ON RANKIN TRIPLE PRODUCT L-FUNCTIONS OVER FUNCTION FIELDS: CENTRAL CRITICAL VALUES

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Abstract. The aim of this article is to study the special values of Rankin triple product L-functions associated to Drinfeld type newforms of equal square-free levels. The functional equation of these L-functions is deduced from a Garrett-type integral representation and the functional equation of Eisenstein series on the group of similitudes of a symplectic vector space of dimension 6. When the associated root number is positive, we present a function field analogue of Gross-Kudla formula for the central critical value. This formula is then applied to the non-vanishing of L-functions coming from elliptic curves over function fields.

Keywords: Function field, Automorphic form, Rankin triple product, Special value of L-function
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Introduction

The purpose of this manuscript is to explore the Rankin triple product L-functions associated to automorphic cusp forms of Drinfeld type. These automorphic forms can be viewed as function field analogue of classical weight 2 modular forms (cf. [7] and [25]). Let N_0 be a square-free ideal of $\mathbb{F}_q[t]$ (with q odd). Let $F = f_1 \otimes f_2 \otimes f_3$, where f_1, f_2, and f_3 are normalized Drinfeld type newforms for the congruence subgroup $\Gamma_0(N_0)$ of $\mathrm{GL}_2(\mathbb{F}_q[t])$. The Rankin triple product L-function $L(F,s)$ is defined by a convergent Euler product in the half-plane $\mathrm{Re}(s) > 5/2$ (cf. §2). Following idea of Piatetski-Shapiro and Rallis [16], we have a Garrett-type integral representation of $L(F,s)$, i.e. $L(F,s)$ can be expressed essentially by an integral of F times the Eisenstein series associated to a suitable section in the Siegel-parabolic induced representation of GSp_3. From the functional equation of Eisenstein series, we get (cf. Theorem 2.1):

$$L(F,s) = \varepsilon \cdot q^{(2-s)\cdot(5\deg N_0 - 11)} \cdot L(F, 4-s).$$

Here $\varepsilon = \varepsilon(F) = -\prod_{\text{prime } P \mid N_0} \varepsilon_P$ is called the *root number*, where $\varepsilon_P = -c_P(f_1)c_P(f_2)c_P(f_3)$ ($\in \{\pm 1\}$) and $c_P(f_i)$ is the eigenvalue of the Hecke operator T_P associated to f_i.

The root number ε determines the parity of the vanishing order at the central critical point $s = 2$. It is natural to study first the central critical value $L(F,2)$ when the root number ε is positive. The main result of this article is the following analogue of Gross-Kudla formula:

Theorem 0.1. *Let N_0 be a square-free ideal of $\mathbb{F}_q[t]$ and let γ_{N_0} be the number of prime factors of N_0. Let $F = f_1 \otimes f_2 \otimes f_3$, where f_i is a normalized Drinfeld type newform for $\Gamma_0(N_0)$ for each i. Suppose the root number $\varepsilon(F) = 1$. Let $N_0^- = \prod_{\varepsilon_P = -1} P$ and $N_0^+ = N_0/N_0^-$. Then the central critical value $L(F,2)$ is equal to*

$$\frac{(F,F)^{\otimes 3}}{q|N_0|_\infty 2^{\gamma_{N_0}-1}} \cdot < \Delta_F, \Delta_F >^{\otimes 3}.$$

Here $(F,F)^{\otimes 3}$ is the "Petersson norm" of F; Δ_F is the F-component of the "diagonal cycle" in $\mathrm{Pic}(X_{N_0^+,N_0^-})^{\otimes 3}$; $X_{N_0^+,N_0^-}$ is the "definite" Shimura curve of type (N_0^+, N_0^-); and $< \cdot,\cdot >^{\otimes 3}$ is the Gross height pairing on $\mathrm{Pic}(X_{N_0^+,N_0^-})^{\otimes 3}$.

Each object in the above formula is defined in §3.4. One ingredient in the proof is a Siegel-Weil formula over function fields. This formula connects the Eisenstein series appearing in the integral representation of $L(F,s)$ and theta series from the associated definite quaternion algebra B. By strong multiplicity one theorem and Jacquet-Langlands correspondence between GL_2 and $B^\times$ (cf. [24]), the integral representation of $L(F,s)$ is then expressed in terms of the "periods" coming from F and the Gross height of the corresponding cycle Δ_F in the definite Shimura curves $X_{N_0^+,N_0^-}$. An immediate consequence from the above formula is that $L(F,2)$ is always non-negative, and the Gross height of Δ_F determines the non-vanishing of $L(F,2)$. In the case when the root number ε is negative, the central critical derivative $L'(F,2)$ will be treated in a subsequent paper.

Let E be an elliptic curve over $\mathbb{F}_q(t)$ which is of conductor $N_0\infty$ and has split multiplicative reduction at ∞. From the works of Weil, Jacquet-Langlands, and Deligne, it is well known that there exists a normalized Drinfeld type newform f_E for $\Gamma_0(N_0)$ such that the Hasse-Weil L-function $L(E,s)$ is equal to $L(f_E,s-1)$. Let $F_E = f_E \otimes f_E \otimes f_E$. Then we have

$$L(F_E,s) = L(E,s-1)^2 \cdot L(\mathrm{Sym}^3 E,s),$$

where $L(\mathrm{Sym}^3 E,s)$ is the L-function associated to the symmetric cube representation $\mathrm{Sym}^3 E$. The works of Deligne [3] and Lafforgue [14] implies that $L(\mathrm{Sym}^3 E,s)$ is entire. Suppose the root number $\varepsilon(F_E)$ is positive. Then the non-vanishing of the Gross height of Δ_{F_E} guarantees the non-vanishing of $L(E,1)$. Note that the Gross height of Δ_{F_E} is only determined by the elliptic curve E. We expect that, after further works, the value $< \Delta_{F_E}, \Delta_{F_E} >^{\otimes 3}$ could be interpreted geometrically by the invariants of E.

The structure of this article is organized as follows. We set up the general notation in the first section, and review basic facts about automorphic forms of Drinfeld type which are needed for our purpose. The second section consists of analytic properties of the Rankin triple product L-functions associated to Drinfeld type newforms with square-free level. The functional equation is formulated in §2.1, and the proof is given at the end of §2 by using the local results in §2.2 and §2.3. In §3, we establish the analogue of Gross-Kudla formula for the central critical value. After a brief review of the Weil representation in §3.1, we recall the Siegel-Eisenstein series and state the Siegel-Weil formula in §3.2. The central critical value $L(F,2)$ is then expressed as an integral of F times a theta series in §3.3. In §3.4, we introduce a Hecke module homomorphism from the Picard group of definite Shimura curves to the space of Drinfeld type automorphic forms. This homomorphism relates the theta series to the diagonal cycle of the associated definite Shimura curve $X_{N_0^+,N_0^-}$, which leads us to the main result in Theorem 3.10. An application to the non-vanishing of Hasse-Weil L-values associated to elliptic curves is given in §4. Finally, two examples from the elliptic curves is given in §4.1.

1. Preliminary

In this section, we start with the general setting, and give a brief review of Drinfeld type automorphic forms. For further details, we refer to Gekeler-Revesat [7], also Weil [27].

1.1. **Notation.** Let $\mathbb{F}_q$ be the finite field with q elements and the characteristic of $\mathbb{F}_q$ is denoted by p. We always assume that p is odd. Let k be the rational function field $\mathbb{F}_q(t)$ with one variable t, and denote by A the polynomial ring $\mathbb{F}_q[t]$. We denote by ∞ the place of k at infinity, i.e. the place corresponding to the degree valuation. Recall the degree valuation $\mathrm{ord}_\infty(a)$ of any element a in A is $-\deg a$. Let k_∞ be the completion of k at ∞ and O_∞ the valuation ring in k_∞. Set π_∞ to be t^{-1}, which is a uniformizer in O_∞. Then $O_\infty = \mathbb{F}_q[[\pi_\infty]]$ and $k_\infty = \mathbb{F}_q((\pi_\infty))$. For any element $\alpha \in k_\infty$, the absolute value $|\alpha|_\infty := q^{-\mathrm{ord}_\infty(\alpha)}$. We fix the following additive character ψ_∞ from k_∞ to $\mathbb{C}^\times$:

$$\psi_\infty(\sum_i a_i \pi_\infty^i) := \exp\left(\frac{2\pi\sqrt{-1}}{p}\,\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(-a_1)\right).$$

1.2. **Automorphic forms of Drinfeld type.** Let $\mathcal{K}_\infty$ be the Iwahori subgroup of $\mathrm{GL}_2(O_\infty)$, i.e.

$$\mathcal{K}_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_\infty) \;\middle|\; c \equiv 0 \bmod \pi_\infty O_\infty \right\}.$$

For each non-zero ideal N of A, let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(A) \;\middle|\; c \equiv 0 \bmod N \right\}.$$

An *automorphic form of Drinfeld type for* $\Gamma_0(N)$ is a $\mathbb{C}$-valued function f on the double coset space

$$\mathbb{Y}_0(N) := \Gamma_0(N)\backslash \mathrm{GL}_2(k_\infty)/Z(k_\infty)\mathcal{K}_\infty$$

(where Z is the center of GL_2) satisfying the so-called *harmonic property*: for $g \in \mathrm{GL}_2(k_\infty)$,

$$f(g) + f\left(g\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right) = 0 \text{ and } \sum_{\kappa \in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} f(g\kappa) = 0.$$

Let $\mathcal{T}_\infty$ be the Bruhat-Tits tree corrsponding to the equivalent classes of rank 2 lattices in the vector space k_∞^2 (cf. [20] or [7] (1.3)). Then the double coset space $\mathbb{Y}_0(N)$ can be identified with the set of oriented edges in the quotient graph $\Gamma_0(N)\backslash\mathcal{T}_\infty$. Under this identification, automorphic forms of Drinfeld type for $\Gamma_0(N)$ are also called $\mathbb{C}$-*valued harmonic cochains on* $\Gamma_0(N)\backslash\mathcal{T}_\infty$ (cf. [7] §3).

1.3. **Petersson inner product.** An automorphic form f of Drinfeld type for $\Gamma_0(N)$ is called a *cusp form* if f is compactly supported modulo $Z(k_\infty)\cdot\Gamma_0(N)$, i.e. f vanishes except for finitely many double cosets in $\mathbb{Y}_0(N)$. Suppose two Drinfeld type automorphic forms f_1 and f_2 for $\Gamma_0(N)$ are given. If one of them is a cusp form, the *Petersson inner product of* f_1 *and* f_2 is

$$(f_1, f_2) := \int_{\mathbb{Y}_0(N)} f_1\overline{f_2} = \sum_{[g]\in\mathbb{Y}_0(N)} f_1(g)\overline{f_2}(g)\mu([g]).$$

Here the measure $\mu([g])$ for each $g \in \mathrm{GL}_2(k_\infty)$ is defined by

$$\mu([g]) := \frac{q-1}{2}\cdot\frac{1}{\#\left(g^{-1}\Gamma_0(N)g \cap \mathcal{K}_\infty\right)}.$$

A Drinfeld type cusp form f for $\Gamma_0(N)$ is called an *old form* if f is a linear combination of the forms

$$f'\left(\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} g_\infty\right)$$

for $g_\infty \in \mathrm{GL}_2(k_\infty)$, where f' is a Drinfeld type cusp form for $\Gamma_0(M)$, $M|N$, $M \neq N$, and $d|(N/M)$. A Drinfeld type cusp form f for $\Gamma_0(N)$ is called a *new form* if f is orthogonal (with respect to the Petersson inner product) to any old form for $\Gamma_0(N)$.

1.4. **Fourier expansion and L-functions.** Let f be an automorphic form of Drinfeld type for $\Gamma_0(N)$. For $r \in \mathbb{Z}$ and $u \in k_\infty$, recall the *Fourier expansion*

$$f\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} = \sum_{\lambda\in A} f^*(r,\lambda)\psi_\infty(\lambda u),$$

where

$$f^*(r,\lambda) = \int_{A\backslash k_\infty} f\begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix}\psi_\infty(-\lambda u)du.$$

Note that $f^*(r,\lambda) = f^*(r,\epsilon\lambda)$ for any $\epsilon \in \mathbb{F}_q^\times$ and $f^*(r,\lambda)$ vanishes when $\deg\lambda > r+2$. If $\deg\lambda \leq r+2$, the harmonic property of f implies that $f^*(r+1,\lambda) = q^{-1}\cdot f^*(r,\lambda)$.

