ON METAPLECTIC FORMS OVER FUNCTION FIELDS

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ABSTRACT. We propose a concept of half integral weight in the global function field context, and construct natural families of functions with given weight. An analogue of Shimura correspondence (between weight 2 functions and weight $\frac{3}{2}$ functions) via theta series from "definite" quaternion algebras over function fields is then established. From the study of Fourier coefficients of these theta series, we arrive at a Waldspurger-type formula. This formula is then applied to L-series coming from elliptic curves over function fields.
Keywords: Function field, Half integral weight, Metaplectic form, Theta series, Special value of L-series
MSC code: MSC 11R58, MSC 11F37, MSC 11F27, MSC 11F67

[1]

INTRODUCTION

Let k be the rational function field $\mathbb{F}_q(t)$ with odd characteristic. Fix the infinite place ∞ of k, and denote by k_∞ the completion of k at ∞. The Kubota 2-cocycle on $\mathrm{GL}_2(k_\infty)$, which is defined via Hilbert n-th symbol for $n \mid q-1$, gives non-trivial central extension of $\mathrm{GL}_2(k_\infty)$, called *metaplectic group*. The aim of this paper is to study families of functions on the metaplectic group $\mathbb{G}$, so-called *metaplectic forms*. We focus on the case when $n = 2$ and present an analogue of classical theory of half integral weight modular forms.

In the function field context, we take the Iwahori Hecke operator at ∞ to be our "non-Euclidean Laplacian." Functions on $\mathbb{G}$ of weight $\frac{\kappa}{2}$ are the eigenfunctions of this operator with eigenvalue $q^{1-\frac{\kappa}{4}}$. From the norm form on lattices of pure quaternions in "definite" quaternion algebra over k (i.e. ramified at ∞), we construct theta series which are functions on $\mathbb{G}$ of weight $\frac{3}{2}$. The action of Hecke operators on these theta series can be expressed by so-called *Brandt matrices*. As in Eichler's theory, these integral matrices are in fact representation of Hecke operators on the space of Drinfeld type "new" forms. This allows us to define a Shimura map **Sh** in §4.3 from Drinfeld type "new" forms to functions of weight $\frac{3}{2}$.

From knowledge about central critical values of L-series, we arrive at the following main theorem, which gives a function field analogue of Waldspurger's formula (cf. §4.3 for further details):

Theorem 0.1. *Let N_0 be a square-free ideal of A with odd number ℓ_{N_0} of prime factors. Given a "normalized" Drinfeld type newform f for $\Gamma_0(N_0)$. Suppose for each prime factor P of N_0, the eigenvalue of the Hecke operator T_P on f is one. Then given any irreducible polynomial D in $A - k_\infty^2$ satisfying Legendre symbol $\left(\frac{D}{P}\right) = -1$ for all primes P dividing N_0, we have*

$$L(f,0)L(f\otimes\varepsilon_D,0) = (q^{\frac{(-1)^{\deg D}-1}{4}}) \cdot \frac{(3-(-1)^{\deg D})\cdot(f,f)}{2\cdot|D|^{\frac{1}{2}}\cdot 4^{(\ell_{N_0}-1)}} \cdot m(f,D)^2.$$

Here $L(f,s)$ is the L-series attached to f, $L(f\otimes\varepsilon_D,s)$ is its twist by the quadratic character ε_D; $(\cdot,\cdot)$ denotes Petersson inner product on the space of Drinfeld type cusp forms; and $m(f,D)$ is the $(-D)$-th Fourier coefficient of the weight $\frac{3}{2}$ function $\mathbf{Sh}(f)$.

With suitable choice of D, the Fourier coefficient $m(f,D)$ determines the non-vanishing of the above central critical value of L-series. This theorem can be applied to L-series coming

[1]This research was partially supported by National Science Council.

from elliptic curves over k having split multiplicative reduction at an even number of places (including ∞) and with good reduction elsewhere.

The contents of this article are as follows. We first pick the automorphy factor from the transformation law of a specific theta series in §1.1. After a brief review of the metaplectic group $\mathbb{G}$, the Iwahori Hecke operator at ∞ is introduced in §1.3. Similar to the classical case, integral weight functions on $\mathbb{G}$ are comes from functions on $\mathrm{GL}_2(k_\infty)$.

In §2 we study examples of functions having integral or half integral weight. Theta series from "imaginary" (with respect to ∞) quadratic extensions over k are of weight 1. Automorphic forms of Drinfeld type, which can be viewed as an analogue of classical weight 2 modular forms (cf. [2] and [11]), are functions of weight 2. Furthermore, Eisenstein series give us examples of higher integral weight functions. The theta series of weight $\frac{3}{2}$ from definite quaternion algebra over k are introduced in §2.2. The Fourier coefficients of these series, which can be viewed as representation numbers of "three squares," are expressed explicitly in terms of numbers of optimal embeddings of quadratic orders into the definite quaternion algebra in question.

In §3 we introduce Hecke operators $T_{P^2,\frac{\kappa}{2}}$ for finite primes P on functions f of half integral weight $\frac{\kappa}{2}$ with κ odd. The Fourier coefficients of $T_{P^2,\frac{\kappa}{2}}f$ can be computed as in classical theory. The Brandt-matrix representation for the action of Hecke operators on the above theta series of weight $\frac{3}{2}$ is established at the end of §3. In §4.1 we recall briefly properties of the so-called *definite Shimura curves* over k. The construction of the Shimura map **Sh** is in §4.2, and the proof of our main theorem is given in §4.3. Finally we apply our theorem to families of elliptic curves over k in §4.4.

Notation

We fix the following notations:

- k : the rational function field $\mathbb{F}_q(t)$, $q = p^{\ell_0}$ where p is an odd prime.
- A : the polynomial ring $\mathbb{F}_q[t]$.
- ∞ : the infinite place, which corresponds to the degree valuation v_∞.
- $|\cdot|$: the absolute value on k_∞: for $a \in k_\infty$, $|a| := q^{-v_\infty(a)}$.
- π_∞ : t^{-1}, a fixed uniformizer of ∞.
- k_∞ : $\mathbb{F}_q((t^{-1}))$, i.e. the completion of k at ∞.
- $\mathcal{O}_\infty$: $\mathbb{F}_q[[t^{-1}]]$, i.e. the valuation ring in k_∞.
- P : a finite prime (place) of k.
- k_P : the completion of k at the finite prime P.
- A_P : the closure of A in k_P.
- $\hat{k}$: $\prod'_P k_P$, the finite adele ring of k.
- $\hat{A}$: $\prod_P A_P$.
- ψ_∞ : a fixed additive character on k_∞: for $y = \sum_i a_i \pi_\infty^i \in k_\infty$, we define $\psi_\infty(y) := \exp\left(\frac{2\pi\sqrt{-1}}{p} \cdot \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-a_1)\right)$.

We identify non-zero ideals of A with the monic polynomials in A by using the same notation.

1. Half integral weight

In the classical case, the theta series

$$\sum_{n\in\mathbb{Z}} \exp(2\pi\sqrt{-1}n^2 z)$$

is a modular form of weight $\frac{1}{2}$ for the congruence subgroup $\Gamma_0(4)$. Shimura ([12]) uses the automorphy factor of this theta series to develop the theory of half integral weight modular forms. In this section, starting from an explicit theta series, we propose a concept of half integral weight in the function field setting which has direct applications to arithmetic of function fields.

1.1. **Theta series.** For $(x, y) \in k_\infty^\times \times k_\infty$, define

$$\theta(x,y) := \sum_{b \in A} \phi_\infty(b^2 x t^2) \psi_\infty(b^2 y)$$

Here ϕ_∞ is the characteristic function of $\mathcal{O}_\infty$. Given (x, y) in $k_\infty^\times \times k_\infty$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}_2(A)$ with $cy + d \neq 0$. The *Möbius transform*

$$\gamma \circ (x,y) := \left(\frac{(ad-bc)x}{(cy+d)^2}, \frac{ay+b}{cy+d} \right).$$

To state the transformation law of θ, we recall the following symbols:

1. For $\beta \in k_\infty^\times$, the *root number* of β

$$\omega(\beta) := \begin{cases} 1 & \text{if } v_\infty(\beta) \text{ is even,} \\ \varepsilon \cdot \mathrm{sgn}(\beta)^{\frac{q-1}{2}} & \text{if } v_\infty(\beta) \text{ is odd.} \end{cases}$$

Here sgn is a *sign function* on $k_\infty^\times$ (with respect to π_∞): for $y \in k_\infty^\times$

$$y = \mathrm{sgn}(y) \cdot u \cdot \pi_\infty^{v_\infty(y)}$$

where $u \in 1 + \pi_\infty \mathcal{O}_\infty$, $\mathrm{sgn}(y) \in \mathbb{F}_q^\times$; ε is the sign of the following *Gauss sum*:

$$\varepsilon := q^{-\frac{1}{2}} \sum_{\epsilon \in \mathbb{F}_q} \epsilon^{\frac{q-1}{2}} \exp\left(\frac{2\pi\sqrt{-1}}{p} \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\epsilon) \right).$$

We point out that $\omega(-\beta) = \omega(\beta)^{-1}$ and $\omega(\beta)^2 = (-1)^{\frac{q-1}{2} v_\infty(\beta)}$.

