UNIQUENESS OF CLOSED SELF-SIMILAR SOLUTIONS TO σ_k^α-CURVATURE FLOW

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Abstract. By adapting the test functions introduced by Choi-Daskaspoulos [11] and Brendle-Choi-Daskaspoulos [9] and exploring properties of the k-th elementary symmetric functions σ_k intensively, we show that for any fixed k with $1 \le k \le n-1$, any strictly convex closed hypersurface in $\mathbb{R}^{n+1}$ satisfying $\sigma_k^\alpha = \langle X, \nu \rangle$, with $\alpha \ge \frac{1}{k}$, must be a round sphere.

1. Introduction

Let $X : M \to \mathbb{R}^{n+1}$ be a smooth embedding of a closed, orientable hypersurface in $\mathbb{R}^{n+1}$ with $n \ge 2$, satisfying

$$\sigma_k^\alpha = \langle X, \nu \rangle \tag{1.1}$$

where ν is the outward unit normal vector field of M, $\alpha > 0$, $1 \le k \le n$ and σ_k is the k-th elementary symmetric functions of principal curvatures of M.

This type of equation is important for the following curvature flow

$$\tilde{X}_t = -\sigma_k^\alpha \nu. \tag{1.2}$$

Actually, if X is a solution of (1.1), then

$$\tilde{X}(x,t) = ((k\alpha+1)(T-t))^{\frac{1}{1+k\alpha}} X(x)$$

gives rise to the solution of (1.2) up to a tangential diffeomorphism [20]. So in the same spirit, we call the solutions of (1.1) self-similar solutions of (1.2).

For $k = 1$, G. Huisken proved the following famous result:

Theorem 1.1 (Huisken, [18])**.** *If M is a closed hypersurface in $\mathbb{R}^{n+1}$, with non-negative mean curvature σ_1 and satisfies the equation*

$$\sigma_1 = \langle X, \nu \rangle,$$

then M must be a round sphere.

For $k = n$, very recently, Choi-Daskalopoulos [11], further, Brendle-Choi-Daskalopoulos [9] proved the following remarkable result:

Theorem 1.2 (Choi-Daskalopoulos [11], Brendle-Choi-Daskalopoulos [9])**.** *Let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying*

$$\sigma_n^\alpha = \langle X, \nu \rangle.$$

If $\alpha > \frac{1}{n+2}$, then M must be a round sphere; if $\alpha = \frac{1}{n+2}$, then M is an ellipsoid.

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Remark 1.3. The results of convergence of σ_n^α-curvature flow could implies Theorem 1.2. In case $\alpha = \frac{1}{n}$, Theorem 1.2 was contained in the results of B. Chow in [12]. In case $n = 2$, Theorem 1.2 was proved by B. Andrews for $\alpha = 1$ in [3], by B. Andrews and X. Chen for $\frac{1}{2} \leq \alpha \leq 1$ in [6]. In case $\alpha = \frac{1}{n+2}$, Theorem 1.2 was proved by B. Andrews in [2]. The more properties of σ_n^α-curvature flow were studied by W. J. Firey [14], B. Chow [12], K. Tso [21], B. Andrews [3], P.-F. Guan and L. Ni [17], B. Andrews, P.-F. Guan and L. Ni [7], etc.

From Theorem 1.1 and Theorem 1.2, the following natural question arises:

Question. For any fixed k with $1 \leq k \leq n-1$, let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying (1.1) with $\alpha \geq \frac{1}{k}$. Can we conclude that M must be a round sphere?

In this paper, we give an affirmative answer to the above question by proving the following result:

Theorem 1.4. *For any fixed k with $1 \leq k \leq n-1$, let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying*

$$\sigma_k^\alpha = \langle X, \nu \rangle$$

with $\alpha \geq \frac{1}{k}$. Then M must be a round sphere.

Remark 1.5. Theorem 1.1 implies Theorem 1.4 for the case $k = 1$ and $\alpha = 1$. For $\alpha = \frac{1}{k}$, Theorem 1.4 was contained in the results of B. Chow [12, 13] and B. Andrews [1, 2, 4, 5]. For general k and α, there are some partial results under certain pinching condition of the principal curvatures of hypersurface, see [20], [8] and [15].

In fact, we prove the following two theorems:

Theorem A. *For any fixed k with $1 \leq k \leq n$, let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying*

$$\sigma_k^\alpha + C = \langle X, \nu \rangle \tag{1.3}$$

with constants α and C. If either $1 \leq k \leq n-1$, $C \leq 0$, $\alpha \geq \frac{1}{k}$, or, $k = n$, $C < 0$, $\alpha \geq \frac{1}{n+2}$, then M must be a round sphere.

Remark 1.6. Choose $C = 0$, Theorem A reduces to Theorem 1.4. When $k = \alpha = 1$, Theorem A implies the uniqueness of closed $\lambda-$hypersurfaces introduced by Cheng-Wei [10].

Let $S_k(\lambda)$ denote the k-th power sum of the principal curvatures $\lambda_1, \cdots, \lambda_n$, defined by $S_k(\lambda) = \sum_{i=1}^n \lambda_i^k$.

Theorem B. *For any fixed k with $k \geq 1$, let M be a closed, strictly convex hypersurface in $\mathbb{R}^{n+1}$ satisfying*

$$S_k^\alpha + C = \langle X, \nu \rangle \tag{1.4}$$

with constants α and C. If $\alpha \geq \frac{1}{k}$ and $C \leq 0$, then M must be a round sphere.

In this paper we first consider the following general equation

$$F + C = \langle X, \nu \rangle, \tag{1.5}$$

where $F = F(h)$ is a homogeneous smooth symmetric function of the second fundamental form $h = (h_{ij})$ of degree β and C is a constant. We also suppose $F > 0$

and $(\frac{\partial F}{\partial h_{ij}})$ is positive definite. In the spirit of Choi-Daskaspoulos [11] and Brendle-Choi-Daskaspoulos [9], we consider the quantities

$$Z = F\,\mathrm{tr} b - \frac{n(\beta-1)}{2\beta}|X|^2, \tag{1.6}$$

$$\tilde{W} = F\,\lambda_{\min}^{-1} - \frac{\beta-1}{2\beta}|X|^2, \tag{1.7}$$

where $b = (b^{ij})$ denotes the inverse of the second fundamental form $h = (h_{ij})$ and $\lambda_{\min}$ is the smallest principal curvature of the hypersurface. In case $F = \sigma_k^\alpha$ or $F = S_k^\alpha$, by exploring properties of σ_k and S_k intensively, we find that the techniques in Choi-Daskaspoulos [11] and Brendle-Choi-Daskaspoulos [9] can be carried out effectively by using the strong maximum principle of $\mathcal{L} = \frac{\partial F}{\partial h_{ij}}\nabla_i\nabla_j$ for Z and by using the maximum principle for W (see Section 4 for definition of W).

