, we define $\beta_{D,Y}:=\beta_{\mathcal{O}(D),Y}$, where $\mathcal{O}(D)$ is the line sheaf associated to D.

Definition 1.3. *We say that close subschemes $Y_1,\cdots,Y_q$ of a projective variety V are in ℓ-sub-general position if, for any $x\in V$, there are at most ℓ subschemes among $Y_1,\dots,Y_q$ which contain x.*

Remark 1.4. In the case $Y_1=y_1,\dots,Y_q=y_q$ are the points (this is what McKinnon and Roth dealt with in [14]), the condition that $y_1,\dots,y_q$ *are distinct* implies that $Y_1,\dots,Y_q$ *are in* 1*-sub-general position (i.e with* $\ell=1$*).*

We establish the following result.

Main Theorem. *Let k be a number field and M_k be the set of places on k. Let $S\subset M_k$ be a finite subset containing all archimedean places. Let V be a projective variety defined over k and $Y_1,\cdots,Y_q$ be closed subschemes of V defined over k in ℓ-sub-general position. For any $v\in S$, choose a local Weil function $\lambda_{Y_j,v}$ for each $Y_j, 1\leq j\leq q$. Let $\mathcal{L}$ be a line sheave with $h^0(V,\mathcal{L}^N)\geq 1$ for N big enough. Then for any $\epsilon>0$*

$$\sum_{v\in S}\sum_{i=1}^{q}\lambda_{Y_i,v}(x)\leq\ell(\max_{1\leq i\leq q}\{\beta_{\mathcal{L},Y_i}^{-1}\}+\epsilon)h_{\mathcal{L}}(x)\tag{1.1}$$

holds for all x outside a proper Zariski-closed subset Z of $V(k)$.

The main result of McKinnon and Roth in [14] easily follows from the above main theorem (with subschemes being the points) as the following Corollary and the proof will be given in the last section.

Corollary 1.5 (cf. Theorem 6.1 in [14])**.** *Let V be a projective variety over k. Then for any ample line bundle L and any $x \in V(\bar{k})$ either*

(a) $\alpha_x(L) \geq \beta_{L,x}$ *or*
(b) *There exists a proper subvariety $Z \subset V$, irreducible over $\bar{k}$, with $x \in Z(\bar{k})$ so that $\alpha_{x,V}(L) = \alpha_{x,Z}(L|Z)$, i.e. "$\alpha_x(L)$ is computed on a proper subvariety of V",*

where $\alpha_x(L)$ is the approximation constant defined in [14, Definition 2.8]*, and $\beta_{L,x}$ is defined in Definition 1.2 (with Y taken as a point x).*

We will show in Lemma 2.2 that for any line bundle L, $x \in V$

$$\beta_{L,x} \geq \frac{n}{n+1}\epsilon_x(L), \tag{1.2}$$

where $n = \dim V$. We note that the Seshadri constant $\epsilon_x(L)$ does not decrease when restricting to a subvariety (see [14, Proposition 3.4]), so we can use induction to further get, from Corollary 1.5 and Eq. (1.2), the following result.

Corollary 1.6 (cf. Theorem 6.2, alternative statement in [14])**.** *Let V be a projective variety over k. For any ample line bundle L and choose $x \in X(\bar{k})$. Then for any $\delta > 0$, thre are only finitely many solutions $y \in X(k)$ to*

$$d_v(x,y) < H_L(y)^{-\left(\frac{n+1}{n\epsilon_x(L)}+\delta\right)}.$$

In the case when $V = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(1)$, we have $\epsilon_x(L) = 1$ for all $x \in \mathbb{P}^n$ (see [14, Lemma 3.3]). Therefore the above result generalizes the theorem of Roth.

We now turn to another extreme case when the subschemes $Y_1, \ldots, Y_q$ are divisors $D_1, \ldots, D_q$. Let $D := D_1 + \cdots + D_q$. Assume that each D_j is linearly equivalent to a fixed ample divisor A. Then we have the following relation of height functions $h_D = qh_A + O(1)$. On the other hand, by the Riemann-Roch theorem, with $n = \dim V$,

$$h^0(ND) = h^0(qNA) = \frac{(qN)^n A^n}{n!} + o(N^n)$$

and

$$h^0(ND - mD_j) = h^0((qN-m)A) = \frac{(qN-m)^n A^n}{n!} + o(N^n).$$

Thus

$$\sum_{m\geq 1} h^0(ND-mD_j)=\frac{A^n}{n!}\sum_{l=0}^{qN-1} l^n+o(N^{n+1})=\frac{A^n(qN-1)^{n+1}}{(n+1)!}+o(N^{n+1}).$$

Hence

$$\beta_{D,D_j}=\lim_{N\to\infty}\frac{\frac{A^n(qN-1)^{n+1}}{(n+1)!}+o(N^{n+1})}{N\frac{(qN)^nA^n}{n!}+o(N^{n+1})}=\frac{q}{n+1}.$$

Thus the Main Theorem, together with the above computation, implies the following result of Chen–Ru-Yan [3] (see also [5]).

