KRONECKER LIMIT FORMULA OVER GLOBAL FUNCTION FIELDS

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Abstract. The aim of this paper is to present a function field analogue of the classical Kronecker limit formula. We first introduce a "non-holomorphic" Eisenstein series on the Drinfeld half plane, and connect its "second term" with Gekeler's discriminant function. One application is to express the Taguchi height of rank 2 Drinfeld modules with complex multiplication in terms of the logarithmic derivative of the corresponding zeta functions. Moreover, from the integral form of the Rankin-type L-function associated to two "Drinfeld-type" newforms, we then derive a formula for a non-central special derivative of the L-function in question. Adapting the classical approach, we also obtain a Kronecker-type solution for Pell's equation over function fields.

1. Introduction

For $z \in \mathbb{C}$ with the imaginary part $\mathrm{Im}(z) > 0$, recall the non-holomorphic Eisenstein series:

$$E(z,s) := \sum_{c,d \in \mathbb{Z}}{}' \frac{\mathrm{Im}(z)^s}{|cz+d|^{2s}}, \quad \mathrm{Re}(s) > 1.$$

Here $'$ means that c and d are not both zero. Extending $E(z,s)$ to a meromorphic function on the complex s-plane, the classical Kronecker limit formula is stated as follows:

$$E(z,s) = \frac{\pi}{s-1} + \pi\big(2\gamma - 2\ln 2 - \ln(\mathrm{Im}(z)|\Delta(z)|^{\frac{2}{12}})\big) + O(s-1). \tag{1.1}$$

Here γ is the Euler constant and $\Delta(z)$ is the modular discriminant function (a weight 12 modular cusp form for $\mathrm{SL}_2(\mathbb{Z})$). From the functional equation of $E(z,s)$:

$$\pi^{-s}\Gamma(s)E(z,s) =: \widetilde{E}(z,s) = \widetilde{E}(z,1-s),$$

the equation (1.1) can be reformulated to:

$$E(z,0) = -1 \quad \text{and} \quad \frac{\partial}{\partial s}E(z,s)\Big|_{s=0} = -\ln(\mathrm{Im}(z)|\Delta(z)|^{\frac{2}{12}}). \tag{1.2}$$

This formula is applied (also by Kronecker) to a "modular" solution of Pell's equations. Together with the Chowla-Selberg formula, the equation (1.2) connects the "Faltings height" of elliptic curves having complex multiplication with special Γ-values (cf. Colmez [2], Section 0.6). Moreover, by Rankin-Selberg method, $E(z,s)$ shows up in the integral form of the Rankin-type L-function $L(f_1 \times f_2, s)$ associated to two normalized Hecke eigenforms f_1 and f_2 with the same weight for $\mathrm{SL}_2(\mathbb{Z})$. Via the equation (1.2), the non-central derivative $\frac{\partial}{\partial s}L(f_1 \times f_2, s)\big|_{s=0}$ can be expressed as the Petersson inner product $\langle f_1 \cdot \ln(\mathrm{Im}(\cdot)|\Delta|^{\frac{2}{12}}), \overline{f_2}\rangle$. In this paper, we shall present a precise analogue of the Kronecker limit formula in the function field context, and explore its various arithmetic applications.

Let k be an arbitrary global function field with finite constant field $\mathbb{F}_q$. Fix a place ∞ of k, regarded as the place at infinity, and the corresponding absolute value is denoted by $|\cdot|_\infty$.

2010 *Mathematics Subject Classification.* 11M36, 11G09, 11R58.
Key words and phrases. Function field, Kronecker limit formula, Eisenstein series, Drinfeld modules.
This work was supported by the grants from Ministry of Science and Technology, Taiwan.

Let k_∞ be the completion of k at ∞ and $\mathbb{C}_\infty$ be the completion of an algebraic closure of k_∞. We set $\mathfrak{H} := \mathbb{C}_\infty - k_\infty$, called the *Drinfeld half plane*. For each $z \in \mathfrak{H}$, the "imaginary part" of z is the distance between z and k_∞:

$$|z|_i := \inf_{\alpha \in k_\infty} |z - \alpha|_\infty.$$

Let A be the subring of k consisting of the functions regular away from ∞. For each (fractional) ideal $\mathfrak{a}$ of A, we introduce the following "non-holomorphic" Eisenstein series

$$\mathbb{E}^{\mathfrak{a}}(z,s) := \sideset{}{'}\sum_{c \in A, d \in \mathfrak{a}} \frac{|z|_i^s}{|cz+d|_\infty^{2s}}, \quad \mathrm{Re}(s) > 1.$$

Our main theorem is stated in the following (cf. Theorem 3.1):

Theorem 1.1. (1) $\mathbb{E}^{\mathfrak{a}}(z,s)$ *has meromorphic continuation to the whole s-plane.*
(2) *Let* $\widetilde{\mathbb{E}}^{\mathfrak{a}}(z,s)$ *be the modified Eisenstein series defined in* Theorem 3.1 (2). *We have the functional equation*

$$\widetilde{\mathbb{E}}^{\mathfrak{a}}(z,s) = \widetilde{\mathbb{E}}^{\mathfrak{a}^{-1}}(z^{-1}, 1-s).$$

(3) *(Kronecker limit formula)*

$$\mathbb{E}^{\mathfrak{a}}(z,0) = -1 \quad \text{and} \quad \frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,s)\Big|_{s=0} = -\ln\left(|z|_i \cdot |\Delta^{\mathfrak{a}}(z)|_\infty^{\frac{2}{q^{2\deg\infty}-1}}\right).$$

Here $\Delta^{\mathfrak{a}}(z)$ *is Gekeler's discriminant function* (cf. Section 2.4)*, which is a Drinfeld modular form of weight* $q^{2\deg\infty} - 1$ *for* $\mathrm{GL}(\Lambda^{\mathfrak{a}})$ *where* $\Lambda^{\mathfrak{a}} := A \oplus \mathfrak{a} \subset k_\infty^2$.

We remark that Theorem 1.1 (3) provides a geometric interpretation of the special derivative of the non-holomorphic Eisenstein series in question. Indeed, according to the theory of Drinfeld, the lattice $\Lambda^{\mathfrak{a}}(z) := Az + \mathfrak{a} \subset \mathbb{C}_\infty$ for $z \in \mathfrak{H}$ corresponds to a unique rank 2 Drinfeld A-module $\phi^{\mathfrak{a},z}$ over $\mathbb{C}_\infty$ (see Section 2.3). Given $z_0 \in \mathfrak{H}$, suppose $\phi^{\mathfrak{a},z_0}$ is isomorphic (over $\mathbb{C}_\infty$) to a Drinfeld A-module $\widetilde{\phi}^{\mathfrak{a},z_0}$ defined over the algebraic closure $\bar{k}$ of k in $\mathbb{C}_\infty$. Then the discriminant $\Delta^{\mathfrak{a}}(z_0)$ of $\phi^{\mathfrak{a},z_0}$ is in fact an algebraic multiple of a "period" of $\widetilde{\phi}^{\mathfrak{a},z_0}$ raised to the $(q^{2\deg\infty} - 1)$-th power (cf. Chang [3]), which is transcendental over k by Yu [23].

Our formula is closely parallel to the classical one, however, the approach is quite different. Note that $\mathbb{E}^{\mathfrak{a}}(z,s)$ is a $\mathbb{C}$-valued function but $\Delta^{\mathfrak{a}}(z)$ lies in the positive characteristic world. Via the building map from the Drinfeld half plane $\mathfrak{H}$ to the Bruhat-Tits tree associated to $\mathrm{GL}_2(k_\infty)$, we first connect $\mathbb{E}^{\mathfrak{a}}(z,s)$ with "adelic" Eisenstein series on GL_2 (cf. Proposition 3.4). Since the analytic behavior of adelic Eisenstein series is well-understood, the meromorphic continuation and the functional equation of $\mathbb{E}^{\mathfrak{a}}(z,s)$ are then easily verified. To prove Theorem 1.1 (3), we introduce the following Eisenstein series of "Jacobi-type":

$$\mathbb{E}^{\mathfrak{a}}(z,w,s) := \sum_{\lambda \in \Lambda^{\mathfrak{a}}(z)} \frac{|z|_i^s}{|\lambda - w|_\infty^{2s}}, \quad \forall z \in \mathfrak{H} \text{ and } w \in \mathbb{C}_\infty - \Lambda^{\mathfrak{a}}(z),$$

which satisfies

$$\mathbb{E}^{\mathfrak{a}}\left(\frac{az+b}{cz+d}, \frac{w}{cz+d}, s\right) = \mathbb{E}^{\mathfrak{a}}(z,w,s), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(\Lambda^{\mathfrak{a}}).$$

The non-archimedean property of $|\cdot|_\infty$ implies that $\mathbb{E}^{\mathfrak{a}}(z,s) - \mathbb{E}^{\mathfrak{a}}(z,w,s)$ is entire with respect to the variable s, which gives the meromorphic continuation of $\mathbb{E}^{\mathfrak{a}}(z,w,s)$. Our Kronecker limit formula then follows from a product expansion of Gekeler's discriminant function in Lemma 2.3 and the equality below (cf. Section 3.2)

$$(|a|_\infty^{2s} - 1)\mathbb{E}^{\mathfrak{a}}(z,s) = \sum_{0 \neq w \in \frac{1}{a}\Lambda^{\mathfrak{a}}(z)/\Lambda^{\mathfrak{a}}(z)} \mathbb{E}^{\mathfrak{a}}(z,w,s), \quad \forall a \in A - \mathbb{F}_q.$$

There have been other analogues of the Kronecker limit formula for different types of Eisenstein series over function fields. Gekeler [7] first observed that, when the base field k is rational, the *van der Put logarithmic derivatives* $r(\Delta^{\mathfrak{a}})$ of $\Delta^{\mathfrak{a}}$ is related to a conditionally convergent complex-valued Eisenstein series on the Bruhat-Tits tree. In [15], Kondo studies the Jacobi-type Eisenstein series of "arbitrary rank," and connects the special derivative of this series with the discriminant function multiplied with the Drinfeld exponential function of the given A-lattice. His formula is a general version of (3.1) for $\mathbb{E}^{\mathfrak{a}}(z, w, s)$ in *Remark* 3.5 (2) for arbitrary rank case. Kondo's argument in [15] is to compare the Fourier coefficients of both sides directly. Thus our limit formula gives an alternative approach to Kondo's result for the case of rank 2. In [16, Section 4], Pál introduces another type of Eisenstein series, and connects the values of these series at $s = 0$ with the van der Put logarithmic derivatives of a special family of invertible holomorphic functions on $\mathfrak{H}$ (cf. [16, Kronecker limit formula 4.8]). One application of Pál's formula in [16] is to determine the Fourier coefficients of $r(\Delta^{\mathfrak{a}})$ (cf. [16, Proposition 5.8]). From our limit formula in Theorem 1.1 (3), we may express $r(\Delta^{\mathfrak{a}})$ explicitly in terms of the derivative of an adelic Eisenstein series at $s = 0$ (cf. *Remark* 3.5 (4)). This generalizes Gekeler's formula in [7, 2.8 COROLLARY] to arbitrary global function fields. Consequently, the standard results on the Whittaker functions of adelic Eisenstein series then provides an alternative way to calculate the Fourier coefficients of $r(\Delta^{\mathfrak{a}})$.

One application of Theorem 1.1 is to derive a Colmez-type formula for the "Taguchi height" of rank 2 Drinfeld A-modules with "complex multiplication." This height (introduced in [21] for the case when A is a polynomial ring) is viewed as a natural analogue of the Faltings height of abelian varieties over number fields. Let K/k be a quadratic field extension which is "imaginary" (i.e. K/k is separable and ∞ does not split in K) and O_K be the integral closure of A in K. For each integral ideal $\mathfrak{c}$ of A, let $O_{\mathfrak{c}} := A + \mathfrak{c}O_K$ be the quadratic A-order of conductor $\mathfrak{c}$. Applying our Kronecker limit formula, we obtain that (cf. Corollary 4.5):

Corollary 1.2. *For every rank 2 Drinfeld A-module ϕ over $\overline{k}$ with complex multiplication by $O_{\mathfrak{c}}$, the "Taguchi height" of ϕ is equal to*

$$\widetilde{h}_{\mathrm{Tag}}(\phi) = -\frac{1}{4}\ln \|\mathfrak{d}(O_{\mathfrak{c}}/A)\| - \frac{1}{2}\cdot \frac{\zeta'_{O_{\mathfrak{c}}}(0)}{\zeta_{O_{\mathfrak{c}}}(0)}.$$

Here $\|\mathfrak{a}\| := \#(A/\mathfrak{a})$ for every integral ideal $\mathfrak{a}$ of A,

$$\mathfrak{d}(O_{\mathfrak{c}}/A) := \mathfrak{c}^2 \cdot \prod_{\substack{\text{prime } \mathfrak{p} \lhd A \\ \mathfrak{p} \text{ is ramified in } K}} \mathfrak{p}, \quad \text{and} \quad \zeta_{O_{\mathfrak{c}}}(s) := \sum_{\substack{\text{invertible ideal} \\ \mathfrak{A} \subset O_{\mathfrak{c}}}} \#(O_{\mathfrak{c}}/\mathfrak{A})^{-s}.$$

Note that $\mathfrak{d}(O_{\mathfrak{c}}/A)$ is the discriminant ideal for $O_{\mathfrak{c}}/A$ when q is odd. In particular, suppose k is a rational function field $\mathbb{F}_q(t)$ with q odd and $A = \mathbb{F}_q[t]$. Let $D \in A - \mathbb{F}_q$ be a square-free polynomial such that $K = k(\sqrt{D})$ is imaginary. We have

$$\frac{\zeta'_{O_K}(s)}{\zeta_{O_K}(s)} = \frac{\zeta'_A(s)}{\zeta_A(s)} + \frac{L'_A(s, (\frac{D}{\cdot}))}{L_A(s, (\frac{D}{\cdot}))}$$

where $\left(\frac{D}{\cdot}\right)$ is the Legendre quadratic symbol and

$$L_A(s, \left(\frac{D}{\cdot}\right)) \quad := \quad \sum_{\text{monic } m \in A} \left(\frac{D}{m}\right) q^{-\deg m s}.$$

Then for every rank 2 Drinfeld A-module ϕ over $\overline{k}$ with complex multiplication by O_K, the Taguchi height $\widetilde{h}_{\mathrm{Tag}}(\phi)$ can be written as follows (cf. *Remark* 4.6 (2)):

$$\widetilde{h}_{\mathrm{Tag}}(\phi) = \frac{1}{4}\ln |D|_\infty - \frac{q \ln q}{2(q-1)} + \frac{1}{2\#\operatorname{Pic}(O_K)} \sum_{\substack{\text{monic } m \in A \\ \deg m < \deg D}} \left(\frac{D}{m}\right) \cdot \ln \left|\frac{m}{D}\right|_\infty .$$

By Ihara's estimation of the Euler-Kronecker constant of K in [13, upper bound (0.6) and lower bound(0.12)], we thereby obtain the following asymptotic formula:

$$\widetilde{h}_{\mathrm{Tag}}(\phi) = \frac{1}{4}\ln|D|_\infty + O(\ln\ln|D|_\infty) \quad \text{ for } |D|_\infty \gg 0.$$

Another application of Theorem 1.1 is on special values of Rankin-type L-functions associated to "Drinfeld-type" automorphic forms. These forms, viewed as analogue of classical weight 2 modular forms, are very useful tools in function field arithmetic (cf. [4], [8], and [22]). We refer the readers to Section 5.1 for the precise definitions and the analytic properties to be used. Let f_1 and f_2 be two normalized Drinfeld-type newforms of square-free levels $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively. The Rankin-Selberg method enables us to express the Rankin-type L-function $L(f_1 \times f_2, s)$ as a Petersson inner product $\langle f_1 \mathcal{E}_{\mathfrak{N}_1,\mathfrak{N}_2}, \overline{f_2}\rangle$, where the function $\mathcal{E}_{\mathfrak{N}_1,\mathfrak{N}_2}$ comes from our non-holomorphic Eisenstein series (see Section 5.2). Therefore, the Kronecker limit formula in Theorem 1.1 (3) leads us to (cf. Theorem 5.5):

Corollary 1.3. *Given two normalized Drinfeld-type newforms f_1 and f_2 with square-free levels $\mathfrak{N}_1$ and $\mathfrak{N}_2$, respectively, let $\Lambda(f_1 \times f_2, s)$ be the Rankin-type L-function modified as in* Theorem 5.5. *Then*

$$\Lambda(f_1 \times f_2, 0) = -\langle f_1, \overline{f_2}\rangle \quad \text{ and } \quad \frac{\partial}{\partial s}\Lambda(f_1 \times f_2, s)\big|_{s=0} = \langle f_1 \cdot \eta_{\mathfrak{N}_1,\mathfrak{N}_2}, \overline{f_2}\rangle.$$

Here $\eta_{\mathfrak{N}_1,\mathfrak{N}_2}$ is an automorphic form on GL_2 *induced from Gekeler's discriminant function.*

Following the classical story, we are also able to derive a function field version of Kronecker's solution of Pell's equations. Let L/k be a quadratic field extension which is "real" (i.e. ∞ splits in L). It is always possible to take an imaginary quadratic field K/k such that LK is a subfield of the Hilbert class field H_{O_K} of O_K. We denote the integral closure of A in L by O_L. Let $\xi_L : \mathrm{Pic}(O_K) \to \{\pm 1\}$ be the quadratic character associated to LK (i.e. $\ker \xi_L \cong \mathrm{Gal}(H_{O_K}/LK)$ via the Artin map). By a Shimura-type "reciprocity law" for Gekeler's discriminant function (cf. Proposition 6.3), we observe that

$$u_L := \prod_{[\mathfrak{A}] \in \mathrm{Pic}(O_K)} \left[\left(\frac{\Delta(\mathfrak{A})\Delta(\mathfrak{A}^{-1})}{\Delta(O_K)^2}\right)^{\xi_L(\mathfrak{A})} \cdot \left(\frac{\Delta(\overline{\mathfrak{A}})\Delta(\overline{\mathfrak{A}}^{-1})}{\Delta(O_K)^2}\right)^{\xi_L(\overline{\mathfrak{A}})}\right] \in O_L^\times.$$

Here $\bar{\cdot}$ is the non-trivial automorphism of K/k, and $\Delta(\mathfrak{A}) := \alpha^{1-q^{2\deg\infty}}\Delta^{\mathfrak{a}}(z)$ when writing $\mathfrak{A}$ as the form $\alpha(Az+\mathfrak{a})$ for an ideal $\mathfrak{a}$ of A, $z \in \mathfrak{H}$, and $\alpha \in \mathbb{C}_\infty^\times$. Moreover, Theorem 1.1 (3) tells us that:

Corollary 1.4. *u_L is not a root of unity. Moreover,*

$$\#\left(\frac{O_L^\times}{< u_L >}\right) = \frac{\#\,\mathrm{Pic}(O_L)\cdot\#\,\mathrm{Pic}(O_{K'})}{\#\,\mathrm{Pic}(A)^2} \cdot \frac{2\cdot\#(O_K^\times)\cdot\#(\mathbb{F}_q^\times)^2\cdot(q_\infty^2-1)}{\#(O_{K'}^\times)}.$$

Here K'/k is the imaginary quadratic subfield of LK/k different from K.