The structure of this paper is as follows. In Section 2, we give some properties of the elementary symmetric functions σ_k and prove our key lemma (Lemma 2.7). In Section 3, we derive some fundamental formulas for the closed hypersurfaces which satisfies self-similar equation (1.5) with the general homogeneous symmetric function F. In Section 4, we do analysis at the maximum point of W. In Section 5 and 6, we present the proofs of Theorem A and Theorem B, respectively.

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2. SOME PROPERTIES OF ELEMENTARY SYMMETRIC FUNCTIONS AND KEY LEMMA

We first collect some basic notations, definitions and properties of elementary symmetric functions, which are needed in our investigation of the σ_k^α self-similar solutions.

Let $\lambda = (\lambda_1, \cdots, \lambda_n)$ denote the principal curvatures of M. Throughout this paper, we assume that $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Denote

$$\sigma_k(\lambda) = \sigma_k(\lambda(A)) = \sum_{1\le i_1<i_2\cdots<i_k\le n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.$$

For convenience, we set $\sigma_0(\lambda) = 1$ and $\sigma_k(\lambda) = 0$ for $k > n$ or $k < 0$. Let $\sigma_{k;i}(\lambda)$ denote the symmetric function $\sigma_k(\lambda)$ with $\lambda_i = 0$ and $\sigma_{k;ij}(\lambda)$, with $i \ne j$, denote the symmetric function $\sigma_k(\lambda)$ with $\lambda_i = \lambda_j = 0$. So $\frac{\partial\sigma_k(\lambda)}{\partial\lambda_i} = \sigma_{k-1;i}$, $\frac{\partial^2\sigma_k(\lambda)}{\partial\lambda_i\partial\lambda_j} = \sigma_{k-2;ij}$. Remark that without causing ambiguity we omit λ in the notations of $\sigma_k(\lambda)$ for simplicity.

Definition 2.1. A hypersurface M is said to be *strictly convex* if $\lambda \in \Gamma_+ = \{\mu \in \mathbb{R}^n | \mu_1 > 0, \mu_2 > 0, \cdots, \mu_n > 0\}$ for any point in M.

The following basic properties related to σ_k will be used directly.

Proposition 2.2 (See, for example, [19]). *For $0 \leq k \leq n$ and $1 \leq i \leq n$, the following equalities hold:*

$$\begin{aligned}\sigma_{k+1} &= \sigma_{k+1;i} + \lambda_i \sigma_{k;i},\\ \sum_{i=1}^n \lambda_i \sigma_{k;i} &= (k+1)\sigma_{k+1},\\ \sum_{i=1}^n \sigma_{k;i} &= (n-k)\sigma_k,\\ \sum_{i=1}^n \lambda_i^2 \sigma_{k;i} &= \sigma_1 \sigma_{k+1} - (k+2)\sigma_{k+2}.\end{aligned}$$

Lemma 2.3. *If $\lambda \in \Gamma_+$ and $i \neq j$, then*

$$\frac{\sigma_{k-1;i}(\lambda_i - \lambda_1)^2 - \sigma_{k-1;j}(\lambda_j - \lambda_1)^2}{\lambda_i - \lambda_j} \geq 0.$$

Proof. Since $\sigma_{k-1;i} = \sigma_{k-1;ij} + \lambda_j \sigma_{k-2;ij}$, we have

$$\begin{aligned}&\sigma_{k-1;i}(\lambda_i - \lambda_1)^2 - \sigma_{k-1;j}(\lambda_j - \lambda_1)^2\\ =&\ \sigma_{k-1;ij}(\lambda_i - \lambda_j)(\lambda_i + \lambda_j - 2\lambda_1) + \sigma_{k-2;ij}(\lambda_i - \lambda_j)(\lambda_i\lambda_j - \lambda_1^2)\\ =&\ (\lambda_i - \lambda_j)\Big(\sigma_{k-1;ij}(\lambda_i + \lambda_j - 2\lambda_1) + \sigma_{k-2;ij}(\lambda_i\lambda_j - \lambda_1^2)\Big).\end{aligned}$$

Then

$$\begin{aligned}&\frac{\sigma_{k-1;i}(\lambda_i - \lambda_1)^2 - \sigma_{k-1;j}(\lambda_j - \lambda_1)^2}{\lambda_i - \lambda_j}\\ =&\ \sigma_{k-1;ij}(\lambda_i + \lambda_j - 2\lambda_1) + \sigma_{k-2;ij}(\lambda_i\lambda_j - \lambda_1^2) \geq 0.\end{aligned}$$

□

Lemma 2.4. *For $\lambda \in \Gamma_k = \{\mu \in \mathbb{R}^n | \sigma_1(\mu) > 0, \cdots, \sigma_k(\mu) > 0\}$, we have*

$$\sigma_k \geq \frac{n}{k}\lambda_1 \sigma_{k-1;1}.$$

Proof. By using Proposition 2.2 and $\sigma_{k;i} \leq \sigma_{k;1}$, we have

$$k\sigma_k = \sum_i \lambda_i \sigma_{k-1;i} = \sum_i (\sigma_k - \sigma_{k;i}) \geq n(\sigma_k - \sigma_{k;1}) = n\lambda_1 \sigma_{k-1;1}.$$

□

We now turn to prove our key lemma of σ_k. First we show two lemmas. Let $D_m^{(k)}(\lambda) = (d_{ij})$, $i, j = 0, \cdots, m$, denote the following symmetric $(m+1) \times (m+1)$-matrix

$$\begin{pmatrix} \sigma_k & \sigma_{k;1} & \sigma_{k;2} & \cdots & \sigma_{k;m}\\ \sigma_{k;1} & \sigma_{k;1} & \sigma_{k;12} & \cdots & \sigma_{k;1m}\\ \sigma_{k;2} & \sigma_{k;21} & \sigma_{k;2} & \cdots & \sigma_{k;2m}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \sigma_{k;m} & \sigma_{k;m1} & \sigma_{k;m2} & \cdots & \sigma_{k;m} \end{pmatrix},$$

i.e., $d_{ij} = d_{ji}$ and

$$d_{ij} = \begin{cases} \sigma_k(\lambda), & \text{if } i = j = 0, \\ \sigma_{k;j}(\lambda), & \text{if } i = 0,\ 1 \le j \le m, \\ \sigma_{k;i}(\lambda), & \text{if } 1 \le i = j \le m, \\ \sigma_{k;ij}(\lambda), & \text{if } 1 \le i < j \le m. \end{cases}$$

Lemma 2.5. *If $\lambda \in \Gamma_+$ and $n \ge 2$, then $D_n^{(k)}(\lambda)$ is semi-positive definite for $1 \le k \le n$.*

Proof. First, since $\sigma_{n;i} = \sigma_{n;pq} = 0$ for $1 \le i, p, q \le n$, it is clear that $D_n^{(n)}$ is semi-positive definite.

