Brandt Matrices and Theta Series over Global Function Fields

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Abstract

The aim of this article is to give a complete account of the Eichler-Brandt theory over function fields and the basis problem for Drinfeld type automorphic forms. Given arbitrary function field k together with a fixed place ∞, we construct a family of theta series from the norm forms of "definite" quaternion algebras, and establish an explicit Hecke-module homomorphism from the Picard group of an associated definite Shimura curve to a space of Drinfeld type automorphic forms. The "compatibility" of these homomorphisms with different square-free levels is also examined. These Hecke-equivariant maps lead to a nice description of the subspace generated by our theta series, and thereby contributes to the so-called basis problem. Restricting the norm forms to pure quaternions, we obtain another family of theta series which are automorphic functions on the metaplectic group, and results in a Shintani-type correspondence between Drinfeld type forms and metaplectic forms.

Key words and phrases. Function fields, Brandt matrices, automorphic forms on GL_2, theta series, metaplectic forms.

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Contents

CHAPTER I

Introduction

The aim of this article is to study the family of the so-called *Brandt matrices.* In the number field case, the entries of these matrices are essentially the representation number of positive integers by reduced norm forms on definite quaternion algebras over $\mathbb{Q}$. Let N_0^- be a square-free positive integer with an odd number of prime factors. Let $D = D_{N_0^-}$ be the definite quaternion algebra over $\mathbb{Q}$ which is ramified precisely at the prime factors of N_0^-. Let N_0^+ be another square-free positive integer prime to N_0^-. Take an Eichler order R of type (N_0^+, N_0^-), i.e. R is an order in D satisfying that $R_p := R \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal $\mathbb{Z}_p$-order in $D_p := D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for every $p \nmid N_0^+$, and for $p \mid N_0^+$

$$R_p \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}_p) \;\middle|\; c \in N_0^+ \mathbb{Z}_p \right\}.$$

Choose $\{I_1, ..., I_n\}$ to be a complete set of representatives of locally-principal right ideal classes of R. For each positive integer m and $1 \leq i, j \leq n$, set

$$B_{ij}(m) := \frac{\#\{b \in I_i I_j^{-1} : \mathrm{Nr}(b) N_{ij}^{-1} = m\}}{\#(R_j^\times)} \in \mathbb{Z}_{\geq 0},$$

where R_j is the left order of I_j, $\mathrm{Nr}(b)$ is the reduced norm of b, and N_{ij} is the positive generator of the fraction ideal $\mathrm{Nr}(I_i)\,\mathrm{Nr}(I_j)^{-1}$ in $\mathbb{Q}$. We call $B(m) := (B_{ij}(m))_{1 \leq i,j \leq n}$ the m-th Brandt matrix associated to R. For convenience, set $B_{ij}(0) := 1/\#(R_j^\times)$.

It is known that for each pair (i,j), $1 \leq i, j \leq n$, the following theta series

$$\sum_{m \geq 0} B_{ij}(m) \exp(2\pi\sqrt{-1} m z)$$

is a weight-2 modular form of level $N_0^+ N_0^-$. Recall that the basis problem (cf. [**3**]) is about finding a "natural" basis for the space of modular forms. Here "natural" means that these linearly independent forms are arithmetically distinguished and whose Fourier coefficients are known or easy to obtain. The celebrated Eichler's trace formula (cf. [**2**] and [**3**]) connects the Brandt matrices with the Hecke operators on the space of weight-2 modular forms. This implies, among other things, that the theta series from definite quaternion algebras over $\mathbb{Q}$ generate the whole space of weight-2 modular forms of the corresponding level. In other words, these theta series provide us a natural basis and give a solution of the basis problem for weight-2 modular forms.

By a global function field k, we mean k is a finitely generated field extension of transcendence degree one over a finite field. Fix a place ∞ of k, we are interested in *Drinfeld type* automorphic forms, which are automorphic forms on GL_2 satisfying the so-called harmonic property with respect to ∞ (cf. Chapter III Section 3). In particular, let T_∞ be the Iwahori Hecke operator at ∞, i.e. T_∞ corresponds to the double coset

$$\mathcal{K}_\infty \begin{pmatrix} \pi_\infty & 0 \\ 0 & 1 \end{pmatrix} \mathcal{K}_\infty$$

where π_∞ is a uniformizer at ∞, $\mathcal{K}_\infty$ is the Iwahori subgroup of $\mathrm{GL}_2(O_\infty)$ and O_∞ is the valuation ring of the completion k_∞ of k at ∞. Then the harmonicity of these forms implies that they are fixed by T_∞. From the point of view of the representation theory, these forms correspond (at ∞) to the new forms in the special representation $\sigma(|\cdot|_\infty^{1/2}, |\cdot|_\infty^{-1/2})$. It is natural to view these forms as analogue of classical weight 2 modular forms.

From the work of Deligne, Drinfeld, Jacquet-Langlands, Weil, and Zarhin, the "Drinfeld modularity" always exists for every non-isotrivial elliptic curve over k. Here we call an elliptic curve over k *(non-)isotrivial* if its j-invariant is (not) in the constant field of k. More precisely, let E be a such elliptic curve over k which has split multiplicative reduction at ∞. Denote by $\mathfrak{N}\infty$ the conductor of E. Then there is a surjective homomorphism over k (cf.

[**4**])

$$J_0(\mathfrak{N}) \twoheadrightarrow E,$$

where $J_0(\mathfrak{N})$ is the Jacobian of the Drinfeld modular curve $X_0(\mathfrak{N})$. In particular, there exists a unique Drinfeld type automorphic cusp form F_E of level $\mathfrak{N}$ such that its L-function coincides with the Hasse-Weil L-function of E over k. This development motivates us to work out Eichler-Brandt theory for any pair (k, ∞), in order to collect as much as possible explicit information for Drinfeld type automorphic forms. In other words, we target at the basis problem for Drinfeld type automorphic forms.

From the reduced norm forms on "definite" (with respect to ∞) quaternion algebras over k, we construct a particular family of theta series which are Drinfeld type automorphic forms. The action of the Hecke operators on these theta series can be read off from Brandt matrices. It is observed that the Brandt matrices also represent the action of the Hecke correspondences on an associated definite Shimura curve (cf. Proposition II.4). We then establish a Hecke-equivariant homomorphism from the Picard group of the definite Shimura curve in question to a space of Drinfeld type automorphic forms. More precisely, let A be the ring of functions in k regular outside ∞, and X is the definite Shimura curve of type $(\mathfrak{N}_0^+, \mathfrak{N}_0^-)$ where $\mathfrak{N}_0^+$ is square-free (cf. Definition II.3). The first main result of this article is the following (cf. Chapter III Section 4 and 5):

THEOREM I.1. *There is a $\mathbb{Z}$-bilinear map*

$$\Phi : \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee \longrightarrow \mathcal{M}_0(\mathfrak{N}_0),$$

where $\mathrm{Pic}(X)^\vee := \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$ *and* $\mathcal{M}_0(\mathfrak{N}_0)$ *is the space of Drinfeld type automorphic forms of level* $\mathfrak{N}_0 = \mathfrak{N}_0^+\mathfrak{N}_0^-$, *such that for each ideal M of A prime to $\mathfrak{N}_0^+$ and each pair* $(e, e') \in \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee$,

$$T_M\Phi(e, e') = \Phi(t_M e, e') = \Phi(e, t_M^* e').$$

Here T_M is the Hecke operator on $\mathcal{M}_0(\mathfrak{N}_0)$, and t_M^* is the adjoint of the Hecke correspondence t_M on X. Moreover, for every normalized Drinfeld *type newform F of level $\mathfrak{N}_0$, there exists a unique (up to a scalar multiple) element e_F in* $\mathrm{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ *such that*

$$\Phi(e, e_F) = < e, e_F > F, \quad \forall e \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C},$$

where $< \cdot, \cdot >$ is the Gross height pairing.

This theorem can be viewed as a function field analogue of Proposition 5.6 in [**6**]. In the number field case, the existence and the uniqueness of e_F essentially follows from Eichler's trace formula. Here we use Jacquet-Langlands correspondence instead (cf. Theorem III.13 and III.14). The Gross height pairing is introduced in [**6**] §4. For convenience, we recall the definition in the following. Note that (cf. Chapter II Section 4) the definite Shimura curve X of type $(\mathfrak{N}_0^+, \mathfrak{N}_0^-)$ is a disjoint union of genus-0 curves, and the components $e_1, ..., e_n$ of X correspond canonically to the locally-principal right ideal classes $[I_1], ..., [I_n]$ of an Eichler A-order R of type $(\mathfrak{N}_0^+, \mathfrak{N}_0^-)$. Let R_i be the left order of I_i and $w_i := \#(R_i^\times / \mathbb{F}_k^\times)$ (where $\mathbb{F}_k$ denotes the constant field of k). Then the Gross height pairing on $\mathrm{Pic}(X)$ is simply defined by

$$< e_i, e_j > := \begin{cases} w_i & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and extended bi-additively. Via this pairing, $\mathrm{Pic}(X)^\vee$ is considered as a subgroup of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We refer the reader to Chapter II Section 4.2 for further details.

Take a place v_0 of k which is prime to $\mathfrak{N}_0\infty$, and let X_{v_0} be the definite Shimura curve over k of type $(v_0\mathfrak{N}_0^+, \mathfrak{N}_0^-)$. The canonical map pr from X_{v_0} to X induces group homomorphisms

$$pr_* : \mathrm{Pic}(X_{v_0}) \to \mathrm{Pic}(X)$$

and

$$pr^* : \mathrm{Pic}(X)^\vee \to \mathrm{Pic}(X_{v_0})^\vee.$$

Let

$$\Phi_{v_0} : \mathrm{Pic}(X_{v_0}) \times \mathrm{Pic}(X_{v_0})^\vee \to \mathcal{M}_0(v_0\mathfrak{N}_0)$$

be the corresponding Hecke module homomorphism for X_{v_0}. The problem of "compatibility" of Φ_{v_0} with Φ then arrives, see Chapter III Section 4.1.

The tool for our construction of theta series is of course Weil representations. We select special test functions according to the arithmetic data from ideal class representatives of a given Eichler A-order of type $(\mathfrak{N}_0^+, \mathfrak{N}_0^-)$. An explicit description for the space generated by our theta series falls out from Theorem I.1. In particular, let $\mathcal{S}_0(1, \mathfrak{N}_0)$ be the subspace of the old forms coming from the Drinfeld type cusp forms of full level. Then every Drinfeld type cusp form of level $\mathfrak{N}_0$ which is orthogonal (with respect to the Petersson inner product) to $\mathcal{S}_0(1, \mathfrak{N}_0)$ can be generated by our theta series. Contrary to the case of classical weight 2 modular forms, the space $\mathcal{S}_0(1, \mathfrak{N}_0)$ is not trivial in general. One can find an elliptic curve E over a suitable function field k which has split multiplicative reduction at ∞ and good reduction elsewhere (cf. Example III.17). Then from the Drinfeld modularity, there exists a Drinfeld type newform F_E in $\mathcal{S}_0(1, \mathfrak{N}_0)$, which does not come from our theta series. In other words, we are unable to find a natural basis for the whole space of Drinfeld type cusp forms of level $\mathfrak{N}_0$ via only the theta series from definite quaternion algebras because of the non-triviality of $\mathcal{S}_0(1, \mathfrak{N}_0)$.

In the second part of this article (Chapter IV), assume the characteristic of k is odd. We are interested in automorphic functions on the *metaplectic group* (cf. Section 1.1). Using the Weil representations of the metaplectic group as a tool, we construct yet another family of theta series from the reduced norm forms, considering pure quaternions inside definite quaternion algebras. These theta series are, in particular, eigenfunctions of the Iwahori Hecke operator T_∞ at ∞, with equal eigenvalue $q_\infty^{1-3/4}$. Taking this operator to be our "non-Euclidean Laplacian", we view these theta series as metaplectic forms of weight-3/2. Moreover, the Fourier coefficients of these theta series contain arithmetic information from pure quaternions. Consequently, the action of the Hecke operators on these theta series can also be described by Brandt matrices (cf. Theorem IV.16). We then obtain a Shintani-type correspondence **Sh** between the space of Drinfeld type cusp forms and the space of weight-3/2 metaplectic forms (cf. Theorem IV.18):

THEOREM I.2. *Let* $\mathcal{S}_0^{(\mathfrak{N}_0^-)-\mathrm{new}}(\mathfrak{N}_0)$ *be the space of Drinfeld type "*$\mathfrak{N}_0^-$*-new" forms, and* $\mathcal{M}_0^{(3/2)}(\Omega^2\mathfrak{N}_0)$ *be the space of "weight-3/2" metaplectic forms of level* $\Omega^2\mathfrak{N}_0$ *(where* $\Omega = \Omega_\psi$ *is the "extra level" coming from the choice of non-trivial additive character* ψ *on the adele class group of* k*). There exists a linear map*

$$\mathbf{Sh} : \mathcal{S}_0^{(\mathfrak{N}_0^-)-\mathrm{new}}(\mathfrak{N}_0) \longrightarrow \mathcal{M}_0^{(3/2)}(\Omega^2\mathfrak{N}_0)$$

satisfying that for each place v *of* k *with* $\mathrm{ord}_v(\Omega^2\mathfrak{N}_0\infty) = 0$,

$$\mathbf{Sh}(T_vF) = T_{v^2,3/2}\ \mathbf{Sh}(F), \quad \forall F \in \mathcal{S}_0^{(\mathfrak{N}_0^-)-\mathrm{new}}(\mathfrak{N}_0).$$

In the number field case, the theory of half-integral weight modular forms has been well developed with an analogous Hecke theory (starting with Shimura's work in [**13**]). Moreover, in [**13**] Shimura established a Hecke-equivariant lifting from half-integral weight modular forms to integral weight forms (via converse theorem). The adjoint lifting of Shimura's (from integral weight modular forms to half-integral weight forms) is provided by Kohnen [**8**], [**9**], and Shintani [**14**]. It is natural to ask for the adjoint lifting of our **Sh**, i.e. a Shimura-type correspondence from $\mathcal{S}_0^{(3/2)}(\Omega^2\mathfrak{N}_0)$ to $\mathcal{S}_0(\mathfrak{N}_0)$. We will study this topic in a future work.

When k is a rational function field, this map **Sh** was first constructed in [**19**]. Based on the results of [**18**] concerning the central critical values of Rankin-type L-functions, a function field analogue of a Waldspurger-type formula is also derived in that paper. It follows that for a normalized Drinfeld type newform F, the λ-th Fourier coefficient of $\mathbf{Sh}(F)$, where λ is irreducible in A, determines the non-vanishing of the central critical value of the L-function of F twisted by a quadratic character χ_λ. We expect that such phenomenon can be found over arbitrary function fields.

We include in the last chapter a detailed study of the trace formula of Brandt matrices in the function field context. This formula expresses the traces of Brandt matrices in terms of modified Hurwitz class numbers of quadratic A-orders. In other words, this connects, in the definite case, the arithmetic of quaternary quadratic forms with that of binary quadratic forms. Similar to the number field case, our method for establishing this formula comes from Eichler's theory of optimal embeddings adapted to function fields of positive characteristic, which is recalled in Section 1 of Chapter V. In the number field case, this formula, together with the trace computation of Hecke operators on weight-2 modular forms, indicates that the algebra generated by Brandt matrices is isomorphic to the Hecke algebra on weight-2 modular forms (with corresponding level). This is a key step in Eichler's argument for the basis problem. We refer readers to **[2]**, **[3]**, and **[15]** for further discussions in this topic.

This article is organized as follows. In Chapter II, after fixing the notations in Section 1, we review basic properties of quaternion algebras over global function fields. In Section 3, we introduce the Brandt matrices associated to a given Eichler A-order in definite quaternion algebras. In Section 4 we introduce the definite Shimura curves, and connects Brandt matrices with the Hecke correspondences on these curves. Also included in Section 4 is the Gross height pairing on the definite Shimura curves, which is crucial to the construction of the Heck module homomorphism Φ in Theorem I.1.

In Chapter III, we recall first the Weil representation, and then construct a family of theta series from the reduced norm forms on definite quaternion algebras over k in Section 2. Also proved there is the harmonicity of the theta series, which shows that these theta series are Drinfeld type automorphic forms. In Section 3 we verify that the Fourier coefficients of our theta series are essentially the entries of Brandt matrices. We then construct the Hecke module map Φ in Section 4. The "compatibility" between these Hecke module maps as the square-free levels varying is discussed in Section 4.1. We finally treat the basis problem for Drinfeld type cusp forms of square-free levels in Section 6, and describe explicitly the subspace generated by our theta series.

In Chapter IV we assume the characteristic of the base field k is odd, and explore the connection between Brandt matrices and Hecke operators on metaplectic forms. Utilizing Weil representations for the metaplectic group, we first construct a particular theta series $\widetilde{\Theta}$, which is an analogue of the most classical weight-1/2 theta series in our context. The theta series from pure quaternions are then constructed in Section 1.2. In Section 2 we show that the Hecke operators acting on these theta series can also be represented by Brandt matrices (via the technical result in Section 3). The Shintani-type correspondence $\mathbf{Sh}$ is finally established at the end of Section 2.

In the last chapter, we first recast Eichler's theory of optimal embeddings by introducing chains of local lattices. The trace formula of our generalized Brandt matrices is then established in Section 2.

CHAPTER II

Brandt matrices and definite Shimura curves

Given a global function field k together with a fixed place ∞, we consider "definite" quaternion algebras over k and introduce the Brandt matrices. The entries are non-negative integers which count the number of ideals of given norm in an ideal class of an Eichler A-order. Then we define the definite Shimura curves associated to Eichler A-orders and introduce Hecke correspondences on these definite Shimura curves. We describe the connection between these correspondences and Brandt matrices.

1. Basic setting

Let k be a global function field with finite constant field $\mathbb{F}_q$, i.e. k is a finitely generated field extension of transcendence degree one over $\mathbb{F}_q$ and $\mathbb{F}_q$ is algebraically closed in k. For each place v of k, the completion of k at v is denoted by k_v, and O_v is the valuation ring in k_v. We choose a uniformizer π_v in O_v and set $\mathbb{F}_v := O_v/\pi_v O_v$, the residue field of k_v. The degree $\deg v$ of v is $[\mathbb{F}_v : \mathbb{F}_q]$, and the cardinality of $\mathbb{F}_v$ is denoted by q_v. For each $a_v \in k_v$, the absolute value $|a_v|_v$ of a_v is normalized to be $q_v^{-\mathrm{ord}_v(a_v)}$. Let $\mathbb{A}$ be the adele ring of k, which is the restricted direct product $\prod_v' k_v$ with respect to O_v. The maximal compact subring $\prod_v O_v$ of $\mathbb{A}$ is denoted by $O_{\mathbb{A}}$. The idele group $\mathbb{A}^\times$ of k is the restricted direct product $\prod_v' k_v^\times$ with respect to $O_v^\times$, and for $a = (a_v)_v \in \mathbb{A}^\times$ we set

$$|a|_{\mathbb{A}} := \prod_v |a_v|_v.$$

Embedding k into $\mathbb{A}$ diagonally, the product formula says that

$$|\alpha|_{\mathbb{A}} = 1, \quad \forall \alpha \in k^\times.$$

Let $\mathrm{Div}(k)$ be the divisor group of k. We adopt the multiplicative notation so that every element $\mathfrak{m}$ in $\mathrm{Div}(k)$ is written as

$$\mathfrak{m} = \prod_v v^{\mathrm{ord}_v(\mathfrak{m})}.$$

Given $\mathfrak{m} \in \mathrm{Div}(k)$, we define

$$\|\mathfrak{m}\| := \prod_v q_v^{\mathrm{ord}_v(\mathfrak{m})} = q^{\deg \mathfrak{m}},$$

where

$$\deg \mathfrak{m} := \sum_v \deg v \cdot \mathrm{ord}_v(\mathfrak{m})$$

is the degree of $\mathfrak{m}$. There is a canonical group epimorphism

$$\mathrm{div} : \mathbb{A}^\times \to \mathrm{Div}(k)$$

defined by

$$a = (a_v)_v \longmapsto \mathrm{div}(a) := \prod_v v^{\mathrm{ord}_v(a_v)},$$

with kernel $O_{\mathbb{A}}^\times$. It is clear that for any $a \in \mathbb{A}^\times$,

$$|a|_{\mathbb{A}} = \|\mathrm{div}(a)\|^{-1}.$$

We fix a section $s : \mathrm{Div}(k) \to \mathbb{A}^\times$ to be

$$s(\mathfrak{m}) := (\pi_v^{\mathrm{ord}_v(\mathfrak{m})})_v$$

for each divisor $\mathfrak{m} \in \mathrm{Div}(k)$.

Fix a place ∞ of k, referred as the place at infinity; and others are referred as finite places of k. Let A be the ring of functions in k regular outside ∞. Each finite place v of k corresponds to a maximal ideal P_v $(= A \cap \pi_v O_v)$ of A. Let $\mathrm{Div}_f(k)$ be the subgroup of $\mathrm{Div}(k)$ generated by finite places of k. There is a natural group isomorphism between $\mathrm{Div}_f(k)$ and the group $\mathcal{I}(A)$ of fractional ideals of A:

$$\begin{array}{rcl} \mathrm{Div}_f(k) & \cong & \mathcal{I}(A) \\ \mathfrak{m} & \longmapsto & M_{\mathfrak{m}} := \prod_v P_v^{\mathrm{ord}_v(\mathfrak{m})}, \\ \prod_v v^{\mathrm{ord}_{P_v}(M)} =: \mathfrak{m}_M & \longleftarrow & M. \end{array}$$

For each place v of k, let

$$\mathbb{A}^v := \prod_{v' \neq v}{}' \; k_{v'}.$$

We denote by $O_{\mathbb{A}^v}$ the maximal compact subring $\prod_{v' \neq v} O_{v'}$ of $\mathbb{A}^v$. In particular, $\mathbb{A}^\infty$ is called the *finite adele ring of* k. The finite idele group of k is the multiplicative group $\mathbb{A}^{\infty,\times}$, with the maximal compact subgroup $O_{\mathbb{A}^\infty}^\times$. Let ι^v and ι_v be the canonical embeddings from $\mathbb{A}^v$ and k_v into $\mathbb{A} = \mathbb{A}^v \times k_v$, i.e. for every $a^v \in \mathbb{A}^v$ and $a_v \in k_v$,

$$\iota^v(a^v) := (a^v, 0) \quad \text{and} \quad \iota_v(a_v) := (0, a_v).$$

We also define

$$\iota^{v,\times} : \mathbb{A}^{v,\times} \hookrightarrow \mathbb{A}^\times = \mathbb{A}^{v,\times} \times k_v^\times \quad \text{and} \quad \iota_v^\times : k_v^\times \hookrightarrow \mathbb{A}^\times$$

respectively by

$$\iota^{v,\times}(a^v) := (a^v, 1), \quad \text{and} \quad \iota_v^\times(a_v) := (1, a_v), \quad \forall (a^v, a_v) \in \mathbb{A}^{v,\times} \times k_v^\times.$$

The section s induces a section (also denoted by s) from $\mathrm{Div}_f(k)$ into $\mathbb{A}^{\infty,\times}$. Embedding $k^\times$ into $\mathbb{A}^{\infty,\times}$ diagonally, the map $\mathrm{div} \circ \iota^{\infty,\times}$ induces a natural isomorphism from $\mathbb{A}^{\infty,\times}/O_{\mathbb{A}^\times}^\times$ (respectively, $k^\times \backslash \mathbb{A}^{\infty,\times}/O_{\mathbb{A}^\infty}^\times$) onto $\mathcal{I}(A)$ (respectively, $\mathrm{Pic}(A)$, i.e. the ideal class group of A).

A divisor $\mathfrak{m}$ is called *positive* if $\mathrm{ord}_v(\mathfrak{m}) \geq 0$ for every place v of k. The set of positive divisors is denoted by $\mathrm{Div}_{\geq 0}(k)$, and we let

$$\mathrm{Div}_{f,\geq 0}(k) := \mathrm{Div}_f(k) \cap \mathrm{Div}_{\geq 0}(k),$$

which is identified with the set of integral ideals of A.

Finally, we fix a non-trivial additive character $\psi : \mathbb{A} \to \mathbb{C}^\times$ such that

$$\psi(\alpha) = 1, \quad \forall \alpha \in k,$$

and let $\delta = \delta_\psi \in \mathrm{Div}(k)$ be the *canonical divisor* associated to ψ. For each place v of k, let ψ_v be the additive character on k_v such that

$$\psi_v(a_v) := \psi(\iota_v(a_v)), \quad \forall a_v \in k_v.$$

Then $\mathrm{ord}_v(\delta)$ is the maximal integer r such that $\pi_v^{-r}O_v$ is contained in the kernel of ψ_v. It is known that $\deg\delta = 2g_k - 2$, where g_k is the genus of k. To keep records on δ, we introduce

$$\epsilon_v(\delta) := \begin{cases} 0 & \text{if } \mathrm{ord}_v(\delta) \text{ is even,} \\ 1 & \text{if } \mathrm{ord}_v(\delta) \text{ is odd,} \end{cases}$$

and

$$\Omega := \prod_{v\neq\infty} v^{\epsilon_v(\delta)} \in \mathrm{Div}_{f,\geq 0}(k).$$

2. Definite quaternion algebra over function fields

Let D be a definite (with respect to ∞) quaternion algebra, i.e. D is a central simple algebra over k with $\dim_k D = 4$ and $D\otimes_k k_\infty$ is a division algebra. Let $\mathfrak{N}^- = \mathfrak{N}_D^- \in \mathrm{Div}_{f,\geq 0}(k)$ be the product of finite places v of k where D is ramified, i.e. $D_v := D\otimes_k k_v$ is division. For each place v of k, we choose an element Π_v in $D_v^\times$ such that $\Pi_v^2 = \pi_v$.

Given a positive divisor $\mathfrak{N}^+ \in \mathrm{Div}_{f,\geq 0}(k)$ which is prime to $\mathfrak{N}^-$. We call a ring R an *Eichler A-order of type* $(\mathfrak{N}^+, \mathfrak{N}^-)$ if R is an A-order of D such that $R_v := R\otimes_A O_v$ is a maximal O_v-order for each $v \nmid \mathfrak{N}^+$; and when $v \mid \mathfrak{N}^+$, there exists an isomorphism $i : D_v \cong \mathrm{Mat}_2(k_v)$ such that

$$i(R_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(O_v) \;\middle|\; c \in \pi_v^{\mathrm{ord}_v(\mathfrak{N}^+)} O_v \right\}.$$

We note that R is unique up to local conjugacy. Since D is definite, the cardinality of the multiplicative group $R^\times$ of R is finite.

A *locally-principal* (*fractional*) *right ideal* I *of* R is an A-lattice in D such that $I\cdot R = I$ and for each finite place v of k, there exists α_v in $D_v^\times$ such that

$$I_v (:= I\otimes_A O_v) = \alpha_v R_v.$$

Two locally-principal right ideals I_1 and I_2 are called equivalent if there exists an element b in $D^\times$ such that $I_1 = b \cdot I_2$. Let

$$D_{\mathbb{A}^\infty} := D \otimes_k \mathbb{A}^\infty \quad \text{and} \quad \widehat{R} := R \otimes_A O_{\mathbb{A}^\infty}.$$

Then the set of locally-principal right ideal classes of R can be identified with the finite double coset space $D^\times \backslash D_{\mathbb{A}^\infty}^\times / \widehat{R}^\times$. More precisely, let $b_1, ..., b_n \in D_{\mathbb{A}^\infty}^\times$ be representatives of the double cosets. Then

$$\{I_i := D \cap b_i \widehat{R} \mid 1 \leq i \leq n\}$$

is a set of representatives of locally-principal right ideal classes of R.

For $1 \leq i \leq n$, let R_i be the left order of I_i, i.e.

$$R_i := \{b \in D : bI_i \subset I_i\}.$$

Then R_i is also an Eichler A-order of type $(\mathfrak{N}^+, \mathfrak{N}^-)$, as

$$R_i = D \cap b_i \widehat{R} b_i^{-1}.$$

Given an element b in D, its reduced trace and the reduced norm are denoted by $\mathrm{Tr}(b)$ and $\mathrm{Nr}(b)$, respectively. We call b is a *pure quaternion* if $\mathrm{Tr}(b) = 0$. Define σ to be the permutation of $\{1, ..., n\}$ such that for $1 \leq i \leq n$, $\bar{I}_i^{-1}$ is equivalent to $I_{\sigma(i)}$. Here

$$\bar{I}_i := \{\bar{b} : b \in I_i\}$$

and $\bar{b} := \mathrm{Tr}(b) - b$ is the conjugate involution of D. It is clear that $\sigma^2 = 1$.