We point out that if $\mathfrak{a}O_K$ is a principal ideal of O_K for every ideal $\mathfrak{a}$ of A, then

$$\tilde{u}_L := \prod_{[\mathfrak{A}] \in \mathrm{Pic}(O_K)} \left(\frac{\Delta(\mathfrak{A})\Delta(\mathfrak{A}^{-1})}{\Delta(O_K)^2}\right)^{\xi_L(\mathfrak{A})} \in O_L^\times$$

and $u_L = \tilde{u}_L^2$. In particular, suppose $k = \mathbb{F}_q(t)$ with q odd and $A = \mathbb{F}_q[t]$. Take a monic square-free polynomial $D \in A$ with even degree and let $L := k(\sqrt{D})$. We can choose $K = k(\sqrt{\epsilon D})$ where $\epsilon \in \mathbb{F}_q^\times$ is a non-square element. Writing $\tilde{u}_L$ as $\tilde{u}_L = a + b\sqrt{D}$ where $a, b \in A$, we then obtain a non-trivial solution (a, b) for the Pell's equation $X^2 - DY^2 = 1$.

This paper is organized as follows. In Section 2, we set up notations and recall the needed properties of rank 2 Drinfeld modules, including Gekeler's discriminant function and the building map on the Drinfeld half plane. Our analogue of the Kronecker limit formula is derived in Section 3. We connect $\mathbb{E}^{\mathfrak{a}}(z,s)$ with adelic Eisensteins series on GL_2 in Section 3.1 and prove Theorem 1.1 in Section 3.2. In Section 4, we first introduce the Taguchi height of Drinfeld A-modules for general A, and then express the height of rank 2 Drinfeld A-modules having complex multiplication by $O_{\mathfrak{c}}$ as the logarithmic derivative of the zeta function $\zeta_{O_{\mathfrak{c}}}(s)$. In Section 5, we study the non-central special value of the Rankin-type L-functions associated to Drinfeld type automorphic forms. Basic properties of Drinfeld type forms are recalled in Section 5.1, and Corollary 1.3 is shown in Section 5.2. In Section 6, we apply Theorem 1.1 to get a Kronecker-type solution of Pell's equations over function fields. Verifying a Shimura-type reciprocity law for Gekeler's discriminant function, Corollary 1.4 is carried out in Section 6.2.

Acknowledgements. The author is deeply grateful to Jing Yu for his steady interest and encouragements, and to Ming-Lun Hsieh for his useful comments. He wish to thank the referee for valuable suggestions, and to Chih-Yun Chuang for helpful discussions on analytic number theory. This work is started during the Japan-Taiwan Joint conference on Number Theory 2014 in Kesennuma Oshima island, sponsored by Research Institute for Mathematical Sciences. The author would also like to thank the organizers for the invitation.

2. Preliminaries

2.1. **Basic settings.** Let $\mathbb{F}_q$ be the finite field with q elements. Let k be a global function field with constant field $\mathbb{F}_q$, i.e. k is a finitely generated field extension over $\mathbb{F}_q$ with transcendence degree one and $\mathbb{F}_q$ is its algebraic closure in k. For each place v of k, the completion of k at v is denoted by k_v, and O_v denotes the valuation ring in k_v. Choosing a uniformizer π_v in O_v once and for all, we set $\mathbb{F}_v := O_v/\pi_v O_v$, the residue field at v, and q_v to be the cardinality of $\mathbb{F}_v$. Let $\deg v := [\mathbb{F}_v : \mathbb{F}_q]$, called the degree of v. The absolute value on k_v is normalized to:

$$|\alpha_v|_v := q_v^{-\operatorname{ord}_v(\alpha_v)} = q^{-\deg v \operatorname{ord}_v(\alpha_v)}, \quad \forall \alpha_v \in k_v.$$

Let $\mathbb{A} := \prod'_v k_v$, the adele ring of k and $O_{\mathbb{A}} := \prod_v O_v$, the maximal compact subring of $\mathbb{A}$. For each element $\alpha = (\alpha_v)_v$ in the idele group $\mathbb{A}^\times$, the norm $|\alpha|_{\mathbb{A}}$ is defined to be

$$|\alpha|_{\mathbb{A}} := \prod_v |\alpha_v|_v.$$

We can embed k (resp. $k^\times$) into $\mathbb{A}$ (resp. $\mathbb{A}^\times$) diagonally, and have the product formula: $|\alpha|_{\mathbb{A}} = 1$ for all $\alpha \in k^\times$. Throughout this paper, we fix a non-trivial additive character $\psi = \otimes_v \psi_v : \mathbb{A} \to \mathbb{C}^\times$ which is trivial on k (here $\psi_v(a_v) := \psi(0, ..., 0, a_v, 0, ...)$, for all a_v in k_v). For each place v of k, let δ_v be the "conductor of ψ at v," i.e. the maximal integer r such that $\pi_v^{-r} O_v$ is contained in the kernel of ψ_v. It is known that $\sum_v \delta_v \deg v = 2g_k - 2$, where g_k is the genus of k. We call $\delta = (\pi_v^{\delta_v})_v \in \mathbb{A}^\times$ a *differential idele of k associated to ψ.*

Fix a place ∞ of k, regarded as the place at infinity. We set $\mathbb{A}^\infty := \prod'_{v \neq \infty} k_v$, called the finite adele ring of k, and $O_{\mathbb{A}^\infty} := \prod_{v\neq\infty} O_v$. Let A be the ring of functions in k regular away from ∞. Then the finite places of k (i.e. the place not equal to ∞) are canonically identified with non-zero prime ideals of A. For a (fractional) ideal $\mathfrak{a}$ of A, we denote by $\mathfrak{a} \lhd A$ if $\mathfrak{a}$ is an integral ideal. In this paper, every ideal is assumed to be non-zero. For each ideal I of A, writing $I = \mathfrak{a}^{-1}\mathfrak{b}$ where $\mathfrak{a}, \mathfrak{b} \lhd A$ we set

$$\|I\| := \frac{\#(A/\mathfrak{b})}{\#(A/\mathfrak{a})}.$$

In particular, $\|I\| = |\alpha|_\infty$ when $I = \alpha A$ for $\alpha \in k^\times$. Given two ideals $\mathfrak{a}_1, \mathfrak{a}_2$ of A, let

$$[\mathfrak{a}_1, \mathfrak{a}_2] := \mathfrak{a}_1 \cap \mathfrak{a}_2 \quad \text{and} \quad (\mathfrak{a}_1, \mathfrak{a}_2) := \mathfrak{a}_1 + \mathfrak{a}_2.$$

Finally, we put $\pi^{\mathfrak{a}} := (\pi_v^{\mathrm{ord}_v(\mathfrak{a})})_{v \neq \infty} \in \mathbb{A}^{\infty,\times}$ for each ideal $\mathfrak{a}$ of A.

2.2. **Drinfeld modules.** Let (F, ι) be an A-*field*, i.e. F is a field together with a ring homomorphism $\iota : A \to F$. The $\mathbb{F}_q$-linear endomorphism ring $\mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a/F})$ is isomorphic to the *twisted polynomial ring* $F\{\tau\}$, where $\tau : \mathbb{G}_{a/F} \to \mathbb{G}_{a/F}$ is the Frobenius map ($x \mapsto x^q$) and $\tau a = a^q \tau$ for every $a \in F$.

Definition 2.1. Suppose an A-field (F, ι) and a positive integer r is given.
(1) A *Drinfeld A-module over F of rank* r is a ring homomorphism $\phi : A \to F\{\tau\}$ satisfying that

$$\phi_a = \iota(a) + \sum_{i=1}^{r \deg a} l_i(\phi_a)\tau^i \in F\{\tau\}, \quad \text{with } l_{r \deg a}(\phi_a) \neq 0 \ \ \forall a \in A.$$

(2) Given two Drinfeld A-modules ϕ_1 and ϕ_2 over F, a homomorphism $f : \phi \to \phi'$ over F is an element in $F\{\tau\}$ satisfying $f \cdot \phi_a = \phi'_a \cdot f$ for every $a \in A$. f is called an isogeny if f is not zero. We denote the set of homomorphisms from ϕ to ϕ' (resp. endomorphism ring of ϕ) over F by $\mathrm{Hom}_A(\phi/F, \phi'/F)$ (resp. $\mathrm{End}_A(\phi/F)$).

2.3. **Drinfeld half plane.** Let $\mathbb{C}_\infty$ be the completion of a chosen algebraic closure of k_∞, and consider $\mathbb{C}_\infty$ as an A-field via the natural embedding $A \hookrightarrow \mathbb{C}_\infty$. Given a rank r Drinfeld A-module ϕ over $\mathbb{C}_\infty$. There exists a unique $\mathbb{F}_q$-linear entire function $\exp_\phi$ on $\mathbb{C}_\infty$ satisfying

$$\exp_\phi(aw) = \phi_a\big(\exp_\phi(w)\big), \quad \forall a \in A \text{ and } w \in \mathbb{C}_\infty.$$

It is known that $\Lambda_\phi := \{\lambda \in \mathbb{C}_\infty : \exp_\phi(\lambda) = 0\}$ is a discrete projective A-submodule of rank r in $\mathbb{C}_\infty$ (i.e. a rank r A-lattice in $\mathbb{C}_\infty$). We call Λ_ϕ the *A-lattice associated to* ϕ. On the other hand, given a rank r A-lattice $\Lambda \subset \mathbb{C}_\infty$, set

$$\exp_\Lambda(w) := w \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{w}{\lambda}\right).$$

This uniquely determines a rank r Drinfeld A-module ϕ^Λ over $\mathbb{C}_\infty$ satisfying that

$$\exp_\Lambda(aw) = \phi_a^\Lambda\left(\exp_\Lambda(w)\right), \quad \forall a \in A \text{ and } w \in \mathbb{C}_\infty. \tag{2.1}$$

In other words, we have the following bijection (cf. [5, Proposition 3.1])

$$\{\text{rank } r \text{ Drinfeld } A\text{-modules over } \mathbb{C}_\infty\} \cong \{\text{rank } r \ A\text{-lattices in } \mathbb{C}_\infty\}.$$

For our purpose, we now focus on the case when $r = 2$ and recall the analytic description of the moduli space for rank 2 Drinfeld A-modules over $\mathbb{C}_\infty$. Let $\mathfrak{H} := \mathbb{C}_\infty - k_\infty$, called the *Drinfeld half plane*. We have an action of $\mathrm{GL}_2(k_\infty)$ on $\mathfrak{H}$ defined by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}, \quad \forall z \in \mathfrak{H} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k_\infty).$$

Take an ideal $\mathfrak{a}$ of A and set $\Lambda^{\mathfrak{a}} := A \oplus \mathfrak{a} \subset k^2$. For $z \in \mathfrak{H}$, let $\Lambda^{\mathfrak{a}}(z) := Az + \mathfrak{a} \subset \mathbb{C}_\infty$ and $\phi^{\mathfrak{a},z}$ denotes the rank 2 Drinfeld A-module over $\mathbb{C}_\infty$ corresponding to $\Lambda^{\mathfrak{a}}(z)$.

Theorem 2.2. (cf. [8, Section 2.5]) *The map* $(\mathfrak{a}, z) \mapsto \phi^{\mathfrak{a},z}$ *induces the following bijection*

$$\left(\coprod_{[\mathfrak{a}] \in \mathrm{Pic}(A)} \mathrm{GL}(\Lambda^{\mathfrak{a}}) \backslash \mathfrak{H}\right) \longleftrightarrow \{\text{rank-2 Drinfeld } A\text{-modules over } \mathbb{C}_\infty\} / \cong .$$

Here

$$\mathrm{GL}(\Lambda^{\mathfrak{a}}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k_\infty) \;\middle|\; a, d \in A,\ b \in \mathfrak{a},\ c \in \mathfrak{a}^{-1},\ ad - bc \in \mathbb{F}_q^\times \right\}.$$

2.4. **Gekeler's discriminant function.** Given an ideal $\mathfrak{a}$ of A and $a \in A$, let $\Delta_a^{\mathfrak{a}}(z)$ be the analytic function on $\mathfrak{H}$ satisfying

$$\phi_a^{\mathfrak{a},z}(x) = ax + \cdots + \Delta_a^{\mathfrak{a}}(z) x^{q^{2\deg a}}, \quad \forall x \in \mathbb{C}_\infty.$$

In other words, $\Delta_a^{\mathfrak{a}}(z) = l_{2\deg a}(\phi_a^{\mathfrak{a},z})$. It is observed that (cf. [6, Chapter V, 3.4 Example])

$$\Delta_a^{\mathfrak{a}}(\gamma z) = (cz+d)^{q^{2\deg a}-1} \cdot \Delta_a^{\mathfrak{a}}(z), \quad \forall \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{GL}(\Lambda^{\mathfrak{a}}).$$

Moreover, the equation (2.1) implies

$$\phi_a^{\mathfrak{a},z}(x) = \Delta_a^{\mathfrak{a}}(z) \cdot x \cdot \prod_{0 \neq w \in \frac{1}{a}\Lambda^{\mathfrak{a}}(z)/\Lambda^{\mathfrak{a}}(z)} \big(x - \exp_{\Lambda^{\mathfrak{a}}(z)}(w)\big).$$

Therefore we have:

Lemma 2.3. *For every $a \in A$,*

$$\Delta_a^{\mathfrak{a}}(z) = (-1)^{q^{2\deg a}-1} \cdot a \cdot \prod_{0 \neq w \in \frac{1}{a}\Lambda^{\mathfrak{a}}(z)/\Lambda^{\mathfrak{a}}(z)} \exp_{\Lambda^{\mathfrak{a}}(z)}(w)^{-1}.$$

Since $\phi_a^{\mathfrak{a},z} \cdot \phi_b^{\mathfrak{a},z} = \phi_{ab}^{\mathfrak{a},z} = \phi_b^{\mathfrak{a},z} \cdot \phi_a^{\mathfrak{a},z}$ for $a, b \in A$, one gets

$$\Delta_a^{\mathfrak{a}}(z)\big(\Delta_b^{\mathfrak{a}}(z)\big)^{q^{2\deg a}} = \Delta_{ab}^{\mathfrak{a}}(z) = \Delta_b^{\mathfrak{a}}(z)\big(\Delta_a^{\mathfrak{a}}(z)\big)^{q^{2\deg b}}. \tag{2.2}$$

Given $\alpha = \sum_{i \geq \mathrm{ord}_\infty(\alpha)} u_i \pi_\infty^i \in k_\infty^\times$ where $u_i \in \mathbb{F}_\infty$, we call α monic if $u_{\mathrm{ord}_\infty(\alpha)} = 1$. Take two monic elements $a, b \in A$ such that $\gcd(\mathrm{ord}_\infty(a), \mathrm{ord}_\infty(b)) = 1$, and choose $\ell_1, \ \ell_2 \in \mathbb{Z}$ such that $\ell_1\big(q_\infty^{2\,\mathrm{ord}_\infty(a)} - 1\big) + \ell_2\big(q_\infty^{2\,\mathrm{ord}_\infty(b)} - 1\big) = q_\infty^2 - 1$. Gekeler [6] introduces the following discriminant function

$$\Delta^{\mathfrak{a}}(z) := \Delta_a^{\mathfrak{a}}(z)^{\ell_1} \Delta_b^{\mathfrak{a}}(z)^{\ell_2},$$

which is a nowhere-zero analytic function on $\mathfrak{H}$ satisfying that

$$\Delta^{\mathfrak{a}}(\gamma z) = (cz+d)^{q_\infty^2 - 1} \cdot \Delta^{\mathfrak{a}}(z), \quad \forall \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{GL}(\Lambda^{\mathfrak{a}}).$$

Proposition 2.4. (cf. [6, Chapter IV, Proposition 5.15]) $\Delta^{\mathfrak{a}}$ *is independent of the chosen monic $a, b \in A$ and $\ell_1, \ell_2 \in \mathbb{Z}$. In particular, $(\Delta^{\mathfrak{a}})^{q^{2\deg a}-1} = (\Delta_a^{\mathfrak{a}})^{q_\infty^2 - 1}$ for every $a \in A - \{0\}$.*

2.5. **Building map.** Let $\mathcal{T}$ be the Bruhat-Tits tree associated to $\mathrm{PGL}_2(k_\infty)$. The set $V(\mathcal{T})$ of vertices of $\mathcal{T}$ is the collection of homothety classes of rank-2 O_∞-lattices in k_∞^2, and the set $\vec{E}(\mathcal{T})$ (resp. $E(\mathcal{T})$) of (non-)oriented edges of $\mathcal{T}$ consists of (non-)order pairs $([L], [L'])$ where $[L], \ [L'] \in V(\mathcal{T})$ with $\pi_\infty L' \subsetneq L \subsetneq L'$. It is known that the realization $\mathcal{T}(\mathbb{R})$ of $\mathcal{T}$ is identified with the equivalent classes of norms on k_∞^2 as follows: suppose $P \in \mathcal{T}(\mathbb{R})$ belongs to the edge $([L], [L'])$ with $\pi L' \subsetneq L \subsetneq L'$, say $P = (1-\epsilon)[L] + \epsilon[L']$, $0 \leq \epsilon < 1$, then the norm ν_P on k_∞^2 corresponding to P is

$$\nu_P(x) := \sup\{\nu_L(x), q_\infty^\epsilon \nu_{L'}(x)\} \quad \text{ with } \quad \nu_L(x) := \inf\{|a|_\infty : a \in k_\infty, x \in aL\}.$$

Definition 2.5. The *building map* $\lambda : \mathfrak{H} \to \mathcal{T}(\mathbb{Q})$ is defined by

$$z \in \mathfrak{H} \longmapsto \nu_z := \big((c,d) \in k_\infty^2 \mapsto |cz+d|_\infty\big).$$

The right action of $\mathrm{GL}_2(k_\infty)$ on k_∞^2 yields a left action on the set of norms on k_∞^2 and then on $\mathcal{T}(\mathbb{R})$.