Theorem 1.7 ([3] or [5]). *Let k be a number fields and M_k is the set of places on k. Let $S\subset M_k$ be a subset. Let V be a projective variety of dimension n defined over k. Let $D_1,\cdots,D_q$ be effective Cartier divisors in ℓ-sub-general position. Assume that each $D_j, 1\leq j\leq q$, is linearly equivalent to a fixed ample divisor A. For any $v\in S$, choose a Weil function $\lambda_{D_j,v}$ for each $D_j, 1\leq j\leq q$. Then for any $\epsilon>0$*

$$\sum_{v\in S}\sum_{i=1}^{q}\lambda_{D_i,v}(x)\leq \ell(n+1+\epsilon)h_A(x) \tag{1.3}$$

holds for all x outside a proper Zariski-closed subset Z of $V(k)$. In particular, if $D_1,\ldots,D_q$ are in general position on V, then the inequality

$$\sum_{v\in S}\sum_{i=1}^{q}\lambda_{D_i,v}(x)\leq n(n+1+\epsilon)h_A(x) \tag{1.4}$$

holds for all but finitely many $x\in V(k)$.

In the general case when $D_1,\ldots,D_q$ are only assumed to be big and nef, we can also compute β_{D,D_j}. The details will be carried out in the next section.

We note that recently the first named author and P. Vojta [20] obtained the following sharp result in the case when $D_1,\ldots,D_q$ in general position and when V is Cohen-Macaulay (for example V is smooth).

Theorem 1.8 (Ru-Vojta). *Let k be a number fields and M_k is the set of places on k. Let $S\subset M_k$ be a finite subset. Let V be a projective variety defined over k. Assume that V is Cohen-Macaulay (for example V is smooth). Let $D_1,\cdots,D_q$ be effective Cartier divisors in general position. For any $v\in S$, choose a Weil function $\lambda_{D_j,v}$ for each $Y_j, 1\leq j\leq q$. Let L be a line bundle on V with $h^0(V,NL)\geq 1$ for N big enough. Then for any $\epsilon>0$*

$$\sum_{v\in S}\sum_{i=1}^{q}\lambda_{D_i,v}(x)\leq (\max_{1\leq i\leq q}\{\beta_{L,D_i}^{-1}\}+\epsilon)h_L(x) \tag{1.5}$$

holds for all x outside a proper Zariski-closed subset Z of $V(k)$.

The above theorem, together with the above computation, recovers the result of Evertse-Ferretti(cf. [6] and[7]) in the case when V is smooth.

2. Computation of the constant $\beta_{L,Y}$

We first compute the constant $\beta_{L,y}$, i.e. $Y = y$ is a point in $V(k)$. The following lemma is a reformulation of Lemma 4.1 in [14].

Lemma 2.1. *Let V be a projective variety and x be a point in V. Let $\pi : \tilde{V} \to V$ be the blow-up along x, and E be the exceptional divisor. Let L be an ample line bundle L and m a positive integer. Then*

(i) $h^0(\tilde{V}, N\pi^*L - mE) = 0$ *if* $m > N \cdot \gamma_{\mathrm{eff},x}$*, where* $\gamma_{\mathrm{eff},x}$ *is defined in* [14].
(ii) $h^0(\tilde{V}, N\pi^*L - mE) \geq h^0(V, NL) - \frac{\mathrm{mult_x V}}{n!} m^n + O(N^{n-1})$ *for* $N >> 0$

Proof. Write $h^0(\tilde{V}, N\pi^*L - mE) = h^0(\tilde{V}, N\pi^*L - N \cdot \gamma E)$, where $\gamma = m/N$. The argument in [14] shows that $h^0(\tilde{V}, N\pi^*L - mE) \geq h^0(V, NL) - \frac{\mathrm{mult_x V}}{n!} m^n + O(N^{n-1})$. □

The following is a restatement of Corollary 4.2 in [14].