3. Brandt matrices

Let R be an Eichler A-order of type $(\mathfrak{N}^+, \mathfrak{N}^-)$. Let $I_1, ..., I_n$ be representatives of locally-principal right ideal classes of R. For $1 \leq i \leq n$, let R_i be the left order of I_i. We denote the reduced ideal norm of I_i by $\mathrm{Nr}(I_i)$, i.e. $\mathrm{Nr}(I_i)$ is the fractional ideal of A generated by $\mathrm{Nr}(b)$ for all elements b in I_i. For $1 \leq i, j \leq n$, set

$$N_{ij} := \mathrm{Nr}(I_i)\,\mathrm{Nr}(I_j)^{-1}.$$

Then for every $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$, the $\mathfrak{m}$-*th Brandt matrix* $B(\mathfrak{m})$ is defined to be $\big(B_{ij}(\mathfrak{m})\big)_{1\leq i,j\leq n} \in \mathrm{Mat}_n(\mathbb{Z})$, where

$$B_{ij}(\mathfrak{m}) := \frac{\#\{b \in I_i I_j^{-1} : \mathrm{Nr}(b) N_{ij}^{-1} = M_{\mathfrak{m}}\}}{\#(R_j^{\times})}.$$

Recall that $M_{\mathfrak{m}}$ is the ideal of A corresponding to $\mathfrak{m}$. It is clear that $B_{ij}(\mathfrak{m})$ depends only on the ideal classes of I_i, I_j, and the divisor $\mathfrak{m}$. For each divisor $\mathfrak{a} \in \mathrm{Div}_f(k)$, We set the permutation matrix

$$L(\mathfrak{a}) := \big(L_{ij}(\mathfrak{a})\big)_{1\leq i,j\leq n} \in \mathrm{Mat}_n(\mathbb{Z})$$

where

$$L_{ij}(\mathfrak{a}) = \begin{cases} 1, & \text{if } M_{\mathfrak{a}} I_i \text{ is equivalent to } I_j, \\ 0, & \text{otherwise.} \end{cases}$$

Then it is observed that

PROPOSITION II.1. (1) *For every* $\mathfrak{m}$ *and* $\mathfrak{m}'$ *in* $\mathrm{Div}_{f,\geq 0}(k)$ *which are relatively prime,*

$$B(\mathfrak{m}\mathfrak{m}') = B(\mathfrak{m})B(\mathfrak{m}').$$

(2) *For* $\mathfrak{m}$ *in* $\mathrm{Div}_{f,\geq 0}(k)$ *and* $\mathfrak{a}$ *in* $\mathrm{Div}_f(k)$*,*

$$B(\mathfrak{m})L(\mathfrak{a}) = L(\mathfrak{a})B(\mathfrak{m}).$$

(3) *When* $v \nmid \mathfrak{N}^+\mathfrak{N}^-$*,*

$$B(v^{r+2}) = B(v^{r+1})B(v) - q_v L(v) B(v^r).$$

(4) $B(v^r) = B(v)^r$ *if* $v \mid \mathfrak{N}^-$*.*

(5) $L(\mathfrak{a}) = L(\mathfrak{a}')$ *if* $M_{\mathfrak{a}}$ *and* $M_{\mathfrak{a}'}$ *are in the same ideal classes of* A*.*

(6) *The summation* $\sum_j B_{ij}(\mathfrak{m})$ *is independent of the choice of* i*. Moreover, let*

$$b(\mathfrak{m}) := \sum_j B_{ij}(\mathfrak{m}),$$

we get

$$\begin{cases} b(\mathfrak{m}\mathfrak{m}') = b(\mathfrak{m})b(\mathfrak{m}') & \text{when } \mathfrak{m} \text{ and } \mathfrak{m}' \text{ are relatively prime,} \\ b(v^n) = \dfrac{q_v^{n+1}-1}{q_v-1} & \text{if } v \nmid \mathfrak{N}^+\mathfrak{N}^-, \\ b(v^n) = 1 & \text{if } v \mid \mathfrak{N}^-, \\ b(v^n) = 2\cdot\dfrac{q_v^{n+1}-1}{q_v-1} - 1 & \text{if } \mathrm{ord}_v(\mathfrak{N}^+) = 1. \end{cases}$$

PROOF. For $1 \leq i, j, \ell \leq n$ and $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$, let

$$\Phi_{ij}(\mathfrak{m}) := \{b \in I_i I_j^{-1} : \mathrm{Nr}(b)N_{ij}^{-1} = M_{\mathfrak{m}}\}.$$

Then we have the following map for $1 \leq \ell \leq n$ and $\mathfrak{m}$, $\mathfrak{m}' \in \mathrm{Div}_{f,\geq 0}(k)$:

$$\begin{array}{ccc} \Phi_{i\ell}(\mathfrak{m}) \times \Phi_{\ell j}(\mathfrak{m}') & \longrightarrow & \Phi_{ij}(\mathfrak{m}\mathfrak{m}') \\ (b_1 \quad , \quad b_2) & \longmapsto & b_1 b_2. \end{array}$$

Suppose $\mathfrak{m}$ and $\mathfrak{m}'$ are relatively prime. Take $b \in \Phi_{ij}(\mathfrak{m}\mathfrak{m}')$. There we can find $b_1 \in \Phi_{i\ell}(\mathfrak{m})$ for a unique ℓ such that

$$bI_j \subset b_1 I_\ell \subset I_i.$$

Moreover, for any element $b_1' \in \Phi_{i\ell}(\mathfrak{m})$ with

$$bI_j \subset b_1' I_\ell \subset I_i,$$

there exists a unique $u \in R_\ell^\times$ such that $b_1' = b_1 u$. Therefore

$$\sum_{\ell=1}^{n} B_{i\ell}(\mathfrak{m})B_{\ell j}(\mathfrak{m}') = B_{ij}(\mathfrak{m}),$$

which proves (1).

We note that for $\mathfrak{a} \in \mathrm{Div}_f(k)$ and $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$,

$$\begin{aligned} & \sum_{\ell=1}^{n} L_{i\ell}(\mathfrak{a})B_{\ell j}(\mathfrak{m}) \\ = \;& \#(R_j^\times)^{-1} \cdot \#\{b \in (M_{\mathfrak{a}} I_i)I_j^{-1} : \mathrm{Nr}(b)M_{\mathfrak{a}}^{-2}N_{ij}^{-1} = M_{\mathfrak{m}}\} \\ = \;& \#(R_j^\times)^{-1} \cdot \#\{b \in I_i(M_{\mathfrak{a}^{-1}} I_j)^{-1} : \mathrm{Nr}(b)M_{\mathfrak{a}}^{-2}N_{ij}^{-1} = M_{\mathfrak{m}}\} \\ = \;& \sum_{\ell=1}^{n} B_{i\ell}(\mathfrak{m})L_{\ell j}(\mathfrak{a}). \end{aligned}$$

This shows (2).

Now, we take a place v of k with $v \neq \infty$. If D is ramified at v (i.e. $v \mid \mathfrak{N}^-$), then for $1 \leq i \leq n$ and $r \in \mathbb{Z}_{\geq 0}$, there exists a unique right ideal $I \subset I_i$ with $\mathrm{Nr}(I) = P_v^r \,\mathrm{Nr}(I_i)$. In particular, $I = P_v^{r/2} I_i$ if r is even. This implies that

$$B(v^r) = B(v)^r \quad \text{and} \quad B(v)^2 = L(v),$$

so (4) follows.

Suppose $v \nmid \mathfrak{N}^+\mathfrak{N}^-$. For $r \in \mathbb{Z}_{\geq 0}$, let

$$B(v^r)^0 := \big(B_{ij}(v^r)^0\big)_{1 \leq i,j \leq n},$$

where

$$B_{ij}(v^r)^0 := \frac{\#\{b \in \Phi_{ij}(v^r):\ b \notin P_v I_i I_j^{-1}\}}{\#(R_j^\times)}.$$

Then

$$B(v^r) = \sum_{\nu \in \mathbb{Z}_{\geq 0},\ r \geq 2\nu} L(v^\nu) \cdot B(v^{r-2\nu})^0.$$

It is observed that for $r = 1$,

$$\begin{aligned} B(v)^0 B(v) &= B(v^2)^0 + (q_v + 1) \cdot L(v) \\ &= B(v^2) + q_v L(v); \end{aligned}$$

for $r > 1$,

$$B(v^r)^0 B(v) = B(v^{r+1})^0 + q_v L(v) B(v^{r-1})^0$$

Therefore

$$B(v^r) B(v) = B(v^{r+1}) + q_v L(v) B(v^{r-1}),$$

which completes the proof of (3).

The definition of $L(\mathfrak{a})$ implies (5) directly, and (6) follows from the following description of $b(\mathfrak{m})$:

$$b(\mathfrak{m}) = \#\{\text{locally-principal right ideals } J \subset I_i \text{ with } \mathrm{Nr}(J) = M_{\mathfrak{m}} \,\mathrm{Nr}(I_i)\}.$$

□

Recall that σ is the permutation of $\{1, ..., n\}$ such that for $1 \leq i \leq n$, $\bar{I}_i^{-1}$ is equivalent to $I_{\sigma(i)}$.

LEMMA II.2. *Given* $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$, *we have*

$$\#(R_j^\times)B_{ij}(\mathfrak{m}) = \#(R_{\sigma(i)}^\times)B_{\sigma(j)\sigma(i)}(\mathfrak{m}), \quad \textit{for } 1 \leq i, j \leq n.$$

PROOF. The conjugate involution of D induces a bijection between

$$\{b \in I_i I_j^{-1} : \mathrm{Nr}(b) N_{ij}^{-1} = M_\mathfrak{m}\}$$

and

$$\{b \in \bar{I}_j^{-1} \bar{I}_i : \mathrm{Nr}(b) N_{ij}^{-1} = M_\mathfrak{m}\}.$$

Therefore the result holds. □

4. Definite Shimura curves

Let Y be the genus zero curve over k associated to the given definite quaternion algebra D, which is defined by the following: the points of Y over any k-algebra S are

$$Y(S) = \{x \in D \otimes_k S : x \neq 0, \mathrm{Tr}(x) = \mathrm{Nr}(x) = 0\}/S^\times,$$

where the action of $S^\times$ on $D \otimes_k S$ is by multiplication on S, Tr and Nr are respectively the reduced trace and the reduced norm on D. The group $D^\times$ acts on Y (from the left) by conjugation.

Recall that $\mathfrak{N}^-$ is the product of the finite ramified places of D, and R denotes a given Eichler A-order of type $(\mathfrak{N}^+, \mathfrak{N}^-)$.

DEFINITION II.3. The *definite Shimura curve* $X = X_{\mathfrak{N}^+,\mathfrak{N}^-}$ of type $(\mathfrak{N}^+, \mathfrak{N}^-)$ is defined as

$$X = D^\times \backslash \left(Y \times (D_{\mathbb{A}^\infty}^\times / \widehat{R}^\times)\right).$$

Let $b_1, ..., b_n$ be representatives for $D^\times \backslash D_{\mathbb{A}^\infty}^\times / \widehat{R}^\times$. For $1 \leq i \leq n$, let

$$\Gamma_i := b_i \widehat{R}^\times b_i^{-1} \cap D^\times.$$

Then X is equal to the disjoint union $\coprod_{i=1}^{n} X_i$, where $X_i := \Gamma_i\backslash Y$. More precisely, each point of X has a representative $(y, b_i\widehat{R}^\times) \in Y \times (D^\times_{\mathbb{A}^\infty}/\widehat{R}^\times)$ and the map

$$\begin{array}{ccc} X & \longrightarrow & \coprod_{i=1}^{n} \Gamma_i\backslash Y \\ [y, b_i\widehat{R}^\times] & \longmapsto & \Gamma_i y \end{array}$$

is the desired bijection. Moreover, the component X_i of X corresponds canonically to the ideal class of R represented by $I_i = D \cap b_i\widehat{R}$, and the Picard group of X can be written as

$$\mathrm{Pic}(X) = \bigoplus_{i=1}^{n} \mathbb{Z}e_i,$$

where e_i is the class of the component X_i. In the following, we refer $\{e_1, ..., e_n\}$ as a canonical basis of $\mathrm{Pic}(X)$.

4.1. Hecke correspondences. Let v_0 be a finite place of k. Suppose $v_0 \nmid \mathfrak{N}^-$. Then the isomorphism between $\mathrm{Mat}_2(k_{v_0})$ and D_{v_0} induces a natural embedding i_{v_0} from $\mathrm{GL}_2(k_{v_0})$ into $D^\times_{\mathbb{A}^\infty}$. We define the Hecke correspondence t_{v_0} on X as follows:

$$\begin{aligned} t_{v_0}([y, b\widehat{R}^\times]) \ := \ & \left(\sum_{u \in \mathbb{F}_{v_0}} [y, b \cdot i_{v_0}\begin{pmatrix} \pi_{v_0} & u \\ 0 & 1 \end{pmatrix}\widehat{R}^\times]\right) \\ & + \mu_{\mathfrak{N}^+}(v_0) \cdot [y, b \cdot i_{v_0}\begin{pmatrix} 1 & 0 \\ 0 & \pi_{v_0} \end{pmatrix}\widehat{R}^\times]. \end{aligned}$$

Here $\mu_{\mathfrak{N}^+}(v_0) := 1$ if $v_0 \nmid \mathfrak{N}^+$ and 0 otherwise.

Now suppose v_0 divides $\mathfrak{N}^-$. Choose an element $\Pi_{v_0} \in R_{v_0}$ such that $\mathrm{Nr}(\Pi_{v_0}) = \pi_{v_0}$. We define the *Atkin-Lehner involution*

$$w_{v_0}([y, b\widehat{R}^\times]) := ([y, b'\widehat{R}^\times]), \quad \text{for } [y, b\widehat{R}^\times] \in X,$$

where $b' = (b'_v)_{v \neq \infty} \in D^\times_{\mathbb{A}^\times}$ with $b'_v = b_v$ if $v \neq v_0$ and $b'_{v_0} = \Pi_{v_0} b_{v_0}$.

To proceed further, for each $\mathfrak{a} \in \mathrm{Div}_f(k)$, one associates a correspondence $l_{\mathfrak{a}}$ defined by

$$l_{\mathfrak{a}}([y, b\widehat{R}^\times]) := [y, s(\mathfrak{a})b\widehat{R}^\times],$$

where $s : \mathrm{Div}_f(k) \to \mathbb{A}^{\infty,\times}$ is the section introduced in Chapter II Section 1.

It is observed that these correspondences commute with each other, and $l_{\mathfrak{a}} = l_{\mathfrak{a}'}$ if the associated ideals $M_{\mathfrak{a}}$ and $M_{\mathfrak{a}'}$ are in the same ideal class of A. Therefore we can define Hecke correspondence $t_{\mathfrak{m}}$ for every $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ in the following way:

$$\begin{cases} t_{\mathfrak{m}\mathfrak{m}'} = t_{\mathfrak{m}} \cdot t_{\mathfrak{m}'} & \text{if } \mathfrak{m} \text{ and } \mathfrak{m}' \text{ are relatively prime,} \\ t_{v^{\ell+2}} = t_v \cdot t_{v^{\ell+1}} - \mu_{\mathfrak{N}^+}(v) q_v \cdot l_v \cdot t_{v^\ell} & \text{if } v \nmid \mathfrak{N}^-, \\ t_{v^\ell} = w_v^\ell & \text{if } v \mid \mathfrak{N}^-. \end{cases}$$

The correspondences $t_{\mathfrak{m}}$ and $l_{\mathfrak{a}}$ induce endomorphisms of the group $\mathrm{Pic}(X)$. With respect to the canonical basis $\{e_1, ..., e_n\}$, these endomorphisms can in fact be represented by Brandt matrices $B(\mathfrak{m})$ and the permutation matrices $L(\mathfrak{a})$:

Proposition II.4. *Given $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ and $\mathfrak{a} \in \mathrm{Div}_f(k)$, suppose that $\mathfrak{m}$ and $\mathfrak{N}^+$ are relatively prime. Then we have*

$$t_{\mathfrak{m}} e_i = \sum_{j=1}^n B_{ij}(\mathfrak{m}) e_j \quad \text{and} \quad l_{\mathfrak{a}} e_i = \sum_{j=1}^n L_{ij}(\mathfrak{a}) e_j.$$

Proof. For each divisor $\mathfrak{a} \in \mathrm{Div}_f(k)$, $l_{\mathfrak{a}} e_i = e_{i'}$ where $I_{i'}$ and $M_{\mathfrak{a}} I_i$ are in the same ideal class of R. On the other hand, we have

$$\sum_{j=1}^n L_{ij}(\mathfrak{a}) e_j = e_{i'} = l_{\mathfrak{a}} e_i.$$

Therefore from the definition of $t_{\mathfrak{m}}$ and the recurrence relations of $B(\mathfrak{m})$ in Proposition II.1, it suffices to prove the case when $\mathfrak{m} = v$ with $v \nmid \mathfrak{N}^+ \infty$.

For $1 \leq i \leq n$, it is clear that $t_v e_i = \sum_{j=1}^n \alpha_{ij} e_j$ where

$$\begin{aligned} \alpha_{ij} &= \#\{\text{locally-principal right } R\text{-ideals } J \subset I_i : \ J \sim I_j \text{ and } \mathrm{Nr}(J) = P_v \, \mathrm{Nr}(I_i)\} \\ &= B_{ij}(v). \end{aligned}$$

Therefore the proof is complete. □

Let v divide $\mathfrak{N}^+$ with $\mathrm{ord}_v(\mathfrak{N}^+) = 1$. Then for each point $[y, b\widehat{R}^\times]$ in X, we introduce the following pseudo-involution on X:

$$w'_v([y, b\widehat{R}^\times]) := [y, b \cdot i_v \begin{pmatrix} 0 & 1 \\ \pi_v & 0 \end{pmatrix} \widehat{R}^\times].$$

It is clear that $w_v'^2 = l_v$. Set $W_v := t_v + w_v' + w_v'^{-1} t_v w_v'$, and

$$W_{v^{\ell+2}} := (W_v - q_v w_v') W_{v^{\ell+1}} - q_v l_v W_{v^\ell}, \qquad \text{for } \ell \geq 0.$$

One can deduce that

Lemma II.5. *Suppose* $\mathrm{ord}_v(\mathfrak{N}^+) = 1$. *Then*

$$W_{v^\ell} e_i = \sum_{j=1}^{n} B_{ij}(v^\ell) e_j.$$

Proof. For $1 \leq i, j \leq n$ and $\ell \geq 0$, set

$$S_{ij}(\ell) := \{J \subset I_i : J \sim I_j, \mathrm{Nr}(J) = P_v^\ell \,\mathrm{Nr}(I_i)\}.$$

It is clear that the cardinality of $S_{ij}(\ell)$ is $B_{ij}(v^\ell)$. Consider

$$S'(\ell) := \{a_v R_v^\times \in D_v^\times / R_v^\times : a_v \in R_v, \mathrm{Nr}(a_v) \in \pi_v^\ell O_v^\times\} \subset D_v^\times / R_v^\times.$$

There is a natural bijection

$$\begin{array}{ccc} S'(\ell) & \cong & \displaystyle\coprod_{j=1}^{n} S_{ij}(\ell) \\ a_v R_v^\times & \longmapsto & D \cap b_i \cdot i_v(a_v) \widehat{R}. \end{array}$$

It suffices to show that for each $[y, b\widehat{R}^\times] \in X$,

$$W_{v^\ell}([y, b\widehat{R}^\times]) = \sum_{a_v R_v^\times \in S'(\ell)} [y, b \cdot i_v(a_v) \widehat{R}^\times],$$

which is straightforward. □

Now, take a place v_0 of k with $\mathrm{ord}_{v_0}(\mathfrak{N}^+ \mathfrak{N}^- \infty) = 0$. Let $R_{(v_0)}$ be an Eichler A-order of type $(v_0 \mathfrak{N}^+, \mathfrak{N}^-)$ contained in R. Let

$$X_{v_0} := D^\times \backslash \left(Y \times (D_{\mathbb{A}^\times}^\times / \widehat{R}_{(v_0)}^\times) \right),$$

the definite Shimura curve over k of type $(v_0 \mathfrak{N}^+, \mathfrak{N}^-)$. Denote the canonical morphism from X_{v_0} to X by pr, i.e.

$$pr([y, b\widehat{R}_{(v_0)}^\times]) = [y, b\widehat{R}^\times]$$

for any $[y, b\widehat{R}_{(v_0)}^\times] \in X_{v_0}$. Then pr induces a group homomorphism pr_* from $\mathrm{Pic}(X_{v_0})$ to $\mathrm{Pic}(X)$.

PROPOSITION II.6. *For each $e \in \mathrm{Pic}(X_{v_0})$ and a non-negative integer r,*

$$pr_*(W_{v_0^r}e) = t_{v_0^r}pr_*(e) + q_v t_{v_0^{r-1}}pr_*(w'_{v_0}e).$$

PROOF. It is observed that

$$pr_*(W_{v_0}e) = t_{v_0}pr_*(e) + q_v pr_*(w'_{v_0}e), \quad \forall e \in \mathrm{Pic}(X_{v_0}).$$

Therefore the result holds when $r = 0$ and 1.

We prove this proposition by induction. Suppose it holds for r and $r-1$ with $r > 0$. For $e \in \mathrm{Pic}(X_{v_0})$,

$$\begin{aligned} pr_*(W_{v_0^{r+1}}e) &= pr_*\Big(\big((W_{v_0} - q_v w'_{v_0})W_{v_0^r} - q_{v_0}l_{v_0}W_{v_0^{r-1}}\big)e\Big) \\ &= t_{v_0}pr_*(W_{v_0^r}e) - q_{v_0}l_{v_0}pr_*(W_{v_0^{r-1}}e) \\ &= t_{v_0}\Big(t_{v_0^r}pr_*(e) + q_v t_{v_0^{r-1}}pr_*(w'_{v_0}e)\Big) \\ &\quad -q_{v_0}l_{v_0}\Big(t_{v_0^{r-1}}pr_*(e) + q_v t_{v_0^{r-2}}pr_*(w'_{v_0}e)\Big) \\ &= t_{v_0^{r+1}}pr_*(e) + q_v t_{v_0^r}pr_*(w'_{v_0}e). \end{aligned}$$

This completes the proof. □

4.2. Gross height pairing. For $1 \leq i \leq n$, recall that R_i is the left order of I_i. Set $w_i := \#(R_i^\times)/q - 1$. Then the *Gross height pairing* $< \cdot, \cdot >$ on $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined by setting

$$\begin{cases} < e_i, e_j > := 0 & \text{if } i \neq j, \\ < e_i, e_i > := w_i, \end{cases}$$

and extending bi-linearly. Therefore $\mathrm{Pic}(X)^\vee := \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$ can be viewed as a subgroup of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with basis

$$\{\check{e}_i := e_i/w_i : i = 1, ..., n\}$$

via this pairing. Note that the permutation σ introduced in Section 3 induces an endomorphism on $\mathrm{Pic}(X)$ by setting

$$\sigma e_i := e_{\sigma(i)}.$$

Then by Lemma II.2 we get:

PROPOSITION II.7. *Given classes e and e' in* $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$*,*

$$< t_{\mathfrak{m}} e, e' > = < e, t_{\mathfrak{m}}^* e' >$$

for $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ *prime to* $\mathfrak{N}^+$*, where* $t_{\mathfrak{m}}^* = \sigma^{-1} t_{\mathfrak{m}} \sigma$*.*

REMARK. We point out that for $\mathfrak{a} \in \mathrm{Div}_f(k)$,

$$< l_{\mathfrak{a}} e, e' > = < e, l_{\mathfrak{a}}^* e' >$$

where $l_{\mathfrak{a}}^* = \sigma^{-1} l_{\mathfrak{a}} \sigma = l_{\mathfrak{a}^{-1}}$. When $\mathrm{ord}_v(\mathfrak{N}^+) = 1$, we also get

$$< W_{v^\ell} e, e' > = < e, W_{v^\ell}^* e' >,$$

where $W_{v^\ell}^* = \sigma^{-1} W_{v^\ell} \sigma$.

CHAPTER III

The basis problem for Drinfeld type automorphic forms

Let D be a definite quaternion algebra over k with $\mathfrak{N}^-$ equal to the product of finite ramified places of D. For $\mathfrak{N}^+ \in \mathrm{Div}_{f,\geq 0}(k)$, take R to be an Eichler A-order of type $(\mathfrak{N}^+, \mathfrak{N}^-)$ having right ideal class number n. For each pair (i,j), $1 \leq i,j \leq n$, with the help of the Weil representation of GL_2 we construct a theta series Θ_{ij} which is a Drinfeld type automorphic form of level $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$ having Fourier coefficients given by the (i,j)-entries of Brandt matrices. Using these theta series, we write down the Hecke equivariant map Φ in Theorem I.1, and describe explicitly the image of Φ inside the space of Drinfeld type automorphic forms.

1. Weil representation

Let (V, Q_V) be the quadratic space (D, Nr_D) over k where $\mathrm{Nr}_D = \mathrm{Nr}$ is the reduced norm form on D. The bilinear form associated to Q_V is

$$< x, y >_V = \mathrm{Tr}_D(x\bar{y}), \quad \forall x, y \in V,$$

where $\mathrm{Tr}_D = \mathrm{Tr}$ is the reduced trace on D and $\bar{y} = \mathrm{Tr}(y) - y$ is the conjugate involution of D. Denote by $\mathrm{O}(V)$ the orthogonal group of V, i.e.

$$\mathrm{O}(V) := \{h \in \mathrm{GL}(V) : Q_V(hx) = Q_V(x),\ \forall x \in V\}.$$

In this section we recall the Weil representations of the groups $\mathrm{SL}_2 \times \mathrm{O}(V)$, $\mathrm{G}\big(\mathrm{SL}_2 \times \mathrm{O}(V)\big)$, and GL_2, and choose a particular family of sections for the construction our theta series in the next section.

1.1. Weil representation of $\mathrm{SL}_2 \times \mathrm{O}(V)$**.** For each place v of k, let $V(k_v) := V \otimes_k k_v$, and denote by $S(V(k_v))$ the space of Schwartz functions on $V(k_v)$. Recall that the local Weil representation $\omega_v(=\omega_{\psi_v})$ of $\mathrm{SL}_2(k_v)\times \mathrm{O}(V)(k_v)$ on the space $\mathrm{S}(V(k_v))$ is defined by the following: for a function $\phi_v \in \mathrm{S}(V(k_v))$ and $x_v \in V(k_v)$,

$$\begin{aligned}
\big(\omega_v(h_v)\phi_v\big)(x_v) &:= \phi_v(h_v^{-1}(x_v)),\ \forall h_v \in \mathrm{O}(V)(k_v),\\
\left(\omega_v\begin{pmatrix} a_v & 0 \\ 0 & a_v^{-1}\end{pmatrix}\phi_v\right)(x_v) &:= |a_v|_v^2 \cdot \phi_v(ax_v),\ \forall a_v \in k_v^\times\\
\left(\omega_v\begin{pmatrix} 1 & u_v \\ 0 & 1\end{pmatrix}\phi_v\right)(x_v) &:= \psi_v(u_v\mathrm{Nr}(x_v))\cdot \phi_v(x_v),\ \forall u_v \in k_v\\
\left(\omega_v\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\phi_v\right)(x_v) &:= \varepsilon_v\widehat{\phi}_v(x_v).
\end{aligned}$$

Here $\varepsilon_v = -1$ if D is ramified at v and 1 otherwise; $\widehat{\phi}_v$ is the Fourier transform of ϕ_v (with respect to ψ_v):

$$\widehat{\phi}_v(x_v) := \int_{V(k_v)} \phi_v(y_v)\psi_v(\mathrm{Tr}(x_v\bar{y}_v))dy_v.$$

The Haar measure dy_v is chosen to be self-dual with respect to the pairing

$$(x_v, y_v) \longmapsto \psi_v(\mathrm{Tr}(x_v\bar{y}_v)).$$

The global Weil representation $\omega(=\omega_\psi)$ of $\mathrm{SL}_2(\mathbb{A})\times \mathrm{O}(V)(\mathbb{A})$ is defined to be $\otimes_v\omega_v$ on the space $\mathrm{S}(V(\mathbb{A}))$.