Proposition 2.6. (cf. [8, Proposition 1.5.3]) *The building map λ is $\mathrm{GL}_2(k_\infty)$-equivariant.*

Given $z \in \mathfrak{H}$, the *imaginary part of* z is defined by

$$|z|_i := \sup\{|z-u|_\infty : u \in k_\infty\}.$$

It is observed that (cf. [8, (1.1.5) Lemma])

$$|\gamma z|_i = \frac{|\det \gamma|_\infty}{|cz+d|_\infty^2} \cdot |z|_i, \quad \forall \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k_\infty).$$

Via the left action of $\mathrm{GL}_2(k_\infty)$ on $\mathcal{T}(\mathbb{R})$, $V(\mathcal{T})$ is identified with $\mathrm{GL}_2(k_\infty)/k_\infty^\times \mathrm{GL}_2(O_\infty)$. More precisely, let $L_0 := O_\infty^2 \subset k_\infty^2$. Then

$$V(\mathcal{T}) = \{[L_0 g^{-1}] : g \in \mathrm{GL}_2(k_\infty)/k_\infty^\times \mathrm{GL}_2(O_\infty)\}.$$

Corollary 2.7. (cf. [7, 1.8 Lemma]) *Given $z \in \mathfrak{H}$ with $|z|_i = |z-u|_\infty = q_\infty^{-r+\epsilon}$ for some $u \in k_\infty$, $r \in \mathbb{Z}$, and $0 \leq \epsilon < 1$, we have $\lambda(z) = \nu_{P_z}$ where*

$$P_z = (1-\epsilon)[L_0 g_z^{-1}] + \epsilon[L_0 \tilde{g}_z^{-1}] \in \mathcal{T}(\mathbb{Q}), \quad \text{with } g_z = \begin{pmatrix} \pi_\infty^r & u \\ 0 & 1 \end{pmatrix} \text{ and } \tilde{g}_z = \begin{pmatrix} \pi_\infty^{r-1} & u \\ 0 & 1 \end{pmatrix}.$$

3. Kronecker limit formula

Let $\mathfrak{a}$ be an ideal of A. For $z \in \mathfrak{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, we are interested in the following "non-holomorphic" Eisenstein series

$$\mathbb{E}^{\mathfrak{a}}(z,s) := \sideset{}{'}\sum_{c \in A, d \in \mathfrak{a}} \frac{|z|_i^s}{|cz+d|_\infty^{2s}}.$$

Here $'$ means that c and d are not both zero. It is clear that $\mathbb{E}^{\mathfrak{a}}(\gamma z, s) = \mathbb{E}^{\mathfrak{a}}(z,s)$ for each $\gamma \in \mathrm{GL}(\Lambda^{\mathfrak{a}})$. The main theorem of this section is stated below.

Theorem 3.1. (1) $\mathbb{E}^{\mathfrak{a}}(z,s)$ *converges absolutely for* $\mathrm{Re}(s) > 1$ *and has meromorphic continuation to the whole s-plane.*
(2) *Suppose $|z|_i = |z-u|_\infty = q_\infty^{-r+\epsilon}$ with $u \in k_\infty$, $r \in \mathbb{Z}$, and $0 \leq \epsilon < 1$. We let* $\tilde{z} := \pi_\infty^{1-2\epsilon}(z-u) + u \in \mathbb{C}_\infty$ (which says $|\tilde{z}|_i = |\tilde{z}-u|_\infty = q_\infty^{-r+1-\epsilon}$), *and*

$$\widetilde{\mathbb{E}}^{\mathfrak{a}}(z,s) := \|\mathfrak{a}\|^s \cdot \begin{cases} \dfrac{q^{(2g_k-2+\deg\infty)s}}{q_\infty^s - q_\infty^{-s}} \cdot \mathbb{E}^{\mathfrak{a}}(z,s) & \text{if } \epsilon = 0, \\ \dfrac{q^{(2g_k-2+\deg\infty)s}}{q_\infty^{\epsilon s} + q_\infty^{(1-\epsilon)s} - q_\infty^{-\epsilon s} - q_\infty^{(\epsilon-1)s}} \cdot \left(\mathbb{E}^{\mathfrak{a}}(z,s) + \mathbb{E}^{\mathfrak{a}}(\tilde{z},s)\right) & \text{if } 0 < \epsilon < 1. \end{cases}$$

We have the following functional equation

$$\widetilde{\mathbb{E}}^{\mathfrak{a}}(z,s) = \widetilde{\mathbb{E}}^{\mathfrak{a}^{-1}}(z^{-1}, 1-s).$$

(3) *(Kronecker limit formula) For every $z \in \mathfrak{H}$, $\mathbb{E}^{\mathfrak{a}}(z,0) = -1$ and*

$$\frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,s)\Big|_{s=0} = -\ln\left(|z|_i \cdot |\Delta^{\mathfrak{a}}(z)|_\infty^{\frac{2}{q_\infty^2-1}}\right).$$

We first connect $\mathbb{E}^{\mathfrak{a}}(z,s)$ with "adelic" Eisenstein series in Section 3.1, and give the proof of Theorem 3.1 in Section 3.2.

3.1. **Adelic Eisenstein series.** Let χ be a character on $\mathrm{Pic}(A) \cong k^\times \backslash \mathbb{A}^\times / k_\infty^\times O_{\mathbb{A}}^\times$. The principal series $I(s,\chi)$ consists of smooth (i.e. locally constant) functions $\Phi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ satisfying that for $a, b \in \mathbb{A}^\times$ and $g \in \mathrm{GL}_2(\mathbb{A})$,

$$\Phi\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g\right) = \chi(b)|a|_{\mathbb{A}}^s |b|_{\mathbb{A}}^{-s}\Phi(g).$$

Let $\Phi^0_\chi(\cdot, s) \in I(s,\chi)$ be the function defined by

$$\Phi^0_\chi\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix}\kappa, s\right) := \chi(b)|a|^s_{\mathbb{A}}|b|^{-s}_{\mathbb{A}}, \quad \forall a,b \in \mathbb{A}^\times \text{ and } \kappa \in \mathrm{GL}_2(O_{\mathbb{A}}).$$

The *adelic Eisenstein series associated to* Φ^0_χ is

$$\mathcal{E}(g,s,\chi) := \sum_{\gamma \in \mathrm{B}(k)\backslash \mathrm{GL}_2(k)} \Phi^0_\chi(\gamma g, s), \quad \forall g \in \mathrm{GL}_2(\mathbb{A}).$$

Let

$$L_A(s,\chi) := \sum_{\mathfrak{a} \lhd A} \frac{\chi(\mathfrak{a})}{\|\mathfrak{a}\|^s}, \quad \mathrm{Re}(s) > 1.$$

It is known that $L_A(s,\chi)$ has meromorphic continuation to the whole s-plane and satisfies the following functional equation (cf. [17, 7-19 Theorem])

$$\widetilde{L}(s,\chi) = \chi(\delta)\widetilde{L}(1-s,\chi^{-1}),$$

where δ is a (any) differential idele of k, $\widetilde{L}(s,\chi) := q^{(g_k-1)s}(1-q_\infty^{-s})^{-1}L_A(s,\chi)$ and g_k is the genus of k. In particular, $L_A(s,\chi)$ is holomorphic at $s=0$ and vanishes if and only if χ is non-trivial. We recall the analytic behavior of $\mathcal{E}(g,s,\chi)$ as follows.

Theorem 3.2. (cf. [1, Theorem 3.7.1 and 3.7.2, Proposition 3.7.2 and 3.7.5])

(1) $\mathcal{E}(g,s,\chi)$ *converges absolutely for* $\mathrm{Re}(s) > 1$ *and has meromorphic continuation to the whole s-plane with the following functional equation*

$$\widetilde{\mathcal{E}}(g,s,\chi) = \chi(\det g)\widetilde{\mathcal{E}}(g,1-s,\chi^{-1}),$$

where $\widetilde{\mathcal{E}}(g,s,\chi) := \widetilde{L}(2s,\chi^{-1})\mathcal{E}(g,s,\chi)$.

(2) *For every* $g \in \mathrm{GL}_2(\mathbb{A})$, $\mathcal{E}(g,s,\chi)$ *is holomorphic at* $s=0$. *In particular,* $\mathcal{E}(g,s,\chi) = 1$ *if* χ *is trivial.*

The expression of $\mathcal{E}(g,s,\chi)$ below will be used later.

Lemma 3.3. *Given an ideal* $\mathfrak{a}$ *of* A, $r \in \mathbb{Z}$, *and* $u \in k_\infty$, *let* $g = (g^\infty_{\mathfrak{a}}, g_\infty) \in \mathrm{GL}_2(\mathbb{A}^\infty) \times \mathrm{GL}_2(k_\infty) = \mathrm{GL}_2(\mathbb{A})$ *where*

$$g^\infty_{\mathfrak{a}} = \begin{pmatrix} 1 & 0 \\ 0 & \pi^{\mathfrak{a}^{-1}} \end{pmatrix} \quad \text{and} \quad g_\infty = \begin{pmatrix} \pi^r_\infty & u \\ 0 & 1 \end{pmatrix}.$$

We have

$$\mathcal{E}(g,s,\chi) = \frac{\|\mathfrak{a}\|^s}{q-1} \sum_{[\mathfrak{n}] \in \mathrm{Pic}(A)} \chi(\mathfrak{n}^{-1}) \cdot \left(\frac{1}{\|\mathfrak{n}\|^{2s}} \sideset{}{'}\sum_{\substack{c \in \mathfrak{n}^{-1}, d \in \mathfrak{n}^{-1}\mathfrak{a} \\ c\mathfrak{a}+dA = \mathfrak{n}^{-1}\mathfrak{a}}} \frac{|\pi^r_\infty|^s_\infty}{\max(|c\pi^r_\infty|_\infty, |cu+d|_\infty)^{2s}} \right).$$

Proof. Without loss of generality, assume $\mathfrak{a} \lhd A$. It is observed that we have the following two bijections (cf. [19, Proposition 10 in §2.3]):

$$\begin{array}{ccc} \mathrm{Pic}(A) & \cong & \mathrm{B}(k)\backslash \mathrm{GL}_2(k)/\mathrm{GL}(\Lambda^{\mathfrak{a}}) \\ [\mathfrak{n}] & \longmapsto & \gamma_{\mathfrak{n}} := \begin{pmatrix} 0 & 1 \\ x_{\mathfrak{n}} & 1 \end{pmatrix} \end{array}$$

where $x_{\mathfrak{n}} \in k^\times$ such that $x_{\mathfrak{n}} \cdot A = \frac{\mathfrak{m}}{\mathfrak{n}'}$ with $\mathfrak{m}, \mathfrak{n}' \lhd A$, $(\mathfrak{m},\mathfrak{n}') = A$, and $[\mathfrak{n}', \mathfrak{a}] = \mathfrak{n}$; and

$$\begin{array}{ccc} \left(\gamma_{\mathfrak{n}}^{-1}\mathrm{B}(k)\gamma_{\mathfrak{n}} \cap \mathrm{GL}(\Lambda^{\mathfrak{a}})\right)\backslash \mathrm{GL}(\Lambda^{\mathfrak{a}}) & \cong & \{(0,0) \neq (c,d) \in \mathfrak{n}^{-1} \times \mathfrak{n}^{-1}\mathfrak{a} : c\mathfrak{a} + dA = \mathfrak{n}^{-1}\mathfrak{a}\}/\mathbb{F}_q^\times \\ \begin{pmatrix} a & b \\ \alpha & \beta \end{pmatrix} \in \mathrm{GL}(\Lambda^{\mathfrak{a}}) & \longmapsto & (ax_{\mathfrak{n}} + \alpha, bx_{\mathfrak{n}} + \beta). \end{array}$$

Here $\mathbb{F}_q^\times$ acts diagonally on $\mathfrak{n}^{-1}\times\mathfrak{n}^{-1}\mathfrak{a}$ by multiplication. We then obtain that

$$\begin{aligned}\mathcal{E}(g,s,\chi) &= \|\mathfrak{a}\|^s\cdot\sum_{[\mathfrak{n}]\in\mathrm{Pic}(A)}\frac{\chi(\mathfrak{n}^{-1})}{\|\mathfrak{n}\|^{2s}}\cdot\left(\sum_{\gamma\in\gamma_\mathfrak{n}^{-1}\mathrm{B}(k)\gamma_\mathfrak{n}\cap\mathrm{GL}(\Lambda^\mathfrak{a})\backslash\mathrm{GL}(\Lambda^\mathfrak{a})}\Phi_\chi^0\left((1,\gamma_\mathfrak{n}\gamma\begin{pmatrix}\pi_\infty^r & u\\ 0 & 1\end{pmatrix}),s\right)\right)\\ &= \frac{\|\mathfrak{a}\|^s}{q-1}\cdot\sum_{[\mathfrak{n}]\in\mathrm{Pic}(A)}\chi(\mathfrak{n}^{-1})\cdot\left(\frac{1}{\|\mathfrak{n}\|^{2s}}\sideset{}{'}\sum_{\substack{c\in\mathfrak{n}^{-1},d\in\mathfrak{n}^{-1}\mathfrak{a}\\ c\mathfrak{a}+dA=\mathfrak{n}^{-1}\mathfrak{a}}}\frac{|\pi_\infty^r|_\infty^s}{\max(|c\pi_\infty^r|_\infty,|cu+d|_\infty)^{2s}}\right).\end{aligned}$$

□

Let $\widehat{\mathrm{Pic}(A)}$ be the group of characters on $\mathrm{Pic}(A)$ and set

$$\mathcal{E}(g,s):=\frac{(q-1)}{\#\,\mathrm{Pic}(A)}\sum_{\chi\in\widehat{\mathrm{Pic}(A)}}L_A(2s,\chi^{-1})\mathcal{E}(g,s,\chi).$$

The connection between $\mathbb{E}^\mathfrak{a}(z,s)$ and the adelic Eisenstein series $\mathcal{E}(g,s)$ is described in the following.

Proposition 3.4. *For each ideal $\mathfrak{a}$ of A and $z\in\mathfrak{H}$ with $|z|_i=|z-u|_\infty=q_\infty^{-r+\epsilon}$ for some $u\in k_\infty$, $r\in\mathbb{Z}$, and $0\le\epsilon<1$, we have*

$$\|\mathfrak{a}\|^s\mathbb{E}^\mathfrak{a}(z,s)=\frac{q_\infty^{(\epsilon-1)s}-q_\infty^{(1-\epsilon)s}}{q_\infty^{-s}-q_\infty^{s}}\mathcal{E}(g_{\mathfrak{a},z},s)+\frac{q_\infty^{-\epsilon s}-q_\infty^{\epsilon s}}{q_\infty^{-s}-q_\infty^{s}}\mathcal{E}(g'_{\mathfrak{a},z},s),$$

where $g_{\mathfrak{a},z}$ and $g'_{\mathfrak{a},z}$ are in $\mathrm{GL}_2(\mathbb{A}^\infty)\times\mathrm{GL}_2(k_\infty)=\mathrm{GL}_2(\mathbb{A})$ defined by

$$g_{\mathfrak{a},z}=\left(\begin{pmatrix}1&0\\0&\pi^{\mathfrak{a}^{-1}}\end{pmatrix},\begin{pmatrix}\pi_\infty^r&u\\0&1\end{pmatrix}\right),\ g'_{\mathfrak{a},z}=\left(\begin{pmatrix}1&0\\0&\pi^{\mathfrak{a}^{-1}}\end{pmatrix},\begin{pmatrix}\pi_\infty^{r-1}&u\\0&1\end{pmatrix}\right).$$

Proof. First, we write $\mathbb{E}^\mathfrak{a}(z,s)$ as

$$\begin{aligned}\mathbb{E}^\mathfrak{a}(z,s) &= \sum_{0\neq\mathfrak{c}\lhd A}\sideset{}{'}\sum_{\substack{c\in\mathfrak{c},d\in\mathfrak{c}\mathfrak{a}\\ c\mathfrak{a}+dA=\mathfrak{c}\mathfrak{a}}}\frac{|z|_i^s}{|cz+d|_\infty^{2s}}\\ &= \sum_{[\mathfrak{n}]\in\mathrm{Pic}(A)}\zeta_{[\mathfrak{n}^{-1}]}(2s)\cdot\left(\frac{1}{\|\mathfrak{n}\|^{2s}}\sideset{}{'}\sum_{\substack{c\in\mathfrak{n}^{-1},d\in\mathfrak{n}^{-1}\mathfrak{a}\\ c\mathfrak{a}+dA=\mathfrak{n}^{-1}\mathfrak{a}}}\frac{|z|_i^s}{|cz+d|_\infty^{2s}}\right),\\ &= \sum_{[\mathfrak{n}]\in\mathrm{Pic}(A)}\zeta_{[\mathfrak{n}^{-1}]}(2s)\cdot\left(\frac{1}{\|\mathfrak{n}\|^{2s}}\sideset{}{'}\sum_{\substack{c\in\mathfrak{n}^{-1},d\in\mathfrak{n}^{-1}\mathfrak{a}\\ c\mathfrak{a}+dA=\mathfrak{n}^{-1}\mathfrak{a}}}\frac{|z|_i^s}{\max\{|cz|_i,|cu+d|_\infty\}^{2s}}\right),\end{aligned}$$

where $\zeta_{[\mathfrak{n}]}(s)$ is the partial zeta function $\sum_{0\neq\mathfrak{c}\lhd A,\ \mathfrak{c}\in[\mathfrak{n}]}\|\mathfrak{c}\|^{-s}$. Since

$$\zeta_{[\mathfrak{n}^{-1}]}(s)=\frac{1}{\#\,\mathrm{Pic}(A)}\sum_{\chi\in\widehat{\mathrm{Pic}(A)}}\chi(\mathfrak{n}^{-1})L_A(s,\chi^{-1}),$$