Lemma 2.2. *For any ample line bundle L, $x \in V$ and positive integer m, we have*

$$\beta_{L,x} \geq \frac{n}{n+1}\left(\frac{L^n}{\mathrm{mult_x V}}\right)^{\frac{1}{n}} \geq \frac{n}{n+1}\epsilon_x(L).$$

Proof. Choose a sufficiently large N. By Lemma 2.1 and the Riemann-Roch theorem,

$$h^0(\tilde{V}, \tilde{\pi}^*NL - mE) \geq h^0(V, NL)\left(1 - \frac{\mathrm{mult_x V}}{L^n}\left(\frac{m}{N}\right)^n\right) + O(N^{n-1}). \tag{2.1}$$

Moreover, the right hand side of (2.1) is less than zero when $m > u = [N(\frac{L^n}{\mathrm{mult_x V}})^{\frac{1}{n}}]$, and hence

$$\sum_{m=1}^{\infty} h^0(\tilde{V}, \tilde{\pi}^*NL - mE) \geq h^0(V, NL) \sum_{m=1}^{u}\left(1 - \frac{\mathrm{mult_x V}}{L^n}\left(\frac{m}{N}\right)^n\right) + O(N^n). \tag{2.2}$$

Consequently,

$$\begin{aligned}\beta_{L,x} &\geq \frac{1}{N}\sum_{m=1}^{u}(1-\frac{\mathrm{mult_x V}}{L^n}(\frac{m}{N})^n)+O(\frac{1}{N})\\ &= \frac{1}{N}(u-\frac{\mathrm{mult_x V}}{L^n}\cdot\frac{u^{n+1}}{(n+1)N^n})+O(\frac{1}{N})\\ &\geq \frac{nu}{(n+1)N}+O(\frac{1}{N}).\end{aligned}$$

Let N run through all sufficiently large integers. Then we have

$$\beta_{L,x} \geq \frac{n}{n+1}(\frac{L^n}{\mathrm{mult_x V}})^{\frac{1}{n}}. \tag{2.3}$$

□

Next we consider the case when $Y_j = D_j, 1 \leq j \leq q$, are effective big and nef Cartier divisors on V.

Definition 2.3. *Suppose that X is a complete variety of dimension n. Let $D_1, \ldots, D_q$ be effective Cartier divisors on X, and let $D = D_1 + D_2 + \cdots + D_q$. We say that D* ***has equi-degree with respect to*** *$D_1, D_2, \ldots, D_q$ if $D_i.D^{n-1} = \frac{1}{q}D^n$ for all $i = 1, \ldots, q$.*

Lemma 2.4 (Lemma 9.7 in [13])**.** *Let V be a projective variety of dimension n. If $D_j, 1 \leq j \leq q$, are big and nef Cartier divisors, then there exist positive real numbers r_j such that $D = \sum_{j=1}^{q} r_j D_j$ has equi-degree.*

Since divisors r_jD_j and D_j have the same support, the above lemma tells us that we can always make the given big and nef divisors have equi-degree without changing their supports. So now we assume that $D := D_1+\cdots+D_q$ is of equi-degree. To compute β_{D,D_j} for $j = 1, \ldots, q$, we use the following lemma from Autissier [1].

Lemma 2.5 (Lemma 4.2 in [1])**.** *Suppose E is a big and base-point free Cartier divisor on a projective variety X, and F is a nef Cartier divisor on X such that $F - E$ is also nef. Let $\delta > 0$ be a positive real number. Then, for any positive integers N and m with $1 \leq m \leq \delta N$, we have*

$$\begin{aligned}h^0(NF-mE) \quad &\geq \quad \frac{F^n}{n!}N^n - \frac{F^{n-1}.E}{(n-1)!}N^{n-1}m\\ &+\frac{(n-1)F^{n-2}.E^2}{n!}N^{n-2}\min\{m^2, N^2\} + O(N^{n-1}),\end{aligned}$$

where the implicit constant depends on β.

We now compute $\sum_{m\geq 1} h^0(ND - mD_i)$ for each $1 \leq i \leq q$. Let $n = \dim X$, and assume that $n \geq 2$. Let $b = \frac{D^n}{nD^{n-1}.D_i}$ and $A = (n-1)D^{n-2}.D_i^2$. Then, by Lemma 2.5,

$$\begin{aligned}
&\sum_{m=1}^{\infty} h^0(ND - mD_i) \\
&\geq \sum_{m=1}^{[bN]} \left(\frac{D^n}{n!} N^n - \frac{D^{n-1}.D_i}{(n-1)!} N^{n-1} m + \frac{A}{n!} N^{n-2} \min\{m^2, N^2\} \right) + O(N^n) \\
&\geq \left(\frac{D^n}{n!} b - \frac{D^{n-1}.D_i}{(n-1)!} \frac{b^2}{2} + \frac{A}{n!} g(b) \right) N^{n+1} + O(N^n) \\
&= \left(\frac{b}{2} + \frac{A}{D^n} g(b) \right) D^n \frac{N^{n+1}}{n!} + O(N^n) \\
&= \left(\frac{b}{2} + \alpha \right) N h^0(ND) + O(N^n)
\end{aligned}$$