Suppose f is a cusp form. The L-*function associated to* f is

$$L(f,s) := (1-q^{-(s+1)})^{-1}\cdot\sum_{m\in A,\text{ monic}}\frac{f^*(\deg m+2, m)}{|m|_\infty^s},\quad \mathrm{Re}(s) > 1.$$

This L-function can be extended to an entire function on $\mathbb{C}$ (which is in fact a polynomial in q^{-s}). Moreover,

$$L(f,s) = -q^{(3-\deg N)s}\cdot L(f',-s),$$

where f' is the Drinfeld type cusp form for $\Gamma_0(N)$ defined by $f'(g) := f\left(\begin{pmatrix} 0 & 1 \\ N & 0 \end{pmatrix} g\right)$.

1.5. **Hecke operators.** Let f be an automorphic form of Drinfeld type for $\Gamma_0(N)$. For each monic irreducible polynomial P of A, the *Hecke operator* T_P is defined by:

$$T_P f(g) \quad := \sum_{\deg u < \deg P} f\left(\begin{pmatrix} 1 & u \\ 0 & P \end{pmatrix} \cdot g\right) + \mu_N(P) \cdot f\left(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \cdot g\right).$$

Here $\mu_N(P) = 1$ if $P \nmid N$ and 0 otherwise. It is clear that $T_P f$ still satisfies the harmonic property, and the Fourier coefficients of $T_P f$ are of the form:

$$(T_P f)^*(r, \lambda) = |P|_\infty \cdot f^*(r + \deg P, P\lambda) + \mu_N(P) \cdot f^*(r - \deg P, \lambda/P).$$

Here $f^*(\pi_\infty^r, \lambda/P) = 0$ if $P \nmid \lambda$. Note that T_P and $T_{P'}$ commute to each other, we define the Hecke operator T_m for each monic polynomial m in A as follows:

$$\begin{cases} T_{mm'} = T_m T_{m'} & \text{if } m \text{ and } m' \text{ are relatively prime,} \\ T_{P^\ell} = T_{P^{\ell-1}} T_P - \mu_N(P) \cdot |P|_\infty T_{P^{\ell-2}}. \end{cases}$$

Now, suppose f is a cusp form. We call f a *Hecke eigenform* if $T_m f = c_m(f) \cdot f$ where $c_m(f) \in \mathbb{C}$ for all monic polynomial m in A. In this case, we must have

$$c_m(f) \cdot f^*(2, 1) = |m|_\infty \cdot f^*(\deg m + 2, m).$$

Since T_P is self-adjoint with respect to the Petersson inner product for $P \nmid N$, we have $c_P(f) \in \mathbb{R}$ for $P \nmid N$. If f is *normalized*, i.e. $f^*(2, 1) = 1$, then $L(f, s)$ can be written as the following Euler product:

$$(1 - q^{-(1+s)})^{-1} \cdot \prod_{\text{monic irrducible } P \text{ of } A} (1 - c_P(f)|P|_\infty^{-(1+s)} + \mu_N(P)|P|_\infty^{1-2(1+s)})^{-1}.$$

Suppose the Hecke eigenform f is a new form (called a *newform*). It is known for a newform f that:

(1) For $P \mid N$, $c_P(f) \in \{\pm 1\}$ if $P \parallel N$ and 0 otherwise. Therefore $c_m(f) \in \mathbb{R}$ for all monic polynomials m, which implies that f is an $\mathbb{R}$-valued function if f is normalized.
(2) $f' = \varepsilon_N(f) \cdot f$ where $\varepsilon_N(f) \in \{\pm 1\}$.
(3) For $P \nmid N$, the quadratic polynomial $X^2 - c_P(f)X + |P|_\infty$ has two complex conjugate roots (i.e. c_P satisfies the so-called Ramanujan bound: $|c_P(f)| \leq 2|P|_\infty^{1/2}$).

1.6. **Adelic language.** For each place v of k, the completion of k at v is denoted by k_v, and O_v is the valuation ring in k_v. We call v a finite place of k if $v \neq \infty$. For any finite place v, there exists a unique monic irreducible polynomial P_v in A which is a uniformizer in O_v. We set $\pi_v := P_v$ and $\mathbb{F}_v := O_v/\pi_v O_v$. The cardinality of $\mathbb{F}_v$ is denoted by q_v (which is equal to $|P_v|_\infty$). For each $\alpha \in k_v$, $|\alpha|_v := q_v^{-\operatorname{ord}_v(\alpha)}$. For the infinite place ∞, we have chosen a uniformizer $\pi_\infty = t^{-1}$, and the cardinality q_∞ of the residue field $\mathbb{F}_\infty := O_\infty/\pi_\infty O_\infty$ is equal to q. The adele ring of k is denoted by $\mathbb{A}_k$, with the maximal compact subring $\prod_v O_v =: O_{\mathbb{A}_k}$.

Consider the compact subgroup $\mathcal{K}_0(N\infty) := \prod_v \mathcal{K}_v$ of $\mathrm{GL}_2(\mathbb{A}_k)$, where for $v = \infty$, we have defined $\mathcal{K}_\infty$ in §1.2; for $v \nmid N\infty$, $\mathcal{K}_v := \mathrm{GL}_2(O_v)$; for $v \mid N$,

$$\mathcal{K}_v := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v) \;\middle|\; c \equiv 0 \bmod \pi_v^{\operatorname{ord}_v(N)} O_v \right\}.$$

The strong approximation theorm (cf. [23] Chapter III Theorem 4.3) tells that the natural map from the double coset space $\mathbb{Y}_0(N)$ to $\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_k)/Z(\mathbb{A}_k)\mathcal{K}_0(N\infty)$ is a bijection. Therefore every automorphic form f of Drinfeld type for $\Gamma_0(N)$ can be viewed as a function on the double coset space $\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A}_k)/Z(\mathbb{A}_k)\mathcal{K}_0(N\infty)$. The harmonic property of f is equivalent to say that (cf. [7] §4) the space generated by $f_g(\cdot) := f(\cdot g)$ for all $g \in \mathrm{GL}_2(k_\infty)$ is isomorphic (as a representation of $\mathrm{GL}_2(k_\infty)$) to the special representation $\sigma(|\cdot|_\infty^{1/2}, |\cdot|_\infty^{-1/2})$.

1.6.1. *Whittaker functions.* Fix the additive character ψ on $\mathbb{A}_k$ defined by $\psi(a) := \prod_v \psi_v(a_v)$ for $a = (a_v)_v \in \mathbb{A}_k$, where for each place v of k,

$$\psi_v(a_v) := \exp\left(\frac{2\pi\sqrt{-1}}{p}\operatorname{Tr}_{\mathbb{F}_v/\mathbb{F}_p}\left(\operatorname{Res}_v(a_v dt)\right)\right).$$

For each Drinfeld type cusp form f for $\Gamma_0(N)$, the *Whittaker function* W_f associated to f is the following function on $\mathrm{GL}_2(\mathbb{A}_k)$:

$$W_f(g) := \int_{k\backslash\mathbb{A}_k} f\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right)\psi(-u)du.$$

Here the Haar measure is normalized so that $\int_{k\backslash\mathbb{A}_k} 1\, du = 1$. The adelic version of the *"Fourier expansion"* of f is

$$f(g) = \sum_{\alpha\in k^\times} W_f\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right) \quad \forall g \in \mathrm{GL}_2(\mathbb{A}_k).$$

Suppose f is a newform. Let $W_{f,v} := W_f\mid_{\mathrm{GL}_2(k_v)}$. Then

$$W_f(g) = \prod_v W_{f,v}(g_v) \quad \forall g = (g_v)_v \in \mathrm{GL}_2(\mathbb{A}_k).$$

2. Rankin triple product

Let N_0 be a square-free ideal of A. Given f_1, f_2, and f_3 be three normalized Drinfeld type newforms for $\Gamma_0(N_0)$, let $F = f_1 \otimes f_2 \otimes f_3$ be the function on $\mathrm{GL}_2(k_\infty)^3$ defined by

$$F(g_1, g_2, g_3) := f_1(g_1)f_2(g_2)f_3(g_3).$$

The *triple product L-function $L(F,s)$ associated to f_1, f_2, and f_3* is the Euler product

$$L(F,s) := L_\infty(F,s)\cdot \prod_{\text{monic irreducible } P \text{ in } A} L_P(F,s),$$

where each local factor is defined by the following:

(1) $L_\infty(F,s) := (1-q^{-s})^{-1}(1-q^{1-s})^{-2}$.

(2) For $P \mid N_0$, we set $\varepsilon_P := -c_P(f_1)c_P(f_2)c_P(f_3) \in \{\pm1\}$ and

$$L_P(F,s) := (1+\varepsilon_P|P|_\infty^{-s})^{-1}(1+\varepsilon_P|P|_\infty^{1-s})^{-2}.$$

(3) For $P \nmid N_0$, let $\alpha_{P,i}^{(1)}$ and $\alpha_{P,i}^{(2)}$ be two complex conjugate roots of the quadratic polynomial $X^2 - c_P(f_i)X + |P|_\infty$. Then we set

$$L_P(F,s) := \prod_{1\le j_1,j_2,j_3\le 2}\left(1-\alpha_{P,1}^{(j_1)}\alpha_{P,2}^{(j_2)}\alpha_{P,3}^{(j_3)}|P|_\infty^{-s}\right)^{-1}.$$

The Ramanujan bound of $c_P(f_i)$ implies that $L(F,s)$ converges absolutely for $\mathrm{Re}(s) > 5/2$. We remark that the local L-factor $L_v(F,s)$ for each place v is in fact the local L-function associated to $\rho_{f_1,v}\otimes\rho_{f_2,v}\otimes\rho_{f_3,v}$. Here for $1\le i\le 3$, $\rho_{f_i,v}$ is the Weil-Deligne representation corresponding to f_i at v via local Langlands correspondence (cf. [2] Chapter 7, 8).

We set $\varepsilon_\infty := -1$. The root number of $L(F,s)$ is, by definition, equal to $\varepsilon := \varepsilon_\infty \cdot \prod_{P\mid N_0}\varepsilon_P$. Let $\Lambda(F,s) := q^{-8(s-\frac{3}{2})}\cdot L(F,s)$. Then

Theorem 2.1. *The function $\Lambda(F,s)$ can be extended to an entire function (in fact, a polynomial in q^{-s}), and satisfies the following functional equation:*

$$\Lambda(F,s) = \varepsilon\cdot(|N_0|_\infty\cdot q)^{5\cdot(2-s)}\cdot\Lambda(F,4-s).$$

Remark. The functional equation implies that $L(F,s)$ is a polynomial in q^{-s} of degree $5\deg N_0 - 11$ and the constant coefficient is 1. Moreover,

$$\varepsilon = (-1)^{\mathrm{ord}_{s=2} L(F,s)},$$

which means that the root number ε tells us the parity of the vanishing order of $L(F,s)$ at $s=2$.

The proof of Theorem 2.1 is in §2.3, by using the local results in §2.1 and §2.2.

2.1. **Zeta integrals.** Let $G := \mathrm{GSp}_3$, i.e. the set of R-points of G for any algebra R is

$$\mathrm{GSp}_3(R) := \left\{ g \in \mathrm{GL}_6(R) \;\middle|\; {}^t g \cdot \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \cdot g = \ell_g \cdot \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \text{ for some } \ell_g \in R^\times \right\}.$$

The center Z_G of G consists of scalar matrices in GL_6. There is a canonical embedding from

$$H := \{(g_1,g_2,g_3) \in (\mathrm{GL}_2)^3 \mid \det g_1 = \det g_2 = \det g_3\}$$

into G:

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \right) \longmapsto \begin{pmatrix} a_1 & & & b_1 & & \\ & a_2 & & & b_2 & \\ & & a_3 & & & b_3 \\ c_1 & & & d_1 & & \\ & c_2 & & & d_2 & \\ & & c_3 & & & d_3 \end{pmatrix}.$$

Let $P_G = N_G \cdot M_G$ be the Siegel parabolic subgroup of G, where the set of R-points of N_G is

$$N_G(R) := \left\{ n(b) := \begin{pmatrix} I_3 & b \\ 0 & I_3 \end{pmatrix} \;\middle|\; b = {}^t b \in \mathrm{Mat}_3(R) \right\};$$

the set of R-points of M_G is

$$M_G(R) := \left\{ m(a,\ell) := \begin{pmatrix} a & 0 \\ 0 & \ell \cdot {}^t a^{-1} \end{pmatrix} \;\middle|\; a \in \mathrm{GL}_3(R), \ell \in R^\times \right\}.$$

Let $K_G := G(O_{\mathbb{A}_k})$. The Bruhat decomposition of G says that

$$G(\mathbb{A}_k) = P(\mathbb{A}_k) \cdot K_G.$$

For $s \in \mathbb{C}$, let $I_{\mathbb{A}_k}(s)$ be the representation of $G(\mathbb{A}_k)$ consisting of smooth functions Φ on $G(\mathbb{A}_k)$ such that

$$\Phi(n \cdot m(a,\ell) \cdot g) = |\det a|_{\mathbb{A}}^{2s+2} \cdot |\ell|_{\mathbb{A}}^{-3s-3} \Phi(g)$$

for all $g \in G(\mathbb{A}_k)$, $n \in N_G(\mathbb{A}_k)$, and $m(a,\ell) \in M_G(\mathbb{A}_k)$. Here $|\alpha|_{\mathbb{A}} := \prod_v |\alpha_v|_v$ for all $\alpha = (\alpha_v) \in \mathbb{A}_k^\times$. Let ϕ be a function on K_G such that

$$\phi(n \cdot m(a,\ell) \cdot g) = \phi(g), \text{ for all } g \in K_G,\ n \in N_G(O_{\mathbb{A}_k}),\ m(a,\ell) \in M_G(O_{\mathbb{A}_k}).$$

Then ϕ gives us a *flat section* Φ_ϕ, i.e. for each $s \in \mathbb{C}$, ϕ can be extended uniquely to a function $\Phi_\phi(\cdot,s)$ in $I_{\mathbb{A}_k}(s)$ such that $\Phi_\phi \mid_{G(O_{\mathbb{A}_k}))} = \phi$. We call Φ a *meromorphic* (*respectively, holomorphic*) *section* if Φ is a linear combination of flat sections where the coefficients are rational functions in q^{-s} (respectively, the coefficients are in $\mathbb{C}[q^{-s}, q^s]$).