By Lemma 3.3, the result is then straightforward. □

3.2. **Proof of Theorem 3.1.** Given $z\in\mathfrak{H}$ with $|z|_i=|z-u|_\infty=q^{-r+\epsilon}$ where $u\in k_\infty$, $r\in\mathbb{Z}$, and $0\le\epsilon<1$, Proposition 3.4 shows that

$$\widetilde{\mathbb{E}}^\mathfrak{a}(z,s)=\frac{(q-1)}{\#\,\mathrm{Pic}(A)}\cdot\begin{cases}\displaystyle\sum_{\chi\in\widehat{\mathrm{Pic}(A)}}\widetilde{\mathcal{E}}(g_{\mathfrak{a},z},s,\chi), & \text{if } \epsilon=0,\\ \displaystyle\sum_{\chi\in\widehat{\mathrm{Pic}(A)}}\widetilde{\mathcal{E}}(g_{\mathfrak{a},z},s,\chi)+\widetilde{\mathcal{E}}(g'_{\mathfrak{a},z},s,\chi), & \text{if } 0<\epsilon<1.\end{cases}$$

On the other hand,

$$\begin{aligned}
\|\mathfrak{a}^{-1}\|^s \cdot \mathbb{E}^{\mathfrak{a}^{-1}}(z^{-1},s) &= \|\mathfrak{a}^{-1}\|^s \cdot \sum_{c\in A, d\in \mathfrak{a}^{-1}} \frac{|z|_i^s}{|c+dz|_\infty^{2s}} \\
&= \sum_{[\mathfrak{n}]\in \operatorname{Pic}(A)} \zeta_{[\mathfrak{n}^{-1}\mathfrak{a}]}(2s) \cdot \frac{\|\mathfrak{a}\|^s}{\|\mathfrak{n}\|^{2s}} \cdot \sum_{\substack{c\in \mathfrak{n}^{-1}\mathfrak{a}, d\in \mathfrak{n}^{-1} \\ cA+d\mathfrak{a} = \mathfrak{n}^{-1}\mathfrak{a}}} \frac{|z|_i^s}{|c+dz|_\infty^{2s}}. \\
&= \frac{(q-1)}{\#\operatorname{Pic}(A)} \sum_{\chi \in \widehat{\operatorname{Pic}(A)}} \chi(\mathfrak{a}^{-1}) L_A(2s,\chi) \\
&\quad \cdot \left(\frac{q_\infty^{(\epsilon-1)s} - q_\infty^{(1-\epsilon)s}}{q_\infty^{-s}-q_\infty^s} \mathcal{E}(g_{\mathfrak{a},z},s,\chi^{-1}) + \frac{q_\infty^{-\epsilon s} - q_\infty^{\epsilon s}}{q_\infty^{-s}-q_\infty^s} \mathcal{E}(g'_{\mathfrak{a},z},s,\chi^{-1})\right),
\end{aligned}$$

which implies that

$$\widetilde{E}^{\mathfrak{a}^{-1}}(z^{-1},s) = \frac{(q-1)}{\#\operatorname{Pic}(A)} \cdot \begin{cases} \displaystyle\sum_{\chi \in \widehat{\operatorname{Pic}(A)}} \chi(\mathfrak{a}^{-1}) \cdot \widetilde{\mathcal{E}}(g_{\mathfrak{a},z},s,\chi^{-1}), & \text{if } \epsilon = 0, \\ \displaystyle\sum_{\chi \in \widehat{\operatorname{Pic}(A)}} \chi(\mathfrak{a}^{-1}) \cdot \left(\widetilde{\mathcal{E}}(g_{\mathfrak{a},z},s,\chi^{-1}) + \widetilde{\mathcal{E}}(g'_{\mathfrak{a},z},s,\chi^{-1})\right), & \text{if } 0<\epsilon<1. \end{cases}$$

Thus the meromorphic continuation and functional equation of $\mathbb{E}^{\mathfrak{a}}(z,s)$ follow from Theorem 3.2 (1). Note that for $\chi \in \widehat{\operatorname{Pic}(A)}$, $L_A(0,\chi) = -\frac{\#\operatorname{Pic}(A)}{q-1}$ if χ is trivial and 0 otherwise. Hence by Theorem 3.2 (2) and Proposition 3.4 we obtain that $\mathbb{E}^{\mathfrak{a}}(z,0) = -1$ for every $z \in \mathfrak{H}$.

Now, for $w \in \mathbb{C}_\infty - \Lambda^{\mathfrak{a}}(z)$ we consider the following Eisenstein series of "Jacobi-type":

$$\mathbb{E}^{\mathfrak{a}}(z,w,s) := \sum_{\lambda \in \Lambda^{\mathfrak{a}}(z)} \frac{|z|_i^s}{|\lambda - w|_\infty^{2s}},$$

which satisfies

$$\mathbb{E}^{\mathfrak{a}}\left(\frac{az+b}{cz+d}, \frac{w}{cz+d}, s\right) = \mathbb{E}^{\mathfrak{a}}(z,w,s), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{GL}(\Lambda^{\mathfrak{a}}).$$

Then

$$\mathbb{E}^{\mathfrak{a}}(z,w,s) - \mathbb{E}^{\mathfrak{a}}(z,s) = \frac{|z|_i^s}{|w|_\infty^{2s}} + \sum_{\substack{0\neq \lambda \in \Lambda^{\mathfrak{a}}(z) \\ |\lambda|_\infty \leq |w|_\infty}} \left(\frac{|z|_i^s}{|\lambda - w|_\infty^{2s}} - \frac{|z|_i^s}{|\lambda|_\infty^{2s}}\right)$$

is entire (with respect to the variable s), which gives the meromorphic continuation of $\mathbb{E}^{\mathfrak{a}}(z,w,s)$. Moreover, $\mathbb{E}^{\mathfrak{a}}(z,w,0) = 0$ and

$$\begin{aligned}
\frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,w,s)\Big|_{s=0} &= \frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,s)\Big|_{s=0} + \ln|z|_i - 2\left(\ln|w|_\infty + \sum_{\substack{0\neq \lambda \in \Lambda^{\mathfrak{a}}(z) \\ |\lambda|_\infty \leq |w|_\infty}} \ln|1-w/\lambda|_\infty\right) \\
&= \frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,s)\Big|_{s=0} + \ln|z|_i - 2\ln|\exp_{\Lambda^{\mathfrak{a}}(z)}(w)|_\infty.
\end{aligned}$$

Taking $a \in A - \mathbb{F}_q$, it is clear that

$$(1-|a|_\infty^{-2s})\mathbb{E}^{\mathfrak{a}}(z,s) = |a|_\infty^{-2s} \cdot \sum_{0\neq w \in \frac{1}{a}\Lambda^{\mathfrak{a}}(z)/\Lambda^{\mathfrak{a}}(z)} \mathbb{E}^{\mathfrak{a}}(z,w,s).$$

Therefore

$$\begin{aligned} 2\ln|a|_\infty \cdot \mathbb{E}^{\mathfrak{a}}(z,0) &= \sum_{0\neq w\in \frac{1}{a}\Lambda^{\mathfrak{a}}(z)/\Lambda^{\mathfrak{a}}(z)} \frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,w,s)\Big|_{s=0} \\ &= (|a|_\infty^2-1)\left(\frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,s)\Big|_{s=0}+\ln|z|_i\right) - 2\sum_{0\neq w\in \frac{1}{a}\Lambda^{\mathfrak{a}}(z)/\Lambda^{\mathfrak{a}}(z)} \ln|\exp_{\Lambda^{\mathfrak{a}}(z)}(w)|_\infty. \end{aligned}$$

By Lemma 2.3, we then obtain that

$$\begin{aligned} \frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,s)\Big|_{s=0} &= -\left[\ln|z|_i + (|a|_\infty^2-1)^{-1}\left(2\ln|a|_\infty - 2\sum_{0\neq w\in \frac{1}{a}\Lambda^{\mathfrak{a}}(z)/\Lambda^{\mathfrak{a}}(z)} \ln|\exp_{\Lambda^{\mathfrak{a}}(z)}(w)|_\infty\right)\right] \\ &= -\ln\left(|z|_i\cdot|\Delta_a^{\mathfrak{a}}(z)|_\infty^{\frac{2}{q^{2\deg a}-1}}\right) \\ &= -\ln\left(|z|_i\cdot|\Delta^{\mathfrak{a}}(z)|_\infty^{\frac{2}{q_\infty^2-1}}\right). \end{aligned}$$

□

Remark 3.5. (1) Different from the classical case, the residue of $\mathbb{E}^{\mathfrak{a}}(z,s)$ at $s=1$ depends upon the imaginary part of z by Proposition 3.4.
(2) For every $z\in\mathfrak{H}$ and $w\in\mathbb{C}_\infty-\Lambda^{\mathfrak{a}}(z)$, we have

$$\frac{\partial}{\partial s}\mathbb{E}^{\mathfrak{a}}(z,w,s)\Big|_{s=0} = -\frac{2}{q_\infty^2-1}\ln|\Delta^{\mathfrak{a}}(z)|_\infty - 2\ln|\exp_{\Lambda^{\mathfrak{a}}(z)}(w)|_\infty. \tag{3.1}$$

When $\mathfrak{a}=A$, this coincides with Kondo's formula in [15, Theorem 1] in the case of rank 2.
(3) Given an arbitrary rank 2 A-lattice $\Lambda\subset\mathbb{C}_\infty$, take $\alpha\in\mathbb{C}_\infty$, $z\in\mathfrak{H}$, and $\mathfrak{a}\lhd A$ such that $\Lambda=\alpha(Az+\mathfrak{a})$. Define

$$\mathbb{E}(\Lambda,s) := \sum_{0\neq\lambda\in\Lambda}|\lambda|_\infty^{-2s} = |z|_i^{-s}|\alpha|_\infty^{-2s}\mathbb{E}^{\mathfrak{a}}(z,s),$$

which also has meromorphic continuation and $\mathbb{E}(\Lambda,0)=-1$. On the other hand, let ϕ^Λ be the rank 2 Drinfeld A-module over $\mathbb{C}_\infty$ associated to Λ. Writing $\phi_a^\Lambda(x)=ax+\cdots+\Delta_a(\Lambda)x^{q^{2\deg a}}$ for each $a\in A$, it is observed that

$$\Delta_a(\Lambda)=\alpha^{-q^{2\deg a}+1}\Delta_a^{\mathfrak{a}}(z).$$

Let $\Delta(\Lambda):=\Delta^{\mathfrak{a}}(z)/\alpha^{q_\infty^2-1}$. Then Theorem 3.1 (3) implies:

$$\frac{\partial}{\partial s}\mathbb{E}(\Lambda,s)\Big|_{s=0} = -\frac{2}{q_\infty^2-1}\ln|\Delta(\Lambda)|_\infty. \tag{3.2}$$

(4) Let $\mathcal{O}(\mathfrak{H})^\times$ be the group of invertible holomorphic functions on $\mathfrak{H}$ and $\mathcal{H}(\mathcal{T},\mathbb{Z})$ be the group of $\mathbb{Z}$-valued harmonic cochains on $\mathcal{T}$. Recall the *van der Put logarithmic derivative* $r:\mathcal{O}(\mathfrak{H})^\times\to\mathcal{H}(\mathcal{T},\mathbb{Z})$ defined by (cf. [8, (1.7.5)]): for $f\in\mathcal{O}(\mathfrak{H})^\times$ and $e\in\vec{E}(\mathcal{T})$ with its origin $o(e)$ and terminus $t(e)$,

$$r(f)(e) := \log_{q_\infty}\left(\sup_{z\in\lambda^{-1}(t(e))}|f(z)|_\infty\right) - \log_{q_\infty}\left(\sup_{z\in\lambda^{-1}(o(e))}|f(z)|_\infty\right).$$

Here $\lambda:\mathfrak{H}\to\mathcal{T}(\mathbb{Q})$ is the building map introduced in Section 2.5. Note that for each $v\in V(\mathcal{T})$, $|\Delta^{\mathfrak{a}}(z)|_\infty$ remains the same when z varies in $\lambda^{-1}(v)$. Therefore our limit formula says

$$r(\Delta^{\mathfrak{a}})(e) = \frac{q_\infty^2-1}{2\ln q_\infty}\cdot\frac{\partial}{\partial s}\Big(\mathbb{E}^{\mathfrak{a}}(z_1,s)-\mathbb{E}^{\mathfrak{a}}(z_2,s)\Big)_{s=0} - \frac{q_\infty^2-1}{2}\cdot\operatorname{sgn}(e),$$

for every $z_1 \in \lambda^{-1}(o(e))$, $z_2 \in \lambda^{-1}(t(e))$, where

$$\mathrm{sgn}(e) = \begin{cases} 1, & \text{if } e \text{ points to the end } \infty \text{ of } \mathcal{T}, \\ -1, & \text{otherwise.} \end{cases}$$

For each $v \in V(\mathcal{T})$, take $g_v \in \mathrm{GL}_2(k_\infty)$ so that $v = [L_0 g_v^{-1}]$ where $L_0 = O_\infty^2 \subset k_\infty^2$ is the standard O_∞-lattice (cf. Section 2.5). Let

$$g^*_{\mathfrak{a},v} := \left(\begin{pmatrix} \pi^{\mathfrak{a}} & 0 \\ 0 & 1 \end{pmatrix}, g_v\right) \in \mathrm{GL}_2(\mathbb{A}^\infty) \times \mathrm{GL}_2(k_\infty) = \mathrm{GL}_2(\mathbb{A}).$$

By Proposition 3.4 and Theorem 3.1 (2), we may write

$$r(\Delta^{\mathfrak{a}})(e) = q^{2g_k - 2 + 2\deg\infty} \cdot \Big(\mathcal{E}(g^*_{\mathfrak{a},o(e)}, s) - \mathcal{E}(g^*_{\mathfrak{a},t(e)}, s)\Big)_{s=1} - \frac{q_\infty^2 - 1}{2} \cdot \mathrm{sgn}(e).$$

Here $\mathcal{E}(g,s)$ is the adelic Eisenstein series defined in the above of Proposition 3.4. This generalizes Gekeler's formula in [7, 2.8 COROLLARY] to arbitrary global function fields. Via the standard results on the Whittaker functions of adelic Eisenstein series (cf. [1, the proof of Theorem 3.7.1]), the Fourier expansion of $r(\Delta^{\mathfrak{a}})$ can be understood accordingly.

4. Application I: Taguchi height of Drinfeld modules

Let F be a finite extension of k (viewing as an A-field via $A \hookrightarrow F$) and ϕ be a Drinfeld A-module of rank r over F. Recall that for each $a \in A$, we write

$$\phi_a = \sum_{i=1}^{r \deg a} l_i(\phi_a)\tau^i \in F\{\tau\}.$$

Denote by O_F the integral closure of A in F. For each prime ideal $\mathcal{P}$ of O_F, put

$$\mathrm{ord}_{\mathcal{P}}(\phi) := \min\{\mathrm{ord}_{\mathcal{P}}(\phi_a) : 0 \neq a \in A\}$$

where $\mathrm{ord}_{\mathcal{P}}(\phi_a) := \min\{\mathrm{ord}_{\mathcal{P}}(l_i(\phi_a))/(q^i - 1) : i \geq 1\}$. Let $\mathcal{L}_\phi$ be the fractional ideal of O_F such that for every prime $\mathcal{P}$ of O_F,

$$\mathrm{ord}_{\mathcal{P}}(\mathcal{L}_\phi) = \lfloor \mathrm{ord}_{\mathcal{P}}(\phi) \rfloor.$$

For each place w of F with $w \nmid \infty$, set the local height at w by

$$h_{\mathrm{Tag},w}(\phi/F) := -[\mathbb{F}_w : \mathbb{F}_q] \cdot \mathrm{ord}_{\mathcal{P}_w}(\mathcal{L}_\phi),$$

where $\mathcal{P}_w \lhd O_F$ is the prime ideal corresponding to w, and $\mathbb{F}_w := O_F/\mathcal{P}_w$.

To define the local height at places of F lying above ∞, we first introduce the "volume" associated to a given A-lattice. Let $\Lambda \subset \mathbb{C}_\infty$ be a rank r A-lattice. Choose a "good" k_∞-basis $\{\lambda_i\}_{1 \leq i \leq r}$ of $k_\infty \cdot \Lambda$ satisfying that

(i) $\lambda_i \in \Lambda$ for $1 \leq i \leq r$;
(ii) $|a_1\lambda_1 + \cdots + a_r\lambda_r|_\infty = \max\{|a_i\lambda_i|_\infty ; 1 \leq i \leq r\}$ for all $a_1, ..., a_r \in k_\infty$.
(iii) $k_\infty \cdot \Lambda = \Lambda + (O_\infty\lambda_1 + \cdots O_\infty\lambda_r)$.