where $\alpha := \frac{A}{D^n} g(b)$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the function given by $g(x) = \frac{x^3}{3}$ if $x \leq 1$ and $g(x) = x - \frac{2}{3}$ for $x \geq 1$. Now from the assumption of equi-degree, $D_i.D^{n-1} = \frac{1}{q} D^n$, so $b = \frac{q}{n}$. Moreover, $\alpha > 0$ since $\dim V \geq 2$ and that the D_i's are big and nef divisors. Hence

$$\beta_{D,D_i} = \sup_N \frac{\sum_{m\geq 1} h^0(ND - mD_i)}{N h^0(ND)} \geq \frac{b}{2} + \alpha.$$

Thus we have proved the following.

Proposition 2.6. *Let V be a projective variety of $\dim V \geq 2$ and $D := \sum_{j=1}^{q} D_j$ has equi-degree respect to $D_1, \ldots, D_q$ which are assumed to be big and nef. Then*

$$\beta_{D,D_i} = \sup_N \frac{\sum_{m\geq 1} h^0(ND - mD_i)}{N h^0(ND)} > \frac{q}{2n} + \alpha,$$

where α is a computable positive number.

This, together with the Main Theorem, implies

Theorem 2.7 (Saud-Ru, [11])**.** *Let k be a number field and let $S \subseteq M_k$ be a finite set containing all archimedean places. Let V be a projective variety of dimension ≥ 2 over k, and let $D_1, \ldots, D_q$ be effective, big, and nef Cartier divisors on V defined over k, located in ℓ-subgeneral position. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^{q} r_i D_i$ has equi-degree (such numbers exist due to Lemma 2.4). Then, for $\epsilon_0 > 0$ small enough, the inequality*

$$\sum_{v\in S} \sum_{j=1}^{q} r_j \lambda_{D_i,v}(x) < \ell \left(\frac{2 \dim V}{q} - \epsilon_0 \right) \left(\sum_{j=1}^{q} r_j h_{D_j}(x) \right)$$

holds for all $x \in V(k)$ outside a proper Zariski-closed subset of V.

3. Proof of the Main Theorem

We first recall some basic properties of local Weil functions associated to closed subschemes from [15, Section 2]. We assume that the readers are familiar with the notion of Weil functions associated to divisors. (See [12, Chapter 10], [10, B.8] or [15, Section 1].)

Let Y be a closed subscheme on a projective variety V defined over k. Then one can associate to each place $v \in M_k$ a function

$$\lambda_{Y,v} : V \setminus \operatorname{supp}(Y) \to \mathbb{R}$$

satisfying some functorial properties (up to a M_k-constant) described in [15, Theorem 2.1]. Intuitively, for each $P \in V, v \in M_k$

$$\lambda_{Y,v}(P) = -\log(v\text{-adic distance from } P \text{ to } Y).$$

The following lemma indicates the existence of local Weil functions.

Lemma 3.1. *Let Y be a closed subscheme of V. There exist effective divisors $D_1, \cdots, D_r$ such that*

$$Y = \cap D_i.$$

Proof. See Lemma 2.2 from [15]. □

Definition 3.2. *Let k be a number field, and M_k be the set of places on k. Let V be a projective variety over k and let $Y \subset V$ be closed subscheme of V. We define the (local) Weil function for Y with respect to $v \in M_k$ as*

$$\lambda_{Y,v} = \min_i \{\lambda_{D_i,v}\}, \tag{3.1}$$

when $Y = \cap D_i$ (such D_i exist according to the above lemma).

Lemma 3.3 (Lemma 2.5.2 in [21] or Theorem 2.1 (h) in [15])**.** *Let Y be a closed subscheme of V, and let $\tilde{V}$ be a blow up of V along Y with exceptional divisor $E = \pi^*Y$, then $\lambda_{Y,v}(\pi(P)) = \lambda_{E,v}(P) + O_v(1)$ for $P \in \tilde{V}$.*

Note that in the original statement of Lemma 2.5.2 in [21], V is assumed to be smooth, but from the proof it is easy to see that it works for general projective variety from Theorem 2.1 (h) in [15].

For our purpose, it suffices to fix a choice of local Weil functions $\lambda_{Y_i,v}$ for each $1 \leq i \leq q$ and $v \in S$.

Lemma 3.4. *Let $Y_1, \cdots, Y_q$ be closed subschemes of a projective variety V in ℓ-sub-general position. Then*

$$\sum_{i=1}^{q} \lambda_{Y_i,v}(x) \leq \max_I \sum_{j \in I} \lambda_{Y_j,v}(x) + O_v(1), \tag{3.2}$$

where I runs over all index subsets of $\{1, \cdots, q\}$ with ℓ elements for all $x \in V(k)$.