Recall that R is a given Eichler A-order in D of type $(\mathfrak{N}^+,\mathfrak{N}^-)$. Denote by O_{D_∞} the maximal compact subring of D_∞. For our purpose, we fix a particular Schwartz function $\varphi = \otimes_v\varphi_v \in S(V(\mathbb{A}))$, where

$$\begin{cases}
\varphi_v := \mathbf{1}_{\Pi_v^{-\mathrm{ord}_v(\delta)}R_v}, & \text{if } v \neq \infty\\
\varphi_\infty := \mathbf{1}_{\Pi_\infty^{-\mathrm{ord}_\infty(\delta)}O_{D_\infty}}, & \text{if } v = \infty.
\end{cases}$$

Here $\delta \in \mathrm{Div}(k)$ is the canonical divisor introduced in Chapter II Section 1, and for each place v of k, Π_v is a chosen element in $D_v^\times$ such that $\Pi_v^2 = \pi_v$. Let $\mathfrak{N} := \mathfrak{N}^+\mathfrak{N}^-$. We obtain that

LEMMA III.1. *Let v be any place v of k. For* $\kappa_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(O_v)$ *with* $c \equiv 0 \bmod \pi_v^{\mathrm{ord}_v(\mathfrak{N}\infty)} O_v$,

$$\omega_v(\kappa_v)\varphi_v = \varphi_v.$$

PROOF. Let Ω_v be the support of φ_v. It is observed that

$$\widehat{\varphi}_v = \begin{cases} \varphi_v, & \text{if } v \nmid \mathfrak{N}^+\mathfrak{N}^-\infty, \\ q_v^{-1} \cdot \mathbf{1}_{\Pi_v^{-1}\Omega_v}, & \text{if } v \mid \mathfrak{N}^+\mathfrak{N}^-\infty. \end{cases}$$

It remains to show that for every $\kappa_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(O_v)$ with $c \in \pi_v O_v$,

$$\omega_v(\kappa_v)\varphi_v = \varphi_v.$$

Write κ_v as

$$\begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}.$$

Hence for $x_v \in V(k_v)$,

$$\begin{aligned} &\omega_v(\kappa_v)\varphi_v(x_v) \\ = \quad &\psi_v\big(bd^{-1}Q(x_v)\big) \cdot \left(\omega_v \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} \varphi_v\right)(d^{-1}x_v). \end{aligned}$$

Since

$$\begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -d^{-1}c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we have

$$\begin{aligned} &\left(\omega_v \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} \varphi_v\right)(d^{-1}x_v) \\ = \quad &\int_{V(k_v)} \psi_v\left(-d^{-1}c \cdot Q(y_v)\right) \psi_v\big(d^{-1}\,\mathrm{Tr}(x_v\bar{y}_v)\big)\widehat{\varphi}_v(-y_v)dy_v. \end{aligned}$$

Note that $\psi_v\left(-d^{-1}c\cdot Q(y_v)\right)=1$ if y_v is in the support of $\widehat{\varphi}_v$. Thus

$$\begin{aligned}
&\left(\omega_v\begin{pmatrix}1&0\\ d^{-1}c&1\end{pmatrix}\varphi_v\right)(d^{-1}x_v)\\
=\;&\int_{V(k_v)}\psi_v\big(\operatorname{Tr}(d^{-1}x_v\bar{y}_v)\big)\widehat{\varphi}_v(-y_v)dy_v\\
=\;&\varphi_v(d^{-1}x_v)=\varphi_v(x_v).
\end{aligned}$$

Since $\psi_v\big(bd^{-1}Q(x_v)\big)=1$ if x_v is in the support of φ_v, the proof is complete. □

1.2. Test functions from arithmetic data. Consider the general orthogonal group of V:

$$\mathrm{GO}(V)=\{h\in \mathrm{GL}(V)\mid Q_V(hx)=\nu(h)\cdot Q_V(x)\ \forall x\in V,\ \text{where } \nu(h)\in\mathbb{G}_m\}.$$

Let

$$\mathrm{G}(\mathrm{SL}_2\times \mathrm{O}(V)):=\{(g,h)\in \mathrm{GL}_2\times\mathrm{GO}(V)\mid \det(g)=\nu(h)\}.$$

We extend ω to a representation ω' of $\mathrm{G}(\mathrm{SL}_2\times\mathrm{O}(V))(\mathbb{A})$ on $S(V(\mathbb{A}))$ by the following: for each pair $(g,h)\in \mathrm{G}(\mathrm{SL}_2\times\mathrm{O}(V))(\mathbb{A})$ and $\phi\in \mathrm{S}(V(\mathbb{A}))$

$$\omega'(g,h)\phi(x):=|v(h)|_{\mathbb{A}}^{-1}\cdot\left[\omega\left(\begin{pmatrix}1&0\\0&\det(g)^{-1}\end{pmatrix}\cdot g\right)\phi\right](h^{-1}x),\ \forall x\in V(\mathbb{A}).$$

Recall that $I_1,...,I_n$ are chosen representatives of locally-principal right ideal classes of the Eichler A-order R, and we let b_i be the corresponding element of I_i in $D_{\mathbb{A}^\infty}^\times$ for each i. Set

$$\Pi^{(\delta)}:=(\Pi_v^{-\operatorname{ord}_v(\delta)})_v\in D_{\mathbb{A}}^\times.$$

Viewing $b_1,...,b_n$ as elements in $D_{\mathbb{A}}^\times(=D_{\mathbb{A}^\infty}^\times\times D_\infty^\times)$, each pair $(b_i\Pi^{(\delta)},b_j)$ induces an element in $\mathrm{GO}(V)(\mathbb{A})$:

$$(b_i\Pi^{(\delta)},b_j)\cdot x:=b_i\Pi^{(\delta)}xb_j^{-1}\quad\forall x\in V(\mathbb{A}).$$

For $1\le i,j\le n$, let

$$\beta_{ij}:=\operatorname{Nr}(b_i)\operatorname{Nr}(b_j)^{-1}\in\mathbb{A}^{\infty,\times}.$$

Then $\operatorname{div}(\beta_{ij}) = \mathfrak{m}_{N_{ij}}$, where $\mathfrak{m}_{N_{ij}} \in \operatorname{Div}_f(k)$ is the divisor associated to $N_{ij} = \operatorname{Nr}(I_i)\operatorname{Nr}(I_j)^{-1}$. Define

$$\varphi_{ij} := \omega'\left(\begin{pmatrix} 1 & 0 \\ 0 & \beta_{ij}s(\delta)\end{pmatrix}, (b_i\Pi^{(\delta)}, b_j)\right)\varphi,$$

where φ is introduced in the last section, and s is the section fixed in Chapter II Section 1. Then for any $x = (x_f, x_\infty) \in V(\mathbb{A}_f) \times V(k_\infty) = V(\mathbb{A})$,

$$\varphi_{ij}(x) = |\beta_{ij}s(\delta)|_{\mathbb{A}}^{-1} \cdot \mathbf{1}_{b_i\widehat{R}b_j^{-1}}(x_f) \cdot \mathbf{1}_{O_{D_\infty}}(x_\infty).$$

Moreover, Lemma III.1 implies directly that

Lemma III.2. *For every element* $\kappa = (\kappa_v)_v \in \mathrm{SL}_2(O_{\mathbb{A}})$ *satisfying that* $\kappa_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v\end{pmatrix}$ *with* $c_v \equiv 0 \bmod \pi_v^{\operatorname{ord}_v(\mathfrak{N}\infty)}O_v$, *we have*

$$\omega\left(\begin{pmatrix} 1 & 0 \\ 0 & \beta_{ij}s(\delta)\end{pmatrix}\kappa\begin{pmatrix} 1 & 0 \\ 0 & \beta_{ij}^{-1}s(\delta)^{-1}\end{pmatrix}\right)\varphi_{ij} = \varphi_{ij}.$$

For each place v of k, we denote by $S(V(k_v)\times k_v^\times)$ the space of functions ϕ_v on $V(k_v)\times k_v^\times$ such that for each $\alpha_v \in k_v^\times$, $\phi_v(\cdot,\alpha_v)$ is in $S(V(k_v))$. Let

$$\phi_v^0 := \varphi_v \otimes \mathbf{1}_{O_v^\times}.$$

The space $S(V(\mathbb{A})\times\mathbb{A}^\times)$ is defined to be the restricted tensor product $\bigotimes_v' S(V(k_v)\times k_v^\times)$ with respect to $\{\phi_v^0\}_v$, i.e. every function in $S(V(\mathbb{A})\times\mathbb{A}^\times)$ is a linear combination of pure-tensors $\phi = \otimes_v\phi_v$, where $\phi_v = \phi_v^0$ for almost all v.

Now, for each place v of k, we extend ω_v to a representation $\tilde{\omega}_v$ of $\mathrm{GL}_2(k_v)$ on $S(V(k_v)\times k_v^\times)$ by the following:

$$\left(\tilde{\omega}_v \begin{pmatrix} 1 & u_v \\ 0 & 1 \end{pmatrix} \phi\right)(x_v, \alpha_v) := \psi_v(u_v \mathrm{Nr}(x_v)\alpha_v)\phi_v(x_v, \alpha_v), \text{ for } u_v \in k_v;$$
$$\left(\tilde{\omega}_v \begin{pmatrix} a_v & 0 \\ 0 & a_v^{-1} \end{pmatrix} \phi_v\right)(x_v, \alpha_v) := |a_v|_v^2 \cdot \phi(a_v x_v, \alpha_v), \text{ for } a_v \in k_v^\times;$$
$$\left(\tilde{\omega}_v \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_v\right)(x_v, \alpha_v) := \varepsilon_v \cdot \mathcal{F}\phi_v(x_v, \alpha_v);$$
$$\left(\tilde{\omega}_v \begin{pmatrix} 1 & 0 \\ 0 & a_v \end{pmatrix} \phi_v\right)(x_v, \alpha_v) := |a|_v^{-1} \phi(x_v, \alpha_v a_v^{-1}), \text{ for } a_v \in k_v^\times$$

Here
$$\mathcal{F}\phi_v(x_v, \alpha_v) := \int_{V(k_v)} \phi_v(y_v, \alpha_v)\psi_v(\alpha_v \mathrm{Tr}(x_v \bar{y}_v)) d_{\alpha_v} y_v$$
where $d_{\alpha_v} y_v$ is the self-dual Haar measure with respect to the pairing
$$(x_v, y_v) \longmapsto \psi_v(\alpha_v \mathrm{Tr}(x_v \bar{y}_v)), \quad \forall x_v,\ y_v \in V(k_v).$$
We set the representation $\tilde{\omega}$ of $\mathrm{GL}_2(\mathbb{A})$ to be $\otimes_v \tilde{\omega}_v$ on $S(V(\mathbb{A}) \times \mathbb{A}^\times)$.

Now, for $1 \leq i, j \leq n$, let $\tilde{\varphi}_{ij} = \tilde{\varphi}_{ij,f} \otimes \tilde{\varphi}_{ij,\infty} \in S(V(\mathbb{A}) \times \mathbb{A}^\times)$ where
$$\tilde{\varphi}_{ij,f}(x_f, \alpha_f) := \mathbf{1}_{b_i \widehat{R} b_j^{-1}}(x_f) \cdot 1_{O_{\mathbb{A}^\infty}^\times}(\alpha_f \cdot \beta_{ij} s(\delta_f))$$
for $(x_f, \alpha_f) \in V(\mathbb{A}^\infty) \times \mathbb{A}^{\infty,\times}$, and
$$\tilde{\varphi}_{ij,\infty}(x_\infty, \alpha_\infty) := \mathbf{1}_{O_\infty}(\mathrm{Nr}(x_\infty) \cdot \alpha_\infty s(\delta_\infty))$$
for $(x_\infty, \alpha_\infty) \in V(k_\infty) \times k_\infty^\times$. Here $\delta_\infty = \infty^{\mathrm{ord}_\infty(\delta)}$, $\delta_f = \delta/\delta_\infty \in \mathrm{Div}_f(k)$ and $\beta_{ij} = \mathrm{Nr}(b_i)\,\mathrm{Nr}(b_j)^{-1} \in \mathbb{A}^{\infty,\times}$. Then an immediate consequence of Lemma III.2 is the following:

LEMMA III.3. (1) *Let $\kappa = (\kappa_v)_v$ be an element in $\mathrm{GL}_2(O_\mathbb{A})$ such that $\kappa_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}$ with $c_v \equiv 0 \bmod \pi_v^{\mathrm{ord}_v(\mathfrak{N}\infty)} O_v$. Then for any element $(x, \alpha) \in V(\mathbb{A}) \times \mathbb{A}^\times$,*
$$\big(\tilde{\omega}(\kappa)\tilde{\varphi}_{ij}\big)(x, \alpha) = \tilde{\varphi}_{ij}(x, \alpha).$$

(2) *For every* $z_\infty \in k_\infty^\times$,

$$\tilde{\omega}\left(\begin{pmatrix} z_\infty & 0 \\ 0 & z_\infty \end{pmatrix}\right)\tilde{\varphi}_{ij,\infty} = \tilde{\varphi}_{ij,\infty}.$$

2. Theta series

For $1 \leq i, j \leq n$, we define

$$\theta_{ij}(g) := \sum_{(x,\alpha)\in V(k)\times k^\times} \big(\tilde{\omega}(g)\tilde{\varphi}_{ij}\big)(x,\alpha), \quad \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

Write g as $(g_f, g_\infty) \in \mathrm{GL}_2(\mathbb{A}^\infty) \times \mathrm{GL}_2(k_\infty)$. It is observed that

$$\theta_{ij}(g) = 0$$

unless $\det(g_f)$ and $\beta_{ij}s(\delta_f)$ represent the same coset in $k^\times\backslash\mathbb{A}_f^\times/O_{\mathbb{A}_f}^\times$. Let $\mathcal{K}_0(\mathfrak{N}\infty)$ be the compact subgroup $\prod_v \mathcal{K}_v$ of $\mathrm{GL}_2(\mathbb{A})$, where for each place v of k,

$$\mathcal{K}_v := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v) \;\middle|\; c \equiv 0 \bmod \pi_v^{\mathrm{ord}_v(\mathfrak{N}\infty)}O_v\right\}.$$

Then Lemma III.3 tells us that

Proposition III.4. *For $1 \leq i, j \leq n$, θ_{ij} can be viewed as a function on the double coset space*

$$\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A})/Z(k_\infty)\mathcal{K}_0(\mathfrak{N}\infty),$$

where Z is the center of GL_2.

Furthermore, these theta series θ_{ij} are *harmonic*, i.e.

Lemma III.5. *For* $g \in \mathrm{GL}_2(\mathbb{A})$,

(1) $\theta_{ij}\left(g\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right) = -\theta_{ij}(g)$ *and* (2) $\displaystyle\sum_{\kappa_\infty\in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} \theta_{ij}(g\kappa_\infty) = 0.$

Here we embed $\mathrm{GL}_2(k_\infty)$ *into* $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}^\infty) \times \mathrm{GL}_2(k_\infty)$ *by*

$$g_\infty \mapsto (1, g_\infty).$$

PROOF. It suffices to show that

$$\tilde{\omega}_\infty\left(\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right)\tilde{\varphi}_{ij,\infty} = -\tilde{\varphi}_{ij,\infty}$$

and

$$\sum_{u\in\mathbb{F}_\infty}\left(\tilde{\omega}_\infty\begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix}\right)\tilde{\varphi}_{ij,\infty} = -\tilde{\varphi}_{ij,\infty}.$$

For $(x,\alpha)\in V(k_\infty)\times k_\infty^\times$, one has

$$\begin{aligned}
&\left(\tilde{\omega}_\infty\left(\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right)\tilde{\varphi}_{ij,\infty}\right)(x,\alpha) \\
= &\left(\tilde{\omega}_\infty\begin{pmatrix} 1 & 0 \\ 0 & \pi_\infty \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\tilde{\varphi}_{ij,\infty}\right)(x,\alpha) \\
= &-q_\infty\int_{V(k_\infty)}\mathbf{1}_{O_\infty}(\mathrm{Nr}(y)\alpha\pi_\infty^{-1+\mathrm{ord}_\infty(\delta)})\cdot\psi_\infty(\mathrm{Tr}(x\bar{y})\alpha\pi_\infty^{-1})d_{\alpha\pi_\infty^{-1}}y.
\end{aligned}$$

Here $d_\epsilon y$ is the self-dual Haar measure with respect to the pairing

$$(x,y)\longmapsto\psi_v(\epsilon\,\mathrm{Tr}(x\bar{y})),\quad \forall x,\ y\in V(k_\infty).$$

The last integral equals to

$$\mathbf{1}_{O_\infty}(\mathrm{Nr}(x)\alpha\pi_\infty^{\mathrm{ord}_\infty(\delta)})\cdot\mathrm{vol}(\Omega),$$

where

$$\Omega=\{y\in D_\infty\mid \mathrm{Nr}(y)\alpha\pi_\infty^{-1+\mathrm{ord}_\infty(\delta)}\in O_\infty\}.$$

From the normalization of the Haar measure, we get $\mathrm{vol}(\Omega)=q_\infty^{-1}$. This completes the proof of (1).

For (2), it follows by

$$\sum_{u\in\mathbb{F}_\infty}\left(\left(\tilde{\omega}_\infty\begin{pmatrix} u & 1\\ 1 & 0\end{pmatrix}\right)\tilde{\varphi}_{ij,\infty}\right)(x,\alpha)$$
$$=\left(-\sum_{u\in\mathbb{F}_\infty}\psi(\alpha\mathrm{Nr}(x)u)\cdot\int_{V(k_\infty)}\mathbf{1}_{O_\infty}(\mathrm{Nr}(y)\alpha\pi_\infty^{\mathrm{ord}_\infty(\delta)})\psi_\infty(\alpha\mathrm{Tr}(x\bar{y}))d_\alpha y\right)$$
$$=-\mathbf{1}_{O_\infty}(\alpha\mathrm{Nr}(x)\alpha\pi_\infty^{1+\mathrm{ord}_\infty(\delta)})\cdot\left[q_\infty^{-1}\cdot\sum_{u\in\mathbb{F}_\infty}\psi(\alpha\mathrm{Nr}(x)u)\right]$$
$$=-\tilde{\varphi}_{ij,\infty}(x,\alpha).$$

□

For $1\leq j\leq n$, let R_j be the left order of the ideal I_j. Then

$$R_j^\times = D^\times\cap b_j\widehat{R}^\times b_j^{-1}$$

is a finite cyclic group. We normalize our theta series as follows:

Definition III.6. For $1\leq i,j\leq n$, set

$$\Theta_{ij}(g) := \frac{1}{\|\delta\|\cdot\#(R_j^\times)}\cdot\theta_{ij}\left(\begin{pmatrix}\beta_{ij}s(\delta) & 0\\ 0 & \beta_{ij}s(\delta)\end{pmatrix}g\right),\qquad \text{for } g\in\mathrm{GL}_2(\mathbb{A}).$$

3. Drinfeld type automorphic forms and Hecke operators

Given $\mathfrak{N}\in\mathrm{Div}_{f,\geq 0}(k)$, recall the compact subgroup $\mathcal{K}_0(\mathfrak{N}\infty)=\prod_v\mathcal{K}_v$ of $\mathrm{GL}_2(\mathbb{A})$, where for each place v of k,

$$\mathcal{K}_v := \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{GL}_2(O_v)\;\middle|\; c\equiv 0 \bmod \pi_v^{\mathrm{ord}_v(\mathfrak{N}\infty)}O_v\right\}.$$

By a *Drinfeld type automorphic form F of level* $\mathfrak{N}$, we mean that F is a $\mathbb{C}$-valued function on the double coset space

$$\mathbb{Y}_0(\mathfrak{N}) := \mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A})/Z(k_\infty)\mathcal{K}_0(\mathfrak{N}\infty)$$

(where Z is the center of GL_2) satisfying the harmonic property: for any $g \in \mathrm{GL}_2(\mathbb{A})$,

$$F\left(g\begin{pmatrix}0 & 1\\ \pi_\infty & 0\end{pmatrix}\right) = -F(g) \quad \text{and} \quad \sum_{\kappa_\infty \in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} F(g\kappa_\infty) = 0.$$

Recall that we embed $\mathrm{GL}_2(k_\infty)$ into $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}^\infty) \times \mathrm{GL}_2(k_\infty)$ by

$$g_\infty \mapsto (1, g_\infty).$$

These forms can be viewed as function field analogue of classical weight 2 modular forms. For further discussions, we refer the reader to [**4**] and [**19**].

Let $\mathcal{M}_0(\mathfrak{N})$ be the space of Drinfeld type automorphic forms of level $\mathfrak{N}$. For each place v of k, the Hecke operator T_v on $\mathcal{M}_0(\mathfrak{N})$ is defined by the following: for $F \in \mathcal{M}_0(\mathfrak{N})$ and $g \in \mathrm{GL}_2(\mathbb{A})$,

$$(T_vF)(g) := \sum_{u \in \mathbb{F}_v} F\left(g\begin{pmatrix}\pi_v & u\\ 0 & 1\end{pmatrix}\right) + \mu_{\mathfrak{N}\infty}(v) \cdot F\left(g\begin{pmatrix}1 & 0\\ 0 & \pi_v\end{pmatrix}\right).$$

Here

$$\mu_{\mathfrak{N}\infty}(v) = \begin{cases} 1 & \text{if } v \nmid \mathfrak{N}\infty, \\ 0 & \text{otherwise.}\end{cases}$$

The harmonicity of F implies that

$$T_\infty F = F, \quad \forall F \in \mathcal{M}_0(\mathfrak{N}).$$

Since T_v and $T_{v'}$ commute to each other for any places v and v', we define the Hecke operator $T_\mathfrak{m}$ for $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ by the following:

$$\begin{cases} T_{\mathfrak{m}\mathfrak{m}'} := T_\mathfrak{m} \cdot T_{\mathfrak{m}'} & \text{for } \mathfrak{m} \text{ and } \mathfrak{m}' \text{ are coprime;} \\ T_{v^{\ell+2}} := T_v T_{v^{\ell+1}} - \mu_\mathfrak{N}(v) q_v \cdot \rho\big(s(v)\big) T_{v^\ell} & \text{for any finite place } v \text{ of } k. \end{cases}$$

For each divisor $\mathfrak{m} \in \mathrm{Div}(k)$, the $\mathfrak{m}$-th Fourier coefficient $F^*(\mathfrak{m})$ of F is

$$F^*(\mathfrak{m}) := \int_{k\backslash\mathbb{A}} F\left(\begin{pmatrix}s(\delta^{-1}\mathfrak{m}) & u\\ 0 & 1\end{pmatrix}\right) \psi(-u)du.$$

Here the Haar measure du is normalized such that the volume of $k\backslash\mathbb{A}$ is one, and δ is the canonical divisor of k introduced in Chapter II Section 1. It is observed that $F^*(\mathfrak{m}) = 0$ unless $\mathfrak{m}$ is positive. Let

$$F_0^*(\mathfrak{m}) := \int_{k\backslash\mathbb{A}} F\left(\begin{pmatrix} s(\delta^{-1}\mathfrak{m}) & u \\ 0 & 1 \end{pmatrix}\right) du.$$

Then the harmonicity of F implies that

$$F^*(\mathfrak{m}) = \|\mathfrak{m}_\infty\|^{-1} F^*(\mathfrak{m}_f) \quad \text{and} \quad F_0^*(\mathfrak{m}) = \|\mathfrak{m}_\infty\|^{-1} F^*(\mathfrak{m}_f).$$

Here

$$\mathfrak{m}_\infty := \infty^{\mathrm{ord}_\infty(\mathfrak{m})}, \quad \mathfrak{m}_f := \mathfrak{m}/\mathfrak{m}_\infty,$$

and $s : \mathrm{Div}(k) \to \mathbb{A}^\times$ is the section fixed in Chapter II Section 1. Moreover, given $\mathfrak{a}$ and $\mathfrak{a}'$ in $\mathrm{Div}_f(k)$, $F_0^*(\mathfrak{a}) = F_0^*(\mathfrak{a}')$ if the corresponding fractional ideals $M_\mathfrak{a}$ and $M_{\mathfrak{a}'}$ are in the same ideal class of A.

For $a_f \in \mathbb{A}^{\infty,\times}$, set

$$\big(\rho(a_f)F\big)(g) := F\left(\begin{pmatrix} \iota^{\infty,\times}(a_f) & 0 \\ 0 & \iota^{\infty,\times}(a_f) \end{pmatrix} g\right).$$

Here $\iota^{\infty,\times} : \mathbb{A}^{\infty,\times} \hookrightarrow \mathbb{A}^\times$ is introduced in Chapter II Section 1. Then the surjectivity of the canonical map from $B(\mathbb{A})$ onto $\mathbb{Y}_0(\mathfrak{N})$ (where B is the standard Borel subgroup of GL_2) implies that

Lemma III.7. *F is uniquely determined by Fourier coefficients*

$$\big(\rho(a_i)F\big)^*(\mathfrak{m}) \quad \textit{and} \quad \big(\rho(a_i)F\big)_0^*(\mathrm{div}(a_j))$$

for $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ and representatives $a_1, ..., a_h$ of $k^\times\backslash\mathbb{A}_f^\times/O_{\mathbb{A}_f}^\times$.

3.1. Fourier coefficients of theta series. Recall that in Chapter III Section 2 we constructed a family of theta series Θ_{ij} (cf. Definition III.6), $1 \leq i, j \leq n$, associated to a given Eichler A-order R of type $(\mathfrak{N}^+, \mathfrak{N}^-)$. Moreover, Proposition III.4 and Lemma III.5 tells us that these series are Drinfeld type automorphic forms of level $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$. In the following, we show that the $\mathfrak{m}$-th Fourier coefficient of Θ_{ij} is essentially the (i, j)-entry of the Brandt matrix $B(\mathfrak{m})$ for every $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$:

PROPOSITION III.8. *For divisors* $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ *and* $\mathfrak{a} \in \mathrm{Div}_f(k)$,

$$\Theta^*_{ij}(\mathfrak{m}) = \frac{B_{ij}(\mathfrak{m})}{\|\mathfrak{m}\|} \quad \text{and} \quad \Theta^*_{ij,0}(\mathfrak{a}) = \frac{\epsilon_{ij}(\mathfrak{a})}{w_j\|\mathfrak{a}\|}.$$

Here

$$\epsilon_{ij}(\mathfrak{a}) := \begin{cases} 1 & \text{if the ideals } M_{\mathfrak{a}} \text{ and } N_{ij}^{-1} \text{ are in the same class of } A, \\ 0 & \text{otherwise;} \end{cases}$$

and $w_j = \#(R_j^\times)/(q-1)$ *is introduced in* Chapter II Section 4.2.