Let Φ be a meromorphic section. The *Eisenstein series* $E(\Phi,s,\cdot)$ on $G(\mathbb{A}_k)$ is defined by:

$$E(\Phi,s,g) := \sum_{\gamma \in P_G(k)\backslash G(k)} \Phi(\gamma \cdot g, s), \ \forall g \in G(\mathbb{A}_k).$$

It is well-known that this series converges for $\mathrm{Re}(s)$ sufficiently large. Moreover, $E(\Phi,s,g)$ has a meromorphic continuation in $s \in \mathbb{C}$ (in fact, a rational function in q^{-s}, cf. [15] IV.1.12).

Definition 2.2. The (global) zeta integral associated to F and a meromorphic section Φ is:

$$Z(F,\Phi,s) := \int_{Z_G(\mathbb{A}_k)H(k)\backslash H(\mathbb{A}_k)} F(h) \cdot E(\Phi,s,h) dh.$$

Here F is viewed as a function on $\mathrm{GL}_2(\mathbb{A}_k)^3$, and the measure dh is induced from the Haar measure on $Z_G(\mathbb{A}_k)\backslash H(\mathbb{A}_k)$ normalized so that the volume of $Z_G(\mathbb{A}_k)\backslash Z_G(\mathbb{A}_k)H(O_{\mathbb{A}_k})$ is 1.

Let U_0 be the following algebraic subgroup of H:

$$U_0 := \left\{ \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & u_3 \\ 0 & 1 \end{pmatrix} \right) \in (\mathrm{GL}_2)^3 \;\middle|\; u_1 + u_2 + u_3 = 0 \right\}.$$

Following Garrett and Harris [6], we choose a particular element

$$\delta := \begin{pmatrix} 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \in \mathrm{GSp}_3 .$$

Then we have

Proposition 2.3. (cf. [16]) *For* $\mathrm{Re}(s)$ *sufficiently large,*

$$Z(F, \Phi, s) = q^{-2} \cdot \int_{Z_G(\mathbb{A}_k)U_0(\mathbb{A}_k)\backslash H(\mathbb{A}_k)} W_F(h) \cdot \Phi(\delta \cdot h, s) dh$$

where $W_F(h_1, h_2, h_3) = W_{f_1}(h_1) \cdot W_{f_2}(h_2) \cdot W_{f_3}(h_3)$ *for any* $h = (h_1, h_2, h_3) \in H(\mathbb{A}_k)$*, and* W_{f_i} *is the Whittaker function associated to* f_i *for* $1 \leq i \leq 3$ *intoduced in §1.6.*

For each place v of k, we set $I_v(s)$ to be the space of smooth functions φ on $G(k_v)$ satisfying that

$$\varphi(n_v m(a_v, \ell_v) g_v) = |\det a_v|_v^{2s+2} \cdot |\ell_v|_v^{-3s-3} \cdot \varphi(g_v)$$

for all $n_v \in N_G(k_v), m(a_v, \ell_v) \in M_G(k_v), g_v \in G(k_v)$. Let $\Phi_{\phi_{G(O_v)}}(\cdot, s) \in I_v(s)$ be the flat section associated to $\phi_{G(O_v)}$ where $\phi_{G(O_v)} \equiv 1$ on $G(O_v)$. Then $I_{\mathbb{A}_k}(s)$ is the restricted tensor product $\otimes_v' I_v(s)$ (w. r. t. $\{\Phi_{\phi_{G(O_v)}}\}_v$). We call a meromorphic section $\Phi \in I_{\mathbb{A}_k}(s)$ is a *pure-tensor* if $\Phi = \otimes_v \Phi_v$ where for each v, $\Phi_v(\cdot, s) \in I_v(s)$ is a meromorphic section, and $\Phi_v = \Phi_{\phi_{G(O_v)}}$ for almost all v.

Lemma 2.4. *For any pure-tensor* $\Phi = \otimes_v \Phi_v \in I_{\mathbb{A}_k}(s)$*, we have* $Z(F, \Phi, s) = q^{-2} \cdot \prod_v Z_v(F, \Phi_v, s)$*, where*

$$Z_v(F, \Phi_v, s) := \int_{Z_G(k_v)U_0(k_v)\backslash H(k_v)} W_{F,v}(h_v) \Phi_v(\delta h_v, s) dh_v$$

and $W_{F,v}(h_{v,1}, h_{v,2}, h_{v,3}) := W_{f_1,v}(h_{v,1}) W_{f_2,v}(h_{v,2}) W_{f_3,v}(h_{v,3})$.

2.2. **Local factors.** When $v \nmid N_0\infty$, the conductor of the fixed additive character ψ_v is trivial. Take $\phi_v = \phi_{G(O_v)}$ where $\phi_{G(O_v)} \equiv 1$ on $G(O_v)$. Then (cf. [16] Theorem 3.1)

$$Z_v(F, \Phi_{\phi_v}, s) = \int_{Z_G(k_v)U_0(k_v)\backslash H(k_v)} W_{F,v}(h_v) \cdot \Phi_{\phi_v}(\delta h_v, s) dh_v = \frac{1}{b_v(s)} \cdot L_v(F, s+2)$$

where $b_v(s) := (1 - q_v^{-2s-2})^{-1}(1 - q_v^{-4s-2})^{-1}$.

Now, suppose $v \mid N_0\infty$. Let $K_0(v)$ be the following compact subgroup in $G(k_v)$:

$$K_0(v) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(O_v) \;\middle|\; A, B, C, D \in \mathrm{Mat}_3(O_v) \text{ and } C \equiv 0 \bmod \pi_v O_v \right\}.$$

For $0 \leq i \leq 3$, let

$$w_i := \begin{pmatrix} I_{3-i} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_i \\ 0 & 0 & I_{3-i} & 0 \\ 0 & -I_i & 0 & 0 \end{pmatrix}.$$

Then the Iwasawa decomposition of G implies

$$G(O_v) = \coprod_{0 \leq i \leq 3} K_0(v) w_i K_0(v).$$

For $0 \le i \le 3$, we set $\phi_v^{(i)}$ to be the characteristic function of $K_0(v)w_iK_0(v)$ on $G(O_v)$. Then $\sum_{0\le i\le 3} \phi_v^{(i)} = \phi_{G(O_v)}$. These four functions $\Phi_{\phi_v^{(i)}}(\cdot, s)$, $0 \le i \le 3$, form a basis for the space $I_v(s)^{K_0(v)}$ of $K_0(v)$-fixed functions in $I_v(s)$.

For each $\Phi \in I_v(s)^{K_0(v)}$, we define $(\omega_v\Phi)(g) := \Phi(g\eta_v)$, where

$$\eta_v := \begin{pmatrix} 0 & I_3 \\ -\pi_v I_3 & 0 \end{pmatrix} \in G(k_v).$$

It is observed that $\omega_v \Phi_{\phi_v^{(i)}} = q_v^{(2i-3)(s+1)} \Phi_{\phi_v^{(3-i)}}$. Set $\widetilde{\Phi}_{\phi_v^{(i)}}(\cdot, s) := q_v^{-i(s+1)} \cdot \Phi_{\phi_v^{(i)}}(\cdot, s) \in I_v(s)$, then one has $\omega_v \widetilde{\Phi}_{\phi_v^{(i)}} = \widetilde{\Phi}_{\phi_v^{(3-i)}}$. Note that

$$\Phi_{\phi_{G(O_v)}} = \sum_{0\le i\le 3} q_v^{i(s+1)} \cdot \widetilde{\Phi}_{\phi_v^{(i)}} \text{ and } \Phi'_{\phi_{G(O_v)}} := \omega_v \Phi_{\phi_{G(O_v)}} = \sum_{0\le i\le 3} q_v^{(3-i)(s+1)} \cdot \widetilde{\Phi}_{\phi_v^{(i)}}.$$

We choose two more functions $\Phi_v^{\pm}(\cdot, s)$ in $I_v(s)^{K_0(v)}$ which are defined by

$$\Phi_v^{\pm}(\cdot, s) := \sum_{0\le i\le 3} (\pm 1)^i \widetilde{\Phi}_{\phi_v^{(i)}}(\cdot, s).$$

Then $\omega_v \Phi_v^{\pm} = \pm \Phi_v^{\pm}$. Suppose $s \ne -1$. Then $\{\Phi_{\phi_{G(O_v)}}, \Phi'_{\phi_{G(O_v)}}, \Phi_v^{\pm}\}$ also form a basis of $I_v(s)^{K_0(v)}$. Next, we calculate the local zeta integral associated to these four functions.

For each $h_v \in H(k_v)$, we have $\omega_v W_{F,v}(h_v) := W_{F,v}(h_v\eta_v) = \varepsilon_v W_{F,v}(h_v)$. Therefore

$$Z_v(F, \omega_v\Phi, s) = \varepsilon_v Z_v(F, \Phi, s) \quad \forall \Phi \in I_v(s)^{K_0(v)}.$$

This tells us that $Z_v(F, \Phi_v^{-\varepsilon_v}, s) = 0$. we also deduce that

$$Z_v(F, \Phi_{\phi_{G(O_v)}}, s) = Z_v(F, \Phi'_{\phi_{G(O_v)}}, s) = 0.$$

The remaining case is the zeta integral $Z_v(F, \Phi_v^{\varepsilon_v}, s)$. Since $\omega_v \widetilde{\Phi}_{\phi_v^{(i)}} = \widetilde{\Phi}_{\phi_v^{(3-i)}}$, we get

$$Z_v(F, \Phi_v^{\varepsilon_v}, s) = -2\varepsilon_v q_v^{s+1} \cdot (1 - \varepsilon_v q_v^{-(s+1)})^2 \cdot Z_v(F, \widetilde{\Phi}_{\phi_v^{(0)}}, s).$$

Proposition 2.5. *If $v \mid N_0\infty$, then we have*

$$Z_v(F, \Phi_{\phi_v^{(0)}}, s) = -q_v^{(1-s)\delta_v} \cdot (q_v + 1)^{-3} \cdot q_v^{-2s-2} \cdot (1 + \varepsilon_v q_v^{-s-2})^{-1} \cdot (1 + \varepsilon_v q_v^{-s-1})^{-2}$$

where $\pi_v^{\delta_v}$ is the conductor of the additive character ψ_v.

Remark. The conductor of ψ_∞ is not trivial. Hence this result is not covered by Proposition 4.2 in [8]. We rework the proof here accordingly.