For $1 \le k \le n-1$, the statement follows by induction on n. In fact, for $n = 2$, the semi-positive-definiteness is proved by directly computation. Now, assum that the statement is true for $n-1$. For $\lambda = (\lambda_1, ..., \lambda_n)$, the assumption implies the following matrices are semi-positive definite

$$D_{n-1;n}^{(k)}(\lambda) = \begin{pmatrix} \sigma_{k;n} & \sigma_{k;1n} & \sigma_{k;2n} & \cdots & \sigma_{k;n-1,n} \\ \sigma_{k;1n} & \sigma_{k;1n} & \sigma_{k;12} & \cdots & \sigma_{k;1,n-1,n} \\ \sigma_{k;2n} & \sigma_{k;21n} & \sigma_{k;2n} & \cdots & \sigma_{k;2,n-1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{k;n-1,n} & \sigma_{k;n-1,1n} & \sigma_{k;n-1,2n} & \cdots & \sigma_{k;n-1,n} \end{pmatrix}$$

for $1 \le k \le n-1$. And, using

$$\sigma_k = \sigma_{k;n} + \lambda_n \sigma_{k-1;n}, \quad \sigma_{k,i} = \sigma_{k;in} + \lambda_n \sigma_{k-1;in} \ (1 \le i \le n-1),$$

we obtain

$$D_n^{(k)}(\lambda) = \lambda_n \begin{pmatrix} D_{n-1;n}^{(k-1)} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} D_{n-1;n}^{(k)} & \eta \\ \eta^T & \sigma_{k;n} \end{pmatrix},$$

where $\eta^T = (\sigma_{k;n}, \sigma_{k;1n}, \sigma_{k;2n}, \cdots, \sigma_{k;n-1,n})$. For

$$\begin{pmatrix} D_{n-1;n}^{(k)} & \eta \\ \eta^T & \sigma_{k;n} \end{pmatrix} = \begin{pmatrix} \sigma_{k;n} & \sigma_{k;1n} & \sigma_{k;2n} & \cdots & \sigma_{k;n-1,n} & \sigma_{k;n} \\ \sigma_{k;1n} & \sigma_{k;1n} & \sigma_{k;12n} & \cdots & \sigma_{k;1,n-1,n}, & \sigma_{k;1n} \\ \sigma_{k;2n} & \sigma_{k;21n} & \sigma_{k;2n} & \cdots & \sigma_{k;2,n-1,n} & \sigma_{k;2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{k;n-1,n} & \sigma_{k;n-1,1,n} & \sigma_{k;n-1,2,n} & \cdots & \sigma_{k;n-1,n} & \sigma_{k;n-1,n} \\ \sigma_{k;n} & \sigma_{k;n1} & \sigma_{k;n2} & \cdots & \sigma_{k;n,n-1} & \sigma_{k;n} \end{pmatrix},$$

by subtracting the first row from the last row and the first column from the last column, we find that $\begin{pmatrix} D_{n-1;n}^{(k)} & \eta \\ \eta^T & \sigma_{k;n} \end{pmatrix}$ is congruent to $\begin{pmatrix} D_{n-1;n}^{(k)} & 0 \\ 0 & 0 \end{pmatrix}$ which is semi-positive definite. So $D_n^{(k)}(\lambda)$ is semi-positive definite. Thus, the proof is completed. □

For $\lambda = (\lambda_1, ..., \lambda_n) \in \Gamma_+$, let $A^{(k)}(\lambda) = (a_{ij})_{n\times n}$ denote the following matrix

$$\begin{pmatrix} \frac{1}{\lambda_1}\sigma_{k-1;1} & \sigma_{k-2;12} & \sigma_{k-2;13} & \cdots & \sigma_{k-2;1n} \\ \sigma_{k-2;21} & \frac{1}{\lambda_2}\sigma_{k-1;2} & \sigma_{k-2;23} & \cdots & \sigma_{k-2;2n} \\ \sigma_{k-2;31} & \sigma_{k-2;32} & \frac{1}{\lambda_3}\sigma_{k-1;3} & \cdots & \sigma_{k-2;3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{k-2;n1} & \sigma_{k-2;n2} & \sigma_{k-2;n3} & \cdots & \frac{1}{\lambda_n}\sigma_{k-1;n} \end{pmatrix},$$

i.e.,

$$a_{ij} = \begin{cases} \dfrac{1}{\lambda_i}\sigma_{k-1;i}(\lambda), & \text{for } i = j, \\ \sigma_{k-2;ij}(\lambda), & \text{for } i \neq j. \end{cases}$$

Lemma 2.6. *Let $\xi^T = (\sigma_{k-1;1}, \sigma_{k-1;2}, ..., \sigma_{k-1;n})$. Then the matrix $\sigma_k A^{(k)} - \xi\xi^T$ is semi-positive definite.*

Proof. Denote $\sigma_k A^{(k)} - \xi\xi^T = (w_{ij})_{n\times n}$. Thus

$$w_{ij} = \begin{cases} \dfrac{\sigma_{k-1;i}}{\lambda_i}\sigma_{k;i}, & \text{for } i = j, \\ \dfrac{1}{\lambda_i\lambda_j}(\sigma_k\sigma_{k;ij} - \sigma_{k;i}\sigma_{k;j}), & \text{for } i \neq j. \end{cases}$$

We divide the proof in three steps.

Step 1. Since the semi-positive-definiteness is preserved under congruent transformation, we multiply λ_i to the i-th row and the i-th column of $\sigma_k A^{(k)} - \xi\xi^T$ for $1 \leq i \leq n$. And, let $\tilde{A}^{(k)} = (\tilde{a}_{ij})_{n\times n}$ denote the new matrix which is defined by

$$\tilde{a}_{ij} = \begin{cases} \sigma_{k;i}(\sigma_k - \sigma_{k;i}), & \text{for } i = j, \\ \sigma_k\sigma_{k;ij} - \sigma_{k;i}\sigma_{k;j}, & \text{for } i \neq j. \end{cases}$$

We will discuss $\tilde{A}^{(k)}$ instead of $\sigma_k A^{(k)} - \xi\xi^T$ in the following.