PROOF. Let

$$\tilde{\varphi}'_{ij} := \tilde{\omega}\begin{pmatrix} \beta_{ij}s(\delta) & 0 \\ 0 & \beta_{ij}s(\delta) \end{pmatrix}\tilde{\varphi}_{ij}$$

where the Weil representation $\tilde{\omega}$ of $\mathrm{GL}_2(\mathbb{A})$ and the Schwartz function $\tilde{\varphi}_{ij}$ are introduced in Chapter III Section 1.2. Then for any pair $(x,\alpha) \in V(\mathbb{A}) \times \mathbb{A}^\times$,

$$\begin{aligned} \tilde{\varphi}'_{ij}(x,\alpha) \quad &= \quad \mathbf{1}_{\beta_{ij}^{-1}s(\delta_f)^{-1}b_i\hat{R}b_j^{-1}}(x_f)\mathbf{1}_{O_{\mathbb{A}^\infty}^\times}(\alpha_f\beta_{ij}^{-1}s(\delta_f)^{-1}) \\ &\quad\quad \cdot\mathbf{1}_{O_\infty}\big(\mathrm{Nr}(x_\infty)\alpha_\infty s(\delta_\infty)\big). \end{aligned}$$

Given $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$, for each $u \in \mathbb{A}$ one has

$$\begin{aligned} &\#(R_j^\times)\cdot\|\mathfrak{m}\|\cdot\Theta_{ij}\left(\begin{pmatrix} s(\delta^{-1}\mathfrak{m}) & u \\ 0 & 1 \end{pmatrix}\right) \\ = &\sum_{(x,\alpha)\in V(k)\times k^\times} \tilde{\varphi}'_{ij}\big(s(\delta^{-1}\mathfrak{m})x, s(\delta\mathfrak{m})^{-1}\alpha\big)\cdot\psi(\mathrm{Nr}(x)u\alpha) \\ = &\sum_{(x,\alpha)\in V(k)\times k^\times} \Bigg[\Big(\mathbf{1}_{\beta_{ij}^{-1}b_i\widehat{R}b_j^{-1}}(s(\mathfrak{m})x_f)\mathbf{1}_{O_{\mathbb{A}^\infty}^\times}(s(\mathfrak{m})^{-1}\alpha_f\beta_{ij}^{-1})\cdot\mathbf{1}_{O_\infty}(\mathrm{Nr}(x_\infty)\alpha_\infty)\Big) \\ &\quad\quad \cdot\psi(\mathrm{Nr}(x)u\alpha)\Bigg]. \end{aligned}$$

Thus

$$\Theta^*_{ij}(\mathfrak{m}) = \Theta^*_{ij,0}(\mathfrak{m}) = 0 = B_{ij}(\mathfrak{m})$$

unless $s(\mathfrak{m})$ and β_{ij}^{-1} represent the same coset in $k^\times\backslash\mathbb{A}^{\infty,\times}/O_{\mathbb{A}^\infty}^\times$. In this case, let $\gamma \in k^\times$ such that $\gamma\cdot\beta_{ij}\cdot s(\mathfrak{m}) \in O_{\mathbb{A}^\infty}^\times$. Then

$$\#(R_j^\times)\cdot\|\mathfrak{m}\|\cdot\Theta^*_{ij,0}(\mathfrak{m}) = q-1$$

and

$$
\begin{aligned}
& \#(R_j^\times)\cdot \|\mathfrak{m}\| \cdot \Theta^*_{ij}(\mathfrak{m}) \\
= \ & \#\big\{(x,\epsilon)\in V(k)\times \mathbb{F}_q^\times \mid \gamma^{-1}x_f \in b_i\widehat{R}b_j^{-1} \text{ and } \mathrm{Nr}(x)\gamma^{-1}\epsilon = 1\big\}. \\
= \ & \#\big\{x\in D\cap b_i\widehat{R}b_j^{-1} \mid \frac{\mathrm{Nr}(x)}{\beta_{ij}} \in s(\mathfrak{m})O_{\mathbb{A}^\infty}^\times\big\} \\
= \ & \#(R_j^\times)\cdot B_{ij}(\mathfrak{m}).
\end{aligned}
$$

Therefore the proof is complete. □

The following proposition says that the action of Hecke operators on the theta series Θ_{ij} can be read off by the Brandt matrices:

PROPOSITION III.9. *For any* $\mathfrak{m}\in \mathrm{Div}_{f,\geq 0}(k)$ *which is prime to* $\mathfrak{N}^+$ *and* $1\leq i,j\leq n$,

$$T_{\mathfrak{m}}\Theta_{ij} = \sum_{\ell=1}^{n} B_{i\ell}(\mathfrak{m})\Theta_{\ell j}.$$

PROOF. First, from the definition of Θ_{ij}, it is clear that

LEMMA III.10. *For* $a_f\in \mathbb{A}^{\infty,\times}$ *and* $g\in GL_2(\mathbb{A})$,

$$\rho(a_f)\Theta_{ij} = \sum_{\ell=1}^{n} L_{i\ell}\big(\mathrm{div}(a_f)\big)\Theta_{\ell j}.$$

Here $L_{ij}(\mathfrak{a})$ *for any divisor* $\mathfrak{a}\in\mathrm{Div}_f(k)$ *is introduced in* §3.

Since the Brandt matrices and the Hecke operators share the same recurrence relation, it suffices to prove the case when $\mathfrak{m}=v$ with $v\nmid\mathfrak{N}^+$.

By Proposition III.8, we obtain that for each divisor $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$,

$$\begin{aligned}
&\sum_{\ell=1}^{n} B_{i\ell}(v)\Theta^*_{\ell j}(\mathfrak{m}) \\
= \ & \|\mathfrak{m}\|^{-1}\cdot\sum_{\ell=1}^{n} B_{i\ell}(v)B_{\ell j}(\mathfrak{m}) \\
= \ & \|\mathfrak{m}\|^{-1}\cdot\left[\sum_{\ell=1}^{n} \mu_{\mathfrak{N}\infty}(v)q_v\cdot L_{i\ell}(v)B_{\ell j}(\frac{\mathfrak{m}}{v}) + B_{ij}(\mathfrak{m}v)\right] \\
= \ & q_v\Theta^*_{ij}(\mathfrak{m}v) + \sum_{\ell=1}^{n} \mu_{\mathfrak{N}\infty}(v)L_{i\ell}(v)\Theta^*_{\ell j}(\frac{\mathfrak{m}}{v}) \\
= \ & (T_v\Theta_{ij})^*(\mathfrak{m}).
\end{aligned}$$

Moreover, it is clear that

$$\sum_{\ell=1}^{n} B_{i\ell}(v)\Theta^*_{\ell j,0}(\mathfrak{m}) = (T_v\Theta_{ij})^*_0(\mathfrak{m}).$$

Therefore by Lemma III.7 and III.10, the proof is complete. □

Remark. Proposition III.8 and Lemma III.10 tell us, in particular, that these theta series Θ_{ij} are $\mathbb{Q}$-valued Drinfeld type automorphic forms. Proposition III.9 leads to a Hecke module homomorphism from the Picard group of definite Shimura curves into the space of $\mathbb{Q}$-valued Drinfeld type automorphic forms. Further discussions are in the next subsection.

4. The Hecke module homomorphism Φ

Recall the definite Shimura curve $X = X_{\mathfrak{N}^+,\mathfrak{N}^-}$ introduced in Chapter II Section 4 and the Gross height pairing

$$< e, \check{e} > = \sum_i a_i a_i',$$

where $(e, \check{e}) \in \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee$ with $e = \sum_i a_i e_i$ and $\check{e} = \sum_i a_i'\check{e}_i$. We let $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$ and denote $\mathcal{M}_0(\mathfrak{N}, \mathbb{Q})$ the space of $\mathbb{Q}$-valued Drinfeld type automorphic forms of level $\mathfrak{N}$. Define the $\mathbb{Z}$-bilinear map

$$\Phi : \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee \to \mathcal{M}_0(\mathfrak{N}, \mathbb{Q})$$

by

$$\Phi(e,\check{e}) := \sum_{1\le i,j\le n} a_i a'_j \Theta_{ij}$$

for any $e \in \mathrm{Pic}(X)$ with $e = \sum_i a_i e_i$ and $\check{e} \in \mathrm{Pic}(X)^\vee$ with $\check{e} = \sum_i a'_i \check{e}_i$. Then for any divisor $\mathfrak{m} \in \mathrm{Div}_{f,\ge 0}(k)$ which is prime to $\mathfrak{N}^+$, by Proposition III.8 and III.9 we get

$$\Phi(e,\check{e})^*(\mathfrak{m}) = \frac{< t_{\mathfrak{m}} e, \check{e} >}{\|\mathfrak{m}\|} = \frac{< e, t^*_{\mathfrak{m}} \check{e} >}{\|\mathfrak{m}\|}.$$

Note that Proposition III.9 implies further that

$$T_{\mathfrak{m}}\big(\Phi(e,\check{e})\big) = \Phi(t_{\mathfrak{m}} e, \check{e}) = \Phi(e, t^*_{\mathfrak{m}} \check{e})$$

for $\mathfrak{m} \in \mathrm{Div}_{f,\ge 0}(k)$ prime to $\mathfrak{N}^+$. Let

$$\mathbb{T}_{\mathbb{Q}} := \mathbb{Q}[t_{\mathfrak{m}} : \mathfrak{m} \in \mathrm{Div}_{f,\ge 0}(k) \text{ prime to } \mathfrak{N}^+].$$

Consider the $\mathbb{T}_{\mathbb{Q}}$-module structure of $\mathrm{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ defined by

$$(t_{\mathfrak{m}}, \check{e}) \mapsto t^*_{\mathfrak{m}} \check{e}.$$

We conclude that

THEOREM III.11. *The map* $\Phi : \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee \longrightarrow \mathcal{M}_0(\mathfrak{N}, \mathbb{Q})$ *satisfies that for any divisor* $\mathfrak{m} \in \mathrm{Div}_{f,\ge 0}(k)$ *which is prime to* $\mathfrak{N}^+$*,*

$$\Phi(e,\check{e})^*(\mathfrak{m}) = \frac{< t_{\mathfrak{m}} e, \check{e} >}{\|\mathfrak{m}\|} = \frac{< e, t^*_{\mathfrak{m}} \check{e} >}{\|\mathfrak{m}\|}$$

and

$$T_{\mathfrak{m}}\Phi(e,\check{e}) = \Phi(t_{\mathfrak{m}} e, \check{e}) = \Phi(e, t^*_{\mathfrak{m}} \check{e}).$$

Moreover, this map induces a homomorphism

$$(\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{T}_{\mathbb{Q}}} (\mathrm{Pic}(X)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}) \longrightarrow \mathcal{M}_0(\mathfrak{N}, \mathbb{Q})$$

as $\mathbb{T}_{\mathbb{Q}}$*-modules.*

REMARK. Suppose $\mathfrak{N}$ is square-free. We change the notation $\mathfrak{N}^+$, $\mathfrak{N}^-$, and $\mathfrak{N}$ to $\mathfrak{N}_0^+$, $\mathfrak{N}_0^-$, and $\mathfrak{N}_0$, respectively. Given $\mathfrak{m} \in \mathrm{Div}_{f,\ge 0}(k)$, write

$\mathfrak{m} = \mathfrak{m}' \prod_{v|\mathfrak{N}_0^+} v^{\mathrm{ord}_v(\mathfrak{m})}$ with $\mathfrak{m}'$ and $\mathfrak{N}_0^+$ coprime. Then by Proposition III.8 and Lemma II.5, we obtain that for $(e, \check{e}) \in \mathrm{Pic}(X) \times \mathrm{Pic}(X)^\vee$,

$$\begin{aligned} \Phi(e,\check{e})^*(\mathfrak{m}) &= \frac{< (t_{\mathfrak{m}'} \prod_{v|\mathfrak{N}^+} W_{v^{\mathrm{ord}_v(\mathfrak{m})}})e, \check{e} >}{\|\mathfrak{m}\|} \\ &= \frac{< e, (t^*_{\mathfrak{m}'} \prod_{v|\mathfrak{N}^+} W^*_{v^{\mathrm{ord}_v(\mathfrak{m})}})\check{e} >}{\|\mathfrak{m}\|}. \end{aligned}$$

4.1. Changing levels. Let X be the definite Shimura curve over k of type $(\mathfrak{N}_0^+, \mathfrak{N}_0^-)$, where $\mathfrak{N}_0 = \mathfrak{N}_0^+\mathfrak{N}_0^-$ is square-free. Choose a place v_0 of k with $\mathrm{ord}_{v_0}(\mathfrak{N}_0^+\mathfrak{N}_0^-\infty) = 0$. Denote by X_{v_0} the definite Shimura curve over k of type $(v_0\mathfrak{N}_0^+, \mathfrak{N}_0^-)$. Then the canonical morphism pr from X_{v_0} to X induces natural group homomorphisms

$$pr_* : \mathrm{Pic}(X_{v_0}) \to \mathrm{Pic}(X) \quad \text{and} \quad pr^* : \mathrm{Pic}(X)^\vee \to \mathrm{Pic}(X_{v_0})^\vee.$$

Define $pr^*_{v_0} : \mathrm{Pic}(X)^\vee \to \mathrm{Pic}(X_{v_0})^\vee$ by

$$< e_{v_0}, pr^*_{v_0}(\check{e}) >_{v_0} := < pr_*(w'_{v_0} e_{v_0}), \check{e} >, \quad \forall (e_{v_0}, \check{e}) \in \mathrm{Pic}(X_{v_0}) \times \mathrm{Pic}(X)^\vee.$$

Here $< \cdot, \cdot >_{v_0}$ and $< \cdot, \cdot >$ are Gross height pairing on $\mathrm{Pic}(X_{v_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, respectively. Recall the Hecke module homomorphism Φ introduced in Theorem III.11, and denote by

$$\Phi_{v_0} : \mathrm{Pic}(X_{v_0}) \times \mathrm{Pic}(X_{v_0})^\vee \to \mathcal{M}_0(v_0\mathfrak{N}_0)$$

the corresponding Hecke module homomorphism for X_{v_0}.

Theorem III.12. *For $\check{e} \in \mathrm{Pic}(X)^\vee$, $e_{v_0} \in \mathrm{Pic}(X_{v_0})$, and $g \in \mathrm{GL}_2(\mathbb{A})$, we have*

$$\Phi_{v_0}\big(e_{v_0}, pr^*(\check{e})\big)(g) = \Phi\big(pr_*(e_{v_0}), \check{e}\big)(g) + \Phi\big(pr_*(w'_{v_0}e_{v_0}), \check{e}\big)\left(g\begin{pmatrix}\pi_{v_0}^{-1} & 0\\ 0 & 1\end{pmatrix}\right)$$

and

$$\Phi_{v_0}\big(e_{v_0}, pr^*_{v_0}(\check{e})\big)(g) = \Phi\big(pr_*(w'_{v_0}e_{v_0}), \check{e}\big)(g) + \Phi\big(l_{v_0}pr_*(e_{v_0}), \check{e}\big)\left(g\begin{pmatrix}\pi_{v_0}^{-1} & 0\\ 0 & 1\end{pmatrix}\right).$$

PROOF. It suffices to prove the equality for the Fourier coefficients, i.e. for $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$, $\mathfrak{a} \in \mathrm{Div}_f(k)$, $\check{e} \in \mathrm{Pic}(X)^\vee$, and $e_{v_0} \in \mathrm{Pic}(X_{v_0})$,

$$\begin{aligned}
(1) \qquad \Phi_{v_0}\big(e_{v_0}, pr^*(\check{e})\big)^*(\mathfrak{m}) &= \Phi\big(pr_*(e_{v_0}), \check{e}\big)^*(\mathfrak{m}) \\
&\quad + \Phi\big(pr_*(w'_{v_0} e_{v_0}), \check{e}\big)^*(\mathfrak{m} v_0^{-1}), \\
(2) \qquad \Phi_{v_0}\big(e_{v_0}, pr^*(\check{e})\big)_0^*(\mathfrak{a}) &= \Phi\big(pr_*(e_{v_0}), \check{e}\big)_0^*(\mathfrak{a}) \\
&\quad + \Phi\big(pr_*(w'_{v_0} e_{v_0}), \check{e}\big)_0^*(\mathfrak{a} v_0^{-1}), \\
(3) \qquad \Phi_{v_0}\big(e_{v_0}, pr^*_{v_0}(\check{e})\big)^*(\mathfrak{m}) &= \Phi\big(pr_*(w'_{v_0} e_{v_0}), \check{e}\big)^*(\mathfrak{m}) \\
&\quad + \Phi\big(l_{v_0} pr_*(e_{v_0}), \check{e}\big)^*(\mathfrak{m} v_0^{-1}), \\
(4) \qquad \Phi_{v_0}\big(e_{v_0}, pr^*_{v_0}(\check{e})\big)_0^*(\mathfrak{a}) &= \Phi\big(pr_*(w'_{v_0} e_{v_0}), \check{e}\big)_0^*(\mathfrak{a}) \\
&\quad + \Phi\big(l_{v_0} pr_*(e_{v_0}), \check{e}\big)_0^*(\mathfrak{a} v_0^{-1}).
\end{aligned}$$

Denote by $\{e_1, ..., e_n\}$ and $\{e_{v_0,1}, ..., e_{v_0,n_{v_0}}\}$ the canonical bases of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X_{v_0})$, respectively. Suppose $pr_*(e_{v_0,i}) = e_{i_0}$. Then by Proposition III.8 it is observed that

$$\begin{aligned}
\Phi_{v_0}\big(e_{v_0,i}, pr^*(\check{e}_j)\big)_0^*(\mathfrak{a}) &= \sum_{e_{v_0,j'} \in \, pr_*^{-1}(e_j)} \frac{\epsilon_{v_0,ij'}(\mathfrak{a})}{w_{v_0,j'} \|\mathfrak{a}\|} \\
&= (q_{v_0} + 1) \frac{\epsilon_{i_0 j}(\mathfrak{a})}{w_j \|\mathfrak{a}\|}.
\end{aligned}$$

On the other hand,

$$\Phi\big(pr_*(e_{v_0,i}), \check{e}_j\big)_0^*(\mathfrak{a}) = \frac{\epsilon_{i_0 j}(\mathfrak{a})}{w_j \|\mathfrak{a}\|} = q_{v_0}^{-1} \Phi\big(pr_*(w'_{v_0} e_{v_0,i}), \check{e}_j\big)_0^*(\mathfrak{a} v_0^{-1}).$$

Therefore the equality (2) holds. Suppose

$$w'_{v_0} e_{v_0,i} = e_{v_0,i'} \quad \text{and} \quad pr_*(e_{v_0,i'}) = e_{i'_0} \quad \text{with } 1 \leq i', i'_0 \leq n.$$

Then

$$\begin{aligned}
\Phi_{v_0}\big(e_{v_0,i}, pr^*_{v_0}(\check{e}_j)\big)_0^*(\mathfrak{a}) &= \sum_{e_{v_0,j'} \in pr_*^{-1}(e_j)} \frac{\epsilon_{v_0,i'j'}(\mathfrak{a})}{w_{v_0,j'} \|\mathfrak{a}\|} \\
&= (q_{v_0} + 1) \frac{\epsilon_{i'_0 j}(\mathfrak{a})}{w_j \|\mathfrak{a}\|}.
\end{aligned}$$

Since

$$q_{v_0}^{-1}\Phi\big(l_{v_0}pr_*(e_{v_0,i}),\check{e}_j\big)_0^*(\mathfrak{a}v_0^{-1}) = \frac{\epsilon_{i_0'j}(\mathfrak{a})}{w_j\|\mathfrak{a}\|} = \Phi\big(pr_*(w_{v_0}'e_{v_0,i}),\check{e}_j\big)_0^*(\mathfrak{a}),$$

the equality (4) holds.

Now, for $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$, write $\mathfrak{m}$ as $\mathfrak{m}'\prod_{v|v_0\mathfrak{N}_0^+} v^{\mathrm{ord}_v(\mathfrak{m})}$ with $\mathfrak{m}'$ and $\mathfrak{N}_0^+$ coprime. The remark of Theorem III.11 says that

$$\Phi_{v_0}\big(e_{v_0},pr^*(\check{e})\big)^*(\mathfrak{m}) = \frac{< \big(t_{\mathfrak{m}'}\prod_{v|\mathfrak{N}_0^+} W_{v^{\mathrm{ord}_v(\mathfrak{m})}}\big)pr_*(W_{v_0^{\mathrm{ord}_{v_0}(\mathfrak{m})}}e_{v_0}),\check{e} >}{\|\mathfrak{m}\|}$$

and

$$\Phi_{v_0}\big(e_{v_0},pr_{v_0}^*(\check{e})\big)^*(\mathfrak{m}) = \frac{< \big(t_{\mathfrak{m}'}\prod_{v|\mathfrak{N}_0^+} W_{v^{\mathrm{ord}_v(\mathfrak{m})}}\big)pr_*(W_{v_0^{\mathrm{ord}_{v_0}(\mathfrak{m})}}w_{v_0}'e_{v_0}),\check{e} >}{\|\mathfrak{m}\|}.$$

By Proposition II.6, we get

$$\begin{aligned}
&\Phi_{v_0}\big(e_{v_0},pr^*(\check{e})\big)^*(\mathfrak{m})\\
= &\frac{< \big(t_{\mathfrak{m}'}\prod_{v|\mathfrak{N}_0^+} W_{v^{\mathrm{ord}_v(\mathfrak{m})}}\big)t_{v_0^{\mathrm{ord}_{v_0}(\mathfrak{m})}}pr_*(e_{v_0}),\check{e} >}{\|\mathfrak{m}\|}\\
&+\frac{q_{v_0}\cdot < \big(t_{\mathfrak{m}'}\prod_{v|\mathfrak{N}_0^+} W_{v^{\mathrm{ord}_v(\mathfrak{m})}}\big)t_{v_0^{\mathrm{ord}_{v_0}(\mathfrak{m})-1}}pr_*(w_{v_0}'e_{v_0}),\check{e} >}{\|\mathfrak{m}\|}\\
= &\Phi\big(pr_*(e_{v_0}),\check{e}\big)^*(\mathfrak{m}) + \Phi\big(pr_*(w_{v_0}'e),\check{e}\big)^*(\mathfrak{m}v_0^{-1}),
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_{v_0}\big(e_{v_0},pr_{v_0}^*(\check{e})\big)^*(\mathfrak{m})\\
= &\frac{< \big(t_{\mathfrak{m}'}\prod_{v|\mathfrak{N}_0^+} W_{v^{\mathrm{ord}_v(\mathfrak{m})}}\big)t_{v_0^{\mathrm{ord}_{v_0}(\mathfrak{m})}}pr_*(w_{v_0}'e_{v_0}),\check{e} >}{\|\mathfrak{m}\|}\\
&+\frac{q_{v_0}\cdot < \big(t_{\mathfrak{m}'}\prod_{v|\mathfrak{N}_0^+} W_{v^{\mathrm{ord}_v(\mathfrak{m})}}\big)t_{v_0^{\mathrm{ord}_{v_0}(\mathfrak{m})-1}}l_{v_0}pr_*(e_{v_0}),\check{e} >}{\|\mathfrak{m}\|}\\
= &\Phi\big(pr_*(w_{v_0}'e_{v_0}),\check{e}\big)^*(\mathfrak{m}) + \Phi\big(l_{v_0}pr_*(e_{v_0}),\check{e}\big)^*(\mathfrak{m}v_0^{-1}).
\end{aligned}$$

Therefore (1) and (3) holds and the proof of this proposition is complete. □

5. Construction of Drinfeld type newforms

Given $\mathfrak{N} \in \mathrm{Div}_{f,\geq 0}(k)$, a Drinfeld type automorphic form F of level $\mathfrak{N}$ is called a *cusp form* if for every $g \in \mathrm{GL}_2(\mathbb{A})$,

$$\int_{k\backslash\mathbb{A}} F\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} g\right) du = 0.$$

Note that this cuspidal condition is equivalent to say that F vanishes at almost all double cosets in $\mathbb{Y}_0(\mathfrak{N})$ (cf. [**7**]). Given two Drinfeld type automorphic forms F_1 and F_2 of level $\mathfrak{N}$, suppose one of them is a cusp form. The *Petersson inner product of* F_1 *and* F_2 is:

$$\begin{aligned}(F_1, F_2) \quad &:= \int_{Z(k_\infty)\,\mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A})} F_1(g)\overline{F_2(g)}dg \\ &= \sum_{[g]\in\mathbb{Y}_0(\mathfrak{N})} F_1(g)\overline{F_2(g)}\mu([g]),\end{aligned}$$

where Z is the center of GL_2 and for each double coset $[g] \in \mathbb{Y}_0(\mathfrak{N})$, the measure $\mu([g])$ is normalized to be

$$\mu([g]) := \frac{q-1}{2\cdot\#(\mathrm{Pic}(A))}\cdot\frac{1}{\#\big(\mathrm{GL}_2(k)\cap g\mathcal{K}_0(\mathfrak{N}\infty)g^{-1}\big)}.$$

A Drinfeld type cusp form F of level $\mathfrak{N}$ is called an *old form* if F is a linear combination of the forms

$$F'\left(g\begin{pmatrix}1 & 0\\ 0 & s(\mathfrak{N}'')\end{pmatrix}\right), \quad \forall g \in \mathrm{GL}_2(\mathbb{A}),$$

where F' is a Drinfeld type cusp form of level $\mathfrak{N}'$ with $\mathfrak{N}'\mathfrak{N}'' \mid \mathfrak{N}$ and $\mathfrak{N}' \neq \mathfrak{N}$. A Drinfeld type cusp form F of level $\mathfrak{N}$ is called a *newform* if F is a Hecke eigenform and $(F, F') = 0$ for any old form F' of level $\mathfrak{N}$.

Now, Suppose $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$, where $\mathfrak{N}^+$ and $\mathfrak{N}^-$ are relatively prime, and $\mathfrak{N}^-$ is the product of finite ramified places of a definite quaternion algebra D over k. Let $X = X_{\mathfrak{N}^+,\mathfrak{N}^-}$ be the definite Shimura curve of type $(\mathfrak{N}^+, \mathfrak{N}^-)$, and denote by

$$\mathrm{Pic}(X)_{\mathbb{C}} := \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{C}, \quad \mathrm{Pic}(X)^\vee_{\mathbb{C}} := \mathrm{Pic}(X)^\vee\otimes_{\mathbb{Z}}\mathbb{C}.$$

Extend $<\cdot,\cdot>$ to a pairing on $\mathrm{Pic}(X)_{\mathbb{C}} \times \mathrm{Pic}(X)_{\mathbb{C}}^{\vee}$ which is linear on the left and conjugate linear on the right. We emphasize that $\mathrm{Pic}(X)_{\mathbb{C}}^{\vee}$ can be identified with the space of $\mathbb{C}$-valued functions on $D^{\times}\backslash D_{\mathbb{A}_f}^{\times}/\widehat{R}^{\times}$:

$$\check{e} \in \mathrm{Pic}(X)_{\mathbb{C}}^{\vee} \longmapsto \big([b_i] \mapsto < e_i, \check{e} >, \quad [b_i] \in D^{\times}\backslash D_{\mathbb{A}_f}^{\times}/\widehat{R}^{\times}\big).$$

For each character $\chi : \mathrm{Pic}(A) \to \mathbb{C}^{\times}$, let

$$e_{0,\chi} := \sum_{i=1}^{n} \chi(\mathrm{Nr}(I_i))^{-1} \cdot \frac{e_i}{w_i} \in \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Then for every divisor $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ which is prime to $\mathfrak{N}^{+}$,

$$t_{\mathfrak{m}} e_{0,\chi} = \chi(\mathfrak{m}) b(\mathfrak{m}) e_{0,\chi}.$$

Recall that $b(\mathfrak{m}) = \sum_{j=1}^{n} B_{ij}(\mathfrak{m})$. Set

$$\mathrm{Pic}_0(X)_{\mathbb{C}}^{\vee} := \{\check{e} \in \mathrm{Pic}(X)_{\mathbb{C}}^{\vee} \mid < e_{0,\chi}, \check{e} >= 0 \quad \forall \chi : \mathrm{Pic}(A) \to \mathbb{C}^{\times}\}.$$

We also denote by $\mathcal{S}_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N})$ the space of *Drinfeld type* $(\mathfrak{N}^-)$*-new forms of level* $\mathfrak{N}$, i.e. every $F \in \mathcal{S}_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N})$ is a linear combination of the forms

$$F'\left(g\begin{pmatrix}1 & 0\\ 0 & s(\mathfrak{N}'')\end{pmatrix}\right),$$

where F' is a Drinfeld type newform of level $\mathfrak{N}'$ with $\mathfrak{N}'\mathfrak{N}'' \mid \mathfrak{N}$ and $\mathfrak{N}^- \mid \mathfrak{N}'$. Then the Jacquet-Langlands correspondence for $D^{\times}$ and GL_2 tells us that

THEOREM III.13. (cf. **[7]**) *There exists an isomorphism* (*as* $\mathbb{C}$*-vector spaces*) JL *from* $\mathrm{Pic}_0(X)_{\mathbb{C}}^{\vee}$ *onto* $\mathcal{S}_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N})$ *such that given* $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ *and* $F \in \mathcal{S}_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N})$,

$$\mathrm{JL}^{-1}(T_{\mathfrak{m}}F) = t_{\mathfrak{m}}^{*}\mathrm{JL}^{-1}(F).$$

Suppose now that $\mathfrak{N}$ is square-free. As in the remark of Theorem III.11, we change the notation $\mathfrak{N}$, $\mathfrak{N}^{+}$, and $\mathfrak{N}^{-}$ respectively to $\mathfrak{N}_0$, $\mathfrak{N}_0^{+}$, and $\mathfrak{N}_0^{-}$. Let $F \in \mathcal{S}_0(\mathfrak{N}_0)$ be a Drinfeld type newform. Define

$$e_F := \overline{\mathrm{JL}^{-1}(F)} \in \mathrm{Pic}_0(X)_{\mathbb{C}}^{\vee}.$$

where $\overline{\check{e}} := \sum_{i=1}^{n} \overline{a_i} \cdot \check{e}_i$ for any $\check{e} = \sum_{i=1}^{n} a_i\check{e}_i \in \mathrm{Pic}_0(X)_{\mathbb{C}}^{\vee}$. Then if F is *normalized*, i.e. $F^{*}(1) = 1$, we get

THEOREM III.14. *For any* $e \in \mathrm{Pic}(X)_{\mathbb{C}}$,

$$\Phi(e, e_F) =< e, e_F > \cdot F.$$

Here Φ *is the map introduced in* Theorem III.11.