Proof. Let

$$U := \left\{ \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix} \right) \right\} \subset \mathrm{GL}_2^3$$

and

$$T := \left\{ \left(\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix}, \begin{pmatrix} a_3 & 0 \\ 0 & a_3^{-1} \end{pmatrix} \right) \right\} \subset \mathrm{GL}_2^3.$$

Fix a place $v \mid N_0\infty$. Let $K_{H_v} := G(O_v) \cap H(k_v)$, and

$$H_v^0 := \{h \in H(k_v) : \mathrm{ord}_v(\det h) \text{ is even}\}.$$

Then $Z_G(k_v), U(k_v), T(k_v), K_{H_v}$ are subgroups in H_v^0. Let $d^\times z$ be the Haar measure on $Z_G(k_v)$ normalized such that $\mathrm{vol}(Z_G(O_v)) = 1$, du be the Haar measure on $U(k_v)$ normalized such that $\mathrm{vol}(U(O_v)) = 1$, $d^\times a (= d^\times a_1 \cdot d^\times a_2 \cdot d^\times a_3)$ be the Haar measure on $T(k_v)$ normalized such that $\mathrm{vol}(T(O_v)) = 1$, and $d\kappa$ be the Haar measure on K_{H_v} such that $\mathrm{vol}(K_{H_v}) = 1$. Then $d^\times z \frac{du}{|a|_v^2} d^\times a d\kappa$

is a Haar measure on H_v^0, where $|a|_v := |a_1a_2a_3|_v$. We can extend this measure to a Haar measure on $H(k_v)$, as

$$H(k_v) = H_v^0 \cup \begin{pmatrix} \pi_v I_3 & 0 \\ 0 & I_3 \end{pmatrix} H_v^0.$$

Now, we embed k_v into $U(k_v)$ by $x \mapsto u(x) := \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, I_2, I_2\right)$. Then the invariant measure dh on $Z_G(k_v)U_0(k_v)\backslash H_v^0$ is

$$dh_0 := \frac{1}{|a|_v^2} dx d^\times a d\kappa,$$

and on $Z_G(k_v)U_0(k_v)\backslash \begin{pmatrix} \pi_v I_3 & 0 \\ 0 & I_3 \end{pmatrix} H_v^0$ is $q_v^2 dh_0$. Hence $Z_v(F, \Phi_{\phi_v^{(0)}}, s) = Z_1 + Z_2$, where

$$Z_1 = \int_{Z_G(k_v)U_0(k_v)\backslash H_v^0} W_F(h_0)\Phi_{\phi_v^{(0)}}(\delta h_0, s) dh_0,$$

and

$$Z_2 = q_v^2 \int_{Z_G(k_v)U_0(k_v)\backslash H_v^0} W_F\left(\begin{pmatrix} \pi_v I_3 & 0 \\ 0 & I_3 \end{pmatrix} h_0\right) \Phi_{\phi_v^{(0)}}\left(\delta \begin{pmatrix} \pi_v I_3 & 0 \\ 0 & I_3 \end{pmatrix} h_0, s\right) dh_0.$$

Since $\Phi_{\phi_v^{(0)}}$ is right invariant by $K_0(v)$, Z_1 is equal to

$$\mathrm{vol}(K_0(v) \cap H(k_v)) \cdot \sum_{\kappa \in K_{H_v}/K_0(v)\cap H(k_v)} Z_1(\kappa),$$

where

$$Z_1(\kappa) := \int_{(k_v^\times)^3} \int_{k_v} W_F\big(u(x)a\kappa\big)\Phi_{\phi_v^{(0)}}\big(\delta u(x)a\kappa, s\big) \frac{dx}{|a|_v^2} d^\times a.$$

Choose the following coset representatives of $K_{H_v}/K_0(v) \cap H(k_v)$:

$$\left\{ \begin{array}{c} 1, \quad \kappa(x_1) := \left(\begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_2, I_2\right), \\ \kappa(x_2) := \left(I_2, \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, I_2\right), \\ \kappa(x_3) := \left(I_2, I_2, \begin{pmatrix} 1 & x_3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \end{array} \;\middle|\; x_1, x_2, x_3 \in \mathbb{F}_v \right\}.$$

When $\kappa = 1$,

$$\Phi_{\phi_v^{(0)}}(\delta u(x)a\kappa, s) = \begin{cases} |a|_v^{2s+2} \cdot |x|_v^{-2s-2} & \text{if } |x|_v > |a_i|_v^2 \text{ for } 1 \le i \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

We then deduce that $Z_1(1)$ is equal to

$$\begin{aligned} &(-\varepsilon_v)^{\delta_v} q_v^{3\delta_v}(1-q_v^{-2s-2})^{-3}(1-q_v^{-2s-4})^{-1} \\ \cdot &\Big[-q_v^{\delta_v(2s+1)} q_v^{-2s-2} q_v^{(-6s-6)\lceil \delta_v/2 \rceil}(1-q_v^{-2s-4}) \\ &+(1-q_v^{-1})q_v^{-6s-6} q_v^{(-2s-4)\lceil \delta_v/2 \rceil} \\ &+(1-q_v^{-1})q_v^{-4s-5} q_v^{(-2s-4)\lfloor \delta_v/2 \rfloor} \Big] \end{aligned}$$

When $\kappa = \kappa(x_i)$ for some $x_i \in \mathbb{F}_v$,

$$\Phi_{\phi_v^{(0)}}(\delta u(x)a\kappa, s) = \begin{cases} |a|_v^{2s+2}|a_i|_v^{-2s-4} & \text{if } |a_i|_v^2 > |x + a_i^2 x_i|_v \text{ and } |a_i|_v > |a_j|_v \text{ for } j \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we get

$$Z_1(\kappa(x_i)) = -q_v^{-1}(-\varepsilon_v)^{\delta_v} q_v^{3\delta_v} q_v^{-4s-5}(1-q_v^{-2s-2})^{-2}(1-q_v^{-2s-4})^{-1} \cdot q_v^{(-2s-4)\lfloor \delta_v/2 \rfloor}.$$

Since the volume $\mathrm{vol}(K_0(v)\cap H(k_v))$ is $(q_v+1)^{-3}$, Z_1 is equal to

$$(-\varepsilon_v)^{\delta_v} q_v^{3\delta_v}(q_v+1)^{-3} q_v^{-2s-2}(1-q_v^{-2s-2})^{-3}(1-q_v^{-2s-4})^{-1}$$
$$\cdot \Big[-3\cdot q_v^{-2s-3} q_v^{(-2s-4)\lfloor \delta_v/2\rfloor}(1-q_v^{-2s-2}) + (-1) q_v^{(2s+1)\delta_v} q_v^{(-6s-6)\lceil \delta_v/2\rceil}(1-q_v^{-2s-4})$$
$$+(1-q_v^{-1}) q_v^{-4s-4} q_v^{(-2s-4)\lceil \delta_v/2\rceil} + (1-q_v^{-1}) q_v^{-2s-3} q_v^{(-2s-4)\lfloor \delta_v/2\rfloor}\Big]$$

Next, we consider Z_2. It is clear that

$$\Phi_{\phi_v^{(0)}}\left(\delta\begin{pmatrix}\pi_v I_3 & 0\\ 0 & I_3\end{pmatrix} h_0, s\right) = q_v^{-s-1}\Phi_{\phi_v^{(0)}}(\delta h_0, s).$$

Note that $\lfloor (\delta_v-1)/2\rfloor = \lfloor \delta_v/2\rfloor - \frac{1+(-1)^{\delta_v}}{2}$. By the same argument, we obtain that Z_2 is equal to

$$(-\varepsilon_v)^{\delta_v} q_v^{3\delta_v}(q_v+1)^{-3} q_v^{-2s-2}(1-q_v^{-2s-2})^{-3}(1-q_v^{-2s-4})^{-1} q_v^{(-2s-4)\lfloor \delta_v/2\rfloor}$$
$$\cdot \quad \left((-\varepsilon_v) q_v^{-3}\cdot q_v^{-s-1}\cdot q_v^{2}\cdot q_v^{(2s+4)(\frac{1+(-1)^{\delta_v}}{2})}\right)$$
$$\cdot \quad \Big[-2q_v^{-2s-3} + q_v^{-2s-4}\left(q_v^{(-4s-5)(\frac{1+(-1)^{\delta_v}}{2})}-1\right) + q_v^{-4s-4} q_v^{(-2s-4)(\frac{1+(-1)^{\delta_v}}{2})}$$
$$+q_v^{-4s-5}\left(3-q_v^{(-2s-4)(\frac{1+(-1)^{\delta_v}}{2})}\right) - q_v^{(-4s-5)(\frac{1+(-1)^{\delta_v}}{2})}\Big].$$

Therefore

$$Z_1+Z_2 = -(q_v+1)^{-3} q_v^{(1-s)\delta_v} q_v^{-2s-2}(1+\varepsilon_v q_v^{-s-2})^{-1}(1+\varepsilon_v q_v^{-s-1})^{-2}$$

where $\pi_v^{\delta_v}$ is the conductor of the additive character ψ_v, which is the desired conclusion. □

For $v \mid N_0\infty$, let

$$\xi_v(s) := 2\varepsilon_v(q_v+1)^{-3}\cdot q_v^{-s-1}\cdot(1-\varepsilon_v q_v^{-s-1})^2\cdot b_v(s).$$

Moreover, we set $b(s) := \prod_v b_v(s)$, and $b^*(s) := q^{-6s-4}\cdot b(s)$. Then one has

Corollary 2.6. (1) *When $v \mid N_0\infty$,*

$$Z_v(F,\Phi_v^{\varepsilon_v}, s) = q_v^{(1-s)\delta_v}\cdot \xi_v(s)\cdot \frac{1}{b_v(s)}\cdot L_v(F, s+2),$$

where $\pi_v^{\delta_v}$ is the conductor of the additive character ψ_v.
(2) *Take $\Phi^\natural = \otimes_v \Phi_v^\natural \in I_{\mathbb{A}_k}(s)$, where*

$$\begin{cases}\Phi_v^\natural = \Phi_{\phi_{G(O_v)}} & \text{for } v\nmid N_0\infty;\\ \Phi_v^\natural := \xi_v(s)^{-1}\cdot\Phi_v^{\varepsilon_v} & \text{for } v\mid N_0\infty.\end{cases}$$

Then

$$Z(F,\Phi^\natural,s) = q^{-2}\cdot q^{-2s+2}\cdot\frac{1}{b(s)}\cdot L(F,s+2) = \frac{1}{b^*(s)}\Lambda(F,s+2).$$

Proof. For $v\nmid N_0\infty$, we have mentioned that

$$Z_v(F,\Phi_{\phi_{G(O_v)}}, s) = \frac{1}{b_v(s)}\cdot L_v(F,s+2).$$

Therefore (2) follows from Proposition 2.3, Lemma 2.4 and (1). For $v\mid N_0\infty$, recall that

$$Z_v(F,\Phi_v^{\varepsilon_v},s) = -2\varepsilon_v q_v^{s+1}\cdot(1-\varepsilon_v q_v^{-(s+1)})^2\cdot Z_v(F,\widetilde{\Phi}_{\phi_v^{(0)}},s).$$

Therefore (1) follows from Proposition 2.5. □

Remark. We point out that the zeta integral $Z(F,\Phi^\natural,s)$ can be extended to a rational function in q^{-s}. Thus the above corollary gives us immediately the meromorphic continuation of $L(F,s)$. In fact, by the main theorem of [10], the triple product L-fucntion $L(F,s)$ is entire. In the next subsection, we shall give the functional equation of $L(F,s)$.

2.3. **Intertwining operator $M(s)$ and the functional equation.** For $\mathrm{Re}(s) > 1$, the global intertwining operator $M(s) : I_{\mathbb{A}_k}(s) \to I_{\mathbb{A}_k}(-s)$ is given by the following integral

$$M(s)\Phi(g) := \int_{N_G(\mathbb{A}_k)} \Phi(w_3 n g) dn.$$

The Haar measure dn is normalized so that $\mathrm{vol}(N_G(k)\backslash N_G(\mathbb{A}_k)) = 1$. It is known that (cf. [16] §4) this integral operator has a meromorphic continuation to the whole s-plane, and it gives the following functional equation of the Eisenstein series $E(\Phi, s, g)$ (cf. [15] IV.1.10):

$$E(\Phi, s, g) = E(M(s)\Phi, -s, g).$$

Replacing Φ by the function $\Phi^\natural$ in Corollary 2.6 (2), the above functional equation implies

$$Z(F, \Phi^\natural, s) = Z(F, M(s)\Phi^\natural, -s).$$

Since $\Phi^\natural = \otimes_v \Phi_v^\natural$ where $\Phi_v^\natural \in I_v(s)$, it is clear that for $\mathrm{Re}(s) > 1$, $M(s)\Phi^\natural = \otimes_v M_v(s)\Phi_v^\natural$ where $M_v(s) : I_v(s) \to I_v(-s)$ is defined by

$$M_v(s)\Phi_v(g_v) = \int_{N_G(k_v)} \Phi_v(w_3 n_v g_v) dn_v, \ \forall \Phi_v \in I_v(s).$$

For each place v, the Haar measure dn_v can be normalized so that $\mathrm{vol}(N_G(O_v)) = q_v^{3\delta_v}$.