To show the existence of a "good" basis, by Riemann-Roch theorem it suffices to find a k-basis $\{\lambda_i\}_{1 \leq i \leq r}$ of $k \cdot \Lambda$ satisfying (ii). Take a non-trivial k-linear functional $f : k \cdot \Lambda \to k$, and extend to a k_∞-linear functional on $k_\infty \cdot \Lambda$ (still denoted by f). Then the map

$$\begin{array}{ccc} k_\infty \cdot \Lambda - \{0\} & \longrightarrow & \mathbb{R}_{\geq 0} \\ z & \longmapsto & \dfrac{|f(z)|_\infty}{|z|_\infty} \end{array}$$

yields a continuous map from the projective space $\mathbb{P}(k_\infty \cdot \Lambda)$ to $\mathbb{R}_{\geq 0}$. Since $k \cdot \Lambda$ is dense in $k_\infty \cdot \Lambda$, there exists $\lambda_1 \in k \cdot \Lambda$ such that the above map takes its maximum at λ_1. By induction, we have a basis $\{\lambda_2, ..., \lambda_r\}$ of $\ker f$ satisfying (ii). Then for $a_1, ..., a_r \in k_\infty$ with $a_1 \neq 0$,

$$\frac{|f(\lambda_1)|_\infty}{|\lambda_1|_\infty} = \frac{|f(a_1\lambda_1)|_\infty}{|a_1\lambda_1|_\infty} \geq \frac{|f(a_1\lambda_1 + \cdots a_r\lambda_r)|_\infty}{|a_1\lambda_1 + \cdots a_r\lambda_r|_\infty} = \frac{|f(a_1\lambda_1)|_\infty}{|a_1\lambda_1 + \cdots a_r\lambda_r|_\infty}.$$

Hence

$$|a_1\lambda_1 + \cdots a_r\lambda_r|_\infty = \begin{cases} |a_1\lambda_1|_\infty & \text{if } |a_2\lambda_2 + \cdots a_r\lambda_r|_\infty \leq |a_1\lambda_1|_\infty \\ \max\{|a_i\lambda_i|_\infty ; 2 \leq i \leq r\} & \text{otherwise.} \end{cases}$$

Therefore the k-basis $\{\lambda_i\}_{1\leq i\leq r}$ of $k \cdot \Lambda$ satisfies (ii).

We define the *A-volume* $D_A(\Lambda)$ associated to a given A-lattice Λ by:

$$D_A(\Lambda) := q^{1-g_k} \cdot \left(\frac{\prod_{1\leq i\leq r} |\lambda_i|_\infty}{\#\big(\Lambda \cap (O_\infty\lambda_1 + \cdots + O_\infty\lambda_r)\big)} \right)^{1/r} = \left(\frac{\prod_{1\leq i\leq r} |\lambda_i|_\infty}{\#\big(\Lambda/(A\lambda_1 + \cdots + A\lambda_r)\big)} \right)^{1/r}.$$

Here $\{\lambda_i\}_{1\leq i\leq r}$ is a "good" basis. The second equality is from the Riemann-Roch Theorem. It is clear that $D_A(\Lambda)$ is independent of the chosen good basis $\{\lambda_i\}_{1\leq i\leq r}$. In particular, given two A-lattices Λ and Λ' of rank r with $\Lambda' \subset \Lambda$, we have $D_A(\Lambda') = D_A(\Lambda) \cdot \#(\Lambda/\Lambda')^{1/r}$.

For each place $\tilde{\infty}$ of F with $\tilde{\infty} \mid \infty$, we embed F into $\mathbb{C}_\infty$ via $\tilde{\infty}$ and let $\Lambda_{\phi,\tilde{\infty}} \subset \mathbb{C}_\infty$ be the A-lattice associated to ϕ. Then $\Lambda_{\phi,\tilde{\infty}}$ is of rank r, and we set

$$h_{\mathrm{Tag},\tilde{\infty}}(\phi/F) := -[F_{\tilde{\infty}} : k_\infty] \cdot \log_q D_A(\Lambda_{\phi,\tilde{\infty}}),$$

Definition 4.1. (cf. [21, Section 5]) The *Taguchi height of* ϕ/F is defined by

$$h_{\mathrm{Tag}}(\phi/F) := \frac{1}{[F:k]} \cdot \left(\sum_{w\nmid\infty} h_{\mathrm{Tag},w}(\phi/F) + \sum_{\tilde{\infty}\mid\infty} h_{\mathrm{Tag},\tilde{\infty}}(\phi/F) \right).$$

Remark 4.2. (1) Let F' be a finite extension F' over F. Given places $\tilde{\infty}$ of F and $\tilde{\infty}'$ of F' with $\tilde{\infty}' \mid \tilde{\infty} \mid \infty$, it is clear that $\Lambda_{\phi,\tilde{\infty}} = \Lambda_{\phi,\tilde{\infty}'} \subset \mathbb{C}_\infty$, and

$$h_{\mathrm{Tag},\tilde{\infty}'}(\phi/F') = [F'_{\tilde{\infty}'} : F_{\tilde{\infty}}] \cdot h_{\mathrm{Tag},\tilde{\infty}}(\phi/F).$$

For each prime $\mathcal{P}$ of O_F, one has $\mathrm{ord}_{\mathcal{P}'}(\phi) = e_{\mathcal{P}'/\mathcal{P}} \cdot \mathrm{ord}_{\mathcal{P}}(\phi)$ for every prime $\mathcal{P}'$ of $O_{F'}$ lying above $\mathcal{P}$, where $e_{\mathcal{P}'/\mathcal{P}}$ is the ramification index of $\mathcal{P}'/\mathcal{P}$. Thus for places w of F and w' of F' with $w' \mid w \nmid \infty$, we get

$$h_{\mathrm{Tag},w'}(\phi/F') \leq [F'_{w'} : F_w] \cdot h_{\mathrm{Tag},w}(\phi/F).$$

In particular, if ϕ has stable reduction at $\mathcal{P}$, then $\mathrm{ord}_{\mathcal{P}}(\phi)$ is an integer, which implies that $h_{\mathrm{Tag},w'}(\phi/F') = [F'_{w'} : F_w] \cdot h_{\mathrm{Tag},w}(\phi/F)$. In conclusion, we have $h_{\mathrm{Tag}}(\phi/F') \leq h_{\mathrm{Tag}}(\phi/F)$, and the equality holds when ϕ has stable reduction everywhere.
(2) Since every Drinfeld A-module ϕ over F has potentially stable reduction everywhere (cf. [11, Proposition 7.2]),

$$\widetilde{h}_{\mathrm{Tag}}(\phi) := \ln q \cdot \lim_{\substack{F': \\ F'/F \text{ finite}}} h_{\mathrm{Tag}}(\phi/F')$$

is well-defined.
(3) Let ϕ and ϕ' be two Drinfeld A-modules over $\overline{k}$ where $\overline{k}$ is the algebraic closure of k in $\mathbb{C}_\infty$. Then

$$\widetilde{h}_{\mathrm{Tag}}(\phi) = \widetilde{h}_{\mathrm{Tag}}(\phi') \quad \text{if } \phi \text{ and } \phi' \text{ are isomorphic over } \overline{k}.$$

4.1. **Rank 1 case.** Let ϕ be a rank 1 Drinfeld A-module over $\overline{k}$. To study $\tilde{h}_{\mathrm{Tag}}(\phi)$, we can assume that ϕ is defined over H_A (the Hilbert class field of A) satisfying that for every $0 \neq a \in A$, $\phi_a = a + \sum_{i=1}^{\deg a} l_i(\phi_a)\tau^n \in O_{H_A}\{\tau\}$ with $l_{\deg a}(\phi_a) \in O_{H_A}^\times$ (cf. [9, Theorem 7.6.4]). Thus $h_{\mathrm{Tag},w}(\phi/H_A) = 1$ for every place w of H_A with $w \nmid \infty$. Now, fix an embedding $H_A \hookrightarrow \mathbb{C}_\infty$. Let $\Lambda_\phi \subset \mathbb{C}_\infty$ be the rank 1 A-lattice associated to ϕ. We may write $\Lambda_\phi = \mathfrak{a}^{-1}\cdot\varpi_\phi$ for an ideal $\mathfrak{a}$ of A and $\varpi_\phi \in \mathbb{C}_\infty^\times$. It is observed that (cf. [12, the formula of the invariant $\xi(\mathfrak{a})$ in the end of p. 237]):

$$\ln(|\varpi_\phi|_\infty \cdot |l_{\deg a}(\phi_a)|_\infty^{\frac{1}{q^{\deg a}-1}}) = \frac{\zeta'_{[\mathfrak{a}]}(0)}{\zeta_{[\mathfrak{a}]}(0)} + \ln(\|\mathfrak{a}\|), \quad \forall a \in A - \mathbb{F}_q,$$

where $\zeta_{[\mathfrak{a}]}(s)$ is the partial zeta function associated to the ideal class $[\mathfrak{a}] \in \mathrm{Pic}(A)$:

$$\zeta_{[\mathfrak{a}]}(s) := \sum_{\substack{\mathfrak{n} \lhd A, \\ \mathfrak{n} \in [\mathfrak{a}]}} \frac{1}{\|\mathfrak{n}\|^s}.$$

On the other hand, $D_A(\Lambda_\phi) = \|\mathfrak{a}\|^{-1} \cdot |\varpi_\phi|_\infty$. Indeed, taking $a \in \mathfrak{a}^{-1}$ with $|a|_\infty$ large enough so that $k_\infty = \mathfrak{a}^{-1} + aO_\infty$ (by Riemann-Roch theorem), we get

$$D_A(\Lambda_\phi) = \frac{|a\varpi_\phi|_\infty}{\#(\mathfrak{a}^{-1}/aA)} = \frac{|\varpi_\phi|_\infty}{\|\mathfrak{a}\|}.$$

Moreover, given $\mathfrak{b} \lhd A$, let $\sigma_\mathfrak{b} \in \mathrm{Gal}(H_A/k)$ be the element corresponding to $[\mathfrak{b}] \in \mathrm{Pic}(A)$. There exists $\varpi_{\phi^{\sigma_\mathfrak{b}}} \in \mathbb{C}_\infty^\times$ such that

$$\Lambda_{\phi^{\sigma_\mathfrak{b}}} = \mathfrak{a}^{-1}\mathfrak{b}^{-1} \cdot \varpi_{\phi^{\sigma_\mathfrak{b}}}.$$

Note that $l_{\deg a}(\phi_a^{\sigma_\mathfrak{b}}) = l_{\deg a}(\phi_a)^{\sigma_\mathfrak{b}} \in O_{H_A}^\times$ for every $a \in A - \mathbb{F}_q$, which says

$$\prod_{[\mathfrak{b}] \in \mathrm{Pic}(A)} |l_{\deg a}(\phi_a^{\sigma_\mathfrak{b}})|_\infty = 1.$$

Let

$$\zeta_A(s) := \sum_{\mathfrak{n} \lhd A} \frac{1}{\|\mathfrak{n}\|^s} = \sum_{[\mathfrak{a}] \in \mathrm{Pic}(A)} \zeta_{[\mathfrak{a}]}(s).$$

Then:

Proposition 4.3. *For every rank 1 Drinfeld A-module ϕ over $\overline{k}$, we have*

$$-\tilde{h}_{\mathrm{Tag}}(\phi) = \frac{1}{\#\,\mathrm{Pic}(A)} \cdot \sum_{[\mathfrak{b}] \in \mathrm{Pic}(A)} \ln\left(\|\mathfrak{a}\mathfrak{b}\|^{-1} \cdot |\varpi_{\phi^{\sigma_\mathfrak{b}}}|_\infty\right) = \frac{\zeta'_A(0)}{\zeta_A(0)}.$$

4.2. **Rank 2 Drinfeld A-modules with complex multiplication.** Suppose an imaginary quadratic field K/k is given (i.e. K/k is a separable quadratic extension and ∞ does not split in K). Let O_K denote the integral closure of A in K. For each ideal $\mathfrak{c} \lhd A$, let $O_\mathfrak{c} := A + \mathfrak{c}O_K$, the quadratic A-order in K with conductor $\mathfrak{c}$. Let ϕ be a rank 2 Drinfeld A-module over $\overline{k}$ with complex multiplication by $O_\mathfrak{c}$ (i.e. $\mathrm{End}_A(\phi/\overline{k}) \cong O_\mathfrak{c}$). To calculate the Taguchi height of ϕ, it suffices to assume that ϕ is defined over the ring class field $H_\mathfrak{c}$ of $O_\mathfrak{c}$ (cf. [11, Theorem 8.10]). Fix an embedding of $H_\mathfrak{c} \hookrightarrow \mathbb{C}_\infty$. Let $\Lambda_\phi \subset \mathbb{C}_\infty$ be the lattice associated to ϕ. Then there exists an invertible ideal $\mathfrak{B}$ of $O_\mathfrak{c}$ and $\alpha \in \mathbb{C}_\infty^\times$ such that $\Lambda_\phi = \alpha \cdot \mathfrak{B}$. For each place w of $H_\mathfrak{c}$ with $w \nmid \infty$, let $\mathcal{P}_w$ be the prime ideal of $O_{H_\mathfrak{c}}$ corresponding to $\mathcal{P}_w$. By [11, Proposition 7.3], we obtain that

$$\mathrm{ord}_{\mathcal{P}_w}(\phi) = \frac{1}{q_\infty^2 - 1}\,\mathrm{ord}_w\left(\Delta(\Lambda_\phi)\right).$$

Therefore

$$\begin{aligned}
\widetilde{h}_{\mathrm{Tag}}(\phi) &= \frac{-\ln q}{[H_{\mathfrak{c}}:k]}\cdot\Bigg[\sum_{\substack{\text{places } w \text{ of } H_{\mathfrak{c}},\\ w\nmid\infty}} \frac{[\mathbb{F}_w:\mathbb{F}_q]}{q_\infty^2-1}\operatorname{ord}_w\big(\Delta(\Lambda_\phi)\big) \\
&\quad + \sum_{[\mathfrak{A}]\in \mathrm{Pic}(O_{\mathfrak{c}})} [K_\infty:k_\infty]\log_q\big(D_A(\Lambda_{\phi^{\sigma_{\mathfrak{A}}}})\big)\Bigg] \\
&= \frac{-1}{\#\,\mathrm{Pic}(O_{\mathfrak{c}})}\sum_{[\mathfrak{A}]\in \mathrm{Pic}(O_{\mathfrak{c}})} \ln\Big(D_A(\Lambda_{\phi^{\sigma_{\mathfrak{A}}}})\cdot|\Delta(\Lambda_{\phi^{\sigma_{\mathfrak{A}}}})|_\infty^{\frac{1}{q_\infty^2-1}}\Big) \\
&= \frac{-1}{\#\,\mathrm{Pic}(O_{\mathfrak{c}})}\sum_{[\mathfrak{A}]\in \mathrm{Pic}(O_{\mathfrak{c}})} \ln\Big(D_A(\mathfrak{B}\mathfrak{A}^{-1})\cdot|\Delta(\mathfrak{B}\mathfrak{A}^{-1})|_\infty^{\frac{1}{q_\infty^2-1}}\Big) \\
&= -\ln\big(D_A(O_{\mathfrak{c}})\big) + \frac{-1}{2\#\,\mathrm{Pic}(O_{\mathfrak{c}})}\sum_{[\mathfrak{A}]\in \mathrm{Pic}(O_{\mathfrak{c}})} \ln\Big(N(\mathfrak{A})\cdot|\Delta(\mathfrak{A})|_\infty^{\frac{2}{q_\infty^2-1}}\Big).
\end{aligned}$$

Here $\mathrm{Pic}(O_{\mathfrak{c}})$ is the class group of the invertible ideals of $O_{\mathfrak{c}}$, which is isomorphic to the Galois group $\mathrm{Gal}(H_{\mathfrak{c}}/K)$ via the Artin map; and $N(\mathfrak{A}) := \#(O_{\mathfrak{c}}/a\mathfrak{A})\cdot|a|_\infty^{-2}$ for $0\neq a\in A$ such that $a\mathfrak{A}\subset O_{\mathfrak{c}}$, which is independent of the chosen a.

On the other hand, for each class $[\mathfrak{A}]\in\mathrm{Pic}(O_{\mathfrak{c}})$, the associated partial zeta function $\zeta_{[\mathfrak{A}]^{-1}}(s)$ is equal to

$$\begin{aligned}
\zeta_{[\mathfrak{A}]^{-1}}(s) &:= \sum_{\substack{\mathfrak{B}\lhd O_{\mathfrak{c}} \text{ invertible},\\ \mathfrak{B}\sim\mathfrak{A}^{-1}}} N(\mathfrak{B})^{-s} \\
&= \frac{1}{\#(O_{\mathfrak{c}}^\times)}\sum_{0\neq\lambda\in\mathfrak{A}}\frac{N(\mathfrak{A})^s}{|\lambda|_\infty^{2s}} \\
&= \frac{N(\mathfrak{A})^s}{\#(O_{\mathfrak{c}}^\times)}\cdot\mathbb{E}(\mathfrak{A},s).
\end{aligned}$$

By our analogue of Kronecker limit formula (the version in *Remark* 3.5 (3)), we have

$$\zeta_{[\mathfrak{A}]^{-1}}(0) = -\#(O_{\mathfrak{c}}^\times)^{-1} \quad\text{and}\quad \frac{\partial}{\partial s}\zeta_{\mathfrak{A}^{-1}}(s)\Big|_{s=0} = -\#(O_{\mathfrak{c}}^\times)^{-1}\left[\ln N(\mathfrak{A}) + \frac{2}{q_\infty^2-1}\ln|\Delta(\mathfrak{A})|_\infty\right].$$

Let

$$\zeta_{O_{\mathfrak{c}}}(s) := \sum_{\text{invertible ideal } \mathfrak{A}\lhd O_{\mathfrak{c}}}\frac{1}{N(\mathfrak{A})^s} = \sum_{[\mathfrak{A}]\in\mathrm{Pic}(O_{\mathfrak{c}})}\zeta_{[\mathfrak{A}]}(s).$$

We then arrive at the following result.