2. For α, $\beta \in k_\infty^\times$, the *Hilbert quadratic symbol*

$$(\alpha, \beta)_\infty := \begin{cases} 1 & \text{if } \alpha X^2 + \beta Y^2 = Z^2 \text{ has a non-trivial solution,} \\ -1 & \text{otherwise.} \end{cases}$$

It can be checked that for $\alpha, \beta \in k_\infty^\times$, $(\alpha, \beta)_\infty = \omega(\alpha)\omega(\beta)/\omega(\alpha\beta)$.

3. Given $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(A)$, the *Kubota symbol* of γ

$$\mu(\gamma) := \begin{cases} \left(\frac{d}{c}\right) & \text{if } cd \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Here $\left(\frac{\cdot}{\cdot}\right)$ is the *Legendre quadratic symbol*.

Basing on Poisson summation formula, the transformation law of θ is worked out:

Proposition 1.1. (cf. [16] IV.2.1) *Let $x \in k_\infty^\times$ and $y \in k_\infty$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(A)$ with $v_\infty(cx) > v_\infty(cy+d)$, one has*

$$\theta(\gamma \circ (x,y)) = |cy+d|^{\frac{1}{2}} \epsilon(\gamma, x, y) \theta(x,y)$$

with

$$\epsilon(\gamma,x,y)=\begin{cases}\mu(\gamma)\omega(cy+d)(c,cy+d)_\infty, & \textit{if } c\neq 0,\\ 1, & \textit{otherwise.}\end{cases}$$

This leads to a function on the *metaplectic cover* $\mathbb{G}$ of $\mathrm{GL}_2(k_\infty)$.

1.2. **Extension $\mathbb{G}$ of $\mathrm{GL}_2(k_\infty)$ by the circle group S^1.** Given $g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ in $\mathrm{GL}_2(k_\infty)$. Let $X(g):=c$ if c is non-zero and d if c is zero. Kubota (cf. [8]) introduced a 2-cocycle $\sigma:\mathrm{GL}_2(k_\infty)\times\mathrm{GL}_2(k_\infty)\to\{\pm 1\}$ which is defined by

$$\sigma(g,g')=\left(\frac{X(gg')}{X(g)},\frac{X(gg')}{\det(g)X(g')}\right)_\infty.$$

This gives us an extension $\mathbb{G}$ of $\mathrm{GL}_2(k_\infty)$ by S^1:

$$1\longrightarrow S^1\longrightarrow\mathbb{G}\longrightarrow\mathrm{GL}_2(k_\infty)\longrightarrow 1$$

where $S^1=\{z\in\mathbb{C},|z|=1\}$.

The map $\tilde{\mu}:\mathrm{SL}_2(A)\to\mathbb{G}$ given by $\gamma\longmapsto\tilde{\gamma}=(\gamma,\mu(\gamma))$ is a group monomorphism (cf. [6]). The group $\mathrm{GL}_2(\mathcal{O}_\infty)$ also can be embedded into $\mathbb{G}$ by the homomorphism $\tilde{\varrho}:\mathrm{GL}_2(\mathcal{O}_\infty)\to\mathbb{G}$ mapping γ_∞ to $\tilde{\gamma}_\infty=(\gamma_\infty,\varrho(\gamma_\infty))$ where for $\gamma_\infty=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathcal{O}_\infty)$,

$$\varrho(\gamma_\infty):=\begin{cases}(c,d/\det\gamma_\infty)_\infty, & \text{if } 0<|c|<1,\\ 1, & \text{otherwise.}\end{cases}$$

Furthermore, since $\sigma(z_1,z_2)=(z_1,z_2)_\infty$ for $z_1,z_2\in k_\infty^\times$, we embed $k_\infty^\times$ into $\mathbb{G}$ via the homomorphism $\tilde{\omega}:k_\infty^\times\to\mathbb{G}$ defined by $z\longmapsto\tilde{z}=(z,\omega(z)^{-1})$. Note that $\tilde{\omega}(k_\infty^\times)$ is not in the center of $\mathbb{G}$.

For any congruence subgroup $\Gamma=\Gamma_0^{(1)}(N)$ in $\mathrm{SL}_2(A)$ where N is a non-zero ideal of A and

$$\Gamma_0^{(1)}(N)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(A):c\equiv 0 \bmod N\right\},$$

one has $\mathrm{GL}_2(k_\infty)=\Gamma\cdot\mathbb{H}_\infty\cdot\Gamma_\infty^1\cdot k_\infty^\times$, where

$$\Gamma_\infty^1:=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(\mathcal{O}_\infty):c\equiv d-1\equiv 0 \bmod \pi_\infty\mathcal{O}_\infty\right\},\ \mathbb{H}_\infty:=\begin{pmatrix}k_\infty^\times & k_\infty\\ 0 & 1\end{pmatrix}.$$

Let $\widetilde{k_\infty^\times}:=\tilde{\omega}(k_\infty^\times)$, $\widetilde{\Gamma}:=\tilde{\mu}(\Gamma)$, $\widetilde{\Gamma_\infty^1}:=\tilde{\varrho}(\Gamma_\infty^1)$, $\overline{\mathbb{H}}_\infty:=\{(h,\xi):h\in\mathbb{H}_\infty,\xi\in S^1\}$. Then $\mathbb{G}=\widetilde{k_\infty^\times}\cdot\widetilde{\Gamma}\cdot\overline{\mathbb{H}}_\infty\cdot\widetilde{\Gamma_\infty^1}$.

Back to the theta series θ in §1.1. Given $\tilde{g}\in\mathbb{G}$, write $\tilde{g}$ as $\tilde{z}\tilde{\gamma}(h,\xi)\tilde{\gamma}_\infty$, where $z\in k_\infty^\times$, $\gamma\in\mathrm{SL}_2(A)$, $\gamma_\infty\in\Gamma_\infty^1$, $h=\begin{pmatrix}x & y\\ 0 & 1\end{pmatrix}$, and $\xi\in S^1$. We extend θ to a function Θ on $\mathbb{G}$ by

$$\Theta(\tilde{g}):=|x|^{\frac{1}{4}}\theta(x,y)\cdot\xi.$$

The transformation law of θ implies that Θ is a well-defined function on $\mathbb{G}$, and for $z\in k_\infty^\times$, $\gamma\in\mathrm{SL}_2(A)$, $\gamma_\infty\in\Gamma_\infty^1$, $\xi\in S^1$

$$\Theta(\tilde{z}\tilde{\gamma}(1,\xi)\tilde{g}\tilde{\gamma}_\infty)=\xi\Theta(\tilde{g}).$$

This leads to a concept of half integral weight in the function field setting.

1.3. **Half integral weight and weight operators.** Here we are interested in functions f on the coset space $\mathbb{G}/\widetilde{\Gamma^1_\infty}$. For $\kappa \in \mathbb{N}$, define

$$\widetilde{T}_{\infty,\frac{\kappa}{2}} f(\tilde{g}) := q^{\frac{\kappa}{4}-1} \cdot \sum_{v\in\mathbb{F}_q} f\left(\tilde{g}\Big(\begin{pmatrix}\pi_\infty & v\\ 0 & 1\end{pmatrix},1\Big)\right).$$

Since

$$\widetilde{\Gamma^1_\infty}\left(\begin{pmatrix}\pi_\infty & 0\\ 0 & 1\end{pmatrix},1\right)\widetilde{\Gamma^1_\infty} = \coprod_{v\in\mathbb{F}_q}\left(\begin{pmatrix}\pi_\infty & v\\ 0 & 1\end{pmatrix},1\right)\widetilde{\Gamma^1_\infty},$$

$\widetilde{T}_{\infty,\frac{\kappa}{2}} f$ is also a function on $\mathbb{G}/\widetilde{\Gamma^1_\infty}$. The metaplectic form Θ is of weight $\frac{1}{2}$ under the following definition:

Definition 1.2. f is of weight $\frac{\kappa}{2}$ if for all $\xi \in S^1$ and $\tilde{g} \in \mathbb{G}$,

$$f((1,\xi)\tilde{g}) = \xi^\kappa f(\tilde{g}) \text{ and } \widetilde{T}_{\infty,\frac{\kappa}{2}} f = f.$$

1.3.1. *Fourier expansion.* Suppose f is a function on $\mathbb{G}/\widetilde{\Gamma^1_\infty}$ satisfying that for $z \in k_\infty^\times$

$$f(\tilde{z}\tilde{g}) = \chi_\infty(z) f(\tilde{g})$$

where χ_∞ is a character $\chi_\infty : k_\infty^\times \to \mathbb{C}^\times$ trivial on $1+\pi_\infty\mathcal{O}_\infty$. We call χ_∞ the "*central" character* of f.