Proof. Let $\{i_1, \cdots, i_q\} = \{1, \cdots, q\}$. Since the Y_i are in ℓ-sub-general position, $\cap_{t=1}^{\ell+1} Y_{i_t} = \emptyset$. Then

$$\min_{1 \leq i \leq \ell+1} \{\lambda_{Y_i,v}\} = \{\lambda_{\cap_{t=1}^{\ell+1} Y_{i_t},v}\} = O_v(1). \tag{3.3}$$

We note that the first equality follows from (3.1), the definition of the local Weil function; and the second equality follows from Corollary 3.3 in [12, Chapter 10]. For x with the following ordering

$$\lambda_{Y_{i_1},v}(x) \geq \lambda_{Y_{i_2},v}(x) \geq \cdots \geq \lambda_{Y_{i_q},v}(x),$$

we have

$$\sum_{i=1}^{q} \lambda_{Y_i,v}(x) = \sum_{i=1}^{\ell} \lambda_{Y_i,v}(x) + O_v(1).$$

Then the assertion (3.2) follows directly as the number of subvarieties under consideration is finite. □

We also need the following generalized Schmidt subspace theorem due to Ru-Vojta [20].

Theorem 3.5 (Theorem 2.7 in [20]). *Let k be a number field, let S be a finite set of places of k containing all archimedean places, let X be a complete variety over k, let D be a Cartier divisor on X, let W be a nonzero linear subspace of $H^0(X, \mathcal{O}(D))$, let $s_1, \ldots, s_q$ be nonzero elements of W, let $\epsilon > 0$, and let $c \in \mathbb{R}$. For each $i = 1, \ldots, q$, let D_j be the Cartier divisor (s_j), and let λ_{D_j} be a Weil function for D_j. Then there is a proper Zariski-closed subset Z of X, depending only on k, S, X, L, W, $s_1, \ldots, s_q$, ϵ, c, and the choices of Weil and height functions, such that the inequality*

$$\sum_{v \in S} \max_J \sum_{j \in J} \lambda_{D_j,v}(x) \leq (\dim W + \epsilon) h_D(x) + c \tag{3.4}$$

holds for all $x \in (X \setminus Z)(k)$. Here the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j \in J}$ are linearly independent.

We are now ready to prove the Main Theorem.

Proof of the Main Theorem. Let $\delta > 0$ be a sufficiently small number. We may choose a sufficiently large integer N such that

$$\sum_{m=1}^{\infty} h^0(V, \mathcal{L}^N \otimes \mathfrak{I}_Y^m) \leq (\beta_{\mathcal{L}, Y_i} - \delta) N h^0(V, \mathcal{L}^N). \tag{3.5}$$

Let L be the associated line bundle of $\mathcal{L}$ and $M = h^0(V, NL) = h^0(V, \mathcal{L}^N)$.

Let $x \in V(k)$ and $v \in S$. Since the Y_i are in ℓ-sub-general position, it follows from Lemma 3.4 that

$$\sum_{i=1}^{q} \lambda_{Y_i, v}(x) \leq \ell \lambda_{Y_{i_0}, v}(x) + O_v(1), \tag{3.6}$$

with $1 \leq i_0 \leq q$, where the constant $O_v(1)$ is independent of x. Note that i_0 depends on the point x, but $O_v(1)$ is independent of x.

We first consider the case when the codimension Y_{i_0} in V is at least 2. Let $\pi : \tilde{V} \to V$ be the blow-up at Y_{i_0} and $E = \pi^{-1}(Y_{i_0})$ be the exceptional divisor of π. We consider the following filtration.

$$H^0(\tilde{V}, \pi^* NL) \supseteq H^0(\tilde{V}, \pi^* NL - E) \supseteq H^0(\tilde{V}, \pi^* NL - 2E) \supseteq \cdots \tag{3.7}$$

Choose regular sections $s_1, \cdots, s_M \in H^0(V, NL)$ successively so that their pull-back $\pi^* s_1, \cdots, \pi^* s_M \in H^0(\tilde{V}, \pi^* NL)$ form a basis associated to this filtration. Equivalently, $s_1, \cdots, s_M \in H^0(V, NL)$ are chosen successively according to the filtration

$$H^0(V, \mathcal{L}^N) \supseteq H^0(V, \mathcal{L}^N \otimes \mathfrak{I}_Y) \supseteq H^0(V, \mathcal{L}^N \otimes \mathfrak{I}_Y^2) \supseteq \cdots \tag{3.8}$$