Step 2. $\tilde{A}^{(k)}$ is semi-positive definite if and only if its principal minors are all non-negative. Let $\tilde{A}^{(k)}_m$ denote the upper-left $m\times m$ sub-matrix of $\tilde{A}^{(k)}$. For the symmetry of the elemental functions, it suffices to show $\det \tilde{A}^{(k)}_m \geq 0$.

Step 3. $\det \tilde{A}^{(k)}_m$ can be calculated as follows.

$$\begin{aligned}
\det \tilde{A}^{(k)}_m &= \det \begin{pmatrix} 1 & \sigma_{k;1} & \sigma_{k;2} & \cdots & \sigma_{k;m} \\ 0 & \sigma_k\sigma_{k;1} - \sigma_{k;1}^2 & \sigma_k\sigma_{k;12} - \sigma_{k;1}\sigma_{k;2} & \cdots & \sigma_k\sigma_{k;1m} - \sigma_{k;1}\sigma_{k;m} \\ 0 & \sigma_k\sigma_{k;12} - \sigma_{k;1}\sigma_{k;2} & \sigma_k\sigma_{k;2} - \sigma_{k;2}^2 & \cdots & \sigma_k\sigma_{k;2m} - \sigma_{k;2}\sigma_{k;m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sigma_k\sigma_{k;m1} - \sigma_{k;m}\sigma_{k;1} & \sigma_k\sigma_{k;m2} - \sigma_{k;m}\sigma_{k;2} & \cdots & \sigma_k\sigma_{k;m} - \sigma_{k;m}^2 \end{pmatrix} \\
&= \det \begin{pmatrix} 1 & \sigma_{k;1} & \sigma_{k;2} & \cdots & \sigma_{k;m} \\ \sigma_{k;1} & \sigma_k\sigma_{k;1} & \sigma_k\sigma_{k;12} & \cdots & \sigma_k\sigma_{k;1m} \\ \sigma_{k;2} & \sigma_k\sigma_{k;12} & \sigma_k\sigma_{k;2} & \cdots & \sigma_k\sigma_{k;2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{k;m} & \sigma_k\sigma_{k;m1} & \sigma_k\sigma_{k;m2} & \cdots & \sigma_k\sigma_{k;m} \end{pmatrix} \\
&= \sigma_k^{-2} \det \begin{pmatrix} \sigma_k^2 & \sigma_k\sigma_{k;1} & \sigma_k\sigma_{k;2} & \cdots & \sigma_k\sigma_{k;m} \\ \sigma_k\sigma_{k;1} & \sigma_k\sigma_{k;1} & \sigma_k\sigma_{k;12} & \cdots & \sigma_k\sigma_{k;1m} \\ \sigma_k\sigma_{k;2} & \sigma_k\sigma_{k;12} & \sigma_k\sigma_{k;2} & \cdots & \sigma_k\sigma_{k;2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_k\sigma_{k;m} & \sigma_k\sigma_{k;m1} & \sigma_k\sigma_{k;m2} & \cdots & \sigma_k\sigma_{k;m} \end{pmatrix} \\
&= \sigma_k^{m-1} \det D^{(k)}_m.
\end{aligned}$$

By Lemma 2.5, we know $\det D^{(k)}_m \geq 0$. So, $\det \tilde{A}^{(k)}_m \geq 0$ which implies $\sigma_k A^{(k)} - \xi\xi^T$ is semi-positive definite. □

With the help of the proceeding two lemmas, we finally obtain our key lemma.

Lemma 2.7. *For $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, the following inequality holds*

$$\sum_{i=1}^n \frac{\sigma_{k-1;i}}{\lambda_i \sigma_k} y_i^2 + \sum_{i \neq j} \frac{\sigma_{k-2;ij}}{\sigma_k} y_i y_j \geq (\sum_{i=1}^n \frac{\sigma_{k-1;i}}{\sigma_k} y_i)^2.$$

Proof. By Lemma 2.6, we know

$$y^T(\frac{1}{\sigma_k} A^{(k)} - \frac{1}{\sigma_k^2} \xi \xi^T) y \geq 0.$$

□

3. Fundamental formulas of self-similar solution with general F

Let $X : M^n \to \mathbb{R}^{n+1}$ be a closed convex hypersurface. Suppose that $e_1, e_2, \cdots, e_n$ is an orthonormal frame on M. Let $h = (h_{ij})$ be the second fundamental form on M with respect to this given frame. And the principal curvatures are the eigenvalues of the second fundamental form h.

Let us first consider the following general equation

$$F(h) + C = \langle X, \nu \rangle,$$

where F is a homogeneous symmetric function of $h = (h_{ij})$ of degree β, C is a constant and ν is the outward normal vector field. And, let $\mathcal{L}$ denote the operator $\mathcal{L} = \frac{\partial F}{\partial h_{ij}} \nabla_i \nabla_j$. We also suppose $F > 0$ and $(\frac{\partial F}{\partial h_{ij}})$ is positive definite. Inspired by [20], [11] and [9], we have the following proposition. The summation convention is used unless otherwise stated.

Proposition 3.1. *Given a smooth function $F : M \to \mathbb{R}^{n+1}$ described as above, the following equations hold:*

$$\mathcal{L}F = \langle X, \nabla F \rangle + \beta F - \frac{\partial F}{\partial h_{ij}} h_{jl} h_{li} (F + C), \tag{1}$$

$$\begin{aligned} \mathcal{L}h_{kl} = {} & h_{klm} \langle X, e_m \rangle + h_{kl} - C h_{km} h_{lm} - \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijk} h_{stl} \\ & - \frac{\partial F}{\partial h_{ij}} h_{mj} h_{mi} h_{kl} + (\beta - 1) F h_{km} h_{ml}, \end{aligned} \tag{2}$$

$$\begin{aligned} \mathcal{L}b^{kl} = {} & \langle X, \nabla b^{kl} \rangle - b^{kl} + C \delta_{kl} + b^{kp} b^{ql} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijp} h_{stq} \\ & + b^{kl} \frac{\partial F}{\partial h_{ij}} h_{mj} h_{mi} - (\beta - 1) F \delta_{kl} + 2 b^{ks} b^{pt} b^{lq} \frac{\partial F}{\partial h_{ij}} h_{sti} h_{pqj}, \end{aligned} \tag{3}$$

$$\begin{aligned} \mathcal{L}(F \mathrm{tr} b) = {} & \langle X, \nabla (F \mathrm{tr} b) \rangle + (\beta - 1) F \mathrm{tr} b - n(\beta - 1) F^2 \\ & + C(nF - \mathrm{tr} b \frac{\partial F}{\partial h_{ij}} h_{jl} h_{li}) + 2 \frac{\partial F}{\partial h_{ij}} \nabla_i F \nabla_j \mathrm{tr} b \\ & + F b^{kp} b^{qk} \frac{\partial^2 F}{\partial h_{ij} \partial h_{st}} h_{ijp} h_{stq} + 2 F b^{ks} b^{pt} b^{kq} \frac{\partial F}{\partial h_{ij}} h_{sti} h_{pqj}, \end{aligned} \tag{4}$$

$$\mathcal{L} \frac{|X|^2}{2} = \sum_i \frac{\partial F}{\partial h_{ii}} - \beta F (F + C). \tag{5}$$