Theorem 4.4. *Let ϕ be a rank 2 Drinfeld A-module over $\overline{k}$ with complex multiplication by $O_{\mathfrak{c}}$, where $O_{\mathfrak{c}}$ is a quadratic A-order of conductor $\mathfrak{c}\lhd A$ in an imaginary quadratic field K/k. Then*

$$-2\widetilde{h}_{\mathrm{Tag}}(\phi) = \ln\big(D_A(O_{\mathfrak{c}})^2\big) + \frac{\zeta_{O_{\mathfrak{c}}}'(0)}{\zeta_{O_{\mathfrak{c}}}(0)}.$$

In particular, we have:

Corollary 4.5. *For each ideal $\mathfrak{c}\lhd A$,*

$$\big(D_A(O_{\mathfrak{c}})^2\big) = \big\|\mathfrak{d}(O_{\mathfrak{c}}/A)\big\|^{\frac{1}{2}}$$

where

$$\mathfrak{d}(O_{\mathfrak{c}}/A) := \mathfrak{c}^2\cdot\prod_{\substack{\textit{prime } \mathfrak{p}\lhd A\\ \textit{ramified in } K}}\mathfrak{p}.$$

Therefore

$$-2\widetilde{h}_{\mathrm{Tag}}(\phi) = \frac{1}{2}\ln\|\mathfrak{d}(O_{\mathfrak{c}}/A)\| + \frac{\zeta'_{O_{\mathfrak{c}}}(0)}{\zeta_{O_{\mathfrak{c}}}(0)}$$

for every rank 2 *Drinfeld* A*-module* ϕ *over* $\overline{k}$ *with complex multiplication by* $O_{\mathfrak{c}}$*.*

Proof. Take $z \in \mathcal{O}_{\mathfrak{c}} \cap \mathfrak{H}$ with $|z|_\infty = |z|_i$. Let $\mathfrak{c}_z \lhd A$ be the conductor of the quadratic A-order $A[z]$ (i.e. $A[z] = O_{\mathfrak{c}_z}$). We have $\mathfrak{c} \mid \mathfrak{c}_z$ and $\#(O_{\mathfrak{c}}/A[z]) = \|\mathfrak{c}_z/\mathfrak{c}\|$. It is observed that

$$D_A(A[z])^2 = |z|_\infty = |N_{K/k}(z)|_\infty^{\frac{1}{2}} = \|\mathfrak{d}(O_{\mathfrak{c}_z}/A)\|^{\frac{1}{2}}.$$

Hence the result follows. □

Remark 4.6. (1) Suppose q is odd. Then $\mathfrak{d}(O_{\mathfrak{c}}/A) \lhd A$ is the discriminant ideal for $O_{\mathfrak{c}}/A$.
(2) Let $k = \mathbb{F}_q(t)$ be a rational function field with q odd and $A = \mathbb{F}_q[t]$. Take a square-free polynomial $D \in A - \mathbb{F}_q$ such that $K := k(\sqrt{D})$ is imaginary over k. Let $O_K = A[\sqrt{D}]$. One has

$$\zeta_{O_K}(s) = \zeta_A(s) L_A(s, \chi_D)$$

where $\chi_D := \left(\frac{D}{\cdot}\right)$, the Legendre quadratic symbol, and

$$\begin{aligned} L_A(s,\chi_D) &:= \sum_{\text{monic } m \in A} \chi_D(m)|m|_\infty^{-s} \\ &= \sum_{\substack{\text{monic } m \in A \\ \deg m < \deg D}} \chi_D(m)|m|_\infty^{-s} \\ &= |D|_\infty^{-s} \sum_{\substack{\text{monic } m \in A \\ \deg m < \deg D}} \chi_D(m)\frac{|D|_\infty^s}{|m|_\infty^s} \end{aligned}$$

We then easily get an analogue of the Chowla-Selberg formula:

$$\frac{L'_A(0,\chi_D)}{L_A(0,\chi_D)} = -\ln|D|_\infty - \frac{1}{\#\operatorname{Pic}(O_K)} \sum_{\substack{\text{monic } m \in A \\ \deg m < \deg D}} \chi_D(m) \ln\left|\frac{m}{D}\right|_\infty.$$

Note that $\zeta_A(s) = (1 - q^{(1-s)})^{-1}$ and $\zeta'_A(0)/\zeta_A(0) = q\ln q/(q-1)$. Therefore for every rank 2 Drinfeld A-module ϕ over $\overline{k}$ with complex multiplication by O_K, we have

$$\widetilde{h}_{\mathrm{Tag}}(\phi) = \frac{1}{4}\ln|D|_\infty - \frac{q\ln q}{2(q-1)} + + \frac{1}{2\#\operatorname{Pic}(O_K)} \sum_{\substack{\text{monic } m \in A \\ \deg m < \deg D}} \chi_D(m) \ln\left|\frac{m}{D}\right|_\infty.$$

Asymptotic behavior. From Ihara's estimation for the Euler-Kronecker constant of K in [13, upper bound (0.6) and lower bound (1.2)], we have

$$\frac{L'_A(1,\chi_D)}{L_A(1,\chi_D)} = O(\ln\ln|D|) \quad \text{for } |D|_\infty \gg 0.$$

The functional equation of $L_A(s,\chi_D)$ then says

$$\frac{L'_A(0,\chi_D)}{L_A(0,\chi_D)} = -\ln|D|_\infty + O(\ln\ln|D|_\infty) \quad \text{for } |D|_\infty \gg 0.$$

Therefore we immediately get

$$\widetilde{h}_{\mathrm{Tag}}(\phi) = \frac{1}{4}\ln|D|_\infty + O(\ln\ln|D|_\infty) \quad \text{for } |D|_\infty \gg 0.$$

5. Application II: On derivative of Rankin-type L-functions

5.1. Drinfeld-type autmorphic forms. Given an ideal $\mathfrak{N} \lhd A$, let

$$\mathcal{K}_0(\mathfrak{N}\infty) := \mathcal{K}_\infty \times \prod_{v \neq \infty} \mathcal{K}_{v^{\mathrm{ord}_v(\mathfrak{N})}}$$

where for $\ell \geq 0$,

$$\mathcal{K}_{v^\ell} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(O_v) \;\middle|\; c \in \pi_v^\ell O_v \right\}.$$

A *Drinfeld-type automorphic form f of level $\mathfrak{N}$* is a $\mathbb{C}$-valued function on the double coset space

$$\mathbb{Y}_0(\mathfrak{N}) := \mathrm{GL}_2(k)\backslash \mathrm{GL}_2(\mathbb{A})/k_\infty^\times \mathcal{K}_0(\mathfrak{N}\infty)$$

satisfying the *harmonic property*: for every $g \in \mathrm{GL}_2(\mathbb{A})$,

$$f\left(g\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}\right) = -F(g) \quad \text{and} \quad \sum_{\kappa \in \mathrm{GL}_2(O_\infty)/\mathcal{K}_\infty} F(g\kappa) = 0.$$

Here we embed $\mathrm{GL}_2(k_\infty)$ into $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}^\infty) \times \mathrm{GL}_2(k_\infty)$ by sending g_∞ to $(1, g_\infty)$. Suppose further that f satisfies

$$f\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} g\right) = \omega_f(a) f(g), \text{ for all } a \in \mathbb{A}^\times.$$

where ω_f is a character on $k^\times\backslash\mathbb{A}^\times/k_\infty^\times O_\mathbb{A}^\times \cong \mathrm{Pic}(A)$, we call ω_f the central character of f. If

$$\int_{k\backslash\mathbb{A}} f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) du = 0, \quad \forall g \in \mathrm{GL}_2(\mathbb{A})$$

then f is called a *cusp form*. Here the Haar measure du is normalize so that the volume of $k\backslash\mathbb{A}$ is 1. In the following, we recall the basic properties of these forms to be used, and refer the reader to [4] for further details.

5.1.1. *Fourier coefficients.* Let f be a Drinfeld-type cusp form of level $\mathfrak{N}$ with central character ω_f. For each non-zero ideal $\mathfrak{m} \lhd A$, the $\mathfrak{m}$-th Fourier coefficient $f^*(\mathfrak{m})$ of f is given by

$$f^*(\mathfrak{m}) := \int_{k\backslash\mathbb{A}} f\left(\begin{pmatrix} \delta^{-1}\pi^\mathfrak{m} & u \\ 0 & 1 \end{pmatrix}\right) \psi(-u) du,$$

where $\pi^\mathfrak{m} = (\pi_v^{\mathrm{ord}_v(\mathfrak{m})})_{v \neq \infty} \in \mathbb{A}^{\infty,\times}$ and δ is the chosen differential idele of k in Section 2.1. It is known that f is uniquely determined by its Fourier coefficients $f^*(\mathfrak{m})$ for finitely many ideals $\mathfrak{m} \lhd A$.

5.1.2. *Hecke operators.* For each place v of k, the Hecke operator T_v on the space $\mathcal{M}_0(\mathfrak{N})$ of Drinfeld type automorphic forms of level $\mathfrak{N} \lhd A$ is defined by the following: for $f \in \mathcal{M}_0(\mathfrak{N})$ and $g \in \mathrm{GL}_2(\mathbb{A})$

$$(T_v f)(g) := \sum_{u \in \mathbb{F}_v} f\left(g\begin{pmatrix} \pi_v & u \\ 0 & 1 \end{pmatrix}\right) + \mu_{\mathfrak{N}\infty}(v) \cdot f\left(g\begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix}\right).$$

Here $\mu_{\mathfrak{N}\infty}(v) = 1$ if $v \nmid \mathfrak{N}\infty$ and 0 otherwise. The harmonicity of f implies that $T_\infty f = f$ for all $f \in \mathcal{M}_0(\mathfrak{N})$. Suppose f has a central character denoted by ω_f. For each ideal $\mathfrak{m} \lhd A$, we define the Hecke operator $T_\mathfrak{m}$ by the following:

$$\begin{cases} T_{\mathfrak{m}\mathfrak{m}'} := T_\mathfrak{m} \cdot T_{\mathfrak{m}'}, & \text{for } \mathfrak{m} \text{ and } \mathfrak{m}' \text{ relatively prime;} \\ T_{v^{l+2}} := T_v T_{v^{l+1}} - \mu_\mathfrak{N}(v)\omega_f(v) \cdot q_v T_{v^l}, & \text{for any finite place } v \text{ of } k. \end{cases}$$

f is called a Hecke eigenform if for each ideal $\mathfrak{m} \lhd A$, there exists $\lambda_{\mathfrak{m}}(f) \in \mathbb{C}$ such that $T_{\mathfrak{m}} f = \lambda_{\mathfrak{m}}(f) \cdot f$. These eigenvalues can be read off by the Fourier coefficients of f:

$$\|\mathfrak{m}\| \cdot f^*(\mathfrak{m}) = \lambda_{\mathfrak{m}}(f) \cdot f^*(A).$$

We call f *normalized* if $f^*(A) = 1$. For our convenience, we let $\lambda_\infty(f) = 1$, which is the Hecke eigenvalue of T_∞ on f.

5.1.3. *Petersson inner product.* Given two Drinfeld-type automorphic forms f_1 and f_2 of level $\mathfrak{N}$, suppose one of them is a cusp form. The *Petersson inner product of* f_1 *and* f_2 is:

$$\langle f_1, f_2 \rangle_{\mathfrak{N}} := \sum_{[g] \in \mathbb{Y}_0(\mathfrak{N})} f_1(g) \overline{f_2(g)} \mu([g]),$$

where for each double coset $[g] \in \mathbb{Y}_0(\mathfrak{N})$, the measure $\mu([g])$ is normalized to be

$$\mu([g]) := \frac{q-1}{\#(\mathrm{Pic}(A))} \cdot \frac{1}{\#(\mathrm{GL}_2(k) \cap g\mathcal{K}_0(\mathfrak{N}\infty) g^{-1})}.$$

A Drinfeld-type cusp form f of level $\mathfrak{N}$ is called an *old form* if f is a $\mathbb{C}$-linear combination of the form

$$f'\left(g \begin{pmatrix} 1 & 0 \\ 0 & \pi^{\mathfrak{N}''} \end{pmatrix}\right), \text{ for } g \in \mathrm{GL}_2(\mathbb{A}),$$

where f' is a Drinfeld-type cusp form of level $\mathfrak{N}'$ with $\mathfrak{N}'\mathfrak{N}'' \mid \mathfrak{N}$ and $\mathfrak{N}' \neq \mathfrak{N}$. Given a Drinfeld-type cusp form f of level $\mathfrak{N}$ which is also a Hecke eigenform, we call f a *newform* if f is orthogonal to all the old forms of level $\mathfrak{N}$.

5.1.4. *Whittaker functions.* Given a normalized Drinfeld type newform f of level $\mathfrak{N}$, the Whittaker function W_f associated to f is defined by: for $g \in \mathrm{GL}_2(\mathbb{A})$,

$$W_f(g) := \int_{k \backslash \mathbb{A}} f\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} g\right) \psi(-u) du.$$

Note that

$$W_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi(x) \cdot W_f(g) \quad \forall x \in \mathbb{A} \quad \text{and} \quad f(g) = \sum_{\alpha \in k^\times} W_f\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Lemma 5.1. (1) *For each ideal* $\mathfrak{m} \lhd A$*,*

$$W_f\begin{pmatrix} \delta^{-1}\pi^{\mathfrak{m}} & 0 \\ 0 & 1 \end{pmatrix} = f^*(\mathfrak{m}) = \frac{\lambda_{\mathfrak{m}}(f)}{\|\mathfrak{m}\|}.$$

Here δ *is the chosen differential idele associated to* ψ *in* Section 2.1.
(2) (cf. [1, Exercise 4.6.2]) *Let* f *be a normalized newform of square-free level* $\mathfrak{N}$ *with central character* ω_f*. For* $v \mid \mathfrak{N}\infty$*,*

$$W_f\left(\begin{pmatrix} \pi_v^{r-\delta_v} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \begin{cases} -\lambda_v(F)^r \cdot q_v^{-r-1}, & \text{for } r \geq -1; \\ 0, & \text{for } r < -1. \end{cases}$$

5.2. **Rankin-type L-functions.** Let f_1, f_2 be two normalized Drinfeld-type newforms of square-free levels $\mathfrak{N}_1$, $\mathfrak{N}_2$ and with central character ω_1, ω_2, respectively. The *Rankin-type* L*-function associated to* f_1 *and* f_2 is defined by

$$\begin{aligned} L(f_1 \times f_2, s) \quad := \quad & L_A^{[\mathfrak{N}_1, \mathfrak{N}_2]}(2s, \omega_1\omega_2) \cdot \left(\prod_{v \mid (\mathfrak{N}_1, \mathfrak{N}_2)\infty} (1 - \lambda_v(f_1)\lambda_v(f_2) q_v^{-s})^{-1}\right) \\ & \cdot (1 - q_\infty^{-1-s})^{-1} \cdot \sum_{0 \neq \mathfrak{m} \lhd A} \frac{\lambda_{\mathfrak{m}}(f_1)\lambda_{\mathfrak{m}}(f_2)}{\|\mathfrak{m}\|^{s+1}}, \end{aligned}$$

where

$$L_A^{[\mathfrak{N}_1,\mathfrak{N}_2]}(2s,\omega_1\omega_2) := L_A(2s,\omega_1\omega_2)\cdot \prod_{v\mid[\mathfrak{N}_1,\mathfrak{N}_2]}(1-\omega_1\omega_2(\pi_v)q_v^{-s}).$$

For each character χ on $\mathrm{Pic}(A)$, let

$$\mathcal{E}_{\mathfrak{N}_1,\mathfrak{N}_2}(g,s,\chi) := \sum_{\mathfrak{n}\mid\frac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)}} \mu(\mathfrak{n})\mathcal{E}\left(g\begin{pmatrix}1 & 0\\ 0 & \pi^{\mathfrak{n}}\end{pmatrix},s,\chi\right),$$

where for square-free ideal $\mathfrak{n}\lhd A$,

$$\mu(\mathfrak{n}) := \prod_{v\mid\mathfrak{n}}(-1).$$

Set

$$\mathcal{E}_{\mathfrak{N}_1,\mathfrak{N}_2}(g,s) := \frac{q-1}{\#\,\mathrm{Pic}(A)}\sum_{\chi\in\widehat{\mathrm{Pic}(A)}} L_A(2s,\chi^{-1})\mathcal{E}_{\mathfrak{N}_1,\mathfrak{N}_2}(g,s,\chi).$$

We have the following integral representation of $L(f_1\times f_2,s)$:

Proposition 5.2.

$$\begin{aligned}
&\langle f_1\cdot\mathcal{E}_{\mathfrak{N}_1,\mathfrak{N}_2}(g,s),\overline{f_2}\rangle_{[\mathfrak{N}_1,\mathfrak{N}_2]}\\
=\;& \frac{q-1}{\#\,\mathrm{Pic}(A)}L_A(2s,\omega_1\omega_2)\cdot\langle f_1\cdot\mathcal{E}_{\mathfrak{N}_1,\mathfrak{N}_2}(g,s,\omega_1^{-1}\omega_2^{-1}),\overline{f_2}\rangle_{[\mathfrak{N}_1,\mathfrak{N}_2]}\\
=\;& \varepsilon(f_1\times f_2)\cdot\frac{q-1}{\#\,\mathrm{Pic}(A)}(q_\infty^{s}-q_\infty^{-s})q^{(g_k-1)(2s-1)}\|[\mathfrak{N}_1,\mathfrak{N}_2]\|^{s}\cdot L(f_1\times f_2,s),
\end{aligned}$$

where $\varepsilon(f_1\times f_2)$ is defined by

$$\varepsilon(f_1\times f_2)=\mu\left(\frac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)}\right)\omega_1^{-1}\left(\frac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)}\right)\omega_2^{-1}\left(\frac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)}\right)\lambda_{(\mathfrak{N}_1,\mathfrak{N}_2)}(f_1)^{-1}\lambda_{(\mathfrak{N}_1,\mathfrak{N}_2)}(f_2)^{-1}.$$

Proof. Put $\omega:=\omega_1\omega_2$. Let $\Phi^0_{\omega^{-1},v}(\cdot,s)$ be the function on $\mathrm{GL}_2(k_v)$ defined by

$$\Phi^0_{\omega^{-1},v}\left(\begin{pmatrix}a & *\\ 0 & b\end{pmatrix}\kappa_v,s\right):=\omega(b)^{-1}\frac{|a|_v^s}{|b|_v^s},\quad \forall a,b\in k_v^\times \text{ and } \kappa_v\in\mathrm{GL}_2(O_v).$$

Then by the Rankin-Selberg method (cf. [1, Proposition 3.8.1 and 3.8.2]) we have

$$\begin{aligned}
&\langle f_1\cdot\mathcal{E}_{\mathfrak{N}_1,\mathfrak{N}_2}(g,s,\omega^{-1}),\overline{f_2}\rangle_{[\mathfrak{N}_1,\mathfrak{N}_2]}\\
=\;& q^{g_k-1}\cdot\prod_v\int_{\mathrm{GL}_2(O_v)}\int_{k_v^\times}W_{f_1}\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\kappa_v\right)\overline{W_{\overline{f_2}}\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\kappa_v\right)}\Phi'_{\omega^{-1},v}(\kappa_v)|t|_v^{s-1}d^\times t\,d\kappa_v,
\end{aligned}$$

where for each place v of k,

$$\Phi'_{\omega^{-1},v}(\kappa_v):=\begin{cases}\Phi^0_{\omega^{-1},v}(\kappa_v)-\Phi^0_{\omega^{-1},v}\left(\kappa_v\begin{pmatrix}1 & 0\\ 0 & \pi_v\end{pmatrix}\right), & \text{if } v\;\Big|\;\dfrac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)};\\ \Phi^0_{\omega^{-1},v}(\kappa_v), & \text{otherwise.}\end{cases}$$