For each place v of k, let $a_v(s) := (1 - q_v^{-2s+1})^{-1} \cdot (1 - q_v^{-4s+1})^{-1}$. Note that for $v \nmid N_0\infty$, $\Phi_v^\natural = \Phi_{\phi_{G(O_v)}}$. Hence (cf. [16] §4)

$$M_v \Phi_v^\natural(s) = \frac{a_v(s)}{b_v(s)} \Phi_v^\natural(-s).$$

Since M_v is $G(k_v)$ intertwining, it carries $I_v(s)^{K_0(v)}$ to $I_v(-s)^{K_0(v)}$. Thus when $v \mid N_0\infty$,

$$Z_v(F, M_v(s)\Phi_v^{\varepsilon_v}, -s) = \alpha_v(s) \cdot Z_v(F, \Phi_v^{\varepsilon_v}, -s)$$

where $\alpha_v(s)$ is a meromorphic function. By the same argument in [8] Proposition 5.1, one gets

Proposition 2.7. *When $v \mid N_0\infty$,*

$$\alpha_v(s) = \varepsilon_v \cdot q^{3\delta_v} \cdot q_v^{-3s-2} \cdot \frac{(1 - \varepsilon_v q_v^{-s-1})^2 (1 + \varepsilon_v q_v^{1-s})(1 + q_v^{1-2s})}{(1 - \varepsilon_v q_v^{1-s})(1 - q_v^{1-4s})},$$

where $\pi_v^{\delta_v}$ is the conductor of the additive character ψ_v.

For each place v of k, we set

$$\eta_v(s) := \begin{cases} 1 & \text{if } v \nmid N_0\infty; \\ q_v^{-3\delta_v} \cdot \dfrac{\xi_v(-s)}{\xi_v(s)} \cdot \dfrac{b_v(s)}{a_v(s)} \cdot \alpha_v(s) = \varepsilon_v q_v^{-5s} & \text{if } v \mid N_0\infty. \end{cases}$$

Then

Corollary 2.8. *For each place v of k, we have*

$$Z_v(F, M_v(s)\Phi_v^\natural, -s) = q_v^{3\delta_v} \cdot \eta_v(s) \cdot \frac{a_v(s)}{b_v(s)} Z_v(F, \Phi_v^\natural, -s).$$

Proof of Theorem 2.1. The remark at the end of §2.2 has already told us that $\Lambda(F, s)$ can be extended to a polynomial in q^{-s}. Let $\eta(s) := \prod_{v \mid N_0\infty} \eta_v(s)$ (which is equal to $\varepsilon \cdot (|N_0|_\infty q)^{-5s}$) and $a(s) := \prod_v a_v(s)$. By Corollary 2.6 and 2.8, we obtain that

$$\frac{1}{b^*(s)} \Lambda(F, s+2) = q^6 \cdot \delta(s) \cdot \frac{a(s)}{b(s)} \frac{1}{b^*(-s)} \Lambda(F, -s+2).$$

Let $\zeta_k^*(s) := q^{-s} \prod_v (1 - q_v^{-s})^{-1}$. Then the functional equation $\zeta_k^*(s) = \zeta_k^*(1-s)$ implies that

$$q^6 \cdot \frac{a(s)}{b(s)} \cdot \frac{1}{b^*(-s)} = \frac{1}{b^*(s)}.$$

Therefore we get the functional equation of $\Lambda(F,s)$ in Theorem 2.1:

$$\Lambda(F,s+2)=\delta(s)\cdot\Lambda(F,-s+2)=\varepsilon\cdot(|N_0|_\infty q)^{-5s}\cdot\Lambda(F,-s+2).$$

□

3. Function field analogue of Gross-Kudla formula

From now on, we assume the root number ε is positive, i.e. $L(F,s)$ does not vanish automatically at $s=2$. In this section, we explore the central critical value $L(F,2)$ and present an analogue of Gross-Kudla formula.

3.1. **Weil representation.** For convenience, we set $\varepsilon_v=1$ for $v\nmid N_0\infty$. Let S be the set consisting of the places v of k such that $\varepsilon_v=-1$. Then the cardinality of S is even. Let B be the unique quaternion algebra over k which is ramified at the places in S and unramified elsewhere. Let (V,Q_V) be the quadratic space (B,Nr_B) over k where $\mathrm{Nr}_B=\mathrm{Nr}$ is the reduced norm form on B. For x,y in V, the bilinear form $<x,y>_V$ associated to Q_V is $\mathrm{Tr}_B(x\bar{y})$, where $\mathrm{Tr}_B=\mathrm{Tr}$ is the reduced trace on B and $\bar{y}:=\mathrm{Tr}(y)-y$ is the main involution of B. We denote by $O(V)$ the orthogonal group of V. Let $G^1:=\mathrm{Sp}_3$, i.e. the following algebraic subgroup of G:

$$\left\{g\in\mathrm{GL}_6\ \middle|\ {}^tg\cdot\begin{pmatrix}0&I_3\\-I_3&0\end{pmatrix}\cdot g=\begin{pmatrix}0&I_3\\-I_3&0\end{pmatrix}\right\}.$$

For each place v of k, we have fixed an additive character ψ_v on k_v in §1.6. Let $V(k_v):=V\otimes_k k_v$ and let $S(V(k_v))$ be the space of *Schwartz functions on* $V(k_v)$, i.e. the space of functions on $V(k_v)$ which are locally constant and compactly supported. The (*local*) *Weil representation* $\omega_v(=\omega_{v,\psi_v})$ of $G^1(k_v)\times O(V)(k_v)$ on the space

$$S(V(k_v))\otimes_{\mathbb{C}}S(V(k_v))\otimes_{\mathbb{C}}S(V(k_v))=S(V(k_v)^3)$$

is defined by the following: for $\varphi=\varphi_1\otimes\varphi_2\otimes\varphi_3\in S(V(k_v)^3)$ and $x=(x_1,x_2,x_3)\in V(k_v)^3$,

$$\begin{aligned}
(\omega_v(h)\varphi)(x)&:=\varphi(h^{-1}x_1,h^{-1}x_2,h^{-1}x_3),\ \forall h\in O(V)(k_v),;\\
\left(\omega_v\begin{pmatrix}a&0\\0&{}^ta^{-1}\end{pmatrix}\varphi\right)(x)&:=|\det a|_v^2\cdot\varphi((x_1,x_2,x_3)\cdot a),\ \forall a\in\mathrm{GL}_3(k_v);\\
\left(\omega_v\begin{pmatrix}I_3&b\\0&I_3\end{pmatrix}\varphi\right)(x)&:=\psi_v\Big(\,Tr\left(b\cdot Q(x)\right)\Big)\cdot\varphi(x),\ \forall b={}^tb\in\mathrm{Mat}_3(k_v);\\
\omega_v(w_i)\varphi&:=(\varepsilon_v)^i\cdot\varphi_1\otimes\cdots\otimes\varphi_{3-i}\otimes\widehat{\varphi}_{3-i+1}\otimes\cdots\otimes\widehat{\varphi}_3,\ 0\le i\le 3.
\end{aligned}$$

Here $Q(x)$ is the 3-by-3 matrix whose (i,j)-entry is $\frac{1}{2}<x_i,x_j>_V$; and $\widehat{\varphi}_i$ is the Fourier transform of φ_i (with respect to ψ_v)

$$\widehat{\varphi}_i(x_i):=\int_{V(k_v)}\varphi_i(y)\psi_v(<x_i,y>_V)dy.$$

The Haar measure dy is chosen to be *self dual*, i.e. $\widehat{\widehat{\varphi}}_i(x_i)=\varphi_i(-x_i)$. Let $V(\mathbb{A}_k):=V\otimes_k\mathbb{A}_k$ and let $S(V(\mathbb{A}_k))$ be the space of Schwartz functions on $V(\mathbb{A}_k)$. Then we have the (global) Weil representation $\omega=\otimes_v\omega_v$ of $G^1(\mathbb{A}_k)\times O(V)(\mathbb{A}_k)$ on the space

$$S(V(\mathbb{A}_k))\otimes_{\mathbb{C}}S(V(\mathbb{A}_k))\otimes_{\mathbb{C}}S(V(\mathbb{A}_k))=S(V(\mathbb{A}_k)^3).$$

Let N_0^- be the product of primes P of A with $\varepsilon_P=-1$, and $N_0^+:=N_0/N_0^-$. Let R be an Eichler A-order of B of type (N_0^+,N_0^-), i.e. for each finite place v, $R_v:=R\otimes_A O_v$ is a maximal O_v-order in $B_v:=B\otimes_k k_v$ if $v\nmid N_0^+$; and

$$R_v\cong\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(O_v)\ \middle|\ c\equiv 0\bmod N_0^+O_v\right\}\quad\text{if } v\mid N_0^+.$$

Let $\varphi=\otimes_v\varphi_v$ be the Schwartz function in $S(V(\mathbb{A}_k))$ where for each $v\neq\infty$,

$$\varphi_v=\text{ the characteristic function of } R_v;$$

$$\varphi_\infty = \text{ the characteristic function of } \pi_\infty O_{B_\infty},$$

where O_{B_∞} is the maximal O_∞-order in $B_\infty := B \otimes_k k_\infty$. Let

$$\tilde{\varphi} := \varphi \otimes \varphi \otimes \varphi \in S(V(\mathbb{A}_k)^3).$$

For $s \in \mathbb{C}$, $g = nm(a,\ell)\kappa \in \mathrm{GSp}_3(\mathbb{A}_k)$ where $n \in N_G(\mathbb{A}_k)$, $m(a,\ell) \in M_G(\mathbb{A}_k)$, $\kappa \in K_G$. We define

$$\Phi_{\tilde{\varphi}}(g,s) := |\ell|_{\mathbb{A}_k}^{-3s-3} \cdot |\det a|_{\mathbb{A}_k}^{2s} \cdot (\omega(g_1)\tilde{\varphi})(0)$$

where $g_1 := \begin{pmatrix} I_3 & 0 \\ 0 & \ell^{-1}I_3 \end{pmatrix} \cdot g$. Although the expression of g as $nm(a,\ell)\kappa$ is not unique, $|\ell|_{\mathbb{A}_k}$ and $|\det a|_{\mathbb{A}_k}$ is uniquely determined by g. Therefore $\Phi_{\tilde{\varphi}}(g,s)$ is well-defined. It is clear that $\Phi_{\tilde{\varphi}}(\cdot,s)$ is in $I_{\mathbb{A}_k}(s)$, and $\Phi_{\tilde{\varphi}} = \otimes_v \Phi_{\tilde{\varphi}_v}$ with $\Phi_{\tilde{\varphi}_v} \in I_v(s)$.

Proposition 3.1. (1) *Suppose $v \nmid N_0\infty$. Then for $g \in G(k_v)$, we have*

$$\Phi_{\tilde{\varphi}_v}(g,0) = \Phi_{\phi_{G(O_v)}}(g,0).$$

(2) *If $v \mid N_0\infty$, we get*

$$\Phi_{\tilde{\varphi}_v}(g,0) = \Phi_v^{\varepsilon_v}(g,0).$$

Here $\Phi_{\phi_{G(O_v)}}$ and $\Phi_v^{\varepsilon_v}$ are introduced in §2.2.

Proof. Recall that $\Phi_{\phi_{G(O_v)}}$ and $\Phi_v^{\varepsilon_v}$ are in $I_v(s)^{K_0(v)}$, and $\Phi_{\phi_{G(O_v)}}(w_i,s) = 1$, $\Phi_v^{\varepsilon_v}(w_i,s) = \varepsilon_v^i q_v^{-i(s+1)}$. It is easy to check that

Lemma 3.2. (1) *For each place $v \neq \infty$,*

$$\widehat{\varphi}_v = \begin{cases} q_v^{-1} \cdot \text{ characteristic function of } \Pi_v^{-1}R_v & \text{if } v \mid N_0, \\ \varphi_v & \text{if } v \nmid N_0. \end{cases}$$

Here when $v \mid N_0^-$, Π_v is a generator of the maximal ideal of R_v; when $v \mid N_0^+$, Π_v is the element of R_v corresponding to $\begin{pmatrix} 0 & 1 \\ \pi_v & 0 \end{pmatrix}$.

(2)

$$\widehat{\varphi}_\infty = q_\infty^{-1} \cdot \text{characteristic function of } \Pi_\infty^{-1}\pi_\infty O_{B_\infty}.$$

Here Π_∞ is a generator of the maximal ideal of O_{B_∞}.