Proof. (1) Differentiating (1.5) gives

$$\nabla_j F = h_{jl}\langle X, e_l\rangle \tag{3.1}$$

and

$$\begin{aligned}\nabla_i\nabla_j F &= h_{jli}\langle X, e_l\rangle + h_{ij} - h_{jl}h_{il}\langle X, \nu\rangle \\ &= h_{jli}\langle X, e_l\rangle + h_{ij} - h_{jl}h_{il}(F+C).\end{aligned}$$

Then, by $\frac{\partial F}{\partial h_{ij}}h_{ij} = \beta F$, we obtain

$$\mathcal{L}F = \nabla_l F\langle X, e_l\rangle + \beta F - \frac{\partial F}{\partial h_{ij}}h_{jl}h_{li}(F+C).$$

(2) By Codazzi equation and Ricci identity, we obtain

$$h_{klji} = h_{kjli} = h_{kjil} + h_{mj}R_{mkli} + h_{km}R_{mjli}.$$

Then, using Gauss equation we have

$$\begin{aligned}\mathcal{L}h_{kl} &= \frac{\partial F}{\partial h_{ij}}(h_{kjil} + h_{mj}R_{mkli} + h_{km}R_{mjli}) \\ &= \nabla_l(\frac{\partial F}{\partial h_{ij}}h_{ijk}) - \frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijk}h_{stl} + \frac{\partial F}{\partial h_{ij}}h_{mj}(h_{ml}h_{ki} - h_{mi}h_{kl}) \\ &\quad + \frac{\partial F}{\partial h_{ij}}h_{km}(h_{ml}h_{ij} - h_{mi}h_{jl}) \\ &= \nabla_l\nabla_k F - \frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijk}h_{stl} - \frac{\partial F}{\partial h_{ij}}h_{mj}h_{mi}h_{kl} + \frac{\partial F}{\partial h_{ij}}h_{km}h_{ml}h_{ij} \\ &= h_{klm}\langle X, e_m\rangle + h_{kl} - h_{km}h_{lm}(F+C) - \frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijk}h_{stl} \\ &\quad - \frac{\partial F}{\partial h_{ij}}h_{mj}h_{mi}h_{kl} + \beta F h_{km}h_{ml}.\end{aligned}$$

(3) Since $h_{km}b^{ml} = \delta_{kl}$, we have

$$\nabla_j b^{kl} = -b^{kp}b^{lq}\nabla_j h_{pq}. \tag{3.2}$$

And,

$$\begin{aligned}\nabla_i\nabla_j b^{kl} &= -\nabla_i(b^{kp}b^{lq}\nabla_j h_{pq}) \\ &= -b^{kp}b^{ql}\nabla_i\nabla_j h_{pq} + b^{ks}b^{pt}b^{lq}\nabla_i h_{st}\nabla_j h_{pq} + b^{kp}b^{ls}b^{qt}\nabla_i h_{st}\nabla_j h_{pq} \\ &= -b^{kp}b^{ql}\nabla_i\nabla_j h_{pq} + 2b^{ks}b^{pt}b^{lq}\nabla_i h_{st}\nabla_j h_{pq}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\mathcal{L}b^{kl} &= -b^{kp}b^{ql}\frac{\partial F}{\partial h_{ij}}\nabla_i\nabla_j h_{pq} + 2b^{ks}b^{pt}b^{lq}\frac{\partial F}{\partial h_{ij}}\nabla_i h_{st}\nabla_j h_{pq} \\ &\quad - b^{kp}b^{ql}\frac{\partial F}{\partial h_{ij}}h_{pm}h_{mq}h_{ij} + 2b^{ks}b^{pt}b^{lq}\frac{\partial F}{\partial h_{ij}}h_{sti}h_{pqj} \\ &= \langle X, \nabla b^{kl}\rangle - b^{kl} + (F+C)\delta_{kl} + b^{kp}b^{ql}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijp}h_{stq} \\ &\quad + b^{kl}\frac{\partial F}{\partial h_{ij}}h_{mj}h_{mi} - \beta F\delta_{kl} + 2b^{ks}b^{pt}b^{lq}\frac{\partial F}{\partial h_{ij}}h_{sti}h_{pqj}.\end{aligned}$$

(4) From (3), we have

$$\frac{\partial F}{\partial h_{ij}}\nabla_i\nabla_j \mathrm{tr}b = \langle X, \nabla \mathrm{tr}b\rangle - \mathrm{tr}b + n(F+C) + b^{kp}b^{qk}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijp}h_{stq} + \mathrm{tr}b\frac{\partial F}{\partial h_{ij}}h_{mj}h_{mi} - n\beta F + 2b^{ks}b^{pt}b^{kq}\frac{\partial F}{\partial h_{ij}}h_{sti}h_{pqj}.$$

Furthermore,

$$\begin{aligned}
\mathcal{L}(F\mathrm{tr}b) &= 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j \mathrm{tr}b + \mathrm{tr}b\frac{\partial F}{\partial h_{ij}}\nabla_i\nabla_j F + F\frac{\partial F}{\partial h_{ij}}\nabla_i\nabla_j \mathrm{tr}b\\
&= 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j \mathrm{tr}b + \mathrm{tr}b\langle X, \nabla F\rangle + \mathrm{tr}b\frac{\partial F}{\partial h_{ij}}h_{ij}\\
&\quad - \mathrm{tr}b\frac{\partial F}{\partial h_{ij}}h_{jl}h_{li}(F+C) + F\langle X, \nabla \mathrm{tr}b\rangle - F\mathrm{tr}b + nF(F+C)\\
&\quad + Fb^{kp}b^{qk}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijp}h_{stq} + F\mathrm{tr}B\frac{\partial F}{\partial h_{ij}}h_{mj}h_{mi} - nF\frac{\partial F}{\partial h_{ij}}h_{ij}\\
&\quad + 2Fb^{ks}b^{pt}b^{kq}\frac{\partial F}{\partial h_{ij}}h_{sti}h_{pqj}\\
&= \langle X, \nabla(F\mathrm{tr}b)\rangle + (\beta-1)F\mathrm{tr}B - n(\beta-1)F^2 + C(nF - \mathrm{tr}B\frac{\partial F}{\partial h_{ij}}h_{jl}h_{li})\\
&\quad + 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j \mathrm{tr}b + Fb^{kp}b^{qk}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijp}h_{stq} + 2Fb^{ks}b^{pt}b^{kq}\frac{\partial F}{\partial h_{ij}}h_{sti}h_{pqj}.
\end{aligned}$$

(5) By direct computation and (1.5), we have

$$\begin{aligned}
\mathcal{L}\frac{|X|^2}{2} &= \frac{\partial F}{\partial h_{ij}}\nabla_i(\langle X, e_j\rangle)\\
&= \sum_i \frac{\partial F}{\partial h_{ii}} - (F+C)\frac{\partial F}{\partial h_{ij}}h_{ij}.
\end{aligned}$$

□

To finish this section, we list the following well-known result (See for example [1] and [16]).