For a section $\pi^* s \in H^0(\tilde{V}, \pi^* NL - mE)$ (regarded as a subspace of $H^0(\tilde{V}, \pi^* NL)$) we have

$$\operatorname{div}(\pi^* s) \geq mE.$$

Hence, $\lambda_{(\pi^* s), v} \geq m \lambda_{E, v} + O_v(1)$. Note that although $O_v(1)$ here depends i_0 (which depends on x), there are q many choices of such i_0 and V is compact, so we can again make $O_v(1)$ independent of x. Therefore, also using Lemma 3.3 and (3.5),

$$\begin{aligned}\sum_{j=1}^{M}\lambda_{(\pi^*s_j),v} &\geq \sum_{m=1}^{\infty} m(h^0(\tilde{V},\pi^*NL-mE)-h^0(\tilde{V},\pi^*NL-(m+1)E))\lambda_{E,v}+O_v(1)\\ &=\sum_{m=1}^{\infty} m(h^0(\tilde{V},\pi^*NL-mE)-h^0(\tilde{V},\pi^*N-(m+1)E))\lambda_{Y_{i_0},v}\circ\pi+O_v(1)\\ &=\sum_{m=1}^{\infty} h^0(\tilde{V},\pi^*NL-mE)\lambda_{Y_{i_0},v}\circ\pi+O_v(1)\\ &\geq(\beta_{L,Y_{i_0}}-\delta)Nh^0(V,NL)\lambda_{Y_{i_0},v}\circ\pi+O_v(1).\end{aligned}$$

Noticing that, by the functorial property of Weil functions implies that $\lambda_{(\pi^*s_j),v}=\lambda_{(s_j),v}\circ\pi+O_v(1)$. Hence, the above inequality, together with (3.6), implies that (3.9)

$$\sum_{i=1}^{q}\lambda_{Y_i,v}(x)\leq\frac{\ell}{N\cdot h^0(V,NL)(\min_{1\leq i\leq q}\{\beta_{L,Y_i}\}-\delta)}\max_J\{\sum_{j\in J}\lambda_{(s_j),v}(x)\}+O_v(1),$$

where J is a subset containing M linearly independent sections taken among the collection of sections $\{s_j(i_0,v)|1\leq i_0\leq q, v\in S\}$ coming from the claim (3.6). Note that when Y_{i_0} is a divisor, by the same proof simply using the filtration (3.8) without doing the blow up, (3.9) still holds. It then follows from Theorem 3.5 and a suitable choice of δ that for given $\epsilon>0$ there exists a proper algebraic subset Z over k such that

$$\sum_{v\in S}\sum_{i=1}^{q}\lambda_{Y_i,v}(x)\leq(\ell\cdot\max_{1\leq i\leq q}\{\beta_{L,Y_i}^{-1}\}+\epsilon)h_L(x) \tag{3.10}$$

for all $x\in V(k)\setminus Z(k)$. □

Proof of Corollary 1.5. Let v be a place of k. The main point of the proof is to reformulate the distance function $d_v(\cdot,\cdot)$ defined on $V(\bar{k})$ [14, Sect. 2] into a product of several distance functions on $V(K)$, where K is a finite extension of k. Following the construction in [14, Sect. 2], we fix an extension of v to $\bar{k}$. The place defines an absolute value $\|\cdot\|_v$ on $\bar{k}$. If $K\subset\bar{k}$ is a finite extension of k, then $d_v(\cdot,\cdot)_K=d_v(\cdot,\cdot)_k^{[K_v:k_v]}$. Here $d_v(\cdot,\cdot)_K$ refers to the distance function defined by using the same embedding and normalizing with respect to K and $d_v(\cdot,\cdot)_k$ the distance function normalized with respect to k. (cf. [14, Proposition 2.1 (b)]) Assume that $V\subset\mathbb{P}^N$ (given by the canonical map associated to the ample line bundle L). For a given fixed point $y=[y_0:\cdots:y_N]\in V(\bar{k})$, let K be the Galois closure of $k(y_0,\ldots,y_N)$ over k. For each $v\in M_k$, the inclusion map $(i_v)|_K:K\to\bar{k}_v$ induces a place $w_0:=v$ of K over v, and other places w of K over v are conjugates by elements $\sigma_w \in Gal(K/k)$ such that $\|\sigma_w(a)\|_w = \|a\|_v$ for all $a \in K$. Then, for $x, y \in K$,

$$\prod_{w\in M_K, w|v} d_w(\sigma_w(x), \sigma_w(y))_K = \prod_{w\in M_K, w|v} d_v(x,y)_K$$
$$= \prod_{w\in M_K, w|v} d_v(x,y)_k^{[K_v:k_v]} = [K:k] d_v(x,y)_k,$$

i.e.