For each function $\phi_v \in S(V(k_v)^3)$ with $\phi_v = \phi_{v,1} \otimes \phi_{v,2} \otimes \phi_{v,3}$, we have that for $0 \leq i \leq 3$,

$$\omega(w_i)\phi_v = \varepsilon_v^i \cdot \phi_{v,1} \otimes \cdots \otimes \varphi_{v,3-i} \otimes \widehat{\phi}_{v,3-i+1} \otimes \cdots \widehat{\phi}_{v,3}.$$

Therefore by Lemma 3.2, $\Phi_{\tilde{\varphi}_v}(w_i,0) = \omega(w_i)\tilde{\varphi}_v(0) = \varepsilon_v^i \cdot \nu_v^i$, where $\nu_v := q_v^{-1}$ if $v \mid N_0$ and 1 otherwise. Moreover, it is observed that $\Phi_{\tilde{\varphi}}$ is in $I_v(s)^{K_0(v)}$. Therefore the proposition holds. $\square$

3.2. **Siegel-Eisenstein series.** The *Siegel-Eisenstein series associated to $\tilde{\varphi}$* is the Eisenstein series $E(g,s,\Phi_{\tilde{\varphi}})$ on $\mathrm{GSp}_3(\mathbb{A}_k)$ associated to the section $\Phi_{\tilde{\varphi}}$. Let

$$\xi(s) := \prod_{v \mid N_0\infty} \xi_v(s)$$

where $\xi_v(s)$ is the rational function in q^{-s} defined in §2.2. Then by Proposition 3.1, we have

Proposition 3.3.

$$E(g,s,\Phi^\natural)|_{s=0} = \xi(0)^{-1} \cdot E(g,s,\Phi_{\tilde{\varphi}})|_{s=0}.$$

For each $g_1 \in G^1(\mathbb{A}_k)$ and $h \in O(V)(\mathbb{A}_k)$, the theta series

$$\theta(g_1,r,\tilde{\varphi}) := \sum_{x \in V(k)^3} (\omega(g_1)\tilde{\varphi})(r^{-1}x)$$

is left $G^1(k)$ invariant as a function of $g_1 \in G^1(\mathbb{A}_k)$ and left $O(V)(k)$ invariant as a function of $r \in O(V)(\mathbb{A}_k)$. We define

$$I(g_1, \tilde{\varphi}) := \int_{O(V)(k)\backslash O(V)(\mathbb{A}_k)} \theta(g_1, r, \tilde{\varphi}) dr.$$

This integral is absolutely convergent, as $O(V)(k)\backslash O(V)(\mathbb{A}_k)$ is compact. We normalized the measure dr so that the total mass is 1. Then the following result, which is called the Siegel-Weil formula for the quadratic space V and Sp_3, connects the Siegel-Eisenstein series associated to $E(g_1, s, \Phi_{\tilde{\varphi}})$ and $I(g_1, \tilde{\varphi})$:

Theorem 3.4. *For every element $g_1 \in G^1(\mathbb{A}_k)$, we have*

$$E(g_1, s, \Phi_{\tilde{\varphi}})|_{s=0} = 2 \cdot I(g_1, \tilde{\varphi}).$$

This result follows from arguments similar to that given in Kudla-Rallis [13], details will be shown in a forthcoming paper [26].

Recall that from Corollary 2.8 (2) and Proposition 3.3, one has

$$L(F, 2) \quad = \quad b(0) \cdot \xi(0)^{-1} \cdot \int_{Z_G(\mathbb{A}_k)H(k)\backslash H(\mathbb{A}_k)} F(h) \Big(E(h, s, \Phi_{\tilde{\varphi}})|_{s=0} \Big) dh.$$

Moreover, the strong approximation theorem for GL_2 and Theorem 3.4 tell us that

Proposition 3.5.

$$L(F, 2) = \frac{b(0)\xi(0)^{-1}}{2\prod_{v|N_0\infty}(q_v + 1)^3} \cdot \sum_{[h] \in \left(\Gamma_0^{(1)}(N_0)\backslash \mathrm{SL}_2(k_\infty)/\mathcal{K}_\infty^{(1)}\right)^3} F(h) I(h, \tilde{\varphi}) \mu_0([h]),$$

where

$$\mu_0([h]) := \prod_{1 \le i \le 3} \frac{2}{\#(h_i^{-1}\Gamma_0^{(1)}(N_0)h_i \cap \mathcal{K}_\infty)}, \quad \forall h = (h_1, h_2, h_3) \in \mathrm{SL}_2(k_\infty)^3.$$

Here $\mathcal{K}_\infty$ is introduced in §1.2 and $\Gamma_0^{(1)}(N_0) = \Gamma_0(N_0) \cap \mathrm{SL}_2(A)$, $\mathcal{K}_\infty^{(1)} := \mathcal{K}_\infty \cap \mathrm{SL}_2(k_\infty)$.

3.3. **Theta series.** Recall that R is a fixed Eichler A-order of type (N_0^+, N_0^-) in the definite quaternion algebra B over k. Let $I_1, ..., I_n$ be representatives of locally-free right ideal classes of R. For $1 \le i, j \le n$, let $N_{ij} \in k^\times$ be the monic generator of the fractional ideal

$$< \mathrm{Nr}(b) : b \in I_i I_j^{-1} >_A .$$

The theta series θ_{ij} associated to $I_i I_j^{-1}$ is a function on $k_\infty^\times \times k_\infty$ defined by:

$$\theta_{ij}(y, x) := \sum_{b \in I_i I_j^{-1}} 1_{O_\infty} \left(\frac{\mathrm{Nr}(b)}{N_{ij}} y t^2 \right) \psi_\infty \left(\frac{\mathrm{Nr}(b)}{N_{ij}} x \right), \quad \forall (y, x) \in k_\infty^\times \times k_\infty.$$

Here 1_{O_∞} is the characteristic function of O_∞. It is known that (cf. [24] §2.1.1) θ_{ij} can be extended to a function $\tilde{\theta}_{ij}$ on $\mathbb{Y}_0^{(1)}(N_0) := \Gamma_0^{(1)}(N_0)\backslash \mathrm{GL}_2(k_\infty)/Z(k_\infty)\mathcal{K}_\infty$ by setting

$$\tilde{\theta}_{ij} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} := |y|_\infty \cdot \theta_{ij}(y, x).$$

Moreover, $\tilde{\theta}_{ij}$ are harmonic, i.e. for $g \in \mathrm{GL}_2(k_\infty)$,

$$\tilde{\theta}_{ij} \left(g \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix} \right) = -\tilde{\theta}_{ij}(g) \text{ and } \sum_{\gamma_\infty \in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} \tilde{\theta}_{ij}(g\gamma_\infty) = 0.$$

For $1 \le i \le n$, let R_i be the left order of I_i and $w_i := \#(R_i^\times/\mathbb{F}_q^\times)$. We have

Proposition 3.6. *For* $h_i = \begin{pmatrix} y_i & y_i^{-1}x_i \\ 0 & y_i^{-1} \end{pmatrix}$ *where* $y_i \in k_\infty^\times$ *and* $x_i \in k_\infty$*,* $1 \leq i \leq 3$*,*

$$I\big((h_1,h_2,h_3),\tilde{\varphi}\big) = \left(\sum_{i=1}^n \frac{1}{w_i}\right)^{-2} \cdot \left(\sum_{1\leq i,j\leq n} \frac{1}{w_iw_j}\tilde{\theta}_{ij}(h_1)\tilde{\theta}_{ij}(h_2)\tilde{\theta}_{ij}(h_3)\right).$$

Proof. Set $\mathbb{A}_k^0 := \prod'_{v\neq\infty} k_v$, the finite adele ring of k. Let $\widehat{R} := \prod_v R\otimes_A O_v$, and $\widehat{B} := B\otimes_k \mathbb{A}_k^0$. The double coset space $B^\times\backslash\widehat{B}^\times/\widehat{R}^\times$ is conanically identified with the set of right ideal classes of R. Here B is embedded into $\widehat{B}$ diagonally. More precisely, let $b_1,...,b_n \in \widehat{B}^\times$ be representatives of the double cosets. Then $I_1,...,I_n$, where $I_i := B\cap b_i\widehat{R}$ are representatives of right ideal classes of R.

Write $\widehat{B}^\times$ as $\coprod_{i=1}^n B^\times b_i\widehat{R}^\times$. We can choose b_i such that $\mathrm{Nr}(b_i) = 1$, as the reduced norm map from B to k is surjective. Take $\epsilon \in B^\times$ such that $\mathrm{Nr}(\epsilon) \in \mathbb{F}_q^\times - (\mathbb{F}_q^\times)^2$ and $\gamma \in \widehat{R}^\times$ such that $\mathrm{Nr}(\gamma) = \mathrm{Nr}(\epsilon)^{-1}$. Let

$$B_{\infty,+}^\times := \{b \in B_\infty^\times = (B\otimes_k k_\infty)^\times : \mathrm{Nr}(b) \text{ is monic with respect to } \pi_\infty\}.$$

Then

$$B^\times B_{\infty,+}^\times(1,b_i)\widehat{R}^\times = B^\times B_{\infty,+}^\times(1,\epsilon b_i\gamma)\widehat{R}^\times \quad\Leftrightarrow\quad R_i^\times = B^\times\cap b_i\widehat{R}^\times b_i^{-1} = \mathbb{F}_{q^2}^\times.$$

Here we embedded B into $B_{\mathbb{A}_k}$ diagonally, $(1,b_i)$ and $(1,\epsilon b_i\gamma)$ are elements in $B_\infty^\times\times\widehat{B}^\times = B_{\mathbb{A}_k}^\times$. Therefore

$$B_{\mathbb{A}_k}^\times = \coprod_{i,\nu_i} B^\times B_{\infty,+}^\times(1,b_i^{(\nu_i)})\widehat{R}^\times$$

where

$$\nu_i \in \begin{cases} \{1\} & \text{if } R_i^\times \cong \mathbb{F}_{q^2}^\times, \\ \{1,2\} & \text{if } R_i^\times \cong \mathbb{F}_q^\times, \end{cases}$$

$b_i^{(1)} = b_i$, $b_i^{(2)} = \epsilon b_i\gamma$. Moreover,

$$\Gamma_i^{\nu_i} := B^\times \cap \Big(B_{\infty,+}^\times \times b_i^{(\nu_i)}\widehat{R}^\times(b_i^{(\nu_i)})^{-1}\Big) = \begin{cases} \{\alpha\in\mathbb{F}_{q^2}^\times : \mathrm{Nr}(\alpha)=1\} & \text{if } R_i^\times \cong \mathbb{F}_{q^2}^\times, \\ \{\pm 1\} & \text{if } R_i^\times \cong \mathbb{F}_q^\times. \end{cases}$$

Let M be the algebraic group defined over k whose S-points for every k-algebra S are

$$\{(b_1,b_2) \in (B\otimes_k S)^\times \times (B\otimes_k S)^\times : \mathrm{Nr}(b_1) = \mathrm{Nr}(b_2)\}.$$

Then we have

Lemma 3.7.

$$M(\mathbb{A}_k) = \coprod_{i,j,\nu_i,\nu_j} M(k)M(k_\infty)_+ m_{i,j}^{(\nu_i,\nu_j)}K_M$$

where $M(k_\infty)_+ := M(k_\infty)\cap(B_{\infty,+}^\times\times B_{\infty,+}^\times)$, $m_{i,j}^{(\nu_i,\nu_j)} := ((1,b_i^{(\nu_i)}),(1,b_j^{(\nu_j)}))$*, and the compact group* $K_M := M(\mathbb{A}_k^0)\cap(\widehat{R}^\times\times\widehat{R}^\times)$*. Moreover,*

$$\Gamma_{i,j}^{(\nu_i,\nu_j)} := M(k)\cap\Big(M(k_\infty)_+\times m_{i,j}^{(\nu_i,\nu_j)}K_M(m_{i,j}^{(\nu_i,\nu_j)})^{-1}\Big) = \Gamma_i^{(\nu_i)}\times\Gamma_j^{(\nu_j)}.$$

We define an involution $\tau : M \to M$ by

$$\tau(\beta_1,\beta_2) \longmapsto (\beta_2\cdot\mathrm{Nr}(\beta_2)^{-1}, \beta_1\cdot\mathrm{Nr}(\beta_1)^{-1}).$$

There is then a surjective homomorphism $\rho : M \rtimes <\tau> \longrightarrow O(V)$ where for each $x \in V = B$,

$$\rho(\beta_1,\beta_2)(x) := \beta_1 x\beta_2^{-1};\ \tau(x) := \bar{x} = Tr(x) - x.$$

This yields the following exact sequence

$$0 \longrightarrow Z_M \longrightarrow M\rtimes <\tau> \longrightarrow O(V) \longrightarrow 0,$$

where Z_M the algebraic subgroup of M whose S-points are $\{(z,z)\in S^\times\times S^\times\}$. This exact sequence is an extension of the isomorphism $M/Z_M \cong SO(V)$ by the involution τ.

For each place v of k, let $\tau_v : M(k_v) \to M(k_v)$ be the involution which extends τ. Let $C := C_\infty \times C^0$, where $C_\infty :=< \tau_\infty >$ and $C^0 := \prod_{v\neq\infty} < \tau_v >$. We note that K_M is preserved by the action of elements in C^0. The homomorphism ρ injects C into $O(V)(\mathbb{A}_k)$, and C is compact with respect to the relative topology.