Lemma 3.2. *If $W = (w_{ij})$ is a symmetric real matrix and $\lambda_m = \lambda_m(W)$ is one of its eigenvalues ($m = 1, \cdots, n$). If $F = F(W) = F(\lambda(W))$, then for any real symmetric matrix $B = (b_{ij})$, we have the following formulas:*

(i) $\displaystyle \frac{\partial F}{\partial w_{ij}}b_{ij} = \frac{\partial F}{\partial \lambda_p}b_{pp}$,

(ii) $\displaystyle \frac{\partial^2 F}{\partial w_{ij}\partial w_{st}}b_{ij}b_{st} = \frac{\partial^2 F}{\partial \lambda_p\partial\lambda_q}b_{pp}b_{qq} + 2\sum_{p<q}\frac{\frac{\partial F}{\partial \lambda_p} - \frac{\partial F}{\partial \lambda_q}}{\lambda_p - \lambda_q}b_{pq}^2$.

Remark 3.3. In the above lemma, $\frac{\frac{\partial F}{\partial \lambda_p} - \frac{\partial F}{\partial \lambda_q}}{\lambda_p - \lambda_q}$ is interpreted as a limit if $\lambda_p = \lambda_q$.

4. Analysis at the maximum points of W

In the recent paper [9], S. Brendle, K. Choi and P. Daskalopoulos proved the following powerful lemma.

Lemma 4.1 ([9]). *Let μ denote the multiplicity of λ_1 at a point x_0, i.e., $\lambda_1(x_0) = \cdots = \lambda_\mu(x_0) < \lambda_{\mu+1}(x_0)$. Suppose that φ is a smooth function such that $\varphi \leq \lambda_1$ everywhere and $\varphi(x_0) = \lambda_1(x_0)$. Then, at x_0, we have*
i) $h_{kli} = \nabla_i \varphi \delta_{kl}$ for $1 \leq k, l \leq \mu$.
ii) $\nabla_i \nabla_i \varphi \leq h_{11ii} - 2\sum_{l>\mu}(\lambda_l - \lambda_1)^{-1} h_{1li}^2$.

Let $\tilde{W} = \frac{F}{\lambda_1} - \frac{\beta-1}{2\beta}|X|^2$ and let x_0 be an arbitrary point where $\tilde{W}$ attains its maximum. In fact, we can choose a smooth function φ such that $W = \frac{F}{\varphi} - \frac{\beta-1}{2\beta}|X|^2$ attains its maximum at x_0. Thus, $\tilde{W}_{\max} = W_{\max}$. Now, we consider W at x_0.

Lemma 4.2. *At x_0, W satisfies the following inequality*

$$\begin{aligned}\mathcal{L}W \geq& \langle X, \nabla(\frac{F}{\varphi})\rangle + 2\frac{\partial F}{\partial h_{ij}}\nabla_i F \nabla_j \frac{1}{\varphi} + 2F\lambda_1^{-3}\frac{\partial F}{\partial \lambda_i}h_{11i}^2\\ &+ F\lambda_1^{-2}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ij1}h_{st1} + 2F\lambda_1^{-2}\frac{\partial F}{\partial \lambda_i}\sum_{l>\mu}(\lambda_l - \lambda_1)^{-1}h_{1li}^2\\ &+ \frac{\beta-1}{\beta}\frac{\partial F}{\partial \lambda_i}(\frac{\lambda_i}{\lambda_1} - 1) - C\frac{\partial F}{\partial \lambda_i}\lambda_i(\frac{\lambda_i}{\lambda_1} - 1).\end{aligned}$$

Proof. At x_0, it follows from Lemma 4.1 and Proposition 3.1 that

$$\begin{aligned}\mathcal{L}\varphi \leq& \mathcal{L}h_{11} - 2\frac{\partial F}{\partial \lambda_i}\sum_{l>\mu}(\lambda_l - \lambda_1)^{-1}h_{1li}^2\\ =& h_{11m}\langle X, e_m\rangle + \lambda_1 - \lambda_1^2 C - \lambda_1\frac{\partial F}{\partial h_{ij}}h_{mj}h_{mi} + \lambda_1^2(\beta-1)F\\ &- \frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ij1}h_{st1} - 2\frac{\partial F}{\partial \lambda_i}\sum_{l>\mu}(\lambda_l - \lambda_1)^{-1}h_{1li}^2.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\mathcal{L}\frac{F}{\varphi} =& 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j\frac{1}{\varphi} + \frac{1}{\varphi}\mathcal{L}F + F\mathcal{L}\frac{1}{\varphi}\\ \geq& 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j\frac{1}{\varphi} + \lambda_1^{-1}\nabla_l F\langle X, e_l\rangle + \lambda_1^{-1}\frac{\partial F}{\partial h_{ij}}h_{ij} - \lambda_1^{-1}\frac{\partial F}{\partial h_{ij}}h_{jl}h_{li}(F+C)\\ &+ 2F\lambda_1^{-3}\frac{\partial F}{\partial \lambda_i}h_{11i}^2 + F\nabla_m\frac{1}{\varphi}\langle X, e_m\rangle - F\lambda_1^{-1} + (1-\beta)F^2 + FC + F\lambda_1^{-1}\frac{\partial F}{\partial h_{ij}}h_{mj}h_{mi}\\ &+ F\lambda_1^{-2}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ij1}h_{st1} + 2F\lambda_1^{-2}\frac{\partial F}{\partial \lambda_i}\sum_{l>\mu}(\lambda_l - \lambda_1)^{-1}h_{1li}^2\\ =& 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j\frac{1}{\varphi} + 2F\lambda_1^{-3}\frac{\partial F}{\partial \lambda_i}h_{11i}^2 + \nabla_m\frac{F}{\varphi}\langle X, e_m\rangle\\ &+ (\beta-1)F\lambda_1^{-1} + (1-\beta)F^2 + C(F - \lambda_1^{-1}\frac{\partial F}{\partial h_{ij}}h_{jl}h_{li})\\ &+ F\lambda_1^{-2}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ij1}h_{st1} + 2F\lambda_1^{-2}\frac{\partial F}{\partial \lambda_i}\sum_{l>\mu}(\lambda_l - \lambda_1)^{-1}h_{1li}^2.\end{aligned}$$