PROOF. By Theorem III.11 and III.13, $\Phi(e, e_F)$ and F share the same eigenvalues of $T_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ prime to $\mathfrak{N}_0^+$. Let v be a place with $v \mid \mathfrak{N}_0^+$. Since F is a newform, we get

$$(t_v + w_v')^* e_F = 0.$$

This implies that $W_{v^\ell}^* e_F = t_{v^\ell}^* e_F$. Therefore

$$\Phi(e, e_F)^*(\mathfrak{m}) =< e, e_F > \cdot F^*(\mathfrak{m}) \quad \text{for any divisor } \mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k).$$

Since e_F is orthogonal to $e_{0,\chi}$ for every character χ of $\mathrm{Pic}(A)$, we obtain

$$\Phi(e, e_F)_0^*(\mathfrak{a}) = 0$$

for any divisor $\mathfrak{a} \in \mathrm{Div}_f(k)$. By Lemma III.7, the proof is complete accordingly. □

REMARK. 1. Suppose $\mathfrak{N}_0^+ = 1$. Then Theorem III.14 tells us that Φ maps $\mathrm{Pic}(X)_{\mathbb{C}} \times \mathrm{Pic}_0(X)_{\mathbb{C}}^{\vee}$ to the space $\mathcal{S}_0^{\mathrm{new}}(\mathfrak{N}_0)$ spanned by Drinfeld type newforms of level $\mathfrak{N}_0$.
2. In general, by Theorem III.12 we obtain that $\Phi(e, \check{e})$ is in fact in $\mathcal{S}_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N}_0)$ for any pair $(e, \check{e}) \in \mathrm{Pic}(X)_{\mathbb{C}} \times \mathrm{Pic}_0(X)_{\mathbb{C}}^{\vee}$.

In the next section, we study the basis problem for Drinfeld type cusp forms of square-free levels.

6. The basis problem

Let $\mathfrak{N}_0 \in \mathrm{Div}_{f,\geq 0}(k)$ be a square-free divisor. Let $\mathcal{S}_0(\mathfrak{N}_0)$ be the space of Drinfeld type cusp forms of level $\mathfrak{N}_0$. Then

$$\mathcal{S}_0(\mathfrak{N}_0) = \mathcal{S}_0(1, \mathfrak{N}_0) \oplus \mathcal{S}_0(1, \mathfrak{N}_0)^{\perp}.$$

Here $\mathcal{S}_0(1, \mathfrak{N}_0)$ denotes the space generated by the old forms

$$F'\left(g \begin{pmatrix} 1 & 0 \\ 0 & s(\mathfrak{N}'') \end{pmatrix}\right),$$

where F' is a Drinfeld type cusp form of level 1 and $\mathfrak{N}'' \mid \mathfrak{N}_0$; $\mathcal{S}_0(1,\mathfrak{N}_0)^{\perp}$ is the othogonal component of $\mathcal{S}_0(1,\mathfrak{N}_0)$ with respect to the Petersson inner product.

For a finite place v of k and $\mathfrak{N}' \in \operatorname{Div}_{f,\geq 0}(k)$ with $\mathfrak{N}'v \mid \mathfrak{N}_0$, the theta series $\Phi(e,e')$ for any $e \in \operatorname{Pic}_0(X_{\mathfrak{N}',v})_{\mathbb{C}}$ and $e' \in \operatorname{Pic}_0(X_{\mathfrak{N}',v})_{\mathbb{C}}^{\vee}$ is in fact in $\mathcal{S}_0(1,\mathfrak{N}_0)^{\perp}$. On the other hand, every $F \in \mathcal{S}_0(1,\mathfrak{N}_0)^{\perp}$ is constructed by newforms of non-trivial levels dividing $\mathfrak{N}_0$. Moreover, for $1 \neq \mathfrak{N}_0' \mid \mathfrak{N}_0$, let v_0 be a finite place dividing $\mathfrak{N}_0'$ and set $\mathfrak{N}_0'' := \mathfrak{N}_0'/v_0$. Then Theorem III.13 and III.14 says that every Drinfeld type newform F of level $\mathfrak{N}_0'$ is equal to

$$< e, e_F >^{-1} \Phi(e, e_F)$$

for $e \in \operatorname{Pic}_0(X_{\mathfrak{N}_0'',v_0})_{\mathbb{C}}$ with $< e, e_F > \neq 0$. Therefore we conclude that

THEOREM III.15. *The space* $\mathcal{S}_0(1,\mathfrak{N}_0)^{\perp}$ *is generated by the family of theta series* $\Phi(e,e')$, *where* $(e,e') \in \operatorname{Pic}_0(X_{\mathfrak{N}_0'^{+},\mathfrak{N}_0'^{-}})_{\mathbb{C}} \times \operatorname{Pic}_0(X_{\mathfrak{N}_0'^{+},\mathfrak{N}_0'^{-}})_{\mathbb{C}}^{\vee}$, *for the pairs* $(\mathfrak{N}_0'^{+},\mathfrak{N}_0'^{-})$ *with* $\mathfrak{N}_0'^{+}\mathfrak{N}_0'^{-}$ *dividing* $\mathfrak{N}_0$, *and the old forms coming from these theta series.*

An immediate consequence is the following:

COROLLARY III.16. *Suppose* $\mathcal{S}_0(1,\mathfrak{N}_0) = 0$. *Then the whole space* $\mathcal{S}_0(\mathfrak{N}_0)$ *is generated by those theta series introduced in* Theorem III.15 *and the old forms coming from them.*

REMARK. 1. When k is a rational function field and ∞ corresponds to the degree valuation, the space $\mathcal{S}_0(1,\mathfrak{N}_0)$ is trivial and hence every Drinfeld type cusp forms of level $\mathfrak{N}_0$ can be generated by our theta series.

2. It is worth pointing out that $\mathcal{S}_0(1,\mathfrak{N}_0)$ is not trivial in general. For example, we might take an elliptic curve E/k which has split multiplicative reduction at ∞ and has good reduction elsewhere. Then from the works of Weil, Jacquet-Langlands, and Deligne, there exists a normalized Drinfeld type newform F_E of level 1 whose L-function is equal to the Hasse-Weil L-function of E/k. The following example indicates the existence of such elliptic curves.

EXAMPLE III.17. (Given by M. Papikian) Let $k = \mathbb{F}_q(t)$ with $(q, 6) = 1$. Consider the following elliptic curve

$$E : y^2 + (t - 1728)xy = x^3 - 36(t - 1728)^3 x - (t - 1728)^5.$$

Then the discriminant of E is $t^2(t - 1728)^9$, and the j-invariant $j(E)$ of E is t. It is observed that E has multiplicative reduction at ∞. Since $j(E)$ is regular outside ∞, E has potentially good reduction at the places t and $t - 1728$. Denote by k_t and k_{t-1728} the completion of k at t and $t - 1728$, respectively. Therefore we can find fields K_1, K_2, K_3, where

$$[K_1 : k_t] = [K_2 : k_{t-1728}] = [K_3 : k_\infty] = 12$$

and E/K_1, E/K_2 are good reduction, E/K_3 is split multiplicative reduction. By Krasner's lemma and approximation theorem, there exists a global function field K with $[K : k] = 12$ and

$$K \otimes_k k_t = K_1,\ K \otimes_k k_{t-1728} = K_2,\ K \otimes_k k_\infty = K_3.$$

Hence E/K has split multiplicative reduction at the unique place ∞_K lying above ∞, and has good reduction elsewhere.

CHAPTER IV

Metaplectic forms and Shintani-type correspondence

In this chapter, we assume that q is **odd**. Following Kubota, we consider a non-trivial central extension of GL_2, the *metaplectic group*. In the function field context, we take the Iwahori Hecke operator at ∞ to be our "non-Euclidean Laplacian," and functions on the metaplectic group of weight $r/2$ are defined to be the eigenfunctions of this operator with eigenvalue $q_\infty^{1-\frac{r}{4}}$. From the norm form on A-lattices of pure quaternions in definite quaternion algebras over the function field k, we construct another family of theta series which are metaplectic forms of weight $3/2$. It turns out that the action of Hecke operators on these theta series can also be expressed by Brandt matrices. This allows us to establish a Shintani-type correspondence **Sh** in Theorem I.2

1. Metaplectic forms

1.1. Metaplectic group. We assumed that q is odd. Let v be a place of k. Recall the *Kubota 2-cocycle* $\sigma'_v : \mathrm{GL}_2(k_v) \times \mathrm{GL}_2(k_v) \to \{\pm 1\}$ defined by (cf. [**10**]):

$$\sigma'_v(g_1, g_2) := \left(\frac{x(g_1g_2)}{x(g_1)}, \frac{x(g_1g_2)}{\det g_1 \cdot x(g_2)}\right)_v, \quad \forall g_1, g_2 \in \mathrm{GL}_2(k_v).$$

Here

$$x\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} c, & \text{if } c \neq 0, \\ d, & \text{if } c = 0; \end{cases}$$

and $(\cdot,\cdot)_v$ is the Hilbert quadratic symbol at v, i.e. for any a, $b \in k_v^\times$,

$$(a,b)_v := \begin{cases} 1, & \text{if } aZ_1^2 + bZ_2^2 = Z_3^2 \text{ has a non-trivial solution,} \\ -1, & \text{otherwise.} \end{cases}$$

Define a map $s_v : \mathrm{GL}_2(k_v) \to \{\pm 1\}$ by setting

$$s_v \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{cases} (c, d/(ad-bc))_v, & \text{if } \mathrm{ord}_v(c) \text{ is odd and } d \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

Let σ_v be the 2-cocycle defined by

$$\sigma_v(g_1, g_2) := \sigma'_v(g_1, g_2) s_v(g_1) s_v(g_2) s_v(g_1 g_2)^{-1}, \quad \forall g_1, g_2 \in \mathrm{GL}_2(k_v).$$

It is known that (cf. [**5**])

$$\sigma_v(\kappa_1, \kappa_2) = 1 \quad \forall \kappa_1, \kappa_2 \in \mathrm{GL}_2(O_v).$$

Hence σ_v induces a central extension $\widetilde{\mathrm{GL}_2(k_v)}$ of $\mathrm{GL}_2(k_v)$ by

$$\mathbb{C}^1 := \{z \in \mathbb{C} : |z| = 1\}$$

which splits on the subgroup $\mathrm{GL}_2(O_v)$. More precisely, the extension $\widetilde{\mathrm{GL}_2(k_v)}$ is identified with $\mathrm{GL}_2(k_v) \times \mathbb{C}^1$ (as sets) with the following group law:

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = \big(g_1 g_2, \xi_1 \xi_2 \sigma_v(g_1, g_2)\big).$$

Globally, we define a 2-cocycle σ on $\mathrm{GL}_2(\mathbb{A})$ by setting $\sigma := \otimes_v \sigma_v$, and denote by $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ the corresponding central extension of $\mathrm{GL}_2(\mathbb{A})$ by $\mathbb{C}^1$. We emphasize that the embeddings

$$\begin{array}{ccc} \mathrm{GL}_2(k) & \longrightarrow & \widetilde{\mathrm{GL}_2(\mathbb{A})} \\ \gamma & \longmapsto & \tilde{\gamma} := (\gamma, s(\gamma)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{GL}_2(O_{\mathbb{A}}) & \longrightarrow & \widetilde{\mathrm{GL}_2(\mathbb{A})} \\ \kappa & \longmapsto & \tilde{\kappa} := (\kappa, 1) \end{array}$$

are group monomorphisms. Here $s(\gamma) := \prod_v s_v(\gamma)$.

Let

$$\mathcal{K}_\infty^+ := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_\infty) \;\middle|\; c \equiv 0 \bmod \pi_\infty O_\infty \text{ and } (\pi_\infty, d)_\infty = 1 \right\}$$

and

$$\widetilde{\mathcal{K}_\infty^+} := \left\{ \tilde{\kappa}_\infty = (\kappa_\infty, 1) \in \widetilde{\mathrm{GL}_2(k_\infty)} : \kappa_\infty \in \mathcal{K}_\infty^+ \right\}.$$

It is observed that

$$\widetilde{\mathcal{K}_\infty^+}\left(\begin{pmatrix}\pi_\infty & 0\\ 0 & 1\end{pmatrix},1\right)\widetilde{\mathcal{K}_\infty^+} = \coprod_{u\in\mathbb{F}_\infty}\left(\begin{pmatrix}\pi_\infty & u\\ 0 & 1\end{pmatrix},1\right)\widetilde{\mathcal{K}_\infty^+}.$$

For each function F on $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ with

$$F(\tilde{g}\tilde{\kappa}_\infty) = F(\tilde{g}), \quad \forall \kappa_\infty \in \mathcal{K}_\infty^+,$$

we define for each integer r, the *weight-$r/2$ operator* (with respect to ∞):

$$T_{\infty,r/2}F(\tilde{g}) := q_\infty^{r/4-1}\cdot\sum_{u\in\mathbb{F}_\infty} F\left(\tilde{g}\cdot\left(\begin{pmatrix}\pi_\infty & u\\ 0 & 1\end{pmatrix},1\right)\right).$$

Definition IV.1. Suppose an integer r is given. A $\mathbb{C}$-valued function F on $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ is call a *weight-$r/2$ metaplectic form* if there exists an open subgroup $\mathcal{K}$ of $\mathrm{GL}_2(O_{\mathbb{A}_f})$ such that

$$F(\tilde{\gamma}\tilde{g}\tilde{\kappa}) = \xi^r F(g,1), \quad \forall \tilde{g} = (g,\xi)\in\widetilde{\mathrm{GL}_2(\mathbb{A})},\ \gamma\in\mathrm{GL}_2(k),\ \kappa\in\mathcal{K}\times\mathcal{K}_\infty^+,$$

and

$$T_{\infty,r/2}F = F.$$

Example IV.2. Let F be a Drinfeld type automorphic forms. Then F induces a function (still denoted by F) on $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ by setting

$$F(g,\xi) := \xi^4\cdot F(g), \quad \forall (g,\xi)\in\widetilde{\mathrm{GL}_2(\mathbb{A})}.$$

The harmonicity of F tells us that

$$T_{\infty,2}F = F.$$

Therefore every Drinfeld type automorphic form can be viewed as a weight-2 metaplectic forms.

In the next subsection, we review the Weil representation of the metaplectic group $\widetilde{\mathrm{GL}_2(\mathbb{A})}$, and construct an explicit family of metaplectic forms having half integral weight.

1.2. Weil representation and theta series from pure quaternions.

1.2.1. *local results.* Recall that (V, Q_V) is the quadratic space (D, Nr), where D is the chosen definite quaternion algebra over k in Chapter III. Then

$$(V, Q_V) = (V_1, Q_1) \oplus (V_3, Q_3),$$

where

$$V_1 := k, \quad V_3 := \{b \in D : \mathrm{Tr}(b) = 0\}, \quad \text{and} \quad Q_i := Q\,|_{V_i}.$$

For each place v of k, Denote by $S(V_r(k_v) \times k_v^\times)$ the space of functions ϕ_v on $V_r(k_v) \times k_v^\times$ such that for each $\alpha_v \in k_v^\times$, $\phi_v(\cdot, \alpha_v)$ is a Schwartz function on $V_r(k_v)$. Define the *Weil index* $\gamma_{\psi_v}(\alpha_v Q_r)$ for the quadratic form $\alpha_v Q_r$:

$$\gamma_{\psi_v}(\alpha_v Q_r) := \int_{L_r} \psi_v(\alpha_v Q_r(u)) d_{\alpha_v} u$$

where L_r is a sufficiently large O_v lattice in $V_r(k_v)$. The Haar measure $d_{\alpha_v} u$ is self-dual with respect to the pairing

$$(x, y) \mapsto \psi_v(\alpha_v \mathrm{Tr}(x\bar{y})), \quad \forall x, y \in V_r(k_v).$$

Note that for $r = 1$ or 3, we define $W_{\psi_v, r} : k_v^\times \longrightarrow \mathbb{C}^1$ by setting

$$W_{\psi_v, r}(\alpha_v) := \frac{\gamma_{\psi_v}(\alpha_v Q_r)}{\gamma_{\psi_v}(Q_r)} \quad \forall \alpha_v \in k_v^\times.$$

Then it is known that

$$W_{\psi_v, 1} \cdot W_{\psi_v, 3} \equiv 1$$

and

$$\frac{W_{\psi_v, r}(\alpha_v) W_{\psi_v, r}(\beta_v)}{W_{\psi_v, r}(\alpha_v \beta_v)} = (\alpha_v, \beta_v)_v, \quad \forall \alpha_v,\ \beta_v \in k_v^\times.$$

In particular, $W_{\psi_v, r}(\alpha_v) = (\pi_v^{\mathrm{ord}_v(\delta)}, \alpha_v)_v$ for any $\alpha_v \in O_v^\times$.

THEOREM IV.3. (Gelbart [**5**]) *There is a representation* $\omega_{r,v}$ *of* $\widetilde{\mathrm{GL}_2(k_v)}$ *on the space* $S(V_r(k_v)\times k_v^\times)$ *satisfying that*

(1) $$\omega_{r,v}(1,\xi)\phi(w,\alpha_v) := \xi^r\phi(w,\alpha_v),\ \xi\in\mathbb{C}^1;$$

(2) $$\omega_{r,v}\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix},1\right)\phi(w,\alpha_v) = \psi_v(x\alpha_v Q_r(w))\phi(w,\alpha_v),\ x\in k_v;$$

(3) $$\omega_{r,v}\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},1\right)\phi(w,\alpha_v) = \gamma_{\psi_v}(\alpha_v Q_r)\mathcal{F}_r(\phi)(w,\alpha_v);$$

(4) $$\omega_{r,v}\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix},1\right)\phi(w,\alpha_v) = |a|_v^{\frac{r}{2}}(a,a)_v^r\frac{\gamma_{\psi_v}(a\alpha_v Q_r)}{\gamma_{\psi_v}(\alpha_v Q_r)}\phi(aw,\alpha_v),\ a\in k_v^\times;$$

(5) $$\omega_{r,v}\left(\begin{pmatrix}1 & 0\\ 0 & \beta\end{pmatrix},1\right)\phi(w,\alpha_v) = |\beta|_v^{-\frac{r}{4}}\phi(w,\alpha_v\beta^{-1}),\ \beta\in k_v^\times.$$

Here

$$\mathcal{F}_r(\phi)(w,\alpha_v) = \int_{V_r(k_v)}\phi(u,\alpha_v)\psi_v(\alpha_v\,\mathrm{Tr}(u\bar{w}))d_{\alpha_v}u,$$

where the Haar measure $d_{\alpha_v}u$ *is normalized so that*

$$\mathcal{F}_r^2(\phi)(w,\alpha_v) = \phi(-w,\alpha_v).$$

Now, consider a particular function in $S(V_1(k_v)\times k_v^\times)$:

$$\varphi_v^{(1)}(x_v,\alpha_v) := \mathbf{1}_{\pi_v^{\lceil -\mathrm{ord}_v(\delta)/2\rceil}O_v}(x_v)\cdot\mathbf{1}_{O_v^\times}(\alpha_v),\quad \forall(x_v,\alpha_v)\in V_1(k_v)\times k_v^\times.$$

Then

LEMMA IV.4. *For any* $\kappa_v = \begin{pmatrix}a_v & b_v\\ c_v & d_v\end{pmatrix}\in\mathrm{GL}_2(O_v)$ *with* $c_v\in\pi_v O_v$, *we have*

$$\omega_{1,v}(\tilde{\kappa}_v)\varphi_v^{(1)} = W_{\psi_v,1}(a_v)\cdot\varphi_v^{(1)}.$$

In particular, suppose $\mathrm{ord}_v(\delta)$ *is even. Then*

$$\omega_{1,v}(\tilde{\kappa}_v)\varphi_v^{(1)} = \varphi_v^{(1)},\quad \forall\kappa_v\in\mathrm{GL}_2(O_v).$$

PROOF. It is observed that

$$\omega_{1,v}\left(\begin{pmatrix}1 & 0\\ 0 & u_v\end{pmatrix},1\right)\varphi_v^{(1)} = \varphi_v^{(1)}$$

for any $u_v \in O_v^\times$, and

$$\mathcal{F}_1(\varphi_v^{(1)})(x_v, \alpha_v) = \mathbf{1}_{\pi_v^{\lfloor -\operatorname{ord}_v(\delta)/2 \rfloor} O_v}(x_v) \cdot \mathbf{1}_{O_v^\times}(\alpha_v) \cdot \begin{cases} q_v^{-1/2} & \text{if } \operatorname{ord}_v(\delta) \text{ is odd,} \\ 1 & \text{if } \operatorname{ord}_v(\delta) \text{ is even.} \end{cases}$$

Let $\kappa_v = \begin{pmatrix} a & b \\ \pi_v c & d \end{pmatrix} \in \mathrm{SL}_2(O_v)$ with $c \in O_v$. Then

$$\kappa_v = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_v d^{-1} c & 1 \end{pmatrix}.$$

Therefore for $(x_v, \alpha_v) \in V_1(k_v) \times k_v^\times$,

$$\begin{aligned} & \omega_{1,v}(\tilde{\kappa}_v)\varphi_v^{(1)}(x_v, \alpha_v) \\ = \ & \psi_v\big(\alpha_v b d^{-1} Q_1(x_v)\big) \cdot \frac{W_{\psi_v,1}(a)}{(a, \alpha_v)_v} \cdot \left(\omega_{1,v}\left(\begin{pmatrix} 1 & 0 \\ \pi_v d^{-1} c & 1 \end{pmatrix}, 1\right)\varphi_v^{(1)}\right)(d^{-1}x_v, \alpha_v). \end{aligned}$$

Since

$$\begin{pmatrix} 1 & 0 \\ \pi_v d^{-1} c & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\pi_v d^{-1} c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

we have

$$\begin{aligned} & \left(\omega_{1,v}\left(\begin{pmatrix} 1 & 0 \\ \pi_v d^{-1} c & 1 \end{pmatrix}, 1\right)\varphi_v^{(1)}\right)(d^{-1}x_v, \alpha_v) \\ = \ & \int_{V_1(k_v)} \psi_v\left(-\alpha_v \pi_v d^{-1} c \cdot Q_1(y_v)\right) \psi_v(2\alpha_v d^{-1} x_v y_v) \mathcal{F}_1(\varphi_v^{(1)})(-y_v, \alpha_v) d_{\alpha_v} y_v. \\ = \ & \mathbf{1}_{O_v^\times}(\alpha_v) \cdot \int_{\pi_v^{\lfloor -\operatorname{ord}_v(\delta)/2 \rfloor} O_v} \psi_v\left(-\alpha_v \pi_v d^{-1} c \cdot Q_1(y_v)\right) \psi_v(2\alpha_v d^{-1} x_v y_v) d_{\alpha_v} y_v \\ & \cdot \begin{cases} q_v^{-1/2} & \text{if } \operatorname{ord}_v(\delta) \text{ is odd,} \\ 1 & \text{if } \operatorname{ord}_v(\delta) \text{ is even.} \end{cases} \\ = \ & \varphi_v(x_v, \alpha_v). \end{aligned}$$

Moreover, for $(x_v, \alpha_v) \in \pi_v^{\lceil -\operatorname{ord}_v(\delta)/2 \rceil} O_v \times O_v^\times$, one has

$$\psi_v\left(\alpha_v b d^{-1} Q_1(x_v)\right) = 1 \quad \text{and} \quad (a, \alpha_v)_v = 1.$$

Therefore

$$\omega_{1,v}(\tilde{\kappa}_v)\varphi_v^{(1)} = W_{\psi_v,1}(a) \cdot \varphi_v^{(1)}$$

for any $\kappa_v = \begin{pmatrix} a & b \\ \pi_v c & d \end{pmatrix} \in \mathrm{GL}_2(O_v)$ with $c \in O_v$.

Suppose $\mathrm{ord}_v(\delta)$ is even. Then $\gamma_{\psi_v}(\alpha_v Q_1) = 1$ for any $\alpha_v \in O_v^\times$ and $\mathcal{F}_1(\varphi_v^{(1)}) = \varphi_v^{(1)}$. Since $\mathrm{GL}_2(O_v)$ is generated by elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \equiv 0 \bmod \pi_v O_v$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the proof of this lemma is complete. □

Recall

$$\epsilon_v(\delta) = \begin{cases} 0 & \text{if } \mathrm{ord}_v(\delta) \text{ is even,} \\ 1 & \text{if } \mathrm{ord}_v(\delta) \text{ is odd.} \end{cases}$$

Then we immediately get

Corollary IV.5. *Let $\widetilde{\varphi}_v^{(1)}$ be the function in $S(V_1(k_v) \times k_v^\times)$ defined by*

$$\widetilde{\varphi}_v^{(1)}(x_v, \alpha_v) := \mathbf{1}_{\pi_v^{-\mathrm{ord}_v(\delta)} O_v}(x_v) \cdot \mathbf{1}_{O_v^\times}(\alpha_v \pi_v^{-\mathrm{ord}_v(\delta) - 2\epsilon_v(\delta)})$$

for any $(x_v, \alpha_v) \in V_1(k_v) \times k_v^\times$. Then for every $\kappa_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \in \mathrm{GL}_2(O_v)$ with $c_v \equiv 0 \bmod \pi_v^{2\epsilon_v(\delta)} O_v$, we have

$$\omega_{1,v}(\tilde{\kappa}_v) \widetilde{\varphi}_v^{(1)} = \widetilde{\varphi}_v^{(1)}.$$

Proof. Note that

$$\begin{aligned} &\omega_{1,v}\left(\left(\begin{pmatrix} \pi_v^{-\lfloor -\mathrm{ord}_v(\delta)/2 \rfloor} & 0 \\ 0 & \pi_v^{-\lfloor -\mathrm{ord}_v(\delta)/2 \rfloor} \end{pmatrix}, 1\right)\right) \varphi_v^{(1)}(x_v, \alpha_v) \\ = \quad & (\pi_v^{-\lfloor -\mathrm{ord}_v(\delta)/2 \rfloor}, \pi_v^{-\lfloor -\mathrm{ord}_v(\delta)/2 \rfloor})_v \frac{W_{\psi_v,1}(\pi_v^{-\lfloor -\mathrm{ord}_v(\delta)/2 \rfloor})}{(\pi_v^{-\lfloor -\mathrm{ord}_v(\delta)/2 \rfloor}, \alpha_v)_v} \\ & \cdot \mathbf{1}_{\pi_v^{-\mathrm{ord}_v(\delta)} O_v}(x_v) \mathbf{1}_{O_v^\times}(\alpha_v \pi_v^{2\lfloor -\mathrm{ord}_v(\delta)/2 \rfloor}). \end{aligned}$$

Hence let

$$\varphi_v'^{(1)}(x_v, \alpha_v) := \mathbf{1}_{\pi_v^{-\mathrm{ord}_v(\delta)} O_v}(x_v) \mathbf{1}_{O_v^\times}(\alpha_v \pi_v^{2\lfloor -\mathrm{ord}_v(\delta)/2 \rfloor}), \quad \forall (x_v, \alpha_v) \in V_1(k_v) \times k_v^\times,$$

we have

$$\omega_{1,v}(\tilde{\kappa}_v) \varphi_v'^{(1)} = W_{\psi_v,1}(a)^{\epsilon_v(\delta)} \cdot \varphi_v'^{(1)}$$

for any $\kappa_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v)$ with $c \equiv 0 \bmod \pi_v^{\epsilon_v(\delta)} O_v$. Here we set $W_{\psi_v,1}(a) := 1$ if $a = 0$. Since

$$\widetilde{\varphi}_v^{(1)} = |\pi_v|_v^{\epsilon_v(\delta)/4} \cdot \omega_{1,v} \left(\begin{pmatrix} 1 & 0 \\ 0 & \pi_v^{\epsilon_v(\delta)} \end{pmatrix}, 1 \right) \varphi_v'^{(1)},$$

we get for any $\kappa_v = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v)$ with $c \equiv 0 \bmod \pi_v^{\epsilon_v(\delta)} O_v$,

$$\omega_{1,v} \left[\left(\begin{pmatrix} 1 & 0 \\ 0 & \pi_v^{\epsilon_v(\delta)} \end{pmatrix}, 1 \right) \tilde{\kappa}_v \left(\begin{pmatrix} 1 & 0 \\ 0 & \pi_v^{-\epsilon_v(\delta)} \end{pmatrix}, 1 \right) \right] \widetilde{\varphi}_v^{(1)} = W_{\psi_v,1}(a)^{\epsilon_v(\delta)} \cdot \widetilde{\varphi}_v^{(1)}.$$

It is clear that

$$\begin{aligned} & \left(\begin{pmatrix} 1 & 0 \\ 0 & \pi_v^{\epsilon_v(\delta)} \end{pmatrix}, 1 \right) \tilde{\kappa}_v \left(\begin{pmatrix} 1 & 0 \\ 0 & \pi_v^{-\epsilon_v(\delta)} \end{pmatrix}, 1 \right) \\ = & \left(\begin{pmatrix} a & \pi_v^{-\epsilon_v(\delta)} b \\ \pi_v^{\epsilon_v(\delta)} c & d \end{pmatrix}, W_{\psi_v,1}(a)^{\epsilon_v(\delta)} \right). \end{aligned}$$

Therefore the proof is complete. □

Let $\widetilde{\varphi}_0^{(1)} := \otimes_v \widetilde{\varphi}_{0,v}^{(1)} \in S(V_1(\mathbb{A}) \times \mathbb{A}^\times)$, where for each v of k with $v \neq \infty$ and $(x_{1,v}, \alpha_v) \in V_1(k_v) \times k_v^\times$,

$$\begin{aligned} \widetilde{\varphi}_{0,v}^{(1)}(x_{1,v}, \alpha_v) &:= \widetilde{\varphi}_v^{(1)}(x_{1,v}, \alpha_v) \\ &= \mathbf{1}_{\pi_v^{-\mathrm{ord}_v(\delta)} O_v}(x_v) \cdot \mathbf{1}_{O_v^\times}(\alpha_v \pi_v^{-\mathrm{ord}_v(\delta) - 2\epsilon_v(\delta)}); \end{aligned}$$

and for $(x_{1,\infty}, \alpha_\infty) \in V_1(k_\infty) \times k_\infty^\times$,

$$\widetilde{\varphi}_{0,\infty}^{(1)}(x_{1,\infty}, \alpha_\infty) := W_{\psi_\infty,1}(\alpha_\infty)^{-1} \mathbf{1}_{O_\infty}(x_{1,\infty}^2 \alpha_\infty \pi_\infty^{\mathrm{ord}_\infty(\delta)}).$$

Let

$$\widetilde{\Theta}(\tilde{g}) := \sum_{(x,\alpha) \in V_1(k) \times k^\times} \left(\omega_1(\tilde{g}) \widetilde{\varphi}_0^{(1)} \right)(x, \alpha), \quad \forall \tilde{g} \in \widetilde{\mathrm{GL}_2(\mathbb{A})}.$$

We have

Proposition IV.6. *The theta series $\widetilde{\Theta}$, is a weight-$1/2$ metaplectic form on $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ which satisfy that*

$$\widetilde{\Theta}(\tilde{g}\tilde{\kappa}) = \widetilde{\Theta}(\tilde{g}), \quad \forall \kappa \in \mathcal{K}_0(\Omega^2 \infty^+).$$

Here $\Omega \in \mathrm{Div}_{f,\geq 0}(k)$ *is introduced at the end of Section 1 in Chapter II and*

$$\mathcal{K}_0(\Omega^2\infty^+) := \left(\prod_{v\neq\infty} \mathcal{K}_v\right) \times \mathcal{K}_\infty^+$$

where

$$\mathcal{K}_v := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v) \;\middle|\; c \equiv 0 \bmod \pi_v^{2\,\mathrm{ord}_v(\Omega)} O_v \right\}$$

for $v \neq \infty$ *and*

$$\mathcal{K}_\infty^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_\infty) \;\middle|\; c \equiv 0 \bmod \pi_\infty O_\infty \text{ and } (\pi_\infty, d)_\infty = 1 \right\}.$$

We emphasize that the theta series $\widetilde{\Theta}$ is viewed as a function field analogue of the theta series

$$\sum_{n\in\mathbb{Z}} e^{2\pi\sqrt{-1}n^2 z},$$

a modular form of weight 1/2 for $\Gamma_0(4)$ (cf. [**13**]).