Given a weight $\frac{\kappa}{2}$ function f on $\mathbb{G}/\widetilde{\Gamma^1_\infty}$ with "central" character χ_∞. Suppose there exists a Dirichlet character $\chi_N : (A/NA)^\times \to \mathbb{C}^\times$ such that for all $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_0^{(1)}(N)$, $f(\tilde{\gamma}\tilde{g}) = \chi_N(d) f(\tilde{g})$. Then for $r \in \mathbb{Z}$ and $u \in k_\infty$, the *Fourier expansion of f* is:

$$f\left(\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix},1\right) = \sum_{\deg\lambda+2\le r} f^*(r,\lambda)\psi_\infty(\lambda u)$$

where

$$f^*(r,\lambda) = \int_{A\backslash k_\infty} f\left(\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix},1\right)\psi_\infty(-\lambda u)du.$$

The Haar measure taken here is normalized so that $\int_{A\backslash k_\infty} du = 1$. Since f is of weight $\frac{\kappa}{2}$, one has $f^*(r+1,\lambda) = q^{-\frac{\kappa}{4}} f^*(r,\lambda)$ for all $\lambda \in A$ with $\deg\lambda + 2 \le r$.

Define function φ_f on $k_\infty^\times \times k_\infty$ by

$$\varphi_f(x,y) := |x|^{-\frac{\kappa}{4}} f\left(\begin{pmatrix}x & y\\ 0 & 1\end{pmatrix},1\right) = \sum_{\lambda\in A,\deg\lambda+2\le v_\infty(x)} \varphi_f^*(\lambda)\psi_\infty(\lambda y)$$

where $\varphi_f^*(\lambda) := q^{\frac{\kappa r}{4}} f^*(r,\lambda)$ for any $r \ge \deg\lambda + 2$. Then we have the following transformation law for φ_f: given $(x,y) \in k_\infty^\times \times k_\infty$ and $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ in $\Gamma_0^{(1)}(N)$ with $v_\infty(cx) > v_\infty(cy+d)$,

$$\varphi_f\big(\gamma\circ(x,y)\big) = \chi_N(d)\chi_\infty(cy+d)^{-1}\left(|cy+d|^{\frac{\kappa}{2}}\epsilon(\gamma,x,y)^\kappa\right)\cdot\varphi_f(x,y).$$

1.3.2. *Integral weight.* When κ is even and h is a weight $\frac{\kappa}{2}$ $(=: \nu)$ function on $\mathbb{G}/\widetilde{\Gamma^1_\infty}$ with "central" character χ_∞, let

$$h'(g) := h(g,1) \text{ for } g \in \mathrm{GL}_2(k_\infty).$$

Then h' is a function on $\mathrm{GL}_2(k_\infty)/\Gamma^1_\infty$ such that for $z \in k_\infty^\times$ and $g \in \mathrm{GL}_2(k_\infty)$

$$h'(zg) = (-1)^{\frac{(q-1)\nu}{2}v_\infty(z)}\cdot\chi_\infty(z)\cdot h'(g),$$

and

$$T_{\infty,\nu}h'(g) := q^{\frac{\nu}{2}-1}\sum_{v\in\mathbb{F}_q} h'\left(g\begin{pmatrix}\pi_\infty & v\\ 0 & 1\end{pmatrix}\right) = h'(g).$$

Conversely, given a function f on $\mathrm{GL}_2(k_\infty)/\Gamma_\infty^1$ such that $f(zg) = \chi_\infty(z)f(g)$ for $z \in k_\infty^\times$, $g \in \mathrm{GL}_2(k_\infty)$ and $T_{\infty,\nu}f = f$. Let

$$f'(g,\xi) = \xi^\nu f(g) \text{ for all } (g,\xi) \in \mathbb{G}.$$

Then f' is a weight ν function with "central" character $\chi_\infty \cdot (-1)^{\frac{(q-1)\nu}{2}v_\infty(\cdot)}$. This tells us that functions of weight ν on $\mathbb{G}/\widetilde{\Gamma}_\infty^1$ are in fact induced from functions on $\mathrm{GL}_2(k_\infty)/\Gamma_\infty^1$ fixed by the "integral weight" operator $T_{\infty,\nu}$.

2. Natural family of functions having weight

In this section we give examples of functions having weight. A function f of integral weight ν can be viewed as a function on $\mathrm{GL}_2(k_\infty)/\Gamma_\infty^1$ which is fixed by the operator $T_{\infty,\nu}$ in §1.3.2. We start with integral weight examples.

2.1. **Functions of integral weight.** 1. *Theta series of imaginary quadratic function fields.* Given a square-free polynomial D in A such that $K = k(\sqrt{D})$ is "imaginary" quadratic field (i.e. ∞ does not split in K). Let $\mathfrak{a}_0$ be a fractional ideal in K such that the ideal norm $N_{K/k}(\mathfrak{a}_0) = (\lambda_0)$ for $\lambda_0 \in k^\times$. The theta series $\theta_{\mathfrak{a}_0,\lambda_0}$ on $k_\infty^\times \times k_\infty$ introduced by Rück [10] is:

$$\theta_{\mathfrak{a}_0,\lambda_0}(x,y) := \sum_{\mu\in\mathfrak{a}_0}\phi_\infty\left(\frac{N_{K/k}(\mu)}{\lambda_0}xt^2\right)\psi_\infty\left(\frac{N_{K/k}(\mu)}{\lambda_0}y\right).$$

For any $g \in \mathrm{GL}_2(k_\infty)$, write g as $\gamma\begin{pmatrix}x & y\\ 0 & 1\end{pmatrix}\gamma_\infty z$ where $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$ in $\Gamma_0^{(1)}(D)$, (x,y) in $k_\infty^\times \times k_\infty$, γ_∞ in Γ_∞^1, and z in $k_\infty^\times$. Let

$$\Theta_{\mathfrak{a}_0,\lambda_0}(g) := |x|^{\frac{1}{2}} \cdot \left(\frac{d}{D}\right)\theta_{\mathfrak{a}_0,\lambda_0}(x,y)\cdot\delta_z.$$

Here $\delta : k_\infty^\times \to \{\pm 1\}$ is the local norm symbol at ∞, i.e. $\delta_z = 1$ if $z \in k_\infty^\times$ is a norm of an element in $K_\infty = k_\infty(\sqrt{D})$ and -1 otherwise. Then from the transformation law of $\theta_{\mathfrak{a}_0,\lambda_0}$ (cf. [10] Proposition 5.1), $\Theta_{\mathfrak{a}_0,\lambda_0}$ is a weight 1 function on $\mathrm{GL}_2(k_\infty)$ satisfying that for $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_0^{(1)}(D)$, $z \in k_\infty^\times$, $g \in \mathrm{GL}_2(k_\infty)$, $\gamma_\infty \in \Gamma_\infty^1$,

$$\Theta_{\mathfrak{a}_0,\lambda_0}(\gamma g\gamma_\infty z) = \left(\frac{d}{D}\right)\Theta_{\mathfrak{a}_0,\lambda_0}(g)\delta_z.$$

2. *Automorphic forms of Drinfeld type.* Given a non-zero ideal N of A. Let

$$\Gamma_0(N) := \left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(A) : c \equiv 0 \bmod N\right\}.$$

An automorphic form f of Drinfeld type for $\Gamma_0(N)$ is a $\mathbb{C}$-valued function on the double coset space $\Gamma_0(N)\backslash\mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$ satisfying the following *harmonic property*: for any element g in $\mathrm{GL}_2(k_\infty)$,

$$f\left(g\begin{pmatrix}0 & 1\\ \pi_\infty & 0\end{pmatrix}\right) = -f(g) \text{ and } \sum_{\mu\in\mathrm{GL}_2(\mathcal{O}_\infty)/\Gamma_\infty} f(g\mu) = 0.$$

Here Γ_∞ is the Iwahori subgroup

$$\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) : c \equiv 0 \bmod \pi_\infty\mathcal{O}_\infty\right\}.$$

The harmonicity implies that f is of weight 2.

Remark. The space of weight 2 functions on $\Gamma_0(N)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$ is generated by automorphic forms of Drinfeld type for $\Gamma_0(N)$ together with a "non-harmonic" function $\mathcal{E}$ on the double coset space $\mathrm{GL}_2(A)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$ satisfying that for $g \in \mathrm{GL}_2(k_\infty)$

$$\mathcal{E}(g) + \mathcal{E}(g \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}) = \frac{q}{1-q},$$

and the Fourier expansion of $\mathcal{E}$ is:

$$\mathcal{E}\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} = q^{-r+2}\left[\frac{1}{1-q^2} + \sum_{\substack{0\neq\lambda\in A \\ \deg\lambda\leq r+2}} \sigma(\lambda)\psi_\infty(\lambda u)\right]$$

where $r \in \mathbb{Z}$ and $u \in k_\infty$. Here σ is the divisor function $\sigma(\lambda) := \sum_{\text{monic } m|\lambda} |m|$. Restricting $\mathcal{E}$ to the subgroup $\mathbb{H}_\infty$, it agrees with the *improper Eisenstein series* of Gekeler (cf. [1] 5.5).