Now, we fix a measure of $M(\mathbb{A}_k) \rtimes C$ as follows. First, we normalize the Haar measure on $M(\mathbb{A}_k^0)$ for which K_M has volume 1. On $M(k_\infty)$, we fix the Haar measure for which $M(k_\infty)_+/Z_M(k_\infty)_+$ has volume 1. Here $Z_M(k_\infty)_+ := Z_M(k_\infty) \cap M(k_\infty)_+$. Finally, we normalize the measure on the compact group C to have volume 1. The homomorphism ρ induces a map

$$Z_M(k_\infty)_+ M(k)\backslash M(\mathbb{A}_k) \rtimes C \longrightarrow O(V)(\mathbb{A}_k),$$

and the measure on $M(\mathbb{A}_k) \rtimes C$ induces an invariant measure $d'r$ on $O(V)(\mathbb{A}_k)$. In particular, the volume of $(O(V)(k)\backslash O(V)(\mathbb{A}_k)$ with respect to $d'r$ is equal to the volume of $Z_M(k_\infty)_+ M(k)\backslash M(\mathbb{A}_k)$, which is

$$\sum_{i,j,\nu_i,\nu_j} \frac{1}{w_{i,j}^{(\nu_i,\nu_j)}}.$$

Here $w_{i,j}^{(\nu_i,\nu_j)} := \#(\Gamma_{i,j}^{(\nu_i,\nu_j)})$. This is from Lemma 3.9, and the volume of $M(k)M(k_\infty)_+ m_{i,j}^{(\nu_i,\nu_j)} K_M$ in $Z_M(k_\infty)_+ M(k)\backslash M(\mathbb{A}_k)$ with respect to $d'r$ is $\frac{1}{w_{i,j}^{(\nu_i,\nu_j)}}$. Let $w_i^{(\nu_i)} := \#(\Gamma_i^{(\nu_i)})$. Then we have

$$\sum_{\nu_i} \frac{1}{w_i^{(\nu_i)}} = \frac{1}{w_i}$$

where $w_i = \#(R_i^\times/\mathbb{F}_q^\times)$. Therefore the volume of $(O(V)(k)\backslash O(V)(\mathbb{A}_k)$ with respect to $d'r$ is

$$\left(\sum_{1\leq i\leq n} \frac{1}{w_i}\right)^2.$$

Note that the function $\tilde{\varphi}$ is invariant under K_M, C, and $M(k_\infty)_+$. So for $g_1 \in G^1(\mathbb{A}_k)$,

$$\begin{aligned} I(g_1,\tilde{\varphi}) &= \left(\sum_{i=1}^n \frac{1}{w_i}\right)^{-2} \cdot \int_{O(V)(k)\backslash O(V)(\mathbb{A}_k)} \theta(g_1, r, \tilde{\varphi}) d'r \\ &= \left(\sum_{i=1}^n \frac{1}{w_i}\right)^{-2} \cdot \int_{Z_M(k_\infty)_+ M(k)\backslash M(\mathbb{A}_k)} \theta(g_1, \rho(m), \tilde{\varphi}) dm \\ &= \left(\sum_{i=1}^n \frac{1}{w_i}\right)^{-2} \cdot \left(\sum_{i,j,\nu_i,\nu_j} \frac{1}{w_i^{(\nu_i)} w_j^{(\nu_j)}} \cdot \theta(g_1, \rho(m_{i,j}^{(\nu_i,\nu_j)}), \tilde{\varphi})\right). \end{aligned}$$

Suppose $g_1 = (h_1,h_2,h_3)$, where $h_\ell = \begin{pmatrix} y_\ell & y_\ell^{-1}x_\ell \\ 0 & y_\ell^{-1}\end{pmatrix} = \begin{pmatrix} 1 & x_\ell \\ 0 & 1\end{pmatrix}\begin{pmatrix} y_\ell & 0 \\ 0 & y_\ell^{-1}\end{pmatrix}$ with $y_\ell \in k_\infty^\times$ and $x_\ell \in k_\infty$ for $1 \leq \ell \leq 3$. We have

$$\begin{aligned} & \theta(g_1, \rho(m_{i,j}^{(\nu_i,\nu_j)}), \tilde{\varphi}) \\ = & |y_1y_2y_3|_\infty^2 \sum_{(\beta_1,\beta_2,\beta_3)\in(B\cap b_i\widehat{R}b_j^{-1})^3} \left(\prod_{1\leq\ell\leq3} 1_{\pi_\infty O_{B_\infty}}(y_\ell\beta_i)\psi_\infty(\mathrm{Nr}(\beta_i)x_\ell)\right) \\ = & \tilde{\theta}_{ij}(h_1)\cdot\tilde{\theta}_{ij}(h_2)\cdot\tilde{\theta}_{ij}(h_3), \end{aligned}$$

which is independent of (ν_i,ν_j). This completes the proof. □

Let $\tilde{\theta} := \sum_{1\leq i,j\leq n} \frac{1}{w_i w_j} \tilde{\theta}_{ij} \otimes \tilde{\theta}_{ij} \otimes \tilde{\theta}_{ij}$, which is a function on $\left(\mathbb{Y}_0^{(1)}(N_0)\right)^3$. The harmonic property of $\tilde{\theta}_{ij}$ and f_i leads to

$$L(F,2) = \frac{b(0)\xi(0)^{-1}}{2\prod_{v|N_0\infty}(q_v+1)^3} \cdot \frac{1}{2^3} \cdot \left(\sum_{i=1}^n \frac{1}{w_i}\right)^{-2} \cdot \sum_{[h]\in\left(\mathbb{Y}_0^{(1)}(N_0)\right)^3} F(h)\tilde{\theta}(h)\mu_0([h]).$$

For $h \in \mathrm{GL}_2(k_\infty)$ and $1 \leq i,j \leq n$, we define

$$\Theta_{ij}(h) := \frac{q^2}{q-1} \cdot \sum_{\alpha\in\mathbb{F}_q^\times} \tilde{\theta}_{ij}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} h\right).$$

Then Θ_{ij} are $\mathbb{Q}$-valued Drinfeld type automorphic forms for $\Gamma_0(N_0)$. Set

$$\tilde{\Theta} := \sum_{1\leq i,j\leq n} \frac{1}{w_i w_j} \Theta_{ij} \otimes \Theta_{ij} \otimes \Theta_{ij}.$$

Then $\tilde{\Theta}$ is a function on $\left(\mathbb{Y}_0(N_0)\right)^3$ and

$$L(F,2) = \frac{b(0)\xi(0)^{-1}}{2\prod_{v|N_0\infty}(q_v+1)^3} \cdot \frac{1}{2^3} \cdot \left(\sum_{i=1}^n \frac{1}{w_i}\right)^{-2} \cdot \frac{2^6}{q^6} \cdot \sum_{[h]\in\left(\mathbb{Y}_0(N_0)\right)^3} F(h)\tilde{\Theta}(h)\mu_1([h])$$

where for $h = (h_1,h_2,h_3) \in \mathrm{GL}_2(k_\infty)^3$, $\mu_1([h]) := \mu([h_1]) \cdot \mu([h_2]) \cdot \mu([h_3])$ and the measure μ on $\mathbb{Y}_0(N_0)$ is introduced in §1.3. Note that

$$b(0) = \left(\frac{q^3}{(q^2-1)(q-1)}\right)^2, \xi(0)^{-1} = \prod_{v|N_0\infty} \frac{(q_v+1)^3(q_v^2-1)^2}{2q_v(q_v-\varepsilon_v)^2},$$

and the mass formula (cf. [4] §1) gives

$$\sum_{i=1}^n \frac{1}{w_i} = \frac{\prod_{v|N_0}(q_v+\varepsilon_v)}{q^2-1}.$$

Let γ_{N_0} be the number of prime factors of N_0. We get

Proposition 3.8. *The central critical value $L(F,2)$ is equal to*

$$\frac{1}{q\cdot|N_0|_\infty\cdot 2^{\gamma_{N_0}-1}} \cdot \sum_{[h]\in\left(\mathbb{Y}_0(N_0)\right)^3} F(h)\overline{\tilde{\Theta}(h)}\mu_1([h]).$$

3.4. **Gross-Kudla type formula.**

3.4.1. *Definite Shimura curves.* We start with a brief review of basic properties of definite Shimura curves. Further details are referred to [24]. Recall that B is the definite quaternion algebra over k which is only ramified at places v of k with $\varepsilon_v = -1$, and R is a chosen Eichler A-order of type (N_0^+, N_0^-). Let Y be the genus zero curve defined over k whose M-points for any k-algebra M are

$$Y(M) = \{x \in B\otimes M : x \neq 0, \mathrm{Tr}(x) = \mathrm{Nr}(x) = 0\}/M^\times.$$

The group $B^\times$ acts on Y (from the right) by conjugation. We define the *definite Shimura curves*

$$X_{N_0^+,N_0^-} := \left(\widehat{R}^\times\backslash\widehat{B}^\times \times Y\right)/B^\times,$$

where $\widehat{B} := B\otimes_k \mathbb{A}_k^0$ ($\mathbb{A}_k^0$ is the finite adele ring of k) and $\widehat{R} := \prod_P R_P$.

We have also chosen representatives $I_1, ..., I_n$ of locally-free right ideal classes of R, and R_i is the left order of I_i. Then $I_i^{-1}, ..., I_n^{-1}$ are representatives of left ideal classes of R, and R_i is the right order of I_i^{-1}. Then $X_{N_0^+,N_0^-}$ is in fact the disjoint union $\coprod_{i=1}^n X_i$, where $X_i := Y/R_i^\times$. Since X_i has genus zero for each i, we have

$$\mathrm{Pic}(X_{N_0^+,N_0^-}) = \bigoplus_{i=1}^n \mathbb{Z}e_i,$$

where the divisor class e_i corresponds to the component X_i.

For each monic polynomial m in A and $1 \leq i, j \leq n$, set

$$B_{ij}(m) := \frac{\#\{b \in I_j I_i^{-1} : (\frac{\mathrm{Nr}(b)}{N_{ij}}) = (m)\}}{(q-1)w_j} \in \mathbb{Z}_{\geq 0}$$

where $w_j = \#(R_j^\times/\mathbb{F}_q^\times)$ and N_{ij} is the monic generator of the fractional ideal generated by $\mathrm{Nr}(b)$ for $b \in I_j I_i^{-1}$. Then for any monic $m \in A$ with $(m, N_0) = 1$, the Hecke correspondence t_m acting on $X_{N_0^+,N_0^-}$ is expressed by

$$t_m e_i = \sum_{i=1}^n B_{ij}(m) e_j.$$

The so-called *Gross height pairing* $< \cdot, \cdot >$ on $\mathrm{Pic}(X_{N_0^+,N_0^-})$ is defined by setting

$$< e_i, e_j > := \begin{cases} 0 & \text{if } i \neq j, \\ w_i & \text{if } i = j, \end{cases}$$

and extending bi-additively. We point out that t_m with $(m, N_0) = 1$ is self-adjoint with respect to this pairing.

Let $M(\Gamma_0(N_0), \mathbb{R})$ be the space of $\mathbb{R}$-valued Drinfeld type automorphic forms for $\Gamma_0(N_0)$. The map

$$\Phi : \mathrm{Pic}(X_{N_0^+,N_0^-}) \times \mathrm{Pic}(X_{N_0^+,N_0^-}) \longrightarrow M(\Gamma_0(N_0), \mathbb{R})$$

which is defined by $\Phi(e_i, e_j) := \Theta_{ij}$ satisfies that for $e, e' \in \mathrm{Pic}(X_{N_0^+,N_0^-})$ and for monic $m \in A$ with $(m, N_0) = 1$,

$$\Phi(t_m e, e') = \Phi(e, t_m e').$$

Moreover, let $\mathbb{T}_\mathbb{R} := \mathbb{R}[t_m : \text{monic } m \in A \text{ with } (m, N_0) = 1]$. Then from the Jacquet-Langlands correspondence and strong multiplicity one theorem (cf. [11]), Φ induces a Hecke module homomorphism (cf. [24] Theorem 2.6)

$$\Phi : \big(\mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_\mathbb{Z} \mathbb{R}\big) \otimes_{\mathbb{T}_\mathbb{R}} \big(\mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_\mathbb{Z} \mathbb{R}\big) \longrightarrow M(\Gamma_0(N_0), \mathbb{R})$$

such that for each newform f for $\Gamma_0(N_0)$, there exists a unique one-dimensional eigenspace $\mathbb{R}e_f$ in $\mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_\mathbb{Z} \mathbb{R}$ satisfying that $t_m e_f = c_m(f) e_f$ for any monic $m \in A$ with $(m, N_0) = 1$. Here $c_m(f)$ is the eigenvalue of T_m associated to f. We point out that if f is normalized, then for each $e' \in \mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_\mathbb{Z} \mathbb{R}$,

$$\Phi(e_f, e') = < e_f, e' > \cdot f.$$

3.4.2. *The diagonal cycle* Δ. Consider $\big(\mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_\mathbb{Z} \mathbb{R}\big)^{\otimes 3}$, with natural action by $\mathbb{T}_\mathbb{R}^{\otimes 3}$. We have an induced pairing $< \cdot, \cdot >^{\otimes 3}$ on $\big(\mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_\mathbb{Z} \mathbb{R}\big)^{\otimes 3}$ by setting

$$< a_1 \otimes a_2 \otimes a_3, a_1' \otimes a_2' \otimes a_3' >^{\otimes 3} := < a_1, a_1' > \cdot < a_2, a_2' > \cdot < a_3, a_3' > .$$

We also have the induced map

$$\Phi^{\otimes 3} : \big(\mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_\mathbb{Z} \mathbb{R}\big)^{\otimes 3} \times \big(\mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_\mathbb{Z} \mathbb{R}\big)^{\otimes 3} \to M(\Gamma_0(N_0), \mathbb{R})^{\otimes 3}$$

by setting

$$\Phi^{\otimes 3}(a_1 \otimes a_2 \otimes a_3, a_1' \otimes a_2' \otimes a_3') := \Phi^{\otimes 3}(a_1, a_1') \otimes \Phi^{\otimes 3}(a_2, a_2') \otimes \Phi^{\otimes 3}(a_3, a_3').$$

Take $\Delta := \sum_{i=1}^{n} \frac{1}{w_i} e_i \otimes e_i \otimes e_i$. Then it is clear that

Lemma 3.9. *The function* $\tilde{\Theta}$ *is equal to* $\Phi^{\otimes 3}(\Delta, \Delta)$.