We have fixed an Eichler A-order R in D of type $(\mathfrak{N}^+, \mathfrak{N}^-)$ and representatives $I_1, ..., I_n$ of locally-principal right ideal classes of R, and denoted by R_i the left order of I_i. For $1 \leq i \leq n$, denote by $R_i^{(p)}$ the set of pure quaternions in R_i, i.e.

$$R_i^{(p)} := \{b \in R_i : \mathrm{Tr}(b) = 0\}.$$

Then

$$R_i = A \oplus R_i^{(p)}$$

and for any $b = b_1 + b_3 \ \in R_i$ with $b_1 \in A$ and $b_3 \in R_i^{(p)}$,

$$\mathrm{Tr}(b) = 2b_1 + \mathrm{Tr}(b_3).$$

For each finite place v of k, let $\varphi_{i,v}^{(3)}$ be the function in $S(V_3(k_v) \times k_v^\times)$ defined by

$$\varphi_{i,v}^{(3)}(x_v, \alpha_v) := \mathbf{1}_{\pi_v^{-\mathrm{ord}_v(\delta)} R_{i,v}^{(p)}}(x_v) \cdot \mathbf{1}_{O_v^\times}(\alpha_v \pi_v^{-\mathrm{ord}_v(\delta_v) - 2\epsilon_v(\delta)}).$$

Here $R_{i,v}^{(p)} := R_i^{(p)} \otimes_A O_v$. It is observed that for any $x_{r,v} \in V_r(k_v)$ and $\alpha_v \in k_v^\times$,

$$\widetilde{\varphi}_v^{(1)}(x_{1,v},\alpha_v)\cdot\varphi_{i,v}^{(3)}(x_{3,v},\alpha_v) = \mathbf{1}_{\pi_v^{-\operatorname{ord}_v(\delta)}R_v}(x_{1,v}+x_{3,v})\mathbf{1}_{O_v^\times}(\alpha_v\pi_v^{-\operatorname{ord}_v(\delta)-2\epsilon_v(\delta)}).$$

Moreover, consider $\widetilde{\varphi}_v^{(1)} \otimes \varphi_{i,v}^{(3)}$ as a function in $S(V(k_v)\times k_v^\times)$, then for any $g \in \mathrm{GL}_2(k_v)$ and $1 \leq i \leq n$,

$$\tilde{\omega}_v(g)\Big[\widetilde{\varphi}_v^{(1)} \otimes \varphi_{i,v}^{(3)}\Big] = \omega_{1,v}(g,1)\widetilde{\varphi}_v^{(1)} \otimes \omega_{3,v}(g,1)\varphi_{i,v}^{(3)}.$$

Recall that $\Omega = \prod_{v\neq\infty} v^{\epsilon_v(\delta)} \in \operatorname{Div}_{f,\geq 0}(k)$. By Corollary IV.5 we get

LEMMA IV.7. *Let v be an arbitrary finite place of k. For each element $\kappa_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \in \mathrm{GL}_2(O_v)$ with $c_v \equiv 0 \bmod \pi_v^{\operatorname{ord}_v(\Omega^2\mathfrak{N}^+\mathfrak{N}^-)}O_v$,*

$$\omega_{3,v}(\tilde{\kappa}_v)\varphi_{i,v}^{(3)} = \varphi_{i,v}^{(3)}, \quad \forall 1 \leq i \leq n.$$

For $(x_{1,\infty}+x_{3,\infty},\alpha_\infty) \in \big(V_1(k_\infty)\oplus V_3(k_\infty)\big)\times k_\infty^\times$, let

$$\varphi_\infty^{(3)}(x_{3,\infty},\alpha_\infty) := W_{\psi_\infty,3}(\alpha_\infty)^{-1}\mathbf{1}_{O_\infty}(\mathrm{Nr}(x_{3,\infty})\alpha_\infty\pi_\infty^{\operatorname{ord}_\infty(\delta)}).$$

Then it is observed that for $x_\infty = x_{1,\infty}+x_{3,\infty} \in V(k_\infty) = V_1(k_\infty)\oplus V_3(k_\infty)$ and $\alpha \in k_\infty^\times$,

$$\widetilde{\varphi}_{0,\infty}^{(1)}(x_{1,\infty},\alpha_\infty)\cdot\varphi_\infty^{(3)}(x_{3,\infty},\alpha_\infty) = \mathbf{1}_{O_\infty}(\mathrm{Nr}(x_\infty)\alpha_\infty\pi_\infty^{\operatorname{ord}_\infty(\delta)}).$$

Similarly, we obtain that

LEMMA IV.8. *For any $\kappa_\infty = \begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty \end{pmatrix} \in \mathrm{GL}_2(O_\infty)$ with c_∞ in $\pi_\infty O_\infty$,*

$$\Big(\omega_{3,\infty}(\tilde{\kappa}_\infty)\varphi_\infty^{(3)}\Big)(x_{3,\infty},\alpha_\infty) = \Big(W_{\psi_\infty,3}(d_\infty)(d_\infty,\alpha_\infty)_\infty\Big)\cdot\varphi_\infty^{(3)}(x_{3,\infty},\alpha_\infty)$$

for any $(x_{3,\infty},\alpha_\infty) \in V_3(k_\infty)\times k_\infty^\times$. In particular, if $(\pi_\infty,d_\infty)_\infty = 1$,

$$\omega_{3,\infty}(\tilde{\kappa}_\infty)\varphi_\infty^{(3)} = \varphi_\infty^{(3)}.$$

Furthermore, it is observed that for $(x_{3,\infty}, \alpha_\infty) \in V_3(k_\infty) \times k_\infty^\times$,

$$\begin{aligned}
&\sum_{u \in \mathbb{F}_\infty} \omega_{3,\infty}\left(\begin{pmatrix} \pi_\infty & u \\ 0 & 1 \end{pmatrix}, 1\right) \varphi_\infty^{(3)}(x_\infty, \alpha_\infty) \\
= \; &\sum_{u \in \mathbb{F}_\infty} \psi_\infty\big(\operatorname{Nr}(x_\infty)\alpha_\infty u\big) \cdot \mathbf{1}_{O_\infty}\big(\operatorname{Nr}(x_\infty)\alpha_\infty \pi_\infty^{1-\operatorname{ord}_\infty(\delta)}\big) \\
&\qquad \cdot \left(|\pi_\infty|_\infty^{-3/4+3/2} (\pi_\infty, \pi_\infty)_\infty \frac{\gamma_{\psi_\infty}(\alpha_\infty Q_3)}{\gamma_{\psi_\infty}(\alpha_\infty \pi_\infty^{-1} Q_3)} W_{\psi_\infty,3}(\alpha_\infty \pi_\infty^{-1})^{-1}\right) \\
= \; & q_\infty^{3/4} \cdot W_{\psi_\infty,3}(\alpha_\infty)^{-1} \mathbf{1}_{O_\infty}\big(\operatorname{Nr}(x_\infty)\alpha_\infty \pi_\infty^{1-\operatorname{ord}_\infty(\delta)}\big) \cdot \sum_{u \in \mathbb{F}_\infty} \psi_\infty\big(\operatorname{Nr}(x_\infty)\alpha_\infty u\big) \\
= \; & q_\infty^{1-3/4} \cdot W_{\psi_\infty,3}(\alpha_\infty)^{-1} \mathbf{1}_{O_\infty}\big(\operatorname{Nr}(x_\infty)\alpha_\infty \pi_\infty^{-\operatorname{ord}_\infty(\delta)}\big) \\
= \; & q_\infty^{1-3/4} \cdot \varphi_\infty^{(3)}(x_\infty, \alpha_\infty).
\end{aligned}$$

Therefore we get

Lemma IV.9.

$$q_\infty^{3/4-1} \cdot \sum_{u \in \mathbb{F}_\infty} \omega_{3,\infty}\left(\begin{pmatrix} \pi_\infty & u \\ 0 & 1 \end{pmatrix}, 1\right) \varphi_\infty^{(3)} = \varphi_\infty^{(3)}.$$

1.2.2. *Theta series of pure quaternions.* Let ω_3 be the Weil representation $\otimes_v \omega_{3,v}$ of $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ on the space $S(V_3(\mathbb{A}) \times \mathbb{A}^\times)$. Recall that $I_i = D \cap b_i \widehat{R}$ with $b_i \in D_{\mathbb{A}^\infty}^\times$. Let $\beta_i = \operatorname{Nr}(b_i) \in \mathbb{A}^{\infty,\times}$. For each finite place v of k, set

$$\widetilde{\varphi}_{i,v}^{(3)}(x_v, \alpha_v) := \varphi_{i,v}^{(3)}(\beta_{i,v} \pi_v^{-2\epsilon_v(\delta)} x_v, \alpha_v \beta_{i,v}^{-2} \pi_v^{4\epsilon_v(\delta)}), \quad \forall (x_v, \alpha_v) \in V_3(k_v) \times k_v^\times.$$

We also let

$$\widetilde{\varphi}_{i,\infty}^{(3)} := \varphi_\infty^{(3)}, \quad 1 \le i \le n.$$

Definition IV.10. For $1 \le i \le n$, let $\widetilde{\Theta}_i$ be the function on $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ defined by

$$\widetilde{\Theta}_i(\tilde{g}) := \|\delta\|^{-3/4} \sum_{(x,\alpha) \in V_3(k) \times k^\times} \left(\omega_3(\tilde{g}) \widetilde{\varphi}_i^{(3)}\right)(x, \alpha), \quad \forall \tilde{g} \in \widetilde{\mathrm{GL}_2(\mathbb{A})},$$

where $\widetilde{\varphi}_i^{(3)} := \otimes_v \widetilde{\varphi}_{i,v}^{(3)} \in S(V_3(\mathbb{A}) \times \mathbb{A}^\times)$.

For each divisor $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$, we set

$$\mathcal{K}_0(\mathfrak{m}\infty^+) = \left(\prod_{v\neq\infty} \mathcal{K}_v\right) \times \mathcal{K}_\infty^+$$

where

$$\mathcal{K}_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v) \;\middle|\; c \equiv 0 \bmod \pi_v^{\mathrm{ord}_v(\mathfrak{m})} O_v \right\}$$

for $v \neq \infty$ and

$$\mathcal{K}_\infty^+ = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_\infty) \;\middle|\; c \equiv 0 \bmod \pi_\infty O_\infty \text{ and } (\pi_\infty, d)_\infty = 1 \right\}.$$

Then by Lemma IV.7, IV.8, and IV.9 we obtain that

PROPOSITION IV.11. *The theta series $\widetilde{\Theta}_i$, $1 \leq i \leq n$, are weight-3/2 metaplectic forms on $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ which satisfy that*

$$\widetilde{\Theta}_i(\tilde{g}\tilde{\kappa}) = \widetilde{\Theta}_i(\tilde{g}), \quad \forall \kappa \in \mathcal{K}_0(\Omega^2\mathfrak{N}\infty^+).$$

Here $\mathfrak{N} = \mathfrak{N}^+\mathfrak{N}^-$. In particular, $\widetilde{\Theta}_i(g,\xi) = 0$ for $(g,\xi) \in \widetilde{\mathrm{GL}_2(\mathbb{A})}$ unless there exists an element $\alpha \in k^\times$ such that for each finite place v of k,

$$\alpha \cdot \det g_v \cdot \beta_{i,v}^2 \cdot \pi_v^{\mathrm{ord}_v(\delta) - 2\epsilon_v(\delta)} \in O_v^\times.$$

1.3. Fourier coefficients of metaplectic theta series. Let F be a weight-$r/2$ metaplectic form satisfying that

$$F(\tilde{g}\tilde{\kappa}) = F(\tilde{g}), \quad \forall \kappa \in \mathcal{K}_0(\Omega^2\mathfrak{N}\infty^+).$$

For $z \in \mathbb{A}^\times$ and $\mathfrak{m} \in \mathrm{Div}(k)$, the Fourier coefficients $F^*(z,\mathfrak{m})$ and $F_0^*(z,\mathfrak{m})$ are defined by

$$F^*(z,\mathfrak{m}) := \int_{k\backslash\mathbb{A}} F\left(\begin{pmatrix} zs(\delta^{-1}\mathfrak{m}) & u \\ 0 & z \end{pmatrix}, 1\right) \psi(-u) du$$

and

$$F_0^*(z,\mathfrak{m}) := \int_{k\backslash\mathbb{A}} F\left(\begin{pmatrix} zs(\delta^{-1}\mathfrak{m}) & u \\ 0 & z \end{pmatrix}, 1\right) du,$$

where the Haar measure du is normalized so that $\int_{k\backslash\mathbb{A}} du = 1$. Then it is observed that $F^*(z,\mathfrak{m}) = 0$ unless $\mathfrak{m} \in \mathrm{Div}_{\geq 0}(k)$, and

$$F^*(z, \mathfrak{m}\infty^\ell) = q_\infty^{-\frac{3}{4}\ell} \cdot F^*(z,\mathfrak{m}), \quad \forall \mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k),\ \ell \in \mathbb{Z}_{\geq 0}.$$

Similarly,

$$F_0^*(z, \mathfrak{m}\infty^\ell) = q_\infty^{-\frac{3}{4}\ell} \cdot F_0^*(z,\mathfrak{m}), \quad \forall \mathfrak{m} \in \mathrm{Div}_{f}(k),\ \ell \in \mathbb{Z}.$$

Now, we focus on the Fourier coefficients of the theta series $\widetilde{\Theta}_i$. For $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$ and $z = (z_f, z_\infty) \in \mathbb{A}^{\infty,\times} \times k_\infty^\times = \mathbb{A}^\times$, let

$$S_i(z,\mathfrak{m}) := \left\{ b \in M_{\mathrm{div}(z_f)} \mathrm{Nr}(I_i) R_i^{(p)} : \begin{array}{c} \mathrm{Nr}(b)\cdot A = M_{\mathrm{div}(z_f)}^2 \mathrm{Nr}(I_i)^2 M_{\Omega^{-2}\mathfrak{m}}, \\ \text{and } \ \mathrm{ord}_\infty(\mathfrak{m}) \geq 0 \end{array} \right\}.$$

Here $M_{\mathrm{div}(z_f)}$ and $M_{\Omega^{-2}\mathfrak{m}}$ are the fractional ideals of A corresponding to the divisors $\mathrm{div}(z_f)$ and $\Omega^{-2}\mathfrak{m}$ in $\mathrm{Div}_f(k)$, respectively. Set

$$W_{\psi,3}(z) := \prod_v W_{\psi_v,3}(z_v), \quad \forall z \in \mathbb{A}^\times.$$

Then we get

Lemma IV.12. *For $1 \leq i \leq n$, $z \in \mathbb{A}^\times$, and $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$,*

$$\begin{aligned} \widetilde{\Theta}_i^*(z,\mathfrak{m}) \quad = \quad & \frac{W_{\psi,3}\big(s(\delta_f^{-1}\mathfrak{m})\big)}{W_{\psi,3}(z)\|\mathfrak{m}\|^{3/4}} \\ & \cdot \sum_{b \in S_i(z,\mathfrak{m})} \Bigg[W_{\psi_\infty,3}\big(\mathrm{Nr}(b)\big)^{-1} \big(z_\infty, \mathrm{Nr}(b)\big)_\infty \\ & \qquad \cdot \prod_{v \neq \infty} \big(z_v \pi_v^{\mathrm{ord}_v(\delta^{-1}\mathfrak{m})}, \mathrm{Nr}(b)\big)_v \Bigg]. \end{aligned}$$

Here $\delta_f = \delta\infty^{-\mathrm{ord}_\infty(\delta)} \in \mathrm{Div}_f(k)$. In particular, $\widetilde{\Theta}_i^*(z,\mathfrak{m}) = 0$ unless $\Omega^2 \mid \mathfrak{m}$.

Proof. It is clear that $S_i(z,\mathfrak{m})$ is empty unless $\Omega^2 \mid \mathfrak{m}$. By definition of $\widetilde{\Theta}_i$ one has

$$\begin{aligned}
&\|\mathfrak{m}\|^{3/4}\cdot\widetilde{\Theta}_i\left(\begin{pmatrix} zs(\delta^{-1}\mathfrak{m}) & zu\\ 0 & z\end{pmatrix},1\right)\\
=\;&\sum_{(x,\alpha)\in V_3(k)\times k^\times}\Bigg[\psi\big(\operatorname{Nr}(x)\alpha u\big)\\
&\qquad\cdot\left(\frac{\gamma_{\psi_\infty}(z_\infty\alpha Q_3)}{\gamma_{\psi_\infty}(Q_3)}\cdot(z_\infty\alpha,-1)_\infty\right.\\
&\qquad\left.\cdot\prod_{v\neq\infty}\frac{\gamma_{\psi_v}(z_v\pi_v^{\operatorname{ord}_v(\delta^{-1}\mathfrak{m})}\alpha Q_3)}{\gamma_{\psi_v}(\alpha Q_3)}\cdot(z_v,-\pi_v^{\operatorname{ord}_v(\delta^{-1}\mathfrak{m})})_v\right)\\
&\qquad\cdot\left(\mathbf{1}_{\beta_i^{-1}\widehat{R}_i^{(p)}}(z_f s(\Omega_f^{-2}\mathfrak{m}_f)x)\cdot\mathbf{1}_{O_{\mathbb{A}^\infty}^\times}\big(\alpha z_f^{-2}\beta_i^{-2}s(\Omega_f^2\mathfrak{m}_f^{-1})\big)\right.\\
&\qquad\left.\cdot\mathbf{1}_{O_\infty}\big(\operatorname{Nr}(x)\alpha\big)\right)\Bigg].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|\mathfrak{m}\|^{3/4}\cdot\widetilde{\Theta}_i^*(z,\mathfrak{m})\\
=\;&\sum_{b\in S_i(z,\mathfrak{m})}\left(\frac{\gamma_{\psi_\infty}(z_\infty\alpha Q_3)}{\gamma_{\psi_\infty}(Q_3)}\cdot(z_\infty\alpha,-1)_\infty\right.\\
&\left.\cdot\prod_{v\neq\infty}\frac{\gamma_{\psi_v}(z_v\pi_v^{\operatorname{ord}_v(\delta^{-1}\mathfrak{m})}\operatorname{Nr}(b)Q_3)}{\gamma_{\psi_v}(\operatorname{Nr}(b)Q_3)}\cdot(z_v,-\pi_v^{\operatorname{ord}_v(\delta^{-1}\mathfrak{m})})_v\right).
\end{aligned}$$

Note that for each place v of k and $a_v\in k_v^\times$,

$$\frac{\gamma_{\psi_v}(z_v a_v\operatorname{Nr}(b)Q_3)}{\gamma_{\psi_v}(\operatorname{Nr}(b)Q_3)}\cdot(z_v,-a_v)_v=\frac{W_{\psi_v,3}(a_v)}{W_{\psi_v,3}(z_v)}\cdot\big(z_v a_v,\operatorname{Nr}(b)\big)_v$$

and

$$\begin{aligned}
&\frac{\gamma_{\psi_\infty}(z_\infty\operatorname{Nr}(b)Q_3)}{\gamma_{\psi_\infty}(Q_3)}\cdot(z_\infty\operatorname{Nr}(b),-1)_\infty\\
=\;&W_{\psi_\infty,3}(z_\infty)^{-1}W_{\psi_\infty,3}\big(\operatorname{Nr}(b)\big)^{-1}\big(z_\infty,\operatorname{Nr}(b)\big)_\infty.
\end{aligned}$$

This completes the proof. □

Similarly, for each divisor $\mathfrak{a} \in \mathrm{Div}_f(k)$, and $z \in \mathbb{A}^\times$, the Fourier coefficient $\widetilde{\Theta}_{i,0}(z,\mathfrak{a})$ is understood by the following result:

COROLLARY IV.13.

$$\begin{aligned}\widetilde{\Theta}^*_{i,0}(z,\mathfrak{a}) &= \frac{W_{\psi,3}\big(s(\delta_f^{-1}\mathfrak{a})\big)}{W_{\psi,3}(z)\|\mathfrak{a}\|^{3/4}} \\ &\quad \cdot \sum_{\substack{\alpha\in k^\times,\\ \alpha z_f^{-2}\beta_i^{-2}s(\Omega^2\mathfrak{a}^{-1})\in O_{\mathbb{A}^\infty}^\times}} \Bigg[W_{\psi_\infty,3}(\alpha)^{-1}(z_\infty,\alpha)_\infty \\ &\qquad\qquad \cdot \prod_{v\neq\infty}\Big((z_v\pi_v^{\mathrm{ord}_v(\delta^{-1}\mathfrak{a})},\alpha)_v\Big)\Bigg].\end{aligned}$$

We emphasize that the Fourier coefficients $\widetilde{\Theta}^*_i(z,\mathfrak{m})$ and $\widetilde{\Theta}^*_{i,0}(z,\mathfrak{a})$ determine uniquely the metaplectic form $\widetilde{\Theta}_i$. From the information of the Fourier coefficients in Lemma IV.12 and Corollary IV.13, we connect the action of Hecke operators on these theta series with Brandt matrices in the next section.

2. Hecke operators and Shintani-type correspondence

Let v be a place of k where $\mathrm{ord}_v(\Omega^2\mathfrak{N}\infty) = 0$. Recall that we embed $\mathrm{GL}_2(O_v)$ into $\widetilde{\mathrm{GL}_2(k_v)}$ by sending any element $\kappa_v \in \mathrm{GL}_2(O_v)$ to $\tilde{\kappa}_v = (\kappa_v, 1)$. Denote by $\widetilde{\mathrm{GL}_2(O_v)}$ the image of this embedding. Then it is observed that

LEMMA IV.14. *For any place v of k with* $\mathrm{ord}_v(\Omega^2\mathfrak{N}\infty) = 0$,

$$\begin{aligned}&\widetilde{\mathrm{GL}_2(O_v)}\left(\begin{pmatrix}\pi_v^2 & 0\\ 0 & 1\end{pmatrix},1\right)\widetilde{\mathrm{GL}_2(O_v)} \\ =\ & \left[\dot{\bigcup_{u \bmod \pi_v^2 O_v}} a_u\ \widetilde{\mathrm{GL}_2(O_v)}\right] \coprod \left[\dot{\bigcup_{h\in\mathbb{F}_v^\times}} b_h\ \widetilde{\mathrm{GL}_2(O_v)}\right] \\ & \coprod c\ \widetilde{\mathrm{GL}_2(O_v)},\end{aligned}$$

where

$$a_u := \left(\begin{pmatrix}\pi_v^2 & u\\ 0 & 1\end{pmatrix},1\right), \quad b_h := \left(\begin{pmatrix}\pi_v & h\\ 0 & \pi_v\end{pmatrix},(\pi_v,h)_v\right),$$

$$c := \left(\begin{pmatrix} 1 & 0 \\ 0 & \pi_v^2 \end{pmatrix}, 1 \right).$$

Let $\mathcal{M}_0^{(r/2)}(\Omega^2\mathfrak{N})$ be the space of weight-$r/2$ metaplectic forms F on $\widetilde{\mathrm{GL}_2(\mathbb{A})}$ satisfying that

$$F(\tilde{g}\tilde{\kappa}) = F(\tilde{g}), \quad \forall \kappa \in \mathcal{K}_0(\Omega^2\mathfrak{N}\infty^+).$$

Definition IV.15. Let v be a place of k where $\mathrm{ord}_v(\Omega^2\mathfrak{N}\infty) = 0$. The Hecke operator $T_{v^2,3/2}$ on $\mathcal{M}_0^{(3/2)}(\Omega^2\mathfrak{N})$ is defined by

$$T_{v^2,3/2}\ F(\tilde{g}) := q_v^{3/2-2} \left[\sum_{u \bmod \pi_v^2 O_v} F(\tilde{g}a_u) + \sum_{h \in \mathbb{F}_v^\times} F(\tilde{g}b_h) + F(\tilde{g}c) \right]$$

for every $F \in \mathcal{M}_0^{(3/2)}(\Omega^2\mathfrak{N})$.

From the pure-algebraic result on pure quaternions in the next section (Theorem IV.22), we obtain that

Theorem IV.16. *For each place v_0 of k with* $\mathrm{ord}_{v_0}(\Omega^2\mathfrak{N}\infty) = 0$,

$$T_{v_0^2,3/2}\ \widetilde{\Theta}_i = \sum_{1 \leq j \leq n} B_{ij}(v_0)\widetilde{\Theta}_j,$$

where $B_{ij}(v_0)$ is the (i,j)-entry of the v_0-th Brandt matrix introduced in Chapter II Section 3.