3. *Principal series and Eisenstein series.* Given an integer ν. Consider the *principal series* $\pi(|\cdot|^{\frac{\nu-1}{2}}, |\cdot|^{\frac{1-\nu}{2}})$ (cf. [7] Chap. I §3), which is the space of smooth functions f on $\mathrm{GL}_2(k_\infty)$ satisfying that for any element $g_\infty \in \mathrm{GL}_2(k_\infty)$, a and $d \in k_\infty^\times$, $b \in k_\infty$,

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g_\infty\right) = |a|^{\frac{\nu}{2}} \cdot |d|^{\frac{-\nu}{2}} f(g_\infty).$$

The subspace of functions f in $\pi(|\cdot|^{\frac{\nu-1}{2}}, |\cdot|^{\frac{1-\nu}{2}})$ such that

$$f(g_\infty \gamma_\infty) = f(g_\infty) \text{ for all } \gamma_\infty \in \mathrm{GL}_2(\mathcal{O}_\infty)$$

is spanned by the function φ_ν where for x and w in $k_\infty^\times$, y in k_∞, γ_∞ in $\mathrm{GL}_2(\mathcal{O}_\infty)$,

$$\varphi_\nu\left(\begin{pmatrix} x & y \\ 0 & w \end{pmatrix} \gamma_\infty\right) := |x|^{\frac{\nu}{2}} |w|^{-\frac{\nu}{2}}.$$

Define

$$\Phi_\nu(g_\infty) := \varphi_\nu(g_\infty) - q^{\frac{\nu}{2}-1}\varphi_\nu(g_\infty \begin{pmatrix} 1 & 0 \\ 0 & \pi_\infty \end{pmatrix})$$

for any $g_\infty \in \mathrm{GL}_2(k_\infty)$. Then Φ_ν is a function of weight ν (with trivial central character) with

$$\Phi_\nu \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = |x|^{\frac{\nu}{2}}(1 - q^{\nu-1}) \text{ and } \Phi_\nu\left(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right) = |x|^{\frac{\nu}{2}}(q^{\frac{\nu}{2}} - q^{\frac{\nu}{2}-1}).$$

Fix an integer $\nu > 2$. Let

$$B(A) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) : c = 0\right\}.$$

Then for any $\gamma \in B(A)$ and $g_\infty \in \mathrm{GL}_2(k_\infty)$, one has

$$\Phi_\nu(\gamma g_\infty) = \Phi_\nu(g_\infty).$$

Consider the Eisenstein series

$$E_\nu(g_\infty) := \sum_{\gamma \in B(A)\backslash \mathrm{GL}_2(A)} \Phi_\nu(\gamma g_\infty), \text{ for } g_\infty \in \mathrm{GL}_2(k_\infty).$$

Then E_ν is a well-defined weight ν function on $\mathrm{GL}_2(A)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$.

2.2. **Functions of half integral weight.** In §1.3 we note that Θ is a function of weight $\frac{1}{2}$. Here we give weight $\frac{3}{2}$ functions via theta series from reduced norm form on lattices of pure quaternions inside quaternion algebras.

Let $\mathcal{D} = \mathcal{D}_{N^-}$ be a "definite" quaternion algebra over k (i.e. $\mathcal{D}$ is ramified at ∞) where N^- is the product of finite ramified primes of $\mathcal{D}$. Choose an ideal N^+ of A which is prime to N^-. An *Eichler order* R_{N^+,N^-} *of type* (N^+, N^-) is an (A-)order in $\mathcal{D}$ such that the (A_P-)order $(R_{N^+,N^-})_P := R_{N^+,N^-} \otimes_A A_P$ is maximal in $\mathcal{D}_P := \mathcal{D} \otimes_k k_P$ for all $P \nmid N^+$, and for $P \mid N^+$ there exist an isomorphism $\varphi_P : \mathcal{D}_P \cong \mathrm{Mat}_2(k_P)$ such that

$$\varphi_P\big((R_{N^+,N^-})_P\big) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(A_P) : c \equiv 0 \bmod N^+ A_P \right\}.$$

A *left ideal* I of R_{N^+,N^-} is a rank 4 A-lattice in $\mathcal{D}$ such that for all finite primes P, $I_P := I \otimes_A A_P = R_{N^+,N^-} \cdot g_P$ for some $g_P \in \mathcal{D}_P^\times$.

Let $I_1, ..., I_n$ be representatives of left ideal classes of R. For each i, let R_i be the right order of I_i. Consider the A-lattice S_i of pure quaternions in R_i, i.e. $S_i := \{b \in R_i : \mathrm{Tr}(b) = 0\}$. For $(x, y) \in k_\infty^\times \times k_\infty$, define the following theta series for S_i:

$$\vartheta_i(x, y) := \frac{1}{2} \sum_{b \in S_i} \phi_\infty(\mathrm{Nr}(b) x t^2) \psi_\infty(\mathrm{Nr}(b) y).$$

Proposition 2.1. (cf. [17] Proposition 3.1) *Let* $N_0 = N^+ \cdot N^-$. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(N_0)$ *with* $v_\infty(cx) > v_\infty(cy + d)$, *we have*

$$\vartheta_i(\gamma \circ (x, y)) = |cy + d|^{\frac{3}{2}} \epsilon(\gamma, x, y)^3 \vartheta_i(x, y).$$

Given $\tilde{g} \in \mathbb{G}$. Write $\tilde{g}$ as $\tilde{z}\tilde{\gamma}(h, \xi)\tilde{\gamma}_\infty$, where z in $k_\infty^\times$, γ in $\Gamma_0^{(1)}(N_0)$, γ_∞ in Γ_∞^1, $h = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ in $\mathbb{H}_\infty$, and ξ in S^1. For each i, we extend ϑ_i to a function Θ_i on $\mathbb{G}$:

$$\Theta_i(\tilde{g}) := |x|^{\frac{3}{4}} \cdot \vartheta_i(x, y) \cdot \xi^3.$$

Then Θ_i is of weight $\frac{3}{2}$ and for $z \in k_\infty^\times$, $\gamma \in \Gamma_0^{(1)}(N_0)$, $\gamma_\infty \in \Gamma_\infty^1$, $\xi \in S^1$

$$\Theta_i(\tilde{z}\tilde{\gamma}(1, \xi)\tilde{g}\tilde{\gamma}_\infty) = \xi^3 \Theta_i(\tilde{g}).$$

Given $\lambda \in A$ with $\lambda \neq 0$, the Fourier coefficient

$$\Theta_i^*(r, \lambda) = \begin{cases} 0 & \text{if } \deg \lambda + 2 > r \text{ or } -\lambda \in (k_\infty^\times)^2, \\ \frac{1}{2} \cdot q^{-\frac{3}{4}r} \cdot \#\{b \in S_i : \mathrm{Nr}(b) = \lambda\} & \text{if } \deg \lambda + 2 \leq r \text{ and } -\lambda \notin k_\infty^2. \end{cases}$$

When $\deg \lambda + 2 \leq r$ and $-\lambda \notin k_\infty^2$, $\Theta_i^*(r, \lambda)$ can be expressed by number of *optimal embeddings* into R_i. Let $d \in A - k_\infty^2$. An optimal embedding of the quadratic order $O_d := A[\sqrt{d}]$ into R_i is an embedding ι of $k(\sqrt{d})$ into $\mathcal{D}$ so that

$$\iota\big(k(\sqrt{d})\big) \cap R_i = \iota(O_d).$$

For any $\alpha \in R_i^\times$, $\alpha^{-1}\iota\alpha$ is also an optimal embedding of O_d into R_i. Let $w_i := \#(R_i^\times/\mathbb{F}_q^\times)$, $u(d) := \#(O_d^\times/\mathbb{F}_q^\times)$, and $h_i(d)$ denote the number of optimal embeddings of O_d into R_i modulo conjugation by elements in $R_i^\times$. We have

Proposition 2.2. *For* $0 \neq \lambda \in A$ *with* $\deg \lambda + 2 \leq r$,

$$\Theta_i^*(r, \lambda) = q^{-\frac{3}{4}r} \cdot \frac{w_i}{2} \sum_{-\lambda = df^2, f \text{ monic}} \frac{h_i(d)}{u(d)}.$$

3. Hecke operators

Here we assume $\kappa = 2\nu + 1$ where $\nu \in \mathbb{Z}_{\geq 0}$ and study Hecke operators on weight $\frac{\kappa}{2}$ functions.

For $0 \neq m \in A$, consider the double coset $\widetilde{\Gamma_1(N)}\left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, 1\right)\widetilde{\Gamma_1(N)}$ in $\mathbb{G}$ where

$$\Gamma_1(N) = \left\{\gamma \in \mathrm{SL}_2(A) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N\right\}.$$

One has

$$\widetilde{\Gamma_1(N)}\left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, 1\right)\widetilde{\Gamma_1(N)} = \coprod_i \widetilde{\Gamma_1(N)}\left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, 1\right) \cdot \tilde{\gamma}_i$$

where $\tilde{\gamma}_i$ are right coset representatives of the following subgroup in $\widetilde{\Gamma_1(N)}$:

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, 1\right)^{-1} \widetilde{\Gamma_1(N)} \left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, 1\right) \bigcap \widetilde{\Gamma_1(N)}.$$

Let f be a weight $\frac{\kappa}{2}$ function f on $\mathbb{G}/\widetilde{\Gamma^1_\infty}$ with "central" character χ_∞ on $k_\infty^\times$. Suppose $f(\tilde{\gamma}\tilde{g}) = f(\tilde{g})$ for all $\gamma \in \Gamma_1(N)$. For each monic polynomial m in A, define

$$f\left(\widetilde{\Gamma_1(N)}\left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, 1\right)\widetilde{\Gamma_1(N)}\tilde{g}\right) := \sum_i f\left(\left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, 1\right)\tilde{\gamma}_i\tilde{g}\right).$$