$$d_v(x,y)_k = \prod_{w\in M_K, w|v} d_w(\sigma_w(x), \sigma_w(y))_K^{\frac{1}{[K:k]}}, \qquad \text{for } x, y \in K. \tag{3.11}$$

To compute $\alpha_y(L)$, we consider any sequence $\{x_i\} \subseteq X(k)$ of distinct points with $d_v(y, x_i) \to 0$. By (3.11), we have $d_v(y,x_i)_k = \prod_{w\in M_K, w|v} d_w(\sigma_w(y), x_i)_K^{\frac{1}{[K:k]}}$. (Here we extend $\sigma_w \in Gal(K/k)$ to the map from $V(K)$ to $V(K)$ by acting on the coordinates of the points.) The distance function $d_w(y,x)$ in [14] is constructed by choosing an embedding $\phi_L : V \to \mathbb{P}^N$ into a projective space via the sections of L and measure the distance in the embedded space. For fixed y, $-\log d_w(y, \cdot)$ is indeed a local Weil function on the embedded space that is denoted by $\lambda_{\phi(y),w}$. We note that this fact can also be proved by slightly modification of Lemma 2.6 in [14]. By the functoriality of Weil functions of closed subschemes [15, Theorem 2.1 (h)] we have $-\log d_w(\sigma_w(y), x) = \lambda_{\sigma_w(y),w}(x) + O(1)$. On the other hand, it is clear from the definition that $\beta_{y,L} = \beta_{\sigma_w(y),L}$ for very $\sigma_w \in \mathrm{Gal}(K/k)$. The Main Theorem (Note that, in this case, $\ell = 1$.) then implies that for any $\epsilon > 0$, for any $x \in K$

$$-\log d_v(y, x_i) = \frac{1}{[K:k]} \sum_{w\in M_K, w|v} -\log d_w(y, x_i) \geq -(\{\beta_{y,L}^{-1}\} + \epsilon) h_L(x_i)$$

holds for all x_i outside a proper Zariski-closed subset Z of $V(K)$. We note that Z is indeed defined over k since all the x_i are in k. The first assertion then follows directly from the definition of $\alpha_y(L)$. The rest of the arguments is the same as the proof of Theorem 6.1 in [14]. □

4. THE COMPLEX CASE

In this section, we state and sketch a proof of the result in Nevanlinna theory which is analogy to our Main Theorem above. We use the standard notation in Nevanlinna theory (see, for example, [18]).

Theorem 4.1. *Let V be a complex projective variety and $Y_1, \cdots, Y_q$ be closed subschemes of V in ℓ-sub-general position. Let $\mathcal{L}$ be a line sheave with $h^0(V, \mathcal{L}^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to V$ be a holomorphic map with Zariski dense image. Then for any $\epsilon > 0$*

$$\sum_{i=1}^{q} m_f(r, Y_i) \leq \ell(\max_{1\leq i\leq q}\{\beta_{\mathcal{L},Y_i}^{-1}\} + \epsilon)T_{f,\mathcal{L}}(r) \ \| \tag{3.12}$$

where $\|$ means the inequality holds for all $r \in (0, +\infty)$ except for a subset $E \subset (0, +\infty)$

In above, for a subcheme Y of V,

$$m_f(r, Y) = \int_0^{2\pi} \lambda_Y(f(re^{i\theta}))\frac{d\theta}{2\pi}$$

where λ_Y is the Weil function of Y defined similarly in the arithmetic case above. To prove the theorem, we need the following result.

Theorem 4.2 (Theorem 2.8 in [20])**.** *Let X be a complex projective variety, let D be a Cartier divisor on X, let V be a nonzero linear subspace of $H^0(X, \mathscr{O}(D))$, and let $s_1, \ldots, s_q$ be nonzero elements of V. For each $i = 1, \ldots, q$, let D_j be the Cartier divisor (s_j), and let λ_{D_j} be a Weil function for D_j. Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then*

$$\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{D_j}(f(re^{i\theta})) \leq (\dim V)T_{f,D}(r) + O(\log^+ T_{f,D}(r)) + o(\log r) \ \|$$

here the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j\in J}$ are linearly independent.

Sketch of the proof of Theorem 4.1: In the same way in deriving (3.9), we can prove that, for any $x \in V$,

$$\sum_{i=1}^{q} \lambda_{Y_i}(x) \leq \frac{\ell}{N \cdot h^0(V, NL)(\min_{1\leq i\leq q}\{\beta_{L,Y_i}\} - \delta)} \max_J\{\sum_{j\in J} \lambda_{(s_j)}(x)\} + O(1).$$

By taking $x = f(re^{i\theta})$, we get

$$\sum_{i=1}^{q} \lambda_{Y_i}(f(re^{i\theta})) \leq \frac{\ell}{N \cdot h^0(V, NL)(\min_{1\leq i\leq q}\{\beta_{L,Y_i}\} - \delta)} \max_J\{\sum_{j\in J} \lambda_{(s_j)}(f(re^{i\theta}))\} + O(1).$$

From here, the theorem can be easily derived by applying Theorem 4.2.