Proof. It suffices to show that for $z \in \mathbb{A}^\times$, $\mathfrak{m} \in \mathrm{Div}(k)$, and $u \in \mathbb{A}$,

$$\begin{aligned} & T_{v_0^2,3/2}\ \widetilde{\Theta}_i \left(\begin{pmatrix} zs(\delta^{-1}\mathfrak{m}) & zu \\ 0 & z \end{pmatrix}, 1 \right) \\ = & \sum_{1 \leq j \leq n} B_{ij}(v_0) \cdot \widetilde{\Theta}_j \left(\begin{pmatrix} zs(\delta^{-1}\mathfrak{m}) & zu \\ 0 & z \end{pmatrix}, 1 \right). \end{aligned}$$

By definition of $T_{v^2,3/2}$, we get

$$\begin{aligned}
& T_{v_0^2,3/2}\ \widetilde{\Theta}_i\left(\begin{pmatrix} zs(\delta^{-1}\mathfrak{m}) & zu \\ 0 & z\end{pmatrix},1\right) \\
= \quad & q_{v_0}^{-1/2}\cdot\Bigg[\sum_{u' \bmod \pi_{v_0}^2 O_{v_0}} \widetilde{\Theta}_i\left(\begin{pmatrix} zs(\delta^{-1}\mathfrak{m}v_0^2) & z(u+\pi_{v_0}^{\mathrm{ord}_{v_0}(\delta^{-1}\mathfrak{m})}u') \\ 0 & z\end{pmatrix},1\right) \\
& \quad + \sum_{h\in\mathbb{F}_{v_0}^\times}(\pi_{v_0}, z_{v_0}\pi_{v_0}^{\mathrm{ord}_{v_0}(\delta^{-1}\mathfrak{m})}h)_{v_0} \\
& \qquad \cdot\widetilde{\Theta}_i\left(\begin{pmatrix} z\pi_{v_0}s(\delta^{-1}\mathfrak{m}) & z\pi_{v_0}(u+\pi_{v_0}^{\mathrm{ord}_{v_0}(\delta^{-1}m)}\frac{h}{\pi_{v_0}}) \\ 0 & z\pi_{v_0}\end{pmatrix},1\right) \\
& \quad + \widetilde{\Theta}_i\left(\begin{pmatrix} zs(\delta^{-1}\mathfrak{m}) & zu\pi_{v_0}^2 \\ 0 & z\pi_{v_0}^2\end{pmatrix},1\right)\Bigg].
\end{aligned}$$

By Corollary IV.13, it is observed that for $z \in \mathbb{A}^\times$ and $\mathfrak{m} \in \mathrm{Div}_f(k)$,

$$\begin{aligned}
& \left(T_{v_0^2,3/2}\ \widetilde{\Theta}_i\right)^*_0(z,\mathfrak{m}) \\
= \quad & q_{v_0}^{3/2}\widetilde{\Theta}^*_{i,0}(z,\mathfrak{m}v_0^2) + q_{v_0}^{-1/2}\widetilde{\Theta}^*_{i,0}(z\pi_{v_0}^2,\mathfrak{m}v_0^{-2}). \\
= \quad & (1+q_{v_0})\cdot\frac{W_{\psi,3}\big(s(\delta_f^{-1}\mathfrak{m})\big)}{W_{\psi,3}(z)\|\mathfrak{m}\|^{3/4}} \\
& \cdot \sum_{\substack{\alpha\in k^\times,\\ \alpha z_f^{-2}\beta_i^{-2}s(\Omega_f^2\mathfrak{m}^{-1}v_0^{-2})\in O_{\mathbb{A}^\infty}^\times}} \left[W_{\psi_\infty,3}(\alpha)^{-1}(z_\infty,\alpha)_\infty\prod_v\Big((z_v\pi_v^{\mathrm{ord}_v(\delta^{-1}\mathfrak{a})},\alpha)_v\Big)\right] \\
= \quad & \sum_{1\le j\le n} B_{ij}(v_0)\widetilde{\Theta}^*_{j,0}(z,\mathfrak{m}).
\end{aligned}$$

Therefore it remains to show that

$$\left(T_{v_0^2,3/2}\ \widetilde{\Theta}_i\right)^*(z,\mathfrak{m}) = \sum_{1\le j\le n} B_{ij}(v_0)\widetilde{\Theta}^*_j(z,\mathfrak{m})$$

for any $z \in \mathbb{A}^\times$ and $\mathfrak{m} \in \mathrm{Div}_{f,\ge 0}(k)$ with $\Omega^2 \mid \mathfrak{m}$.

Fix $z \in \mathbb{A}^\times$ and $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$. Then $\left(T_{v_0^2,3/2}\,\widetilde{\Theta}_i\right)^*(z,\mathfrak{m})$ is equal to

$$\begin{aligned} & q_{v_0}^{3/2}\widetilde{\Theta}_i^*(z,\mathfrak{m}v_0^2) \\ + & \Bigg[\left(q_{v_0}^{-1/2}\sum_{h\in\mathbb{F}_{v_0}^\times}(\pi_{v_0},\pi_{v_0}^{\mathrm{ord}_{v_0}(\delta^{-1}\mathfrak{m})}\frac{h}{\pi_{v_0}})_{v_0}\psi_{v_0}(\pi_{v_0}^{\mathrm{ord}_{v_0}(\delta^{-1}\mathfrak{m})}\frac{h}{\pi_{v_0}})\right) \\ & \cdot\Big((\pi_{v_0},z_{v_0}\pi_{v_0})_{v_0}\widetilde{\Theta}_i^*(z\pi_{v_0},\mathfrak{m})\Big)\Bigg] \\ + & q_{v_0}\cdot\left(q_{v_0}^{-3/2}\widetilde{\Theta}_i^*(z\pi_{v_0}^2,\mathfrak{m}v_0^{-2})\right). \end{aligned}$$

Since $\mathrm{ord}_{v_0}(\delta)$ is even, it is observed that

$$\begin{aligned} & q_{v_0}^{-1/2}\sum_{h\in\mathbb{F}_{v_0}^\times}(\pi_{v_0},\pi_{v_0}^{\mathrm{ord}_{v_0}(\delta^{-1}\mathfrak{m})}\frac{h}{\pi_{v_0}})_{v_0}\psi_{v_0}(\pi_{v_0}^{\mathrm{ord}_{v_0}(\delta^{-1}\mathfrak{m})}\frac{h}{\pi_{v_0}}) \\ = & \begin{cases} 0 & \text{if } \mathrm{ord}_{v_0}(\mathfrak{m})>0, \\ (\pi_{v_0},-1)_{v_0}W_{\psi_{v_0},1}(\pi_{v_0}) & \text{if } \mathrm{ord}_{v_0}(\mathfrak{m})=0. \end{cases} \end{aligned}$$

By Lemma IV.12, $\left(T_{v_0^2,3/2}\,\widetilde{\Theta}_i\right)^*(z,\mathfrak{m})$ is equal to

$$\begin{aligned} & \frac{W_{\psi,3}\big(s(\delta_f^{-1}\mathfrak{m})\big)}{W_{\psi,3}(z)\|\mathfrak{m}\|^{3/4}} \\ & \cdot\Bigg[\sum_{b\in S_i(z,\mathfrak{m}v_0^2)}\left(W_{\psi_\infty,3}\big(\mathrm{Nr}(b)\big)^{-1}\big(z_\infty,\mathrm{Nr}(b)\big)_\infty\prod_{v\neq\infty}\big(z_v\pi_v^{\mathrm{ord}_v(\delta^{-1}\mathfrak{m})},\mathrm{Nr}(b)\big)_v\right) \\ & +\xi_{v_0}(\mathfrak{m})\sum_{b\in S_i(z\pi_{v_0},\mathfrak{m})}\big(\pi_{v_0},-\mathrm{Nr}(b)\big)_{v_0}\cdot\Bigg(W_{\psi_\infty,3}\big(\mathrm{Nr}(b)\big)^{-1}\big(z_\infty,\mathrm{Nr}(b)\big)_\infty \\ & \qquad\cdot\prod_{v\neq\infty}\big(z_v\pi_v^{\mathrm{ord}_v(\delta^{-1}\mathfrak{m})},\mathrm{Nr}(b)\big)_v\Bigg) \\ & +q_{v_0}\cdot\sum_{b\in S_i(z\pi_{v_0}^2,\mathfrak{m}v_0^{-2})}\Bigg(W_{\psi_\infty,3}\big(\mathrm{Nr}(b)\big)^{-1}\big(z_\infty,\mathrm{Nr}(b)\big)_\infty \\ & \qquad\cdot\prod_{v\neq\infty}\big(z_v\pi_v^{\mathrm{ord}_v(\delta^{-1}\mathfrak{m})},\mathrm{Nr}(b)\big)_v\Bigg)\Bigg]. \end{aligned}$$

Here

$$\xi_{v_0}(\mathfrak{m}) := \begin{cases} 1 & \text{if } \mathrm{ord}_{v_0}(\mathfrak{m}) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mu : k^\times \to \mathbb{C}^1$ by

$$\mu(\alpha) := W_{\psi_\infty,3}\big(\alpha\big)^{-1}\big(z_\infty, \alpha\big)_\infty \prod_{v \neq \infty} \big(z_v \pi_v^{\mathrm{ord}_v(\delta^{-1}\mathfrak{m})}, \alpha\big)_v.$$

Then

$$\mu(\beta^2\alpha) = \mu(\alpha), \quad \forall \alpha, \beta \in k^\times.$$

Therefore by Theorem IV.22 in Section 3, we get that for any $z \in \mathbb{A}^\times$ and $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$,

$$\Big(T_{v^2,3/2}\ \widetilde{\Theta}_i\Big)^*(z, \mathfrak{m}) = \sum_{1 \leq j \leq n} B_{ij}(v)\widetilde{\Theta}_j^*(z, \mathfrak{m}).$$

This completes the proof. □

Recall that $X = X_{\mathfrak{N}^+,\mathfrak{N}^-}$ denotes the definite Shimura curve of type $(\mathfrak{N}^+, \mathfrak{N}^-)$, and $\mathrm{Pic}(X)_\mathbb{C} \cong \mathrm{Pic}(X)_\mathbb{C}^\vee$ is generated by $e_1, ..., e_n$ where e_i corresponds to the ideal I_i canonically. We introduce the following map:

$$\begin{array}{rccc} \Psi: & \mathrm{Pic}(X)_\mathbb{C} & \longrightarrow & \mathcal{M}_0^{(3/2)}(\Omega^2\mathfrak{N}) \\ & \sum\limits_{1\leq i\leq n} a_i e_i & \longmapsto & \sum\limits_{1\leq i\leq n} a_i \cdot \widetilde{\Theta}_{\sigma(i)}. \end{array}$$

Here σ is the order-2 permutation on $\{1, ..., n\}$ introduced in Chapter II Section 2. Then Proposition II.4, II.7 and Theorem IV.16 imply immediately that

Proposition IV.17. *For each place v of k with $\mathrm{ord}_v(\Omega^2\mathfrak{N}\infty) = 0$,*

$$\Psi(t_v^* e) = T_{v^2,3/2}\ \Psi(e), \quad \forall e \in \mathrm{Pic}(X)_\mathbb{C}.$$

Recall that the Jacquet-Langlands correspondence JL introduced in Theorem III.13 identifies the space $S_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N})$ with $\mathrm{Pic}_0(X)_\mathbb{C}^\vee$. More precisely, for each finite place v of k,

$$\mathrm{JL}^{-1}(T_v F) = t_v^* \mathrm{JL}^{-1}(F), \quad \forall F \in S_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N}).$$

Therefore

Theorem IV.18. *The linear map*

$$\mathbf{Sh} := \Psi \circ \mathrm{JL}^{-1} : S_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N}) \longrightarrow \mathcal{M}_0^{(3/2)}(\Omega^2\mathfrak{N})$$

satisfies that for each place v of k with $\mathrm{ord}_v(\Omega^2\mathfrak{N}\infty) = 0$,

$$\mathbf{Sh}(T_v F) = T_{v^2,3/2}\ \mathbf{Sh}(F), \quad \forall F \in S_0^{(\mathfrak{N}^-)-\mathrm{new}}(\mathfrak{N}).$$

3. Pure quaternions and Brandt matrices

In this section, we focus on pure quaternions and work out a purely algebraic result (Theorem IV.22). This is the key ingredient for connecting the Brandt matrices with the Hecke operators on metaplectic forms in Theorem IV.16.

Let $D^{(p)}$ be the subspace of pure quaternions in D, i.e.

$$D^{(p)} := \{b \in D : \ \mathrm{Tr}(b) = 0\}.$$

Recall that $I_1, ..., I_n$ are chosen representatives of locally-principal right ideal classes of the Eichler A-order R of type $(\mathfrak{N}^+, \mathfrak{N}^-)$, and R_i is the right order of I_i. Let M_1 be a fractional ideal of A. For any integral ideal M of A and $1 \le i, j \le n$, define

$$S_i(M_1, M) := \{b \in D^{(p)} \cap M_1 \,\mathrm{Nr}(I_i) R_i : \ \mathrm{Nr}(b)A = M_1^2 \,\mathrm{Nr}(I_i)^2 M\},$$

and set

$$S_{ij}(M) := \{\alpha \in I_i I_j^{-1} : \mathrm{Nr}(\alpha)\,\mathrm{Nr}(I_j) = \mathrm{Nr}(I_i) M\}.$$

It is clear that

$$\#(S_{ij}(M)) = \#(R_i^\times) \cdot B_{ij}(\mathfrak{m}_M),$$

where $\mathfrak{m}_M \in \mathrm{Div}_{f,\ge 0}(k)$ is the divisor corresponding to the ideal M. Let v_0 be a finite place of k such that $\mathrm{ord}_{v_0}(\mathfrak{N}^+\mathfrak{N}^-) = 0$. We have the following canonical map:

$$\begin{array}{ccccc} \coprod_{j=1}^{n} \Big(S_{ij}(P_0) & \times & S_j(M_1, M) \Big) & \longrightarrow & S_i(M_1, P_0^2 M) \\ (\alpha & , & b) & \longmapsto & \alpha b \bar{\alpha}. \end{array}$$

Here P_0 is the prime ideal of A corresponding to the finite place v_0.

Take an element $b \in S_i(M_1, P_0^2 M) - S_i(M_1 P_0, M)$. Let J be the unique locally-principal left ideal of R_i satisfying that

$$R_i b \subset J \subset M_1 \operatorname{Nr}(I_i) R_i \quad \text{and} \quad \operatorname{Nr}(b) \cdot A = P_0 \operatorname{Nr}(J).$$

There exist unique j and $\alpha \in S_{ij}(P_0)$, up to the right multiplication by elements in $R_j^\times$, such that $J = I_i I_j^{-1}(\alpha^{-1} b)$. Let $\beta = \alpha^{-1} b$. Since

$$\operatorname{Tr}(b) = b + \bar{b} = 0,$$

we get $b = -\bar{b} = -\bar{\beta}\bar{\alpha}$. Let $J' := M_1 \operatorname{Nr}(I_i) \bar{I}_i^{-1} \bar{I}_j \bar{\alpha}$. Then we also have

$$R_i b \subset J' \subset M_1 \operatorname{Nr}(I_i) R_i \quad \text{and} \quad \operatorname{Nr}(J') = P_0 M_1^2 \operatorname{Nr}(I_i)^2.$$

Since $b \in S_i(M_1, P_0^2 M) - S_i(M_1 P_0, M)$, we get a chain of left ideals of R_i:

$$R_i b = R_i \bar{b} \subset J \subset J' \subset M_1 \operatorname{Nr}(I_i) R_i.$$

Let $b' := \beta \bar{\alpha}^{-1}$. Then $b = \alpha b' \bar{\alpha}$ and $J \subset J'$ implies that $b' \in S_j(M_1, M)$. we conclude that

Lemma IV.19. *For any $b \in S_i(M_1, P_0^2 M) - S_i(M_1 P_0, M)$, there exist unique j and $\alpha \in S_{ij}(P_0)$, up to the right multiplication by elements in $R_j^\times$, such that*

$$b' = \alpha^{-1} b \bar{\alpha}^{-1} \in S_j(M_1, M).$$

Now, take an element b in $S_i(M_1 P_0, M)$ and consider the following two cases:

(1) Suppose $P_0 \mid M$. Then the left ideal $P_0^{-1} R_i b$ of R_i is contained in $M_1 \operatorname{Nr}(I_i) R_i$. If $P_0^{-2} R_i b \not\subset M_1 \operatorname{Nr}(I_i) R_i$, there exists a unique locally-principal left ideal J of R_i contained in $M_1 \operatorname{Nr}(I_i) R_i$ with

$$P_0^{-1} R_i b \subset J \subset M_1 \operatorname{Nr}(I_i) R_i$$

and

$$\operatorname{Nr}(b) \cdot A = P_0^3 \operatorname{Nr}(J).$$

There exist unique j and $\alpha \in S_{ij}(P_0)$, up to the right multiplication by elements in $R_j^\times$, such that

$$P_0 J = I_i I_j^{-1}(\alpha^{-1} b).$$

Therefore we have the following chain of left ideals of R_i:

$$R_i b \subset P_0 J = I_i I_j^{-1}(\alpha^{-1} b) \subset P_0 M_1 \operatorname{Nr}(I_i) R_i \subset M_1 \operatorname{Nr}(I_i) \bar{I}_i^{-1} \bar{I}_j \bar{\alpha} \subset M_1 \operatorname{Nr}(I_i) R_i.$$

This tells us that

$$b' := \alpha^{-1} b \bar{\alpha}^{-1} \in S_j(M_1, M) \quad \text{and} \quad b = \alpha b' \bar{\alpha}.$$

If $P_0^{-2} R_i b \subset M_1 \operatorname{Nr}(I_i) R_i$, we have the ideal chain

$$R_i b \subset P_0^{-1} R_i b \subset P_0 M_1 \operatorname{Nr}(I_i) R_i \subset M_1 \operatorname{Nr}(I_i) R_i.$$

Thus for every j and $\alpha \in S_{ij}(P_0)$,

$$R_i b \subset I_i I_j^{-1}(\alpha^{-1} b) \subset P_0^{-1} R_i b$$

and

$$P_0 M_1 \operatorname{Nr}(I_i) R_i \subset M_1 \operatorname{Nr}(I_i) \bar{I}_i^{-1} \bar{I}_j \bar{\alpha} \subset M_1 \operatorname{Nr}(I_i) R_i,$$

which implies that

$$\alpha^{-1} b \bar{\alpha}^{-1} \in S_j(M_1, M).$$

We conclude that

Lemma IV.20. *Suppose $P_0 \mid M$. Let $b \in S_i(M_1 P_0, M) - S_i(M_1 P_0^2, P_0^{-2} M)$. Then there exist unique j and $\alpha \in S_{ij}(P_0)$, up to the right multiplication by elements in $R_j^\times$, such that*

$$b' = \alpha^{-1} b \bar{\alpha}^{-1} \in S_j(M_1, M).$$

Moreover, if $b \in S_i(M_1 P_0^2, P_0^{-2} M)$, we have that for every j and $\alpha \in S_{ij}(P_0)$,

$$b' = \alpha^{-1} b \bar{\alpha}^{-1} \in S_j(M_1, M).$$

(2) Suppose $P_0 \nmid M$. Then $S_i(M_1 P_0^2, P_0^{-2} M)$ is empty. Take b in $S_i(M_1 P_0, M)$. Let $\mathcal{O}_b$ be the quadratic A-order of $k(b)$ generated by elements in

$$(P_0 M_1 \operatorname{Nr}(I_i))^{-1} b := \{ab : \ a \in (P_0 M_1 \operatorname{Nr}(I_i))^{-1}\} \quad \big(\subset R_i \cap k(b)\big).$$

Suppose that b can be written as $\alpha b' \bar{\alpha}$ where $\alpha \in S_{ij}(P_0)$ for some j and $b' \in S_j(M_1, M)$. Set

$$J := \bar{I}_i^{-1} \bar{I}_j \bar{\alpha} \subset R_i,$$

the left ideal of R_i with $\mathrm{Nr}(J) = P_0$. Then $Ja \subset J$ for any $a \in \mathcal{O}_b$. Therefore we get

$$J = R_i \mathcal{P}_0$$

where $\mathcal{P}_0 = J \cap \mathcal{O}_b$ is a prime ideal of $\mathcal{O}_b$ lying above P_0.

On the other hand, for every prime ideal $\mathcal{P}'$ of $\mathcal{O}_b$ lying above P_0, there exist unique j' and $\alpha' \in S_{ij'}(P_0)$, up to the right multiplication by elements in $R_{j'}^{\times}$, such that

$$\bar{I}_i^{-1} \bar{I}_{j'} \bar{\alpha}' = R_i \mathcal{P}'.$$

It is observed that $\alpha'^{-1} b \bar{\alpha}'^{-1} \in S_{j'}(M_1, M)$ and $R_i\mathcal{P}' \neq R_i\mathcal{P}''$ if $\mathcal{P}' \neq \mathcal{P}''$. We emphasize that the number of prime ideals of $\mathcal{O}_b$ lying above P_0 is

$$1 + \big(\pi_{v_0}, -\mathrm{Nr}(b)\big)_{v_0}.$$

Therefore we conclude that

Lemma IV.21. *Suppose $P_0 \nmid M$. Let $b \in S_i(M_1 P_0, M)$. Then exist exactly $1 + \big(\pi_{v_0}, -\mathrm{Nr}(b)\big)_{v_0}$ choices of the pair (j, α) with $\alpha \in S_{ij}(P_0)$, up to the multiplication by elements in $R_j^{\times}$, such that*

$$b' = \alpha^{-1} b \bar{\alpha}^{-1} \in S_i(M_1, M).$$

Let $\mu : k^{\times} \longrightarrow \mathbb{C}^1$ be a function satisfying that

$$\mu(\beta^2 \alpha) = \alpha, \quad \forall \alpha, \beta \in k^{\times}.$$

Then from Lemma IV.19 $\sim$ IV.21, we arrive at

Theorem IV.22. *Let M be an integral ideal of A and M_1 be a fractional ideal of A. Then for each finite place v_0 of k with $\mathrm{ord}_{v_0}(\mathfrak{N}^+\mathfrak{N}^-) = 0$, we* *have*

$$
\begin{aligned}
&\sum_{j=1}^{n} B_{ij}(v_0) \sum_{b\in S_j(M_1,M)} \mu(\mathrm{Nr}(b)) \\
= \quad & \sum_{b\in S_i(M_1,P_0^2M)} \mu(\mathrm{Nr}(b)) \\
&+\xi_{v_0}(M) \sum_{b\in S_i(M_1P_0,M)} \big(\pi_{v_0}, -\mathrm{Nr}(b)\big)_{v_0} \mu(\mathrm{Nr}(b)) \\
&+q_{v_0} \sum_{b\in S_i(M_1P_0^2,P_0^{-2}M)} \mu(\mathrm{Nr}(b)).
\end{aligned}
$$

Here $\xi_{v_0}(M) := 1$ *if* $P_0 \nmid M$ *and* $\xi_{v_0}(M) := 0$ *otherwise.*

CHAPTER V

Trace formula of Brandt matrices

Following the notations in Chapter II, we fix a pair (k, ∞), where k is a global function field with constant field $\mathbb{F}_q$ and ∞ is a place of k. The ring of functions in k regular outside ∞ is denoted by A. Let D be the definite (with respect to ∞) quaternion algebra over k in Chapter II Section 2. A family of Brandt matrices was introduced in Chapter II Section 3, to encode information from the arithmetic of D. Adapting Eichler's method from the $\mathbb{Q}$ case, we establish here a fine formula expressing the trace of these matrices in terms of class numbers of specific A-orders inside "imaginary" (with respect to ∞) quadratic field extension of k embeddable into D. The proof is based on a detailed study of the so-called *optimal embeddings* from quadratic A-orders into D.

1. Optimal embeddings

Take $\mathfrak{N}_0^- \in \mathrm{Div}_{f,\geq 0}(k)$ be the product of finite places of k where D is ramified. Let R be an Eichler A-order in D of type $(\mathfrak{N}_0^+, \mathfrak{N}_0^-)$ where $\mathfrak{N}_0^+ \in \mathrm{Div}_{f,\geq 0}(k)$ is square-free. Let K be a quadratic extension of k. There exists an embedding ι from K into D if and only if v does not split in K for each place $v \mid \mathfrak{N}_0^- \infty$. Let c be a non-zero ideal of A, and $\mathcal{O}_c$ denotes the quadratic A-order in K with conductor c, i.e.

$$\mathcal{O}_c := A + c\mathcal{O}_K$$

where $\mathcal{O}_K$ is the integral closure of A in K. An embedding $\iota : K \hookrightarrow D$ is called an *optimal embedding* from $\mathcal{O}_c$ into R if

$$\iota(K) \cap R = \iota(\mathcal{O}_c).$$

Fix an embedding $\iota : K \hookrightarrow D$. Let c be a non-zero ideal of A. By Noether-Skolem theorem, the set of optimal embeddings from $\mathcal{O}_c$ into R can be identified with

$$\iota(K^\times)\backslash\{g \in D^\times : g^{-1}\iota(K)g \cap R = g^{-1}\iota(\mathcal{O}_c)g\}.$$

Choose representatives I_1, ..., I_n of locally-principal right ideal classes of R, and R_i denotes the left order of I_i. For $1 \leq i \leq n$, let

$$\mathcal{E}_i(\iota, c, R) := \{g \in D^\times : g^{-1}\iota(K)g \cap R_i = g^{-1}\iota(\mathcal{O}_c)g\}.$$

Then R_i acts on $\mathcal{E}_i(\iota, c, R)$ by right multiplication, and $\iota(K^\times)\backslash\mathcal{E}_i(\iota, c, R)/R_i^\times$ can be identified with the set of the optimal embeddings from $\mathcal{O}_c$ into R_i modulo the conjugation by $R_i^\times$.

Set

$$\widehat{\mathcal{E}}(\iota, c, R) := \{\hat{g} \in D_{\mathbb{A}^\infty}^\times : \iota(K_{\mathbb{A}^\infty}) \cap \hat{g}\widehat{R}\hat{g}^{-1} = \iota(\widehat{\mathcal{O}}_c)\},$$

where

$$K_{\mathbb{A}^\infty} := K \otimes_k \mathbb{A}^\infty \quad \text{and} \quad \widehat{\mathcal{O}}_c := \mathcal{O}_c \otimes_A O_{\mathbb{A}^\infty}.$$

We have the following observation.

Lemma V.1. *There is a bijection between*

$$\begin{array}{ccc} \coprod_{i=1}^{n} \iota(K^\times)\backslash\mathcal{E}_i(\iota, c, R)/R_i^\times & \cong & \iota(K^\times)\backslash\widehat{\mathcal{E}}(\iota, c, R)/\widehat{R}^\times \\ [g_i],\ g_i \in \mathcal{E}_i(\iota, c, R) & \mapsto & [g_i \cdot b_i], \end{array}$$

where $b_i \in D_{\mathbb{A}^\infty}^\times$ *such that* $I_i = D \cap b_i\widehat{R}$.