According to Proposition 3.1 and the homogeneity of F, we have

$$\begin{aligned}&-\frac{\beta-1}{\beta}\mathcal{L}\frac{|X|^2}{2}+(\beta-1)F\lambda_1^{-1}+(1-\beta)F^2+C(F-\lambda_1^{-1}\frac{\partial F}{\partial h_{ij}}h_{jl}h_{li})\\=&\frac{\beta-1}{\beta}\frac{\partial F}{\partial\lambda_i}(\frac{\lambda_i}{\lambda_1}-1)-C\frac{\partial F}{\partial\lambda_i}\lambda_i(\frac{\lambda_i}{\lambda_1}-1),\end{aligned}$$

thus the proof is completed.

∎

Let

$$J_1=\frac{\beta-1}{\beta}\frac{\partial F}{\partial\lambda_i}(\frac{\lambda_i}{\lambda_1}-1)-C\frac{\partial F}{\partial\lambda_i}\lambda_i(\frac{\lambda_i}{\lambda_1}-1).$$

Lemma 4.3. *If $C\le 0$ and $\beta>1$, then $J_1\ge 0$. And $J_1=0$ if and only if $\lambda_1=\cdots=\lambda_n$.*

Proof. The proof directly follows from that $\frac{\partial F}{\partial\lambda_i}>0$ and $\frac{\lambda_i}{\lambda_1}\ge 1$. ∎

Lemma 4.4. *At x_0, we have the following equalities*

$$\langle X,\nabla(\frac{F}{\varphi})\rangle=\frac{\beta-1}{\beta}\sum_i\lambda_i^{-2}(\nabla_iF)^2, \tag{1}$$

$$F\lambda_1^{-2}h_{11j}=(\lambda_1^{-1}-\frac{\beta-1}{\beta}\lambda_j^{-1})\nabla_jF,\ \textit{for } 1\le j\le n, \tag{2}$$

$$\nabla_mF=0,\ \textit{for } 2\le m\le\mu. \tag{3}$$

Proof. (1) Using $\nabla W=0$ and (3.1), we have

$$\begin{aligned}\langle X,\nabla(\frac{F}{\varphi})\rangle&=\langle X,\nabla W\rangle+\frac{\beta-1}{\beta}\sum_m\langle X,e_m\rangle^2\\&=\frac{\beta-1}{\beta}\sum_i\lambda_i^{-2}(\nabla_iF)^2.\end{aligned}$$

(2) Using $\nabla_jW=0$, Lemma 4.1 and (3.1), we have

$$\begin{aligned}0&=F\nabla_j\frac{1}{\varphi}+\frac{1}{\varphi}\nabla_jF-\frac{\beta-1}{\beta}\lambda_j^{-1}\nabla_jF\\&=-F\lambda_1^{-2}h_{11j}+(\lambda_1^{-1}-\frac{\beta-1}{\beta}\lambda_j^{-1})\nabla_jF.\end{aligned}$$

(3) By Lemma 4.1, we have $h_{11m}=0$ if $2\le m\le\mu$. Then, (2) leads to (3). ∎

Lemma 4.5.

$$\begin{aligned}&\frac{\partial^2F}{\partial h_{ij}\partial h_{st}}h_{ij1}h_{st1}+2\frac{\partial F}{\partial\lambda_i}\sum_{l>\mu}(\lambda_l-\lambda_1)^{-1}h_{1li}^2\\=&\frac{\partial^2F}{\partial\lambda_i\partial\lambda_j}h_{ii1}h_{jj1}+2\sum_{i>\mu}\frac{\partial F}{\partial\lambda_i}(\lambda_i-\lambda_1)^{-1}h_{11i}^2+2\sum_{i>\mu}\frac{\partial F}{\partial\lambda_i}(\lambda_i-\lambda_1)^{-1}h_{1ii}^2\\&+2\sum_{i>j>\mu}\frac{\frac{\partial F}{\partial\lambda_i}(\lambda_i-\lambda_1)^2-\frac{\partial F}{\partial\lambda_j}(\lambda_j-\lambda_1)^2}{(\lambda_i-\lambda_1)(\lambda_j-\lambda_1)(\lambda_i-\lambda_j)}h_{ij1}^2.\end{aligned}$$

Proof. Due to

$$\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ij1}h_{st1} = \Big(\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}h_{ii1}h_{jj1} + 2\sum_{i>j}(\lambda_i-\lambda_j)^{-1}(\frac{\partial F}{\partial\lambda_i}-\frac{\partial F}{\partial\lambda_j})h_{ij1}^2\Big)$$
$$= \frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}h_{ii1}h_{jj1} + 2\sum_{i>\mu}(\lambda_i-\lambda_1)^{-1}(\frac{\partial F}{\partial\lambda_i}-\frac{\partial F}{\partial\lambda_1})h_{11i}^2$$
$$+ 2\sum_{i>j>\mu}(\lambda_i-\lambda_j)^{-1}(\frac{\partial F}{\partial\lambda_i}-\frac{\partial F}{\partial\lambda_j})h_{ij1}^2$$

and

$$2\frac{\partial F}{\partial\lambda_i}\sum_{l>\mu}(\lambda_l-\lambda_1)^{-1}h_{1li}^2 = 2\frac{\partial F}{\partial\lambda_1}\sum_{l>\mu}(\lambda_l-\lambda_1)^{-1}h_{11l}^2 + 2\sum_{i>\mu}\frac{\partial F}{\partial\lambda_i}(\lambda_i-\lambda_1)^{-1}h_{1ii}^2$$
$$+ 2\sum_{i>l>\mu}\frac{\partial F}{\partial\lambda_i}(\lambda_l-\lambda_1)^{-1}h_{1li}^2 + 2\sum_{l>i>\mu}\frac{\partial F}{\partial\lambda_i}(\lambda_l-\lambda_1)^{-1}h_{1li}^2,$$

the lemma follows by adding the above two equations. □

Lemma 4.6. *For $\beta \geq 1$, at x_0, W satisfies the following inequality*

$$\mathcal{L}W \geq J_1 + J_2 + J_3,$$

where

$$J_1 = \frac{\beta-1}{\beta}\frac{\partial F}{\partial\lambda_i}(\frac{\lambda_i}{\lambda_1}-1) - C\frac{\partial F}{\partial\lambda_i}\lambda_i(\frac{\lambda_i}{\lambda_1}-1),$$