3.4.3. *Special values.* Let $S(\Gamma_0(N_0))$ be the space of Drinfeld type cusp forms for $\Gamma_0(N_0)$. For $F_1 = f \otimes g \otimes h \in S(\Gamma_0(N_0))^{\otimes 3}$ and $F_2 = f' \otimes g' \otimes h' \in M(\Gamma_0(N_0))^{\otimes 3}$, we extend the Petersson inner product by setting

$$(F_1, F_2)^{\otimes 3} := (f, f') \cdot (g, g') \cdot (h, h').$$

Given any three monic polynomials m_1, m_2, m_3 in A with $(m_1 m_2 m_3, N_0) = 1$, we have a natural action of $T_{m_1} \otimes T_{m_2} \otimes T_{m_3}$ on $M(\Gamma_0(N_0))^{\otimes 3}$ defined by

$$T_{m_1} \otimes T_{m_2} \otimes T_{m_3}(h_1 \otimes h_2 \otimes h_3) := T_{m_1} h_1 \otimes T_{m_2} h_2 \otimes T_{m_3} h_3.$$

Recall that our function $F = f_1 \otimes f_2 \otimes f_3 \in M(\Gamma_0(N_0), \mathbb{R})^{\otimes 3}$, where f_1, f_2, f_3 are three normalized newforms for $\Gamma_0(N_0)$. Let

$$e_F := e_{f_1} \otimes e_{f_2} \otimes e_{f_3} \in \left(\mathrm{Pic}(X_{N_0^+, N_0^-}) \otimes_{\mathbb{Z}} \mathbb{R} \right)^{\otimes 3}$$

where $\mathbb{R} e_{f_i}$ is the eigenspace corresponding to f_i. Let $t_F \in \mathbb{T}_{\mathbb{R}}^{\otimes 3}$ be the projection from the space $\left(\mathrm{Pic}(X_{N_0^+, N_0^-}) \otimes_{\mathbb{Z}} \mathbb{R} \right)^{\otimes 3}$ onto $\mathbb{R} e_F$ with respect to $< \cdot, \cdot >^{\otimes 3}$, i.e.

$$t_F x := \frac{< x, e_F >^{\otimes 3}}{< e_F, e_F >^{\otimes 3}} \cdot e_F.$$

Define $\Delta_F := t_F \Delta$, the component of Δ in the space $\mathbb{R} e_F$. Then

Lemma 3.10. *The component of* $\tilde{\Theta}$ *in the eigenspace* $\mathbb{R}F$ *with respect to* $(\cdot, \cdot)^{\otimes 3}$ *is*

$$< \Delta_F, \Delta_F >^{\otimes 3} F.$$

The above lemma says that $(F, \tilde{\Theta})^{\otimes 3} = (F, F)^{\otimes 3} \cdot < \Delta_F, \Delta_F >^{\otimes 3}$. By Proposition 3.7, we arrive at our main result:

Theorem 3.11. *Let* N_0 *be a square-free ideal of* A *and let* γ_{N_0} *be the number of prime factors of* N_0. *Let* $F = f_1 \otimes f_2 \otimes f_3$, *where* f_1, f_2, f_3 *are normalized Drinfeld type newforms for* $\Gamma_0(N_0)$. *Suppose the root number* $\varepsilon = \prod_{v | N_0 \infty} \varepsilon_v = 1$. *Then we have*

$$L(F, 2) = \frac{(F, F)^{\otimes 3}}{q |N_0|_\infty 2^{\gamma_{N_0} - 1}} \cdot < \Delta_F, \Delta_F >^{\otimes 3}.$$

Remark. 1. The central critical value $L(F, 2)$ is a non-negative real number.
2. Suppose $e_{f_i} = \sum_{j=1}^{n} \beta_{i,j} e_j \in \mathrm{Pic}(X_{N_0}) \otimes_{\mathbb{Z}} \mathbb{R}$ where $\beta_{i,j} \in \mathbb{R}$ for $1 \le i \le 3$. Then

$$< \Delta_F, \Delta_F >^{\otimes 3} = \frac{(\sum_j w_j^2 \beta_{1,j} \beta_{2,j} \beta_{3,j})^2}{(\sum_j w_j \beta_{1,j}^2)(\sum_j w_j \beta_{2,j}^2)(\sum_j w_j \beta_{3,j}^2)}.$$

4. Application to elliptic curves and examples

Let N_0 be a square-free ideal of A. Let E be an elliptic curve over k which is of conductor $N_0 \infty$ and has split multiplicative reduction at an even number of places including ∞. From the work of Weil, Jacquet-Langlands, and Deligne, there exists a normalized Drinfeld type newform f_E for $\Gamma_0(N_0)$ such that

$$L(E, s+1) = L(f_E, s).$$

Here $L(E, s+1)$ is the Hasse-Weil L-function associated to E.

Let $F_E := f_E \otimes f_E \otimes f_E$. Clearly, the root number of $L(F_E, s)$ is positive, and we have

$$L(F_E, s) = L(\mathrm{Sym}^3 E, s) \cdot L(E, s-1)^2,$$

From the work of Deligne [3] and Lafforgue [14], The L-function $L(\mathrm{Sym}^3 E, s)$ is entire. Therefore the special value formula in Theorem 3.10 implies that

Corollary 4.1. *Let E be an elliptic curve over k which is of conductor $N_0\infty$ and has split multiplicative reduction at an even number of places including ∞. Let N_0^- be the product of primes of A where E has split multiplicative reduction and $N_0^+ = N_0/N_0^-$. If we write*

$$e_{f_E} = \sum_{j=1}^{n} \beta_j(E)e_j \in \mathrm{Pic}(X_{N_0^+,N_0^-}) \otimes_{\mathbb{Z}} \mathbb{Q},$$

then the central critical value $L(E,1)$ does not vanish if $A_E := \sum_{j=1}^n w_j^2 \beta_j(E)^3 \neq 0$.

Remark. The Birch and Swinnerton-Dyer conjecture predicts the following equality:

$$\mathrm{ord}_{s=1} L(E,s) \overset{?}{=} \mathrm{rank}_{\mathbb{Z}} E(k).$$

It is known that (cf. [22]) $\mathrm{ord}_{s=1} L(E,s) \geq \mathrm{rank}_{\mathbb{Z}} E(k)$, which means, in particular, that the conjecture holds when $L(E/k,1) \neq 0$. Therefore the non-vanishing of A_E guarantees the finiteness of the Mordell-Weil group $E(k)$.

4.1. **Examples.** We present two examples from elliptic curves in this subsection. Some of the calculations below were performed using the computer package: Sage.

4.1.1. *Example 1.* Let $k = \mathbb{F}_7(t)$ (i.e. $q = 7$). Let E be the following elliptic curve:

$$E : y^2 = x^3 - 3t(t^3+2)x + (-2t^6 + 3t^3 + 1).$$

The conductor of E is $(t^3-2)\infty$. More precisely, E has split multiplicative reduction at (t^3-2) and ∞. Let $N_0 = t^3 - 2$. Let f_E be the normalized Drinfeld type newform for $\Gamma_0(N_0)$ corresponding to E. Let $F_E := f_E \otimes f_E \otimes f_E$. We compute

$$L(F_E, s) = 1 - 28q^{-s} - 1617q^{-2s} - 67228q^{-3s} + 5764801q^{-4s}.$$

This L-function satisfies the functional equation in Theorem 2.1, and the central critical value $L(F_E, 2) = 9/49$.

On the other hand, $\gamma_{N_0} = 1$, and from a formula of Gekeler (cf. [19] Theorem 1.1) we immediately get $(f_E, f_E) = 39$. Such computation can be also checked via the algorithm in [17]. According to [18] Example 19, we have $w_1 = 8$, $w_2 = \cdots = w_8 = 1$, and the corresponding divisor is $e_{f_E} = [1,-4,-1,-1,2,-1,2,2]$. Then we get $< \Delta_{F_E}, \Delta_{F_E} >^{\otimes 3} = 21^2/39^3$. Therefore

$$\frac{(F_E,F_E)^{\otimes 3}}{q|N_0|_\infty 2^{\gamma_{N_0}-1}} \cdot < \Delta_{F_E}, \Delta_{F_E} >^{\otimes 3} = \frac{39^3}{7^4}\cdot\frac{21^2}{39^3} = \frac{9}{49} = L(F_E, 2).$$

4.1.2. *Example 2.* Let $k = \mathbb{F}_3(t)$ (i.e. $q = 3$). For $0 \leq i \leq 2$, let E_i be the following elliptic curve over k:

$$E_i : y^2 = x^3 + ((t+i)^2+1)x^2 + (t+i)^2 x.$$

The conductor of E_i is $(t)(t+1)(t+2)\infty$ for each i. More precisely, E_i has split multiplicative reduction at $(t+i)$ and ∞, and has non-split multiplicative reduction at $(t+j)$ for $j \neq i$.

Let $N_0 = t(t+1)(t-1) = t^3 - t$. Let f_i be the normalized Drinfeld type newform for $\Gamma(N_0)$ associated to E_i. Let $F := f_0 \otimes f_1 \otimes f_2$. We compute the triple product L-function

$$L(F, s) = 1 + 28q^{-s} + 358q^{-2s} + 2268q^{-3s} + 6561q^{-4s}.$$

This L-function satisfies the functional equation in Theorem 2.1, and the central critical value $L(F,2) = 1024/81$. On the other hand, $\gamma_{N_0} = 3$, and we compute that for $0 \leq i \leq 2$, $(f_i, f_i) = 16$. Therefore we get $(F,F)^{\otimes 3} = 2^{12}$. The number γ_{N_0} of prime factors of N_0 is 3. The only value remained is $< \Delta_F, \Delta_F >^{\otimes 3}$.

Let B be the definite quaternion algebra over k ramified at (t), $(t+1)$, and $(t-1)$. Then

$$B = k + k\alpha + k\beta + k\alpha\beta$$

where $\alpha^2 = -1$, $\beta^2 = N_0 = t^3 - t$, and $\beta\alpha = -\alpha\beta$. Let $R := A + A\alpha + A\beta + A\alpha\beta$, which is a maximal A-order in B. the class number (of left ideal classes) is 4, and $w_i = \#(R^\times/\mathbb{F}_3^\times) = 4$ for $1 \leq i \leq 4$. We calculate the following Brandt matrices:

$$\big(B_{ij}(t)\big)_{1\leq i,j\leq 4} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \big(B_{ij}(t{+}1)\big)_{1\leq i,j\leq 4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \big(B_{ij}(t{+}2)\big)_{1\leq i,j\leq 4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The corrsponding divisors e_{f_i} for $0 \leq i \leq 2$ can be chosen by:

$$e_{f_0} := [1, 1, -1, -1],\ \ e_{f_1} := [1, -1, 1, -1],\ \ e_{f_2} := [1, -1, -1, 1].$$

Therefore $< \Delta_F, \Delta_F >^{\otimes 3} = 1$, and

$$\frac{(F,F)^{\otimes 3}}{q|N_0|_\infty 2^{\gamma_{N_0}-1}} \cdot < \Delta_F, \Delta_F >^{\otimes 3} = \frac{2^{12}}{3^4 \cdot 2^2} \cdot 1 = \frac{1024}{81} = L(F, 2).$$

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