Lemma 3.1. *Let m be a monic polynomial in A. Suppose the prime-to-N part of the ideal (m) is not a square ideal of A, then for all $\tilde{g} \in \mathbb{G}$*

$$f\left(\widetilde{\Gamma_1(N)}\left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, 1\right)\widetilde{\Gamma_1(N)}\tilde{g}\right) = 0.$$

Definition 3.2. Let f be a weight $\frac{\kappa}{2}$ function f on $\mathbb{G}/\widetilde{\Gamma^1_\infty}$ with "central" character χ_∞ on $k_\infty^\times$. Suppose $f(\tilde{\gamma}\tilde{g}) = f(\tilde{g})$ for all $\gamma \in \Gamma_1(N)$. Given any prime P of A. We define for $\tilde{g} \in \mathbb{G}$

$$T_{P^2,\frac{\kappa}{2}} f(\tilde{g}) := |P|^{(\frac{\kappa}{2}-2)} \cdot f\left(\widetilde{\Gamma_1(N)}\left(\begin{pmatrix} 1 & 0 \\ 0 & P^2 \end{pmatrix}, 1\right)\widetilde{\Gamma_1(N)}\tilde{g}\right).$$

As in the classical case, we have

Proposition 3.3. *Let $\chi_N : (A/NA)^\times$ be a Dirichlet character. Given a weight $\frac{\kappa}{2}$ function f on $\mathbb{G}$ with "central" character χ_∞ such that $f(\tilde{\gamma}\tilde{g}) = \chi_N(d)f(\tilde{g})$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(N)$ and $\tilde{g} \in \mathbb{G}$. Then for each prime P,*

$$T_{P^2,\frac{\kappa}{2}} f\left(\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix}, 1\right) = \sum_{\deg \lambda + 2 \leq r} (T_{P^2,\frac{\kappa}{2}} f)^*(r, \lambda)\psi_\infty(\lambda u)$$

where

$$\begin{aligned}(T_{P^2,\frac{\kappa}{2}} f)^*(r,\lambda) \;=\; & \chi_\infty(P^2) \cdot \left[|P|^{\frac{\kappa}{2}} f^*(r + 2\deg P, P^2\lambda)\right] \\ & + \chi_N(P)\chi_\infty(P)\left(\frac{(-1)^\nu \lambda}{P}\right)|P|^{(\nu-1)} \cdot \left[f^*(r,\lambda)\right] \\ & + \chi_N(P^2)|P|^{\kappa-2} \cdot \left[|P|^{-\frac{\kappa}{2}} f^*(r - 2\deg P, \frac{\lambda}{P^2})\right].\end{aligned}$$

Here we set $f^*(r - 2\deg P, \frac{\lambda}{P^2}) = 0$ if $P^2 \nmid \lambda$ and $\chi_N(P) = 0$ if $P|N$.

Proof. For sake of completeness we sketch the argument.

Consider the double coset $\Gamma_1(N)\begin{pmatrix}1 & 0\\ 0 & P^2\end{pmatrix}\Gamma_1(N)$, which is equal to

$$\begin{cases} \bigcup_{\deg v<2\deg P}\Gamma_1(N)a_v & \text{if } P|N,\\ \left(\bigcup_{\deg v<2\deg P}\Gamma_1(N)a_v\right)\cup\left(\bigcup_{h\neq 0,\deg h<\deg P}\Gamma_1(N)b_h\right)\cup\Gamma_1(N)\tau & \text{if } P\nmid N.\end{cases}$$

Here $a_v=\begin{pmatrix}1 & v\\ 0 & P^2\end{pmatrix}$, $b_h=\varsigma_P\begin{pmatrix}P & h\\ 0 & P\end{pmatrix}$, $\tau=\varsigma_{P^2}\begin{pmatrix}P^2 & 0\\ 0 & 1\end{pmatrix}$, ς_m is an element in $\mathrm{SL}_2(A)$ such that $\varsigma_m\equiv\begin{pmatrix}m^{-1} & 0\\ 0 & m\end{pmatrix}$ mod N. The contribution of a_v is:

$$\begin{aligned} &|P|^{(\frac{\kappa}{2}-2)}\cdot\sum_{\deg v<2\deg P} f\left(\left(\begin{pmatrix}1 & v\\ 0 & P^2\end{pmatrix},1\right)\cdot\left(\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix},1\right)\right)\\ =&\sum_{\deg\lambda+2\leq r}\chi_\infty(P^2)\cdot\left[|P|^{\frac{\kappa}{2}}f^*(r+2\deg P,P^2\lambda)\right]\psi_\infty(\lambda u).\end{aligned}$$

When P divides N, $\chi_N(P)=\chi_N(P^2)=0$ and so the formula holds.

Assume P is prime to N. The contribution of τ is:

$$\begin{aligned} &|P|^{(\frac{\kappa}{2}-2)}f\left(\tilde{\varsigma}_{P^2}\left(\begin{pmatrix}\pi_\infty^{r-2\deg P} & P^2u\\ 0 & 1\end{pmatrix},1\right)\right)\\ =&\sum_{\deg\lambda+2\leq r}\chi_N(P^2)|P|^{(\kappa-2)}\left[|P|^{-\frac{\kappa}{2}}f^*(r-2\deg P,\frac{\lambda}{P^2})\right]\cdot\psi_\infty(\lambda u).\end{aligned}$$

For the contribution of b_h with $h\neq 0$ and $\deg h<\deg P$, let $\gamma_h,\gamma_h'\in\Gamma_1(N)$ such that $\gamma_h'\begin{pmatrix}1 & 0\\ 0 & P^2\end{pmatrix}\gamma_h=b_h$.

Then $\tilde{\gamma}_h'\left(\begin{pmatrix}1 & 0\\ 0 & P^2\end{pmatrix},1\right)\tilde{\gamma}_h=\tilde{\varsigma}_P\left(\begin{pmatrix}P & h\\ 0 & P\end{pmatrix},\left(\frac{h}{P}\right)\right)$ and

$$\begin{aligned} &|P|^{(\frac{\kappa}{2}-2)}f\left(\left(\begin{pmatrix}1 & 0\\ 0 & P^2\end{pmatrix},1\right)\tilde{\gamma}_h\left(\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix},1\right)\right)\\ =&\left[\omega(P)^\kappa\left(\frac{h}{P}\right)|P|^{-\frac{1}{2}}\psi_\infty(\frac{\lambda h}{P})\right]\cdot\sum_{\deg\lambda+2\leq r}\chi_N(P)\chi_\infty(P)|P|^{(\nu-1)}f^*(r,\lambda)\psi_\infty(\lambda u)\end{aligned}$$

Since $\omega(P)^{2\nu}=\left(\frac{(-1)^\nu}{P}\right)$ and $|P|^{-\frac{1}{2}}\sum_{h\neq 0,\deg h<\deg P}\left(\frac{h}{P}\right)\psi_\infty(\frac{\lambda h}{P})=\left(\frac{\lambda}{P}\right)\omega(P)^{-1}$,

$$\begin{aligned} &|P|^{(\frac{\kappa}{2}-2)}\sum_{h\neq 0,\deg h<\deg P} f\left(\left(\begin{pmatrix}1 & 0\\ 0 & P^2\end{pmatrix},1\right)\tilde{\gamma}_h\left(\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix},1\right)\right)\\ =&\sum_{\deg\lambda+2\leq r}\chi_N(P)\chi_\infty(P)|P|^{(\nu-1)}\left(\frac{(-1)^\nu\lambda}{P}\right)\left[f^*(r,\lambda)\right]\psi_\infty(\lambda u).\end{aligned}$$

Combining these we get the formula for the Fourier coefficients of $T_{P^2,\frac{\kappa}{2}}f$. □

From lattices of pure quaternions in the right orders of left ideals $I_1, \ldots, I_n$ of Eichler order R_{N^+,N^-} in $D_{(N^-)}$ we have introduced weight $\frac{3}{2}$ theta series Θ_i for $1 \leq i \leq n$ in §2.2. Suppose $N^+ = 1$, i.e. $R = R_{N^+,N^-}$ is a maximal order (and $N_0 = N^-$). The action of Hecke operators on Θ_i can be expressed by so-called *Brandt matrices*.

Given a left ideal I of R_{N^+,N^-}, the set $I^{-1} = \{b \in \mathcal{D} : IbI \subset I\}$ is a right ideal for R_{N^+,N^-} whose left order is the right order of I. The reduced ideal norm of I is denoted by $\mathrm{Nr}(I)$, which is the fractional ideal of A generated by the reduced norm of elements in I. For each monic polynomial m in A, the *m-th Brandt matrix* $B(m) := \Big(B_{ij}(m)\Big)_{1\leq i,j\leq n}$ in $\mathrm{Mat}_n(\mathbb{Z})$ where

$$B_{ij}(m) := \frac{\#\{b \in I_j^{-1}I_i : (\mathrm{Nr}(b)/N_{ij}) = (m)\}}{(q-1)w_j}.$$

Here N_{ij} is the monic generator of the reduced ideal norm $\mathrm{Nr}(I_j)^{-1}\,\mathrm{Nr}(I_i)$ of $I_j^{-1}I_i$, and $w_j = \#(R_j^\times/\mathbb{F}_q^\times)$ where R_j is the right order of I_j.