$$J_2 = 2F\lambda_1^{-2}\sum_{i>j>\mu}\frac{\frac{\partial F}{\partial\lambda_i}(\lambda_i-\lambda_1)^2 - \frac{\partial F}{\partial\lambda_j}(\lambda_j-\lambda_1)^2}{(\lambda_i-\lambda_1)(\lambda_j-\lambda_1)(\lambda_i-\lambda_j)}h_{ij1}^2$$

and

$$J_3 = \frac{\beta-1}{\beta}\lambda_1^{-1}\Big(\lambda_1^{-1} - \frac{2}{\beta}F^{-1}\frac{\partial F}{\partial\lambda_1}\Big)(\nabla_1 F)^2 + 2F\lambda_1^{-2}\frac{\partial F}{\partial\lambda_i}\sum_{i>\mu}(\lambda_i-\lambda_1)^{-1}h_{1ii}^2$$
$$+ F\lambda_1^{-2}\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}h_{ii1}h_{jj1}.$$

Proof. By Lemma 4.4, we have

$$\langle X, \nabla(\frac{F}{\varphi})\rangle + 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j\frac{1}{\varphi} + 2F\lambda_1^{-3}\frac{\partial F}{\partial\lambda_i}h_{11i}^2$$
$$= \frac{\beta-1}{\beta}\sum_i\lambda_i^{-2}(\nabla_i F)^2 - 2F^{-1}\frac{\partial F}{\partial\lambda_i}(\lambda_1^{-1} - \frac{\beta-1}{\beta}\lambda_i^{-1})(\nabla_i F)^2 + 2F\lambda_1^{-3}\frac{\partial F}{\partial\lambda_i}h_{11i}^2$$
$$= \frac{\beta-1}{\beta}\lambda_1^{-1}\Big(\lambda_1^{-1} - \frac{2}{\beta}F^{-1}\frac{\partial F}{\partial\lambda_1}\Big)(\nabla_1 F)^2 + \sum_{i>\mu}\Big(\frac{\beta-1}{\beta}\lambda_i^{-2}$$
$$- \frac{2(\beta-1)}{\beta}F^{-1}\frac{\partial F}{\partial\lambda_i}\lambda_1\lambda_i^{-1}(\lambda_1^{-1} - \frac{\beta-1}{\beta}\lambda_i^{-1})\Big)(\nabla_i F)^2.$$

Furthermore, by Lemma 4.4 and 4.5, we have

$$\begin{aligned}\mathcal{L}W \geq\ & \frac{\beta-1}{\beta}\lambda_1^{-1}\Big(\lambda_1^{-1}-\frac{2}{\beta}F^{-1}\frac{\partial F}{\partial\lambda_1}\Big)(\nabla_1 F)^2\\ &+\sum_{i>\mu}\Big\{\frac{\beta-1}{\beta}\lambda_i^{-2}+\frac{2}{\beta}F^{-1}\frac{\partial F}{\partial\lambda_i}\Big(\lambda_i^{-1}+\frac{1}{\beta}\lambda_1\lambda_i^{-2}+\frac{\lambda_1^2}{\beta\lambda_i^2(\lambda_i-\lambda_1)}\Big)\Big\}(\nabla_i F)^2\\ &+2F\lambda_1^{-2}\sum_{i>j>\mu}\frac{\frac{\partial F}{\partial\lambda_i}(\lambda_i-\lambda_1)^2-\frac{\partial F}{\partial\lambda_j}(\lambda_j-\lambda_1)^2}{(\lambda_i-\lambda_1)(\lambda_j-\lambda_1)(\lambda_i-\lambda_j)}h_{ij1}^2+2F\lambda_1^{-2}\frac{\partial F}{\partial\lambda_i}\sum_{i>\mu}(\lambda_i-\lambda_1)^{-1}h_{1ii}^2\\ &+F\lambda_1^{-2}\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}h_{ii1}h_{jj1}+\frac{\beta-1}{\beta}\frac{\partial F}{\partial\lambda_i}(\frac{\lambda_i}{\lambda_1}-1)-C\frac{\partial F}{\partial\lambda_i}\lambda_i(\frac{\lambda_i}{\lambda_1}-1).\end{aligned}$$

Noticing the second term is nonnegative, we finish the proof. □

Lemma 4.7. *For $F=\sigma_k^\alpha$ and $C\leq 0$, if $\alpha>\frac{1}{k}$, then at the maximum point of $\tilde{W}$, $\lambda_1=\cdots=\lambda_n$.*

Proof. By Lemma 4.3, we know $J_1\geq 0$ and the equality occurs if and only if $\lambda_1=\cdots=\lambda_n$. And Lemma 2.3 implies $J_2\geq 0$. Using Lemma 2.7, we have

$$\begin{aligned}J_3 =\ & \frac{\alpha(k\alpha-1)}{k}\sigma_k^{2\alpha}\lambda_1^{-1}\Big(\lambda_1^{-1}-\frac{2}{k}\frac{\sigma_{k-1;1}}{\sigma_k}\Big)(\nabla_1\log\sigma_k)^2\\ &+2\alpha\sigma_k^{2\alpha-1}\lambda_1^{-2}\sigma_{k-1;i}\sum_{i>\mu}(\lambda_i-\lambda_1)^{-1}h_{1ii}^2\\ &+\alpha\sigma_k^{2\alpha-1}\lambda_1^{-2}\sigma_{k-2;ij}h_{ii1}h_{jj1}+\alpha(\alpha-1)\sigma_k^{2\alpha}\lambda_1^{-2}(\nabla_1\log\sigma_k)^2\\ \geq\ & \alpha\sigma_k^{2\alpha}\lambda_1^{-1}\Big(\frac{2k\alpha-k-1}{k}\lambda_1^{-1}-2\frac{(k\alpha-1)}{k^2}\frac{\sigma_{k-1;1}}{\sigma_k}\Big)(\nabla_1\log\sigma_k)^2\\ &+2\alpha\sigma_k^{2\alpha-1}\lambda_1^{-2}\sigma_{k-1;i}\sum_{i>\mu}(\lambda_i-\lambda_1)^{-1}h_{1ii}^2\\ &+\alpha\sigma_k^{2\alpha}\lambda_1^{-2}(\nabla_1\log\sigma_k)^2-\alpha\sigma_k^{2\alpha-1}\lambda_1^{-2}\frac{\sigma_{k-1;i}}{\lambda_i}h_{ii1}^2.\end{aligned}$$

Then using Lemma 4.4, we obtain

$$\begin{aligned}J_