Corollary 4.3. *Let V be a complex projective variety of dimension n and $a_1, \cdots, a_q$ be distinct points on V. Let $\mathcal{L}$ be a line sheave with $h^0(V, \mathcal{L}^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to V$ be a holomorphic map with Zariski dense image. Then for any $\epsilon > 0$*

$$\sum_{i=1}^{q} m_f(r, a_i) \leq \left(\frac{n+1}{n} \max_{1 \leq i \leq q} \{\epsilon_{a_i}^{-1}(\mathcal{L})\} + \epsilon \right) T_{f,\mathcal{L}}(r) \ \|$$

where $\epsilon_x(\mathcal{L})$ is the Seshadri constant of $\mathcal{L}$ at the point $x \in V$.

In particular, if $V = \mathbb{P}^n$, then for any $\epsilon > 0$

$$\sum_{i=1}^{q} m_f(r, a_i) \leq \left(\frac{n+1}{n} + \epsilon \right) T_{f,\mathcal{L}}(r). \ \|$$

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References

[1] P. Autissier, *Géométries, points entiers et courbes entières (French) [Geometry, integral points and integral curves]*, Ann. Sci. Éc. Norm Supér, (4) **42** (2009) 221-239.

[2] E. Bombieri and W. Gubler, *Heights in Diophantine Geometry,* New Mathematical Monograph vol. 4, Cambridge University Press, Cambridge, 2006.

[3] Z. H. Chen, M. Ru and Q. Yan, *The Degenerated Second Main Theorem and Schmidt's Subspace Theorem*, Science in China, **55** (2012), 1367–1380.

[4] P. Corvaja and U. Zannier, *On a general Thue's equation*, Amer. J. Math., **126**(5) (2004),1033–1055.

[5] P. Corvaja and U. Zannier, *Addedum to "On a general Thue's equation"*, Amer. J. Math., **128** (2006),1057–1066.

[6] J.-H. Evertse and R. G. Ferretti, *Diophantine inequalities on projective varieties,* Int. Math. Res. Notices **25** (2002), 1295–1330.

[7] J.-H. Evertse and R. Ferretti, *A generalization of the Subspace Theorem with polynomials of higher degree*, In *Diophantine approximation*, volume 16 of Dev. Math., pages 175–198. SpringerWienNewYork, Vienna, 2008.

[8] G. Faltings and G. Wüstholz, *Diophantine approximations on projective varieties,* Invent. Math. **116** (1994), 109–138.

[9] R. Hartshorne, *Algebraic Geometry,* Grad. Texts in Math. vol. 52, Springer-Verlag, New York, 1997.

[10] M. Hindry and J. H. Silverman, *Diophantine Geometry-an introduction,* Springer-Verlag, New York, 2000.

[11] S. Hussein and M. Ru *A general defect relation and height inequality for divisors in subgeneral position,* preprint.

[12] S. Lang, *Fundamentals of Diophantine Geometry,* Springer-Verlag, New York, 1983.

[13] A. Levin, *Generalizations of Siegel's and Picard's theorems*, Ann. of Math. (2), **170** (2) (2009), 609–655.

[14] D. McKinnon and M. Roth, *Seshadri constants, diophantine approximation and Roth's theorem for arbitrary varieties,* Invent. Math. **200** (2015), 513–583.

[15] J. H. Silverman, *Arithmetic distance functions and height functions in Diophantine geometry,* Math. Ann. **279** (1987), no. 2, 193–216.

[16] M. Ru, *A defect relation for holomorphic curves intersecting hypersurfaces,* Amer. J. Math. **126** (2004), 215–226.

[17] M. Ru, *Holomorphic curves into algebraic varieties,* Annals of Mathematics, **169** (2009), 255–267.
[18] M. Ru, *A defect relation for holomorphic curves intersecting general divisors in projective varieties,* J. of Geometric Analysis, first online: 15 October 2015.
[19] M. Ru, *A general Diophantine inequality*, Functiones et Approximatio, to appear.
[20] M. Ru and P. Vojta, *Birational Nevanlinna constant and its consequences*, Preprint, arXiv:1608.05382 [math.NT].
[21] P. Vojta, *Diophantine Approximation and Value distribution theory*, Volume 1239 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1987.

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