Proposition 3.4. *For every prime P of A,*

$$T_{P^2,\frac{3}{2}}\Theta_i = \sum_j B_{ij}(P)\Theta_j = w_i \sum_j B_{ji}(P)(\frac{1}{w_j}\Theta_j).$$

Proof. For each monic polynomial m in A, $w_jB_{ij}(m) = w_iB_{ji}(m)$ and so the second equality is clear. We only need to show the first equality.

When $P \mid N_0$, $T_{P^2,\frac{3}{2}}\Theta_i^*(r,\lambda) = |P|^{\frac{3}{2}}\Theta_i^*(r + 2\deg P, P^2\lambda)$ for $\lambda \in A$ with $\deg\lambda + 2 \leq r$. By Proposition 2.2 and the fact that $h_i(P^2d) = 0$ for any d and prime $P \mid N_0$, we get $|P|^{\frac{3}{2}}\Theta_i^*(r + 2\deg P, P^2\lambda) = \Theta_i^*(r,\lambda)$ and so $T_{P^2,\frac{3}{2}}\Theta_i = \Theta_i$.

On the other hand, $B(P)$ is a permutation matrix of order 2, and the entry $B_{ij}(P) = 1$ implies $R_i \cong R_j$. Therefore the proposition holds for $P \mid N_0$.

Suppose $P \nmid N_0$. For $\lambda \in A$ and $r \geq \deg\lambda + 2$, let $c_i(\lambda) =: q^{\frac{3}{4}r}\Theta_i(r,\lambda)$. To prove this proposition, we need to show

$$c_i(P^2\lambda) + \Big(\frac{-\lambda}{P}\Big)c_i(\lambda) + |P|c_i(\frac{\lambda}{P^2}) = \sum_j B_{ij}(P)c_j(\lambda).$$

For each j, consider the map

$$\{\alpha \in I_j^{-1}I_i : (\mathrm{Nr}(\alpha)/N_{ij}) = (P)\} \times \{\epsilon a : a \in S_j, \mathrm{Nr}(a) = \lambda, \epsilon \in \mathbb{F}_q^\times\} \longrightarrow S_i$$

which is defined by

$$(\alpha, \epsilon a) \longmapsto b = \alpha^* \cdot \epsilon a \cdot \alpha.$$

Here $\alpha^* := \bar{\alpha}N_{ij}$. Then $\mathrm{Nr}(b) = \lambda P^2\epsilon'^2$ for some $\epsilon' \in \mathbb{F}_q^\times$. Note that

$$\#\{\alpha \in I_j^{-1}I_i : (\mathrm{Nr}(\alpha)/N_{ij}) = (P)\} = (q-1)w_jB_{ij}(P), \text{ and}$$

$$\#\{\epsilon a : a \in S_j, \mathrm{Nr}(a) = \lambda, \epsilon \in \mathbb{F}_q^\times\} = (q-1)c_j(\lambda).$$

Given $b \in S_i$ with $\mathrm{Nr}(b) = \lambda P^2\epsilon^2$ for $\epsilon \in \mathbb{F}_q^\times$. First, we consider the case when $P \nmid b$, i.e. $b \neq Pa$ for any $a \in S_i$. Then there exists a unique $\alpha \in I_j^{-1}I_i$ for some j, up to $R_j^\times$, with $(\mathrm{Nr}(\alpha)/N_{ij}) = (P)$ such that $b = \beta\alpha$ where $\beta \in I_i^{-1}I_j$. Since $\mathrm{Tr}(b) = b + \bar{b} = 0$, $b = -\bar{\alpha}\bar{\beta} = -\alpha^*\beta^*$ and so $\beta^* = \epsilon a\alpha$ with $a \in S_j$, $\mathrm{Nr}(a) = \lambda$ and $\epsilon \in \mathbb{F}_q^\times$. Therefore in this case, $b = \alpha^*(-\epsilon a)\alpha$ for a unique $\alpha \in I_j^{-1}I_i$ up to $R_j^\times$.

Now, we suppose $P \mid b$ and consider the following two cases:
(1) Assume $P \mid \lambda$. Write b as $P \cdot \epsilon a$ for some $a \in S_i$ with $\mathrm{Nr}(a) = \lambda$. If $P \nmid a$, there exist

an element $\alpha \in I_j^{-1}I_i$ for some j, up to $R_j^\times$, such that $(\mathrm{Nr}(\alpha)/N_{ij}) = (P)$ and $a = \beta\alpha$ with $\beta \in I_i^{-1}I_j$. Therefore

$$b = P \cdot \epsilon a = \alpha^* \cdot (\alpha\epsilon\beta) \cdot \alpha.$$

If $P \mid a$, then $a = a'P$ for some $a' \in S_i$ with $\mathrm{Nr}(a') = \frac{\lambda}{P^2}$ and so

$$b = P\epsilon a' P = \alpha^* \cdot (\alpha\epsilon' a' \alpha^*) \cdot \alpha$$

for any j and any $\alpha \in I_j^{-1}I_i$ with $(\mathrm{Nr}(\alpha)/N_{ij}) = (P)$. Hence we obtain

$$\sum_j B_{ij}(P)c_j(\lambda) = \big(c_i(\lambda P^2) - c_i(\lambda)\big) + \big(c_i(\lambda) - c_i(\frac{\lambda}{P^2})\big) + (|P|+1)c_i(\frac{\lambda}{P^2})$$
$$= c_i(\lambda P^2) + |P|c_i(\frac{\lambda}{P^2}).$$

(2) Assume $P \nmid \lambda$. Write b as $\epsilon a P$ where $a \in S_i$ with $\mathrm{Nr}(a) = \lambda$. If $b = \alpha^*\epsilon' a' \alpha$ where $\alpha \in I_j^{-1}I_i$ with $\mathrm{Nr}(a) = \lambda$ and $\epsilon' \in \mathbb{F}_q^\times$, then

$$\alpha^*\epsilon' a' \alpha = P\epsilon a = \alpha^*(\alpha\epsilon a).$$

Let $I = I_i^{-1}I_j\alpha$. Then $I \subset R_i$ with $\mathrm{Nr}(I) = (P)$ and $I \cdot a \subset I$. We deduce that there exist a unique prime ideal $\mathfrak{p}$ of the quadratic order $A[a]$ so that $I = R_i\mathfrak{p}$. Since there are only $1 + \left(\frac{-\lambda}{P}\right)$ ideals of $A[a]$ whose ideal norm is P, b can be written as $\alpha^*\epsilon' a\alpha$ with $\mathrm{Nr}(a) = \lambda$ in $1 + \left(\frac{-\lambda}{P}\right)$ ways, up to $R_j^\times$ if $\alpha \in I_j^{-1}I_i$. Combining these we have that when $P \nmid \lambda$

$$\sum_j B_{ij}(P)c_j(\lambda) = \big(c_i(\lambda P^2) - c_i(\lambda)\big) + (1 + \left(\frac{-\lambda}{P}\right))c_i(\lambda) = c_i(\lambda P^2) + \left(\frac{-\lambda}{P}\right)c_i(\lambda).$$

By (1) and (2) the proposition holds. □

4. Special values and an analogue of Waldspurger's formula

In this section we present a function field analogue of Waldspurger's formula.

4.1. **Definite Shimura curves and automorphic forms.** Let $\mathcal{D} = \mathcal{D}_{N^-}$ be a definite quaternion algebra over k and let N^- be the product of finite ramified primes of $\mathcal{D}$. Choose an ideal N^+ of A prime to N^-. The *definite Shimura curve* $X = X_{N^+,N^-}$ *of type* (N^+, N^-) is

$$\hat{R}^\times_{N^+,N^-}\backslash(\hat{\mathcal{D}}^\times \times Y)/\mathcal{D}^\times.$$

Here $\hat{\mathcal{D}} := \mathcal{D} \otimes_k \hat{k}$; $\hat{R}_{N^+,N^-} := R_{N^+,N^-} \otimes_A \hat{A}$ where R_{N^+,N^-} is an Eichler order of type (N^+, N^-); Y is the curve of genus zero such that for each k-algebra M,

$$Y(M) := \{x \in \mathcal{D} \otimes_k M : x \neq 0, \mathrm{Tr}(x) = \mathrm{Nr}(x) = 0\}/M^\times.$$

The (right) action of $\mathcal{D}^\times$ on Y is by conjugation. It is known that X is a disjoint finite union of genus zero curves, and the components correspond canonically to left ideal classes of R_{N^+,N^-}.

From now on we assume $N^+ = 1$, i.e. $R_{N^+,N^-} = R$ is a maximal order and $N_0 = N^-$. Let $I_1, \dots, I_n$ be representatives of left ideal classes of R, and let X_i be the component of X corresponding to I_i. Denote e_i to be the divisor class in $\mathrm{Pic}(X)$ corresponding to X_i. Then $\mathrm{Pic}(X) = \oplus\mathbb{Z}e_i$. The *Gross height pairing* on $\mathrm{Pic}(X)$ is defined by

$$< e_i, e_j >= \begin{cases} 0 & \text{if } i \neq j, \\ w_i = \#(R_i^\times/\mathbb{F}_q^\times) & \text{if } i = j, \end{cases}$$

and extending bi-additively.