The Haar measure $d^\times t$ and $d\kappa_v$ are normalized so that $\mathrm{vol}(O_v^\times,d^\times t)=1$ and

$$\mathrm{vol}(\mathcal{K}_{v^{\mathrm{ord}_v([\mathfrak{N}_1,\mathfrak{N}_2]\infty)}},d\kappa_v)=1.$$

By Lemma 5.1, we calculate that

$$\begin{aligned}
&\int_{\mathrm{GL}_2(O_v)}\int_{k_v^\times} W_{f_1}\left(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\kappa_v\right)\overline{W_{\overline{f_2}}\left(\begin{pmatrix} t & 0 \\ 0 & 1\end{pmatrix}\kappa_v\right)}\Phi'_{\omega^{-1},v}(\kappa_v)|t|_v^{s-1}d^\times t d\kappa_v \\
= \;& q_v^{\delta_v(s-1)}\cdot\sum_{n\geq 0}\frac{\lambda_{v^n}(f_1)\lambda_{v^n}(f_2)}{q_v^{n(s+1)}}\cdot\begin{cases} 1+\lambda_v(f_1)^{-1}\lambda_v(f_2)^{-1}\cdot q_v^s, & \text{if } v\mid (\mathfrak{N}_1,\mathfrak{N}_2)\infty; \\ -\omega(\pi_v)^{-1}q_v^s+q_v^{-s}, & \text{if } v \mid \frac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)}; \\ 1, & \text{otherwise.}\end{cases} \\
= \;& q_v^{\delta_v(s-1)}\cdot(1-\omega(\pi_v)q_v^{-2s})\cdot\sum_{n\geq 0}\frac{\lambda_{v^n}(f_1)\lambda_{v^n}(f_2)}{q_v^{n(s+1)}} \\
&\cdot\begin{cases} \dfrac{\lambda_v(f_1)^{-1}\lambda_v(f_2)^{-1}\cdot q_v^s}{1-\lambda_v(f_1)\lambda_v(f_2)q_v^{-s}}, & \text{if } v\mid(\mathfrak{N}_1,\mathfrak{N}_2)\infty; \\ -\omega(\pi_v)^{-1}\cdot q_v^s, & \text{if } v\mid\frac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)}; \\ (1-\omega(\pi_v)q_v^{-2s})^{-1}, & \text{otherwise.}\end{cases}
\end{aligned}$$

Therefore the result follows. □

Remark 5.3. By Theorem 3.2 (1), we have the meromorphic continuation of $L(f_1\times f_2,s)$ and the functional equation:

$$\widetilde{L}(f_1\times f_2,s)=\widetilde{L}(\overline{f}_1\times\overline{f}_2,1-s),$$

where

$$\widetilde{L}(f_1\times f_2,s):=q^{(4g_k-4)s}\cdot q_\infty^s\cdot\|[\mathfrak{N}_1,\mathfrak{N}_2]\|^s\cdot L(f_1\times f_2,s).$$

By the strong approximation theorem for GL_2,

$$\{(g_{\mathfrak{a}}^\infty,g_\infty):[\mathfrak{a}]\in\mathrm{Pic}(A),\ [g_\infty]\in\mathrm{GL}(\Lambda^{\mathfrak{a}})\backslash\mathrm{GL}_2(k_\infty)/k_\infty^\times\mathrm{GL}_2(O_\infty)\}$$

forms a set of representatives of $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A})/k_\infty^\times\mathrm{GL}_2(O_{\mathbb{A}})$. Given $g_\infty\in\mathrm{GL}_2(k_\infty)$, take any $z\in\mathfrak{H}$ so that $\lambda(z)=[g_\infty]\in\mathrm{GL}(\Lambda^{\mathfrak{a}})\backslash\mathrm{GL}_2(k_\infty)/k_\infty^\times\mathrm{GL}_2(O_\infty)$. We define

$$\eta(g_{\mathfrak{a}}^\infty,g_\infty):=-\ln\left(\|\mathfrak{a}\|\cdot|z|_i\cdot|\Delta^{\mathfrak{a}}(z)|_\infty^{\frac{2}{q_\infty^2-1}}\right).$$

Extending η to a function on $\mathrm{GL}_2(k)\backslash\mathrm{GL}_2(\mathbb{A})/k_\infty^\times\mathrm{GL}_2(O_{\mathbb{A}})$, Theorem 3.1 (3) implies:

Lemma 5.4.

$$\frac{\partial}{\partial s}\mathcal{E}(g,s)\Big|_{s=0}=\eta(g),\quad\forall g\in\mathrm{GL}_2(\mathbb{A}).$$

Moreover, define

$$\eta_{\mathfrak{N}_1,\mathfrak{N}_2}(g):=\sum_{\mathfrak{n}\mid\frac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)}}\mu(\mathfrak{n})\eta\left(g\begin{pmatrix}1 & 0\\ 0 & \pi^{\mathfrak{n}}\end{pmatrix},s\right),\quad\forall g\in\mathrm{GL}_2(\mathbb{A}).$$

It is observed that when $1\neq[\mathfrak{N}_1,\mathfrak{N}_2]/(\mathfrak{N}_1,\mathfrak{N}_2)$ is not a prime,

$$\eta_{\mathfrak{N}_1,\mathfrak{N}_2}(g_{\mathfrak{a}}^\infty,g_\infty)=-\frac{2}{q_\infty^2-1}\ln\left|\prod_{\mathfrak{n}\mid\frac{[\mathfrak{N}_1,\mathfrak{N}_2]}{(\mathfrak{N}_1,\mathfrak{N}_2)}}\Delta^{\mathfrak{a}\mathfrak{n}^{-1}}(z)^{\mu(\mathfrak{n})}\right|_\infty$$

for every $z\in\mathfrak{H}$ with $\lambda(z)=[g_\infty]$. By Proposition 5.2 we then conclude that:

Theorem 5.5. *Let*

$$\Lambda(f_1 \times f_2, s) := \varepsilon(f_1 \times f_2) \cdot \frac{q-1}{\# \operatorname{Pic}(A)}(1-q_\infty^{-2s})q^{(g_k-1)(2s-1)}\|[\mathfrak{N}_1,\mathfrak{N}_2]\|^s \cdot L(f_1 \times f_2, s).$$

Then

$$\Lambda(f_1 \times f_2, 0) = -\langle f_1, \overline{f_2}\rangle_{[\mathfrak{N}_1,\mathfrak{N}_2]}$$

and

$$\frac{\partial}{\partial s}\Lambda(f_1 \times f_2, s)\Big|_{s=0} = \langle f_1 \cdot \eta_{\mathfrak{N}_1,\mathfrak{N}_2}, \overline{f_2}\rangle_{[\mathfrak{N}_1,\mathfrak{N}_2]}.$$

Remark 5.6. Given a normalized Drinfeld-type newform f of square-free level $\mathfrak{N}$ with central character ω_f, it is known that

$$L(f \times \overline{f}, s) = (1-q_\infty^{-s})^{-1}\zeta_A(s)L(f, \operatorname{Ad}, s),$$

where $L(f, \operatorname{Ad}, s)$ is the *adjoint L-function*

$$\begin{aligned} L(f, \operatorname{Ad}, s) \quad := \quad & \prod_{v \nmid \mathfrak{N}\infty}\Big((1-\alpha_{v,1}(f)\overline{\alpha_{v,2}(f)}q_v^{-1-s})(1-q_v^{-s})(1-\alpha_{v,2}(f)\overline{\alpha_{v,1}(f)}q_v^{-1-s})\Big)^{-1} \\ & \cdot \prod_{v \mid \mathfrak{N}\infty}(1-q_v^{-1-s})^{-1}, \end{aligned}$$

and $\alpha_{v,1}(f), \alpha_{v,2}(f)$ are two roots of $X^2 - \lambda_v(f)X + \omega_f(v)q_v$. Then we have the following functional equation:

$$\widetilde{L}(f, \operatorname{Ad}, s) = \widetilde{L}(f, \operatorname{Ad}, 1-s)$$

where

$$\widetilde{L}(f, \operatorname{Ad}, s) := q^{(3g_k-3)s}q_\infty^s\|\mathfrak{N}\|^s L(f, \operatorname{Ad}, s).$$

Moreover, Theorem 5.5 implies immediately that

$$L(f, \operatorname{Ad}, 0) = \frac{q^{g_k-1}}{2}\langle f, f\rangle_{\mathfrak{N}}$$

and

$$\begin{aligned} \frac{\partial}{\partial s}L(f, \operatorname{Ad}, s)\Big|_{s=0} \quad = \quad & -\frac{q^{g_k-1}}{2} \cdot \langle f \cdot \eta, f\rangle_{\mathfrak{N}} \\ & -\frac{q^{g_k-1}}{4}\Big[4g_k - 3 + 2\ln(q_\infty \cdot \|\mathfrak{N}\|) - 2\frac{\zeta_A'(0)}{\zeta_A(0)}\Big] \cdot \langle f, f\rangle_{\mathfrak{N}}. \end{aligned}$$

6. Application III: Solution of "Pell's equations"

6.1. **Hecke L-functions.** Let K be an imaginary quadratic field over k and O_K be the integral closure of A in K. Given $\mathfrak{c} \lhd A$, let $O_{\mathfrak{c}}$ be the quadratic A-order in K of conductor $\mathfrak{c}$, i.e. $O_{\mathfrak{c}} = A + \mathfrak{c}O_K$. For each invertible ideal $\mathfrak{A}$ of $O_{\mathfrak{c}}$, recall that $N(\mathfrak{A}) = \#(O_{\mathfrak{c}}/a\mathfrak{A}) \cdot |a|_\infty^{-2}$ for any $a \in A \neq 0$ such that $a\mathfrak{A} \subset O_{\mathfrak{c}}$. Fix an embedding $K \hookrightarrow \mathbb{C}_\infty$. Then each non-zero ideal $\mathfrak{A}$ of $O_{\mathfrak{c}}$ can be viewed as a rank 2 A-lattice in $\mathbb{C}_\infty$.

Let $\operatorname{Pic}(O_{\mathfrak{c}})$ denote the class group of invertible ideals of $O_{\mathfrak{c}}$. For each class $[\mathfrak{A}] \in \operatorname{Pic}(O_{\mathfrak{c}})$, recall the associated partial zeta function $\zeta_{[\mathfrak{A}]^{-1}}(s)$:

$$\zeta_{[\mathfrak{A}]^{-1}}(s) := \sum_{\substack{\mathfrak{B} \lhd O_{\mathfrak{c}} \text{ invertible}, \\ \mathfrak{B} \sim \mathfrak{A}^{-1}}} N(\mathfrak{B})^{-s}.$$

In Section 4.2 we pointed out that $\zeta_{[\mathfrak{A}]^{-1}}(0) = -\#(O_{\mathfrak{c}}^\times)^{-1}$ and

$$\frac{\partial}{\partial s}\zeta_{\mathfrak{A}^{-1}}(s)\Big|_{s=0} \quad = \quad -\#(O_{\mathfrak{c}}^\times)^{-1}\left[\ln N(\mathfrak{A}) + \frac{2}{q_\infty^2-1}\ln|\Delta(\mathfrak{A})|_\infty\right].$$

For each character ξ on $\operatorname{Pic}(O_{\mathfrak{c}})$, let

$$L_{O_{\mathfrak{c}}}(s,\xi) := \sum_{\mathfrak{A}\lhd O_{\mathfrak{c}} \text{ invertible}} \frac{\xi(\mathfrak{A})}{N(\mathfrak{A})^s} = \sum_{[\mathfrak{A}]\in \operatorname{Pic}(O_{\mathfrak{c}})} \xi(\mathfrak{A})^{-1}\zeta_{[\mathfrak{A}]^{-1}}(s).$$

We then have:

Theorem 6.1. *$L_{O_{\mathfrak{c}}}(0,\xi) = -\frac{\#\operatorname{Pic}(O_{\mathfrak{c}})}{\#(O_{\mathfrak{c}}^\times)}$ if ξ is trivial and 0 otherwise; and*

$$\frac{\partial}{\partial s} L_{O_{\mathfrak{c}}}(s,\xi)\big|_{s=0} = -\#(O_{\mathfrak{c}}^\times)^{-1} \sum_{[\mathfrak{A}]\in \operatorname{Pic}(O_{\mathfrak{c}})} \xi(\mathfrak{A})^{-1} \ln\left(N(\mathfrak{A})\cdot |\Delta(\mathfrak{A})|_\infty^{\frac{2}{q_\infty^2-1}}\right).$$

Let H_{O_K} be the Hilbert class field of O_K (i.e. H_{O_K}/K is the maximal unramified-everywhere abelian extension of K in which the infinite place of K splits completely) and O_H be the integral closure of O_K in H_{O_K}. The zeta function $\zeta_{O_H}(s)$ is expressed by the following:

$$\zeta_{O_H}(s) = \prod_{\xi\in \widehat{\operatorname{Pic}(O_K)}} L_{O_K}(s,\xi),$$

where $\widehat{\operatorname{Pic}(O_K)}$ is the character group of $\operatorname{Pic}(O_K)$. Thus

$$\zeta_{O_H}(s) = \left[\zeta_{O_K}(0)\cdot \prod_{1\neq \xi\in \widehat{\operatorname{Pic}(O_K)}} \frac{\partial}{\partial s}L_{O_K}(s,\xi)\big|_{s=0}\right] s^{\#\operatorname{Pic}(O_K)-1} + O(s^{\#\operatorname{Pic}(O_K)}).$$

It is known that the order of $\zeta_{O_H}(s)$ at $s=0$ is equal to $\#\operatorname{Pic}(O_K)-1$ (cf. [18, Theorem 14.4]). Therefore:

Corollary 6.2. *For each non-trivial character ξ on $\operatorname{Pic}(O_K)$,*

$$\sum_{[\mathfrak{A}]\in \operatorname{Pic}(O_K)} \xi(\mathfrak{A})^{-1}\ln\left(N(\mathfrak{A})\cdot|\Delta(\mathfrak{A})|_\infty^{\frac{2}{q_\infty^2-1}}\right) = -\#(O_K^\times)\cdot \frac{\partial}{\partial s}L_{O_K}(s,\xi)\Big|_{s=0} \neq 0.$$

6.2. **Units of "real" quadratic function fields.** We first state the following Shimura-type "reciprocity law" for Gekeler's discriminant function Δ:

Proposition 6.3. (1) *For each invertible ideal $\mathfrak{A}\lhd O_{\mathfrak{c}}$, $\Delta(\mathfrak{A}^{-1})/\Delta(O_{\mathfrak{c}})$ is in the ring class field $H_{\mathfrak{c}}$ of $O_{\mathfrak{c}}$ (i.e. $\operatorname{Gal}(H_{\mathfrak{c}}/K)\cong \operatorname{Pic}(O_{\mathfrak{c}})$ via the Artin map).*
(2) *Given two invertible ideals $\mathfrak{A},\mathfrak{B}\lhd O_{\mathfrak{c}}$,*

$$\left(\frac{\Delta(\mathfrak{A}^{-1})}{\Delta(O_{\mathfrak{c}})}\right)^{\sigma_{\mathfrak{B}}} = \frac{\Delta(\mathfrak{A}^{-1}\mathfrak{B}^{-1})}{\Delta(\mathfrak{B}^{-1})},$$

where $\sigma_{\mathfrak{B}}\in \operatorname{Gal}(H_{\mathfrak{c}}/K)$ is the automorphism corresponding to $\mathfrak{B}$.
(3) *Given two invertible ideals $\mathfrak{A},\mathfrak{B}\lhd O_{\mathfrak{c}}$,*

$$u_{\mathfrak{A},\mathfrak{B}} := \frac{\Delta(\mathfrak{A}^{-1})\Delta(\mathfrak{B}^{-1})}{\Delta(O_{\mathfrak{c}})\Delta(\mathfrak{A}^{-1}\mathfrak{B}^{-1})} \text{ lies in } O_{H_{\mathfrak{c}}}^\times,$$

which only depends on the ideal class $[\mathfrak{A}],[\mathfrak{B}]\in \operatorname{Pic}(O_{\mathfrak{c}})$.
(4) *Fix an embedding $H_{\mathfrak{c}}\hookrightarrow K_\infty$ and let $\bar{\cdot}$ be the conjugation on K_∞/k_∞. Then*

$$\overline{\left[\frac{\Delta(\mathfrak{A})}{\Delta(O_{\mathfrak{c}})}\right]} = \frac{\Delta(\bar{\mathfrak{A}})}{\Delta(O_{\mathfrak{c}})}.$$

Proof. Let $\phi^{O_{\mathfrak{c}}}$ be the rank 2 Drinfeld A-module over $\mathbb{C}_\infty$ associated to the lattice $O_{\mathfrak{c}}\subset \mathbb{C}_\infty$. Viewing $\phi^{O_{\mathfrak{c}}}$ as a rank 1 Drinfeld $O_{\mathfrak{c}}$-module, it is known that (cf. [11, Theorem 8.10]) there exists $\alpha\in\mathbb{C}_\infty^\times$ such that $\phi^{\alpha O_{\mathfrak{c}}}(=\alpha\phi^{O_{\mathfrak{c}}}\alpha^{-1})$ is defined over $H_{\mathfrak{c}}$. For each invertible ideal $\mathfrak{A}\lhd O_{\mathfrak{c}}$, let $u_{\mathfrak{A}}^{\alpha O_{\mathfrak{c}}}\in H_{\mathfrak{c}}\{\tau\}$ be the monic generator of the left ideal of $H_{\mathfrak{c}}\{\tau\}$ generated by