For $1 \leq i \leq n$, let

$$h_i(c) := \#\big(\iota(K^\times)\backslash\mathcal{E}_i(\iota, c, R)/R_i^\times\big),$$

and $h(c)$ denotes the class number of the invertible ideals of $\mathcal{O}_c$, i.e.

$$h(c) = \#\left(K^\times\backslash K_{\mathbb{A}^\infty}^\times/\widehat{\mathcal{O}}_c^\times\right).$$

From the natural surjection

$$\iota(K^\times)\backslash\widehat{\mathcal{E}}(\iota, c, R)/\widehat{R}^\times \twoheadrightarrow \iota(K_{\mathbb{A}^\infty}^\times)\backslash\widehat{\mathcal{E}}(\iota, c, R)/\widehat{R}^\times,$$

one has

LEMMA V.2.

$$\begin{aligned}\sum_{i=1}^{n} h_i(c) &= \#\big(\iota(K^\times)\backslash\widehat{\mathcal{E}}(\iota,c,R)/\widehat{R}^\times\big)\\ &= h(c)\cdot\#\big(\iota(K^\times_{\mathbb{A}^\infty})\backslash\widehat{\mathcal{E}}(\iota,c,R)/\widehat{R}^\times\big).\end{aligned}$$

Note that

$$\iota(K^\times_{\mathbb{A}^\infty})\backslash\widehat{\mathcal{E}}(\iota,c,R)/\widehat{R}^\times \cong \prod_{v\neq\infty}\iota(K_v^\times)\backslash\mathcal{E}_v(\iota,c,R)/R_v^\times$$

where

$$K_v := K\otimes_k k_v, \quad \mathcal{O}_{c,v} = \mathcal{O}_c\otimes_A O_v,$$

and

$$\mathcal{E}_v(\iota,c,R) := \{g_v\in D_v^\times : \iota(K_v)\cap g_vR_vg_v^{-1} = \iota(\mathcal{O}_{c,v})\}.$$

We then obtain the number $\sum_{i=1}^n h_i(c)$ by computing $\#\big(\iota(K_v^\times)\backslash\mathcal{E}_v(\iota,c,R)/R_v^\times\big)$ for each finite place v of k:

PROPOSITION V.3. (1) *When* $v\nmid\mathfrak{N}_0^+\mathfrak{N}_0^-\infty$, *we have*

$$\#\big(\iota(K_v^\times)\backslash\mathcal{E}_v(\iota,c,R)/R_v^\times\big) = 1.$$

(2) *Suppose* $v\mid\mathfrak{N}_0^+$. *Then*

$$\#\big(\iota(K_v^\times)\backslash\mathcal{E}_v(\iota,c,R)/R_v^\times\big) = \begin{cases}2, & \textit{if } v \textit{ is split in } K \textit{ or } \mathrm{ord}_v(c)>0,\\ 1, & \textit{if } v \textit{ is ramified in } K \textit{ and } \mathrm{ord}_v(c)=0,\\ 0, & \textit{if } v \textit{ is inert in } K \textit{ and } \mathrm{ord}_v(c)=0.\end{cases}$$

(3) *Suppose* $v\mid\mathfrak{N}_0^-$. *Then*

$$\#\big(\iota(K_v^\times)\backslash\mathcal{E}_v(\iota,c,R)/R_v^\times\big) = \begin{cases}2, & \textit{if } v \textit{ is inert in } K \textit{ and } \mathrm{ord}_v(c)=0,\\ 1, & \textit{if } v \textit{ is ramified in } K \textit{ and } \mathrm{ord}_v(c)=0,\\ 0, & \textit{otherwise.}\end{cases}$$

PROOF. Suppose first that $v \mid \mathfrak{N}_0^-$. Since R_v is the unique maximal O_v-order in D_v, it is clear that

$$\mathcal{E}_v(\iota, c, R) = \begin{cases} D_v^\times, & \text{if } \operatorname{ord}_v(c) = 0, \\ \text{empty}, & \text{otherwise.} \end{cases}$$

Therefore (3) holds. We complete the proof of (1) and (2) by computing the number of local optimal embeddings in Section 1.1. □

For each finite place v of k, set

$$\left\{\frac{c}{v}\right\} = \begin{cases} 1 & \text{if } v \text{ splits in } K \text{ or } \operatorname{ord}_v(c) > 0, \\ -1 & \text{if } v \text{ is inert in } K \text{ and } \operatorname{ord}_v(c) = 0, \\ 0 & \text{if } v \text{ is ramified in } K \text{ and } \operatorname{ord}_v(c) = 0. \end{cases}$$

Then we arrive at:

COROLLARY V.4.

$$\sum_{i=1}^n h_i(c) = h(c) \prod_{v \mid \mathfrak{N}_0^-} \left(1 - \left\{\frac{c}{v}\right\}\right) \prod_{v \mid \mathfrak{N}_0^+} \left(1 + \left\{\frac{c}{v}\right\}\right).$$

1.1. Local optimal embeddings. Let F be a non-archimedean local field. The valuation ring of F is denoted by $\mathcal{O}_F$, and let $\mathcal{P}_F$ be the maximal ideal of $\mathcal{O}_F$. The valuation map $v_F : F^\times \to \mathbb{Z}$ is normalized so that

$$v_F(\alpha) = 1 \quad \text{for } \alpha \in \mathcal{P}_F - \mathcal{P}_F^2.$$

Let L be a quadratic extension over F (or the F-algebra $F \times F$). Further, allow the integral closure of $\mathcal{O}_F$ in L (or $\mathcal{O}_F \times \mathcal{O}_F$ if $L = F \times F$) to be denoted by $\mathcal{O}_L$. Let $R_0 := \operatorname{Mat}_2(\mathcal{O}_F)$. Fix an F-algebra embedding $\iota : L \to \operatorname{Mat}_2(F)$. For $c \in \mathcal{O}_L$, set $\mathcal{O}_{L,c} := \mathcal{O}_F + c\mathcal{O}_L$ and

$$\mathcal{E}(\iota, c, R_0) := \{g \in \mathrm{GL}_2(F) \mid g^{-1}\iota(L)g \cap R_0 = g^{-1}\iota(\mathcal{O}_{L,c})g\}.$$

Then we have the natural bijection

$$\{\text{optimal embeddings from } \mathcal{O}_{L,c} \text{ into } R_0\}/R_0^\times \cong \iota(L^\times) \setminus \mathcal{E}(\iota, c, R_0)/R_0^\times.$$

Let $V := F^2$ and $\lambda^0 := \mathcal{O}_F^2$, the standard $\mathcal{O}_F$-lattice in V. Given an $\mathcal{O}_F$-lattice λ, define

$$\mathrm{End}(\lambda) := \{g \in \mathrm{Mat}_2(F) \mid g\lambda \subset \lambda\}.$$

Then

$$\mathrm{GL}_2(F)/R_0^\times \cong \{\mathcal{O}_F\text{-lattice of } V\} \quad \text{and} \quad \mathrm{End}(g\lambda^0) = g^{-1}R_0 g.$$

Let λ be an $\mathcal{O}_F$-lattice in V. We call λ an $\mathcal{O}_{L,c}$-lattice if

$$\iota(\mathcal{O}_{L,c}) \cdot \lambda \subset \lambda.$$

LEMMA V.5. *Any $\mathcal{O}_{L,c}$-lattice λ is of the form $\mathcal{O}_{L,c'}x$ for $x \in L$ and $c' \in \mathcal{O}_L$ with $v_F(c) \geq v_F(c')$.*

PROOF. First, we consider the case when L is a quadratic field over F. Then V can be viewed as an L-vector space of dimension one. Identifying V with L, every $\mathcal{O}_{L,c}$-lattice in V can be viewed as a (fractional) $\mathcal{O}_{L,c}$-ideal in L. Let $x \in \lambda$ such that

$$v_L(x) = \min\{v_L(\alpha) \mid \alpha \in \lambda\},$$

where v_L is the normalized valuation map on L. Then it is observed that

$$\mathcal{O}_L x = \mathcal{O}_L \lambda \supset \lambda \supseteq \mathcal{O}_{L,c}x.$$

Consequently, there exists an element $c' \in \mathcal{O}_F$ with $v_F(c) \geq v_F(c')$ such that

$$\lambda = \mathcal{O}_{L,c'}x.$$

Now, we consider the case when $L = F \times F$. Similarly, identifying V with L, λ is viewed as an $\mathcal{O}_{L,c}$-lattice in L. Let $x_1, y_2 \in F^\times$ such that

$$v_F(x_1) = \min\{v_F(x) \mid (x,y) \in \lambda\}$$

and

$$v_F(y_2) = \min\{v_F(y) \mid (x,y) \in \lambda)\}.$$

Then there exists $y_1, x_2 \in F$ such that $(x_1, y_1), (x_2, y_2) \in \lambda$. So we have

$$v_F(y_1) \geq v_F(y_2) \quad \text{and} \quad v_F(x_2) \geq v_F(x_1).$$

Choose a particular element $(x_0, y_0) \in \lambda$ by the following:

1. When $v_F(y_1) = v_F(y_2)$ (or $v_F(x_1) = v_F(x_2)$), take

$$(x_0, y_0) := (x_1, y_1) \text{ (or } (x_2, y_2)) \ \in \lambda.$$

2. If $v_F(y_2) > v_F(y_2)$ and $v_F(x_2) > v_F(x_1)$, take

$$(x_0, y_0) := (x_1 + x_2, y_1 + y_2) \in \lambda.$$

Therefore

$$v_F(x) \geq v_F(x_0),\ v_F(y) \geq v_F(y_0) \quad \forall (x, y) \in \lambda.$$

Using the above choice of (x_0, y_0), we get

$$\mathcal{O}_L(x_0, y_0) = \mathcal{O}_L \lambda \supseteq \lambda \supseteq \mathcal{O}_{L,c}(x_0, y_0).$$

Therefore, there exists $c' \in \mathcal{O}_F$ with $v_F(c) \geq v_F(c')$ such that

$$\lambda = \mathcal{O}_{L,c'}(x_0, y_0).$$

□

Suppose λ is an $\mathcal{O}_{L,c}$-lattice. We call λ *optimal* if $\iota(L) \cap \mathrm{End}(\lambda) = \iota(\mathcal{O}_{L,c})$. Equivalently, there exists an element $x \in \lambda$ such that

$$\lambda = \iota(\mathcal{O}_{L,c}) \cdot x.$$

Two optimal $\mathcal{O}_{L,c}$ lattice λ_1 and λ_2 are isomorphic if there exists an element g in $\iota(L^\times)$ such that $g\lambda_1 = \lambda_2$. For $g \in \mathrm{GL}_2(F)$, it is clear that $g\lambda_0$ is an optimal $\mathcal{O}_{L,c}$ lattice if and only if $g \in \mathcal{E}(\iota, c, R_0)$. Let $\mathrm{LA}(L, c)$ be the set of isomorphism classes of optimal $\mathcal{O}_{L,c}$ lattices. We then have the following bijection

$$\begin{array}{ccc} \iota(L^\times) \backslash \mathcal{E}_\iota(\mathcal{O}_{L,c}, R_0)/R_0^\times & \cong & \mathrm{LA}(L, c) \\ \iota(L^\times) g R_0^\times & \mapsto & [g\lambda_0]. \end{array}$$

Therefore Proposition V.3 (1) follows from the following lemma.

LEMMA V.6. *The cardinality of* $\mathrm{LA}(L, c)$ *is one.*

PROOF. This follows directly from Lemma V.5. □

Now, consider the Iwahori $\mathcal{O}_F$-order

$$R := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathcal{O}_F) \;\middle|\; c \in \mathcal{P}_F \right\}.$$

Similarly, we let

$$\mathcal{E}(\iota, c, R) := \{g \in \mathrm{GL}_2(F) \mid g^{-1}\iota(L)g \cap R = g^{-1}\iota(\mathcal{O}_{L,c})g\}.$$

Then

$$\{\text{optimal embeddings from } \mathcal{O}_{L,c} \text{ into } R\}/R^\times \cong \iota(L^\times) \setminus \mathcal{E}(\iota, c, R)/R^\times.$$

Take two $\mathcal{O}_F$-lattices λ_1, λ_2 in $V = F^2$. We call $\lambda_* := \{\lambda_1, \lambda_2\}$ an *$\mathcal{O}_F$-chain in V* if

$$\lambda_1 \supset \lambda_2 \quad \text{and} \quad \lambda_1/\lambda_2 \cong \mathcal{O}_F/\mathcal{P}_F \text{ as } \mathcal{O}_F\text{-modules}.$$

Let $\lambda_*^0 := \{\lambda_1^0, \lambda_2^0\}$ be the standard $\mathcal{O}_F$-chain, i.e.

$$\lambda_1^0 = \mathcal{O}_F^2 \quad \text{and} \quad \lambda_2^0 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{O}_F \;\middle|\; y \in \mathcal{P}_F \right\}.$$

Then for $g \in \mathrm{Mat}_2(F)$, $g\lambda_*^0 \subset \lambda_*^0$ (i.e. $g\lambda_i^0 \subset \lambda_i^0$ for $i = 1, 2$) if and only if $g \in R$. Moreover, for every $\mathcal{O}_F$-chain λ_*, there exists a unique $g_{\lambda_*} \in \mathrm{GL}_2(F)$, up to the right multiplication by elements in $R^\times$, such that $\lambda_* = g_{\lambda_*}\lambda_*^0$. This says that there is a natural bijection

$$\begin{array}{ccc} \mathrm{GL}_2(F)/R^\times & \cong & \{\mathcal{O}_F\text{-chains in } V\} \\ gR^\times & \mapsto & g\lambda_*^0. \end{array}$$

In particular, for each $\mathcal{O}_F$-chain λ_*, let

$$\mathrm{End}(\lambda_*) := \{g \in \mathrm{GL}_2(F) \mid g\lambda_* \subset \lambda_*\}.$$

Then for every $g \in \mathrm{GL}_2(F)$ one has

$$\mathrm{End}(g\lambda_*^0) = gRg^{-1}.$$

A given $\mathcal{O}_F$-chain λ_* of $\mathcal{O}_{L,c}$-lattices is called *an optimal $\mathcal{O}_{L,c}$-chain* if

$$\iota(L) \cap \mathrm{End}(\lambda_*) = \iota(\mathcal{O}_{L,c}).$$

Two optimal $\mathcal{O}_{L,c}$-chains λ_*^1 and λ_*^2 are isomorphic if there exists an element g in $\iota(L^\times)$ such that $g\lambda_*^1 = \lambda_*^2$. Let $\mathrm{CH}(L,c)$ be the set of isomorphism classes of optimal $\mathcal{O}_{L,c}$-chains. We have the following bijection

$$\begin{array}{ccc} \iota(L^\times)\backslash\mathcal{E}_\iota(\mathcal{O}_{L,c},R)/R^\times & \cong & \mathrm{CH}(L,c) \\ \iota(L^\times)gR^\times & \mapsto & [g\lambda_*^0]. \end{array}$$

Thus the computation of the cardinality of $\mathrm{CH}(L,c)$ in Lemma V.7∼V.9 ensures Proposition V.3 (2).

Lemma V.7. *Suppose L is a quadratic field over F. Then for $c \in \mathcal{O}_F^\times$,*

$$\#\big(\mathrm{CH}(L,c)\big) = \begin{cases} 0, & \text{if } L/F \text{ is umramified;} \\ 1, & \text{if } L/F \text{ is ramified.} \end{cases}$$

Proof. Notice that $\mathcal{O}_{L,c} = \mathcal{O}_L$ for every $c \in \mathcal{O}_F^\times$. Identifying V with L (as L-vector space), every $\mathcal{O}_L$-lattice can be viewed as a (fractional) $\mathcal{O}_L$-ideal in L. Therefore there is no optimal $\mathcal{O}_L$-chain if L/F is unramified; when L/F is ramified, every optimal $\mathcal{O}_L$-chain is of the form

$$\{\mathcal{O}_L x, \mathcal{P}_L x\}, \quad x \in L.$$

This completes the proof. □

Lemma V.8. *Suppose L is a quadratic field over F. For every element c in $\mathcal{P}_F$ we have*

$$\#\big(\mathrm{CH}(L,c)\big) = 2.$$

Proof. Let $\lambda_* = \{\lambda_1, \lambda_2\}$ be an $\mathcal{O}_{L,c}$-chain (not necessary optimal) in V. Identifying V with L (as L-vector space), take $x_i \in \lambda_i$ for $i = 1,2$ such that

$$v_L(x_i) = \min\{v(\alpha) | \alpha \in \lambda_i\}.$$

We have $v_L(x_1) \le v_L(x_2)$, and by Lemma V.5 there exist $c_1, c_2 \in \mathcal{O}_L$ with $v_F(c) \ge v_F(c_1), v_F(c_2)$ such that $\lambda_i = \mathcal{O}_{L,c_i} x_i$. We separate into 2 cases.

i. Suppose $v_L(x_1) = v_L(x_2)$. Then $\lambda_1 = \mathcal{O}_{L,c_1} x_2$ and $\lambda_2 = \mathcal{O}_{L,c_2} x_2$. Since $\lambda_1/\lambda_2 \cong \mathcal{O}_F/\mathcal{P}_F$, we must have $c_2 = c_1 \pi_F$ where π_F is a uniformizer of $\mathcal{P}_F$.

ii. Suppose $v_L(x_1) < v_L(x_2)$. From $\lambda_1/\lambda_2 \cong \mathcal{O}_F/\mathcal{P}_F$, we get

$$v_L(x_1) < v_L(x_2) \leq v_L(x_1) + v_L(\pi_F).$$

If $v_F(x_2) = v_F(\pi_F x_1)$, then $\lambda_2 = \mathcal{O}_{L,c_2}\pi_F x_1$ and $c_1 = c_2\pi_F$. Suppose $v_L(x_1) < v_L(x_2) < v_L(x_1) + v_F(\pi_F)$. Then L/F must be ramified and $v_L(x_2) = v_L(x_1)+1$. Write x_2 as αx_1 where $\alpha \in \mathcal{O}_{L,c_1}$ with $v_L(\alpha) = 1$. Thus we must have $c_1 \in \mathcal{O}_F^*$ (i.e. $\mathcal{O}_{L,c_1} = \mathcal{O}_L$). Since

$$\lambda_1/\lambda_2 \cong \mathcal{O}_F/\mathcal{P}_F \cong \mathcal{O}_L x_1/\mathcal{O}_L x_2 \quad \text{and} \quad \lambda_2 \subset \mathcal{O}_L x_2,$$

we get $\lambda_2 = \mathcal{O}_L x_2$. Therefore $\lambda_1 \supset \lambda_2$ is an $\mathcal{O}_L$-chain, which is a contradiction.

We conclude that for every optimal $\mathcal{O}_{L,c}$-chain $\lambda_* = \{\lambda_1, \lambda_2\}$, we can find an element x in λ_1 such that either

$$\lambda_1 = \mathcal{O}_{L,c\pi_F^{-1}} \cdot x \quad \text{and} \quad \lambda_2 = \mathcal{O}_{L,c} \cdot x,$$

or

$$\lambda_1 = \mathcal{O}_{L,c} \cdot x \quad \text{and} \quad \lambda_2 = \mathcal{O}_{L,c\pi_F^{-1}} \cdot \pi_F x.$$

Therefore the proof is complete. □

Lemma V.9. *When $L = F \times F$, the cardinality of* $\mathrm{CH}(L, c)$ *is* 2 *for every* $c \in \mathcal{O}_F$.

Proof. Given an $\mathcal{O}_{L,c}$-chain, $\lambda_1 \supset \lambda_2$ with $\lambda_1/\lambda_2 \cong \mathcal{O}_F/\mathcal{P}_F$, From Lemma V.5, we take $(x_i, x_i') \in \lambda_i$ and $c_i \in \mathcal{O}_F$ with $v_F(c) \geq v_F(c_i)$ such that

$$\lambda_1 = \mathcal{O}_{L,c_1}(x_1, x_1') \quad \text{and } \lambda_2 = \mathcal{O}_{L,c_2}(x_2, x_2').$$

So $v_F(x_1) + 1 \geq v_F(x_2) \geq v_F(x_1)$ and $v_F(x_1') + 1 \geq v_F(x_2') \geq v_F(x_1')$. We separate into 4 cases.

(i) $v_F(x_2) = v_F(x_1)$ and $v_F(x_2') = v_F(x_1')$.
Then $\lambda_1 = \mathcal{O}_{L,c_1}(x_2, x_2')$ and $\lambda_2 = \mathcal{O}_{L,c_2}(x_2, x_2')$. So $c_2 = c_1\pi_F$.

(ii) $v_F(x_2) = v_F(\pi_F x_1)$ and $v_F(x_2') = v_F(\pi_F x_1')$.
Then $\lambda_2 = \mathcal{O}_{L,c_2}(\pi_F x_1, \pi_F x_1')$. Hence we must get $c_1 = c_2\pi_F$.

(iii) $v_F(x_2) = v_F(x_1)$ and $v_F(x_2') = v_F(\pi_F x_1')$.
Take $u \in \mathcal{O}_F^*$ such that $ux_2 = x_1$. Then

$$(x_1, x_1') - u(x_2, x_2') = (0, x_1' - ux_2') \in \lambda_1.$$

Since $v_F(x_2') = v_F(\pi_F x_1')$, then there exist $u' \in \mathcal{O}_F^*$ such that $u'(x_1' - ux_2') = x_1'$. So $(0, x_1') \in \lambda_1$. This implies that $\mathcal{O}_{L,c_1} = \mathcal{O}_L$. Now we have

$$\lambda_1/\lambda_2 \cong \mathcal{O}_F/\mathcal{P}_F \cong \frac{\mathcal{O}_L(x_1, x_1')}{\mathcal{O}_L(x_2, x_2')}$$

and $\lambda_2 \subset \mathcal{O}_L(x_2, x_2')$. So $\lambda_2 = \mathcal{O}_L(x_2, x_2') = \mathcal{O}_L(x_1, \pi_F x_1')$. Hence $\lambda_1 \supset \lambda_2$ is an $\mathcal{O}_L$-chain.

(iv) $v_F(x_2) = v_F(\pi_F x_1)$ and $v_F(x_2') = v_F(x_1')$.
The argument is similar to case (iii), we get $\lambda_1 \supset \lambda_2$ is also an $\mathcal{O}_L$-chain.

Therefore (i) and (ii) asserts the case when $c \in \mathcal{P}_F$. The case when $c \in \mathcal{O}_F^\times$ follows from (iii) and (iv). $\square$

2. Trace formula

A quadratic extension K of k is called *imaginary* if ∞ does non-split in K. From the study of optimal embeddings in the last section, we express the trace of Brandt matrices in terms of the so-called *modified Hurwitz class numbers* of imaginary quadratic fields over k.

Let k^{sep} be a fixed separable closure of k. Define $\wp_q : k^{\text{sep}} \to k^{\text{sep}}$ by the following:

$$\wp_q(x) := \begin{cases} x^2, & \text{if } q \text{ is odd,} \\ x^2 + x, & \text{if } q \text{ is even.} \end{cases}$$

Then for each $d \in k$, $K_d := k(\wp_q^{-1}(d))$ is a separable quadratic extension of k if $\wp_q^{-1}(d) \not\subset k$. In this case, denote by $\mathcal{O}_d$ the integral closure of A in K_d. For each integral ideal c of A, let $\mathcal{O}_{c,d} := A + c \cdot \mathcal{O}_d$, the quadratic A-order in K_d of conductor c. Let $h(c,d)$ be the class number of $\mathcal{O}_{c,d}$ and $u(c,d)$ denotes the cardinality of $\mathcal{O}_{c,d}^\times/\mathbb{F}_q^\times$ if K_d is imaginary.

Suppose q is even, we let K_{in} be the unique (up to k-isomorphism) inseparable quadratic extension of k, and $\mathcal{O}_{\text{in}}$ be the integral closure of A in K_{in}. The quadratic A-order $A + c \cdot \mathcal{O}_{\text{in}}$ is denoted by $\mathcal{O}_{c,\text{in}}$, and we let $h_{\text{in}}(c)$ be the class number of $\mathcal{O}_{c,\text{in}}$.

Recall that R is a given Eichler A-order of type $(\mathfrak{N}_0^+, \mathfrak{N}_0^-)$. For each d in k such that K_d is imaginary and a non-zero integral ideal c of A, the *modified Hurwitz class number* is

$$H(c,d) = \frac{1}{q-1} \sum_{\substack{\text{ideal } c' \subset A,\\ c' \mid c}} \frac{h(c',d)}{u(c',d)} \prod_{v \mid \mathfrak{N}_0^-} \left(1 - \left\{\frac{(c',d)}{v}\right\}\right) \prod_{v \mid \mathfrak{N}_0^+} \left(1 + \left\{\frac{(c',d)}{v}\right\}\right),$$

where

$$\left\{\frac{(c,d)}{v}\right\} = \begin{cases} 1 & \text{if } v \text{ splits in } K_d \text{ or } \operatorname{ord}_v(c) > 0, \\ -1 & \text{if } v \text{ is inert in } K_d \text{ and } \operatorname{ord}_v(c) = 0, \\ 0 & \text{if } v \text{ is ramified in } K_d \text{ and } \operatorname{ord}_v(c) = 0. \end{cases}$$

Similarly, let

$$H_{\text{in}}(c) := \sum_{c' \mid c} \left(h_{\text{in}}(c') \cdot \prod_{v \mid \mathfrak{N}_0^+} \left(1 + \left\{\frac{c'}{v}\right\}_{\text{in}}\right) \right),$$

where

$$\left\{\frac{c'}{v}\right\}_{\text{in}} = \begin{cases} 1 & \text{if } \operatorname{ord}_v(c) > 0, \\ 0 & \text{if } \operatorname{ord}_v(c) = 0. \end{cases}$$

For convenience, set

$$H(1,0) := \sum_{i=1}^{n} \#(R_i^\times)^{-1}$$

and

$$H_{\text{in}}(0) := (q-1) \cdot \sum_{i=1}^{n} \#(R_i^\times)^{-1}.$$

We then arrive at the trace formula of Brandt matrices (associated to R):

THEOREM V.10. *For each divisor* $\mathfrak{m} \in \mathrm{Div}_{f,\geq 0}(k)$, $\mathrm{Trace}\big(B(\mathfrak{m})\big) = 0$ *unless the corresponding ideal* $M_{\mathfrak{m}}$ *of* A *is principal. In this case, let* $m \in A$ *be a generator of the ideal* $M_{\mathfrak{m}}$. *Then*

$$\mathrm{Trace}\big(B(\mathfrak{m})\big) = \epsilon_q H_{\mathrm{in}}(c_1(\mathfrak{m})) + \sum_{\substack{(\nu,s)\in\mathbb{F}_q^\times\times A,\ d(\nu m,s)=0 \text{ or} \\ K_{d(\nu m,s)} \text{ is imaginary}}} H(c(\nu m,s), d(\nu m,s)).$$

Here

- $d = d(\nu m, s) := \begin{cases} s^2 - 4\nu m & \text{if } q \text{ is odd,} \\ \nu m/s^2 & \text{if } q \text{ is even and } s \neq 0; \end{cases}$
- $c(\nu m, s) := 1$ *if* $d = 0$; *if* K_d *is imaginary over* k, $c(\nu m, s)$ *is the conductor of the quadratic order* $A[b] \subset K_d$, *where* $b \in K_d$ *satisfies* $\mathrm{Tr}(b) = s$ *and* $\mathrm{Nr}(b) = \nu m$;
- *when* q *is even,*

$$\epsilon_q = \begin{cases} 1 & \text{if } q \text{ is even,} \\ 0 & \text{otherwise;} \end{cases}$$

  $c_1(\mathfrak{m})$ *denotes the conductor of the order* $A[\sqrt{m}] \subset K_{\mathrm{in}}$ *if* $\sqrt{m} \notin k$ *and* $c_1(\mathfrak{m}) := 0$ *if* $\sqrt{m} \in k$.

PROOF. Without loss of generality, assume the ideal $M_{\mathfrak{m}}$ is generated by $m \in A$. Recall that $I_1, ..., I_n$ are representatives of locally-principal right ideal classes of R, and R_i is the left order of I_i for $1 \leq i \leq n$. Given $\nu \in \mathbb{F}_q^\times$ and $s \in A$, let

$$A_i(\nu, s) = \{b \in R_i | \mathrm{Nr}(b) = \nu m, \mathrm{Tr}(b) = s\}.$$

It is clear that $A_i(\nu, s)$ is a finite set, which is empty if $K_d \neq k$ and ∞ is split in K_d where $d = d(\nu m, s)$. Then

$$\begin{aligned} &\mathrm{Trace}\big(B(\mathfrak{m})\big) \\ &= \sum_{i=1}^n \frac{\#\{b \in R_i | \mathrm{Nr}(b)A = M_{\mathfrak{m}}\}}{\#(R_i^\times)} \\ &= \left(\sum_{i=1}^n \sum_{\nu\in\mathbb{F}_q^\times} \frac{\#A_i(\nu,0)}{\#(R_i^\times)}\right) + \left(\sum_{i=1}^n \sum_{\nu\in\mathbb{F}_q^\times,\ s\in A-\{0\}} \frac{\#A_i(\nu,s)}{\#(R_i^\times)}\right). \end{aligned}$$

When q is odd and $d = 0$, it is clear that

$$\sum_{i=1}^{n} \frac{\#A_i(\nu,s)}{\#(R_i^\times)} = H(1,0).$$

Similarly, when q is even and $\sqrt{m} \in k$,

$$\sum_{i=1}^{n} \sum_{\nu\in\mathbb{F}_q^\times} \frac{A_i(\nu,0)}{\#(R_i^\times)} = H_{\text{in}}(0).$$

Now, suppose that K_d is imaginary. Every $b \in A_i(\nu,s)$ gives rise to an embedding of the order $\mathcal{O}_{c(\nu m,s),d}$ into R_i. The group $\Gamma_i := R_i^\times/\mathbb{F}_q^\times$ acts on $A_i(\nu,s)$ and the set of these embeddings by conjugation. For each non-zero ideal c of A, Let $h_i(c,d)$ be the number of optimal embeddings $\mathcal{O}_{c,d}$ into R_i, modulo conjugation by $R_i^\times$. Then we get

$$\#A_i(\nu,s) = \frac{\#(R_i^\times)}{q-1} \sum_{c'|c(\mu,s)} h_i(c',d)/u(c',d).$$

Therefore Corollary V.4 implies that

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\#A_i(\nu,s)}{\#(R_i^\times)} \\
&= \frac{1}{q-1} \cdot \sum_{c'|c(\nu m,s)} \left(\frac{1}{u(c',d)} \cdot \sum_{i=1}^{n} h_i(c',d) \right) \\
&= \frac{1}{q-1} \cdot \sum_{c'|c(\nu m,s)} \frac{h(c',d)}{u(c',d))} \prod_{v|\mathfrak{N}^-} \left(1 - \left\{\frac{(c',d)}{v}\right\}\right) \prod_{v|\mathfrak{N}^+} \left(1 + \left\{\frac{(c',d)}{v}\right\}\right) \\
&= H(c(\nu m,s), d(\nu m,s)).
\end{aligned}$$

Similarly, when q is even and $\sqrt{m} \notin k$, by Corollary V.4 we get

$$\sum_{i=1}^{n} \sum_{\nu\in\mathbb{F}_q^\times} \frac{\#A_i(\nu,0)}{\#(R_i^\times)} = H_{\text{in}}(c_1(\mathfrak{m})).$$

Therefore the proof is complete. $\square$

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Symbols