As in the case of definite Shimura curves over $\mathbb{Q}$ (cf. [4]), we have *Hecke correspondence* t_m on X for each monic polynomial m in A which satisfies

Proposition 4.1. *For $i = 1, 2, ..., n$,*

$$t_m e_i = \sum_{j=1}^{n} B_{ij}(m) e_j.$$

Here $B(m) = \big(B_{ij}(m)\big)_{1 \leq i,j \leq n}$ is the m-th Brandt matrix.

We point out that the Hecke correspondences are self-adjoint with respect to the Gross height pairing.

Function field analogue of Eichler's theory. Recall that an automorphic form f of Drinfeld type for $\Gamma_0(N)$ is a $\mathbb{C}$-valued function on $\Gamma_0(N)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$ satisfying the harmonic property in §2.1. We say f is a *cusp form* if for all $g_\infty \in \mathrm{GL}_2(k_\infty)$ and $\gamma \in \mathrm{GL}_2(A)$,

$$\int_{A\backslash k_\infty} f\left(\gamma \begin{pmatrix} 1 & h_\gamma u \\ 0 & 1 \end{pmatrix} g_\infty\right) du = 0.$$

Here the Haar measure du is normalized so that $\int_{A\backslash k_\infty} 1\, du = 1$, and h_γ is a generator of the ideal of A which is maximal for the property that

$$\gamma \begin{pmatrix} 1 & h_\gamma A \\ 0 & 1 \end{pmatrix} \gamma^{-1} \subset \Gamma_0(N).$$

For each non-zero ideal N of A, the *Petersson inner product* on the $\mathbb{C}$-vector space $S(\Gamma_0(N))$ of Drinfeld type cusp forms for $\Gamma_0(N)$ is a non-degenerate pairing

$$(f, g) := \int_{G_0(N)} f \cdot \overline{g}.$$

Here $G_0(N) = \Gamma_0(N)\backslash \mathrm{GL}_2(k_\infty)/\Gamma_\infty k_\infty^\times$. The measure of each double coset $[e]$ in $G_0(N)$ is normalized to be

$$d([e]) := \frac{q-1}{2} \cdot \frac{1}{\#(\mathrm{Stab}_{\Gamma_0(N)}(e))}.$$

Definition 4.2. A Drinfeld type cusp form f for $\Gamma_0(N)$ is a *new form* if, with respect to Petersson inner product, f is orthogonal to functions

$$g\left(\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} g_\infty\right)$$

for all Drinfeld type cusp form g for $\Gamma_0(M)$ where $M \mid N$ and $d \mid (N/M)$. We call f a *newform* if f is also a Hecke eigenform.

Let $M^{\mathrm{new}}(\Gamma_0(N_0)) := S^{\mathrm{new}}(\Gamma_0(N_0) \oplus \mathbb{C}\mathcal{E}_{N_0}$, where $S^{\mathrm{new}}(\Gamma_0(N_0)$ is the space of Drinfeld type new forms and $\mathcal{E}_{N_0}$ is an analogue of Eisenstein series with Fourier expansion given by: for $r \in \mathbb{Z}$ and $u \in k_\infty$

$$\mathcal{E}_{N_0}\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} = q^{-r+2}\left(\frac{1}{q^2-1}\prod_{P \mid N_0}(|P|-1) + \sum_{\substack{m \in A,\ \text{monic} \\ \deg m + 2 \leq r}} \sigma_{N_0}(m) \sum_{\epsilon \in \mathbb{F}_q^\times} \psi_\infty(\epsilon m u)\right).$$

Here $\sigma_{N_0}(m)$ is the divisor function $\sigma_{N_0}(m) := \sum_{\substack{m' \mid m,\ \text{monic} \\ (N_0, m') = 1}} |m'|$.

As in classical Eichler's theory, one can establish an isomorphism (cf. [18] Theorem 2.5)

$$\begin{array}{ccc} M^{\mathrm{new}}(\Gamma_0(N_0)) & \longrightarrow & \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C} \\ f & \longmapsto & e_f \end{array}$$

such that $T_m f \mapsto t_m e_f$ for all monic polynomials m in A, and $< e_f, e_f >= 1$ for each newform f *normalized* so that the "first" Fourier coefficient $f^*(2, 1)$ is one. Here the T_m are Hecke operators on automorphic forms for $\Gamma_0(N_0)$.

4.2. **The Shimura map Sh.** Consider the definite Shimura curve $X = X_{1,N^-}$ (i.e. $N^+ = 1$ and $N_0 = N^-$). For any element $e = \sum_i a_i e_i \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$, define

$$\Theta_e := \sum_i a_i \Theta_i$$

where Θ_i are weight $\frac{3}{2}$ function introduced in §2.2. For each $a \in A - k_\infty^2$, let

$$e_a := \sum_{i=1}^{n} \left(\sum_{a=df^2, f \text{ monic}} \frac{h_i(d)}{2u(d)} \right) e_i \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

From Proposition 2.2, the Fourier coefficients of Θ_e are

$$\Theta_e^*(r, \lambda) = q^{-\frac{3}{4}r} \cdot < e, e_{-\lambda} >$$

for $-\lambda \neq 0 \in A - k_\infty^2$ with $\deg \lambda + 2 \leq r$, and $\Theta_e^*(r, 0) = q^{-\frac{3}{4}r} \cdot \deg e/2$.

Given any prime P of A, Proposition 3.4 and Proposition 4.1 tells us that

$$T_{P^2, \frac{3}{2}} \Theta_e = \Theta_{(t_P e)}.$$

Let $M_{\frac{3}{2}}(\Gamma_0^{(1)}(N_0))$ be the space of weight $\frac{3}{2}$ functions on the double coset space

$$\widetilde{k_\infty^\times} \widetilde{\Gamma_0^{(1)}(N_0)} \Big\backslash \mathbb{G} \Big/ \widetilde{\Gamma_\infty^1}.$$

Since $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ and the space $M^{\mathrm{new}}(\Gamma_0(N_0))$ of automorphic "new" forms of Drinfeld type for $\Gamma_0(N_0)$ are isomorphic Hecke modules, we have the following map

$$\begin{array}{rcccl} \mathbf{Sh}: \ M^{\mathrm{new}}(\Gamma_0(N_0)) & \cong & \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C} & \longrightarrow & M_{\frac{3}{2}}(\Gamma_0^{(1)}(N_0)) \\ f & \mapsto & e_f & \longmapsto & \Theta_{e_f} =: \mathbf{Sh}(f) \end{array}$$

such that for all prime P of A

$$\mathbf{Sh}(T_P f) = T_{P^2, \frac{3}{2}} \mathbf{Sh}(f).$$

This map **Sh** can be viewed as an analogue of Shimura correspondence. Let $M^{\mathrm{new}}(\Gamma_0(N_0))^+$ be the eigenspace of T_P with eigenvalue 1 for $P \mid N_0$, and let $M^{\mathrm{new}}(\Gamma_0(N_0))^- \subset S^{\mathrm{new}}(\Gamma_0(N_0))$ be the orthogonal component (with respect to Petersson inner product). Then the kernel of **Sh** contains the subspace $M^{\mathrm{new}}(\Gamma_0(N_0))^-$.

For each $f \in M^{\mathrm{new}}(\Gamma_0(N_0))$, let $e_f \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ be the corresponding divisor from the Hecke module isomorphism as in §4.1. Then

$$q^{\frac{3}{4}r} \cdot \mathbf{Sh}(f)^*(r, \lambda) =< e_f, e_{-\lambda} >$$

is independent of r, for $-\lambda \in A - k_\infty^2$ with $\deg \lambda + 2 \leq r$. This value is denoted by $m(f, -\lambda)$ (i.e. the λ-th Fourier coefficient of the weight $\frac{3}{2}$ function $\mathbf{Sh}(f)$)).

Suppose f is a normalized Drinfeld type newform for $\Gamma_0(N_0)$. Let D be an irreducible polynomial in $A - k_\infty^2$ satisfying $\left(\frac{D}{P}\right) = -1$ for all prime factors P of N_0. Set the divisor $e_{f,D}$ to be $< e_f, e_D > \cdot e_f$. Summarizing, we have

Theorem 4.3. *The Gross height* $< e_{f,D}, e_{f,D} >$ *of the divisor* $e_{f,D}$ *is exactly* $m(f, D)^2$.

4.3. **Function field analogue of Waldspurger's formula.** Let N_0 be a square-free ideal of A with odd number ℓ_{N_0} of prime factors. To an automorphic cusp form f of Drinfeld type for $\Gamma_0(N_0)$ one can attach an L-series $L(f,s)$: let $\mathfrak{m}$ be an effective divisor of k written as $\operatorname{div}(\lambda)_0 + (r - \deg\lambda)\infty$ for a nonzero polynomial λ $(= \lambda(\mathfrak{m}))$ in A, with

$$\operatorname{div}(\lambda)_0 := \sum_{\text{finite prime } P} \operatorname{ord}_P(\lambda)P.$$

Set

$$f^*(\mathfrak{m}) := \int_{A\backslash k_\infty} f\begin{pmatrix}\pi_\infty^{r+2} & u\\ 0 & 1\end{pmatrix}\psi_\infty(-\lambda u)du = f^*(r+2,\lambda).$$

The L-series $L(f,s)$ attached to f is

$$L(f,s) := \sum_{\mathfrak{m}\geq 0} f^*(\mathfrak{m})q^{-\deg(\mathfrak{m})s}, \ \ \operatorname{Re}s > 1.$$

Given a square-free $D \in A - k_\infty^2$. Let ε_D be the following quadratic character on divisors of k:

$$\varepsilon_D(P) = \left(\frac{D}{P}\right) \text{ and } \varepsilon_D(\infty) = \begin{cases} -1 & \text{if } \deg D \text{ is even,}\\ 0 & \text{if } \deg D \text{ is odd.}\end{cases}$$

The *twisted L-series of f by ε_D* is:

$$L(f\otimes\varepsilon_D, s) := \sum_{\mathfrak{m}\geq 0} f^*(\mathfrak{m})\varepsilon_D(\mathfrak{m})q^{-\deg\mathfrak{m}s}.$$

Note that $L(f,s)$ and $L(f\otimes\varepsilon_D,s)$ have analytic continuation to s-plane with functional equation for $s\mapsto -s$ (cf. [19] Chap. VII Theorem 2).