$\phi_a^{\alpha O_\mathfrak{c}}$ for all $a \in \mathfrak{A}$. Then there exists a unique rank 1 Drinfeld $O_\mathfrak{c}$-module $\mathfrak{A} * \phi^{\alpha O_\mathfrak{c}}$ over $H_\mathfrak{c}$ such that

$$u_{\mathfrak{A}}^{\alpha O_\mathfrak{c}} \cdot \phi_a^{\alpha O_\mathfrak{c}} = (\mathfrak{A} * \phi^{\alpha O_\mathfrak{c}})_a \cdot u_{\mathfrak{A}}^{\alpha O_\mathfrak{c}}, \quad \forall a \in O_\mathfrak{c}.$$

Considering $u_{\mathfrak{A}}^{\alpha O_\mathfrak{c}}$ as an endomorphism of $\mathbb{G}_a/H_\mathfrak{c}$, we can express $u_{\mathfrak{A}}^{\alpha O_\mathfrak{c}}$ by

$$u_{\mathfrak{A}}^{\alpha O_\mathfrak{c}}(x) = \prod_{w \in \frac{\alpha \mathfrak{A}^{-1}}{\alpha O_\mathfrak{c}}} \big(x - \exp_{\alpha O_\mathfrak{c}}(w)\big), \quad \forall x \in \mathbb{C}_\infty.$$

Let

$$c_{\mathfrak{A}} = \prod_{0 \neq w \in \frac{\alpha \mathfrak{A}^{-1}}{\alpha O_\mathfrak{c}}} \big(-\exp_{\alpha O_\mathfrak{c}}(w)\big) \in H_\mathfrak{c}.$$

It is then observed that

$$\mathfrak{A} * \phi^{\alpha O_\mathfrak{c}} = c_{\mathfrak{A}}^{-1} \phi^{\alpha \mathfrak{A}^{-1}} c_{\mathfrak{A}}.$$

Therefore

$$\frac{\Delta(\mathfrak{A}^{-1})}{\Delta(O_\mathfrak{c})} = \frac{\Delta(\alpha\mathfrak{A}^{-1})}{\Delta(\alpha O_\mathfrak{c})} = c_{\mathfrak{a}}^{q_\infty^2 - 1} \Delta(\alpha O_\mathfrak{c})^{N(\mathfrak{A})-1} \in H_\mathfrak{c}. \tag{6.1}$$

Given another invertible ideal $\mathfrak{B} \lhd O_\mathfrak{c}$, it is known that (cf. [11, Theorem 8.5]) there exists $\beta \in \mathbb{C}_\infty^\times$ such that

$$(\phi^{\alpha O_\mathfrak{c}})^{\sigma_\mathfrak{B}} = \beta \phi^{\mathfrak{B}^{-1}} \beta^{-1} = \phi^{\beta \mathfrak{B}^{-1}},$$

which means that $\beta\mathfrak{B}^{-1} \subset \mathbb{C}_\infty$ is the lattice corresponding to $(\phi^{\alpha O_\mathfrak{c}})^{\sigma_\mathfrak{B}}$. Note that we have $(\mathfrak{A} * \phi^{\alpha O_\mathfrak{c}})^{\sigma_\mathfrak{B}} = \mathfrak{A} * (\phi^{\alpha O_\mathfrak{c}})^{\sigma_\mathfrak{B}}$ (cf. [11, Proposition 8.1]). Therefore

$$\begin{aligned}
(\phi^{\alpha\mathfrak{A}^{-1}})^{\sigma_\mathfrak{B}} &= c_{\mathfrak{A}}^{\sigma_\mathfrak{B}} \cdot (\mathfrak{A} * \phi^{\alpha O_\mathfrak{c}})^{\sigma_\mathfrak{B}} \cdot (c_{\mathfrak{A}}^{-1})^{\sigma_\mathfrak{B}} \\
&= c_{\mathfrak{A}}^{\sigma_\mathfrak{B}} \cdot \mathfrak{A} * (\phi^{\alpha O_\mathfrak{c}})^{\sigma_\mathfrak{B}} \cdot (c_{\mathfrak{A}}^{-1})^{\sigma_\mathfrak{B}} \\
&= c_{\mathfrak{A}}^{\sigma_\mathfrak{B}} \cdot \mathfrak{A} * \phi^{\beta\mathfrak{B}^{-1}} \cdot (c_{\mathfrak{A}}^{-1})^{\sigma_\mathfrak{B}} \\
&= \phi^{\beta\mathfrak{A}^{-1}\mathfrak{B}^{-1}}.
\end{aligned}$$

We then obtain that

$$\left(\frac{\Delta(\mathfrak{A}^{-1})}{\Delta(O_\mathfrak{c})}\right)^{\sigma_\mathfrak{B}} = \left(\frac{\Delta(\alpha\mathfrak{A}^{-1})}{\Delta(\alpha O_\mathfrak{c})}\right)^{\sigma_\mathfrak{B}} = \frac{\Delta(\beta\mathfrak{A}^{-1}\mathfrak{B}^{-1})}{\Delta(\beta\mathfrak{B}^{-1})} = \frac{\Delta(\mathfrak{A}^{-1}\mathfrak{B}^{-1})}{\Delta(\mathfrak{B}^{-1})}.$$

Since the conjugation $\bar{\cdot}$ on K_∞/k_∞ is continuous, (4) follows from the expression of Δ in Lemma 2.3.

It remains to prove (3). By [11, Proposition 7.3], there exists a finite extension H' of $H_\mathfrak{c}$ so that $\phi^{\alpha O_\mathfrak{c}}$ and $\phi^{\beta\mathfrak{B}^{-1}}$ over H' have good reduction everywhere. Without loss of generality, we assume $\mathfrak{A} = \mathfrak{P} \lhd O_\mathfrak{c}$ is an invertible prime ideal which is unramified in H'. Let $O_{H'}$ be the integral closure of A in H'. For each prime $\mathcal{P}$ of $O_{H'}$, choose $\alpha = \alpha_\mathcal{P}$ and $\beta = \beta_\mathcal{P}$ so that $\Delta(\alpha O_\mathfrak{c}), \Delta(\beta\mathfrak{B}^{-1}) \in (O_{H',\mathcal{P}})^\times$. Then by [11, Proposition 7.6] we have

$$\operatorname{ord}_\mathcal{P}(c_\mathfrak{P}) = \operatorname{ord}_\mathcal{P}\left(c_\mathfrak{P}^{\sigma_\mathfrak{B}}\right) = \begin{cases} 1 & \text{if } \mathcal{P} \mid \mathfrak{P}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\operatorname{ord}_\mathcal{P}\left(\frac{\Delta(\mathfrak{P}^{-1})}{\Delta(O_\mathfrak{c})}\right) = \operatorname{ord}_\mathcal{P}\left(\frac{\Delta(\mathfrak{P}^{-1}\mathfrak{B}^{-1})}{\Delta(\mathfrak{B}^{-1})}\right) = \begin{cases} q_\infty^2 - 1 & \text{if } \mathcal{P} \mid \mathfrak{P}, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $u_{\mathfrak{P},\mathfrak{B}} \in O_{H'} \cap H_\mathfrak{c} = O_{H_\mathfrak{c}}^\times$. □

Remark 6.4. $\Delta(\mathfrak{A}^{-1})/\Delta(O_\mathfrak{c})$ generates the principal ideal $\mathfrak{A}^{q_\infty^2-1}O_{H_\mathfrak{c}}$.

Let $\xi : \mathrm{Pic}(O_\mathfrak{c}) \to \{\pm 1\}$ be a non-trivial quadratic character. Then Theorem 6.1 says that

$$\frac{\partial}{\partial s} L_{O_\mathfrak{c}}(s,\xi)\Big|_{s=0} = \frac{-1}{\#(O_\mathfrak{c}^\times)(q_\infty^2-1)} \sum_{[\mathfrak{A}]\in \mathrm{Pic}(O_\mathfrak{c})} \xi(\mathfrak{A}) \ln\left|\frac{\Delta(\mathfrak{A})\Delta(\mathfrak{A}^{-1})}{\Delta(O_\mathfrak{c})^2}\right|_\infty.$$

Let L_ξ/K be the subfield of $H_\mathfrak{c}$ such that $\mathrm{Gal}(H_\mathfrak{c}/L_\xi) \cong \ker \xi$, and O_{L_ξ} be the integral closure of O_K in L_ξ. Set

$$u_\xi := \prod_{[\mathfrak{A}]\in \mathrm{Pic}(O_\mathfrak{c})} \left(\frac{\Delta(\mathfrak{A})\Delta(\mathfrak{A}^{-1})}{\Delta(O_\mathfrak{c})^2}\right)^{\xi(\mathfrak{A})}.$$

Then for each invertible ideal $\mathfrak{B}$ of $O_\mathfrak{c}$, we get $u_\xi^{\sigma_\mathfrak{B}} = u_\xi^{\xi(\mathfrak{B})}$. In particular, u_ξ lies in $O_{L_\xi}^\times$.

Corollary 6.5. *Suppose $\mathfrak{c}$ is trivial, i.e. $O_\mathfrak{c} = O_K$.*
(1) Let $\xi : \mathrm{Pic}(O_K) \to \{\pm 1\}$ be a non-trivial quadratic character. The subgroup generated by u_ξ in $O_{L_\xi}^\times$ is of finite index.
(2) Let $G_{O_K} \subset H_{O_K}$ be the subfield fixed by $2\,\mathrm{Pic}(O_K)$ and O_G be the integral closure of O_K in G_{O_K}. Then the subgroup of $O_G^\times$ generated by $\big\{u_\xi : 1 \neq \xi : \mathrm{Pic}(O_K) \to \{\pm 1\}\big\}$ is of finite index.

Proof. By Corollary 6.2 we have that $\ln |u_\xi|_\infty \neq 0$, which says that u_ξ is not a root of unity. Thus (1) holds. To show (2), note that the $\mathbb{Z}$-rank of $O_G^\times$ is $\#(\mathrm{Pic}(O_K)/2\,\mathrm{Pic}(O_K)) - 1$. Suppose $\prod_{1\neq\xi:\mathrm{Pic}(O_K)\to\{\pm1\}} u_\xi^{a_\xi}$ is a root of unity for some $a_\xi \in \mathbb{Z}$. Then for $[\mathfrak{A}] \in \mathrm{Pic}(O_K)$,

$$\prod_{1\neq\xi:\mathrm{Pic}(O_K)\to\{\pm1\}} (u_\xi^{a_\xi})^{\sigma_\mathfrak{A}} = \prod_{1\neq\xi:\mathrm{Pic}(O_K)\to\{\pm1\}} u_\xi^{a_\xi\cdot\xi(\mathfrak{A})}$$

is also a root of unity. Therefore $a_\xi = 0$ for every non-trivial quadratic character ξ, which implies (2). □

Now, let L/k be a "real" quadratic extension (i.e. ∞ splits in L) and the integral closure of A in L is denoted by O_L. Suppose there exists an imaginary quadratic extension K/k such that LK is a subfield of H_{O_K}. Let $\xi_L : \mathrm{Pic}(O_K) \to \{\pm 1\}$ be the quadratic character associated to LK/K. We set

$$u_L := u_{\xi_L} \cdot \overline{u_{\xi_L}} \in O_L^\times.$$

Then

$$\frac{\partial}{\partial s} L_{O_K}(s,\xi_L)\Big|_{s=0} = \frac{-\ln|u_{\xi_L}|_\infty}{\#(O_K^\times)(q_\infty^2-1)} = \frac{-\ln|u_L|_\infty}{2\#(O_K^\times)(q_\infty^2-1)}.$$

Note that LK/k is Galois with the Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Let K'/k be the quadratic extension so that $K' \neq K, L$, and $K' \subset LK$. It is clear that K'/k is imaginary. Moreover,

$$\zeta_{O_L}(s) = \frac{L_{O_K}(s,\xi_L)\zeta_A(s)^2}{\zeta_{O_{K'}}(s)}.$$

In particular,

$$\begin{aligned}\frac{\partial}{\partial s}\zeta_{O_L}(s)\Big|_{s=0} &= \frac{\zeta_A(0)^2}{\zeta_{O_{K'}}(0)} \cdot \frac{\partial}{\partial s} L_{O_K}(s,\xi_L)\big|_{s=0} \\ &= \frac{\zeta_A(0)^2}{\zeta_{O_{K'}}(0)} \cdot \frac{-1}{2\#(O_K^\times)(q_\infty^2-1)} \cdot \ln|u_L|_\infty.\end{aligned}$$

On the other hand, let $\mathbb{F}_L$ be the algebraic closure of $\mathbb{F}_q$ in L and $\varepsilon_L \in O_L^\times$ be a fundamental unit so that $|\epsilon_L|_\infty < 1$. Then (cf. [18, Theorem 14.4])

$$\frac{\partial}{\partial s}\zeta_{O_L}(s)\Big|_{s=0} = \frac{\#\,\mathrm{Pic}(O_L)\cdot \ln|\varepsilon_L|_\infty}{\#(\mathbb{F}_L^\times)}.$$

Hence we conclude that:

Corollary 6.6. *Let K/k be an imaginary quadratic field. For each real quadratic field L/k such that LK is a subfield of H_{O_K}, we have*

$$\#\left(\frac{O_L^\times}{< u_L >}\right) = \frac{\#\operatorname{Pic}(O_L)\cdot\#\operatorname{Pic}(O_{K'})}{\#\operatorname{Pic}(A)^2}\cdot\frac{2\#(O_K^\times)\cdot\#(\mathbb{F}_q^\times)^2\cdot(q_\infty^2-1)}{\#(O_{K'}^\times)}.$$

Remark 6.7. (1) Let K/k be an imaginary quadratic field. Assume that $\mathfrak{a}O_\mathfrak{c}$ is principal for every ideal $\mathfrak{a}$ of A. It is observed that for every invertible ideal $\mathfrak{A}\lhd O_\mathfrak{c}$,

$$N(\mathfrak{A})\left|\frac{\Delta(\mathfrak{A})}{\Delta(O_\mathfrak{c})}\right|_\infty^{\frac{1}{q_\infty^2-1}} = N(\mathfrak{A})\left|\frac{\Delta(\overline{\mathfrak{A}})}{\Delta(O_\mathfrak{c})}\right|_\infty^{\frac{1}{q_\infty^2-1}} = \left|\frac{\Delta(\mathfrak{A}^{-1})}{\Delta(O_\mathfrak{c})}\right|_\infty^{\frac{1}{q_\infty^2-1}}.$$

Set

$$u_\mathfrak{A} := \frac{\Delta(\mathfrak{A})\Delta(\mathfrak{A}^{-1})}{\Delta(O_\mathfrak{c})^2}\in O_{H_\mathfrak{c}}^\times.$$

Then for every character ξ on $\operatorname{Pic}(O_\mathfrak{c})$, by Theorem 6.1 we get

$$\frac{\partial}{\partial s}L_{O_\mathfrak{c}}(s,\xi)\Big|_{s=0} = \frac{-1}{\#(O_\mathfrak{c}^\times)(q_\infty^2-1)}\sum_{[\mathfrak{A}]\in\operatorname{Pic}(O_\mathfrak{c})}\xi(\mathfrak{A})^{-1}\ln|u_\mathfrak{A}|_\infty.$$

Suppose $\mathfrak{c}$ is trivial, i.e. $O_\mathfrak{c}=O_K$. Using Dedekind's group determinant for the group $\operatorname{Pic}(O_K)$ we have

$$\begin{aligned}\det\left(\ln|u_\mathfrak{A}^{\sigma_\mathfrak{B}}|_\infty\right)_{[O_K]\neq[\mathfrak{A}],[\mathfrak{B}]\in\operatorname{Pic}(O_K)} &= \prod_{1\neq\xi\in\widehat{\operatorname{Pic}(O_K)}}\sum_{[\mathfrak{A}]\in\operatorname{Pic}(O_K)}\xi(\mathfrak{A})^{-1}\ln|u_\mathfrak{A}|_\infty.\\ &= \prod_{1\neq\xi\in\widehat{\operatorname{Pic}(O_K)}}\left[-\#(O_K^\times)\cdot(q_\infty^2-1)\frac{\partial}{\partial s}L_{O_K}(s,\xi)\Big|_{s=0}\right],\end{aligned}$$

which is not equal to zero by Corollary 6.2. Therefore:

Corollary 6.8. *Let K/k be an imaginary quadratic field satisfying that $\mathfrak{a}O_K$ is principal for every ideal $\mathfrak{a}$ of A. Then the subgroup of $O_H^\times$ generated by $u_\mathfrak{A}$ for all $[\mathfrak{A}]\in\operatorname{Pic}(O_K)$ is of finite index.*

(2) Let L/k be a real quadratic extension so that LK is a subfield of H_{O_K}. If $\mathfrak{a}O_K$ is a principal ideal of O_K for every ideal $\mathfrak{a}$ of A, then

$$\tilde{u}_L := \prod_{[\mathfrak{A}]\in\operatorname{Pic}(O_K)}\left(\frac{\Delta(\mathfrak{A})\Delta(\mathfrak{A}^{-1})}{\Delta(O_K)^2}\right)^{\xi_L(\mathfrak{A})}$$

lies in $O_L^\times$ and $u_L=\tilde{u}_L^2$.

Example 6.9. Let $k=\mathbb{F}_q(t)$ with q odd and $A=\mathbb{F}_q[t]$. Take a monic square-free polynomial $D\in A-\mathbb{F}_q$ with $\deg D$ even, let $L:=k(\sqrt{D})$. Choose $\epsilon\in\mathbb{F}_q-\mathbb{F}_q^2$, and let $K:=k(\sqrt{\epsilon D})$. Then L/k is real, K/k is imaginary, and LK is a subfield of H_{O_K}. In particular, $\mathbb{F}_L=\mathbb{F}_K=\mathbb{F}_q$. By Corollary 6.6, one has that $\#(\mathbb{F}_L^\times)\cdot\#(\mathbb{F}_q^\times)$ divides $\#\left(\frac{O_L^\times}{<\tilde{u}_L>}\right)$, which implies that the norm $\mathrm{N}_{L/k}(\tilde{u}_L)$ of $\tilde{u}_L$ must equals to 1. In other words, writing $\tilde{u}_L=a+b\sqrt{D}\in O_L^\times$ for $a,b\in A$, we then obtain a solution (a,b) for the Pell's equation $x^2-Dy^2=1$.

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