Suppose f is a normalized Drinfeld type newform for $\Gamma_0(N_0)$ and $T_Pf = f$ for all prime factors P of N_0. If $D\in A-k_\infty^2$ is irreducible and satisfies $\left(\frac{D}{P}\right) = -1$ for all prime $P \mid N_0$, the central critical value $L(f,0)L(f\otimes\varepsilon_D,0)$ can be expressed by the Gross height of $e_{f,D}$ times a "period" constant related to f, N_0, and D. More precisely, let

$$C(N_0,f,D) := (q^{\frac{(-1)^{\deg D}-1}{4}})\cdot\frac{(3-(-1)^{\deg D})\cdot(f,f)}{2\cdot|D|^{\frac{1}{2}}\cdot 4^{(\ell_{N_0}-1)}}.$$

Then

$$L(f,0)L(f\otimes\varepsilon_D,0) = C(N_0,f,D)\cdot < e_{f,D}, e_{f,D} > .$$

This formula can be obtained by Rankin's method (cf. [18] Theorem 3.3). Therefore Theorem 4.3 leads to our analogue of Waldspurger's formula in Theorem 0.1:

Corollary 4.4. *Let N_0 be a square-free ideal of A with odd number ℓ_{N_0} of prime factors. Let f be a normalized Drinfeld type newform for $\Gamma_0(N_0)$. Suppose $T_Pf = f$ for all prime factors P of N_0. Then for any irreducible polynomial D in $A-k_\infty^2$ with $\left(\frac{D}{P}\right) = -1$ for all prime factors P of N_0, we have*

$$L(f,0)L(f\otimes\varepsilon_D,0) = C(N_0,f,D)\cdot m(f,D)^2$$

Remark. Recall that for each prime P of A and $f\in M^{\text{new}}(\Gamma_0(N_0))$,

$$\mathbf{Sh}(T_Pf) = T_{P^2,\frac{3}{2}}\,\mathbf{Sh}(f).$$

To know the dimension of the image of **Sh**, it suffices to determine which normalized newforms f fixed by T_P for all $P \mid N_0$ satisfy $\mathbf{Sh}(f)\neq 0$. By Corollary 4.4, $\mathbf{Sh}(f)\neq 0$ if there exists an irreducible polynomial D in $A-k_\infty^2$ such that $\left(\frac{D}{P}\right) = -1$ for all $P\mid N_0$ and the central critical value $L(f,0)L(f\otimes\varepsilon_D,0)$ is non-zero. Adapting methods in [5] Theorem 1, it can be shown that for each normalized newform f, it is always possible to choose an irreducible polynomial D in $A - k_\infty^2$ so that

$$\left(\frac{D}{P}\right) = -1 \text{ for all } P \mid N_0 \text{ and } L(f \otimes \varepsilon_D, 0) \neq 0.$$

Therefore the condition reduces to $L(f,0) = 0$ and we claim that the dimension of the image of **Sh** is equal to the number of newforms f with $L(f,0) \neq 0$ plus one (the image of "Eisenstein series" $\mathcal{E}_{N_0}$).

4.4. **Application to elliptic curves.** Let E be an elliptic curve over a global function field F. Mordell-Weil theorem tells us that the abelian group $E(F)$ of F-rational points of E is finitely generated. The Birch and Swinnerton-Dyer conjecture is the following equality:

$$\mathrm{ord}_{s=1} L(E/F, s) \stackrel{?}{=} \mathrm{rank}_{\mathbb{Z}} E(F).$$

Here $L(E/k, s)$ is the Hasse-Weil L-series of E over k. It is known that (cf. [13])

$$\mathrm{ord}_{s=1} L(E/F, s) \geq \mathrm{rank}_{\mathbb{Z}} E(F).$$

We focus on the case when E is defined over k. From the work of Weil, Jacquet-Langlands, and Deligne, one knows that there exists an automorphic cusp form f_E such that

$$L(E/k, s+1) = L(f_E, s).$$

Suppose the conductor of E is $N_0\infty$ and E has split multiplicative reduction at ∞. Then the automorphic form f_E is of Drinfeld type for $\Gamma_0(N_0)$, which is a normalized newform (cf. [2] §8.3).

Assume N_0 is square-free with odd number of prime factors. Given an irreducible polynomial D in $A - k_\infty^\times$ such that $\left(\frac{D}{P}\right) = -1$ for all prime factors P of N_0 and let $K = k(\sqrt{D})$. Then

$$L(E/K, s+1) = L(f_E, s)L(f_E \otimes \varepsilon_D, s).$$

From Corollary 4.4 above, we have

Proposition 4.5. *$E(K)$ is a finite abelian group if $m(f_E, D) \neq 0$.*

Remark. 1. Let E^D be the *twist of E by D.* Suppose the Weierstrass equation of E is $y^2 = x^3 + ax^2 + bx + c$ where $a, b, c \in k$, then the Weierstrass equation of the twist E^D is $y^2 = x^3 + aDx^2 + bD^2x + cD^3$. Note that E and E^D are isomorphic over K via the map $(x, y) \mapsto (\frac{x}{D}, \frac{y}{D\sqrt{D}})$, and

$$L(E/K, s) = L(E/k, s)L(E^D/k, s).$$

Therefore when $m(f_E, D) \neq 0$, the conjecture of Birch and Swinnerton-Dyer is true for E and E^D over k and so both of $E(k)$ and $E^D(k)$ are finite abelian groups.

2. When $m(f_E, D) \neq 0$, the special value of $L(E/K, s)$ at $s = 1$ can be expressed in terms of invariants of E:

$$L(E/K, 1) = \frac{\#\big(\text{Ш}(E/K)\big) \cdot \tau}{\#\big(E(K)_{\text{tors}}\big)^2}.$$

Here $\text{Ш}(E/K)$ is the Tate-Shafarevitch group of E/K, and τ is a Tamagawa number (an analogue of period). Comparing this formula with our result, the constant $m(f_E, D)$ contains information of $\#\big(\text{Ш}(E/K)\big)$ and $\#\big(E(K)_{\text{tors}}\big)$.

Note that the divisor e_{f_E} is in $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let c_E be the minimal positive integer so that $c_E \cdot e_{f_E} \in \mathrm{Pic}(X)$. Then $c_E \cdot m(f_E, D) \in \mathbb{Z}$. In the special case when N_0 is a prime and E is a strong Weil curve, let $\widetilde{E}_{N_0}$ be the reduction of E at N_0. Then c_E^2 is equal to $\deg \pi_E \cdot \varepsilon_E$, where ε_E is the number of components of $\widetilde{E}_{N_0}$ and π_E is the strong uniformization from the Drinfeld modular curve $X_0(N_0)$ to E. Also in this special case, the relation between the value $m(f_E, D)$ and the cardinality of $\text{Ш}(E/K)$ is given explicitly in [9] §4. We expect this should holds for general N_0.

3. Note that $E(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a $\mathrm{Gal}(K/k)$-module. Let ι be the nontrivial element in $\mathrm{Gal}(K/k)$. Then $E(K) \otimes_{\mathbb{Z}} \mathbb{Q}$ decomposes to a direct sum of eigenspaces for ι with eigenvalues ± 1. It is easy to see that $E(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the eigenspace for ι with eigenvalue 1. From the isomorphism $(x, y) \mapsto (Dx, D\sqrt{D}y)$, we identify $E^D(k) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the eigenspace for ι with eigenvalue -1. Hence

$$E(K) \otimes_{\mathbb{Z}} \mathbb{Q} = (E(k) \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (E^D(k) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

Assume further that E has split multiplicative reduction at all bad primes, i.e. f_E is in $M^{\mathrm{new}}(\Gamma(N_0))^+$. Then when $m(f_E, D) = 0$, we have

$$\mathrm{ord}_{s=1} L(E/K, s) = \mathrm{ord}_{s=1} L(E/k, s) + \mathrm{ord}_{s=1} L(E^D/k, s) \geq 1.$$

Therefore if $L(E^D/k, 1) = L(f_E \otimes \varepsilon_D, 0) \neq 0$, E should have a k-rational point of infinite order according to Birch and Swinnerton-Dyer conjecture.

Acknowledgements. The author is grateful to Jing Yu for his steady encouragements, and Henri Darmon for the hospitality and making the author's stay at McGill in 2009 a pleasant one. The author would also like to thank National Center for Theoretical Sciences for its help during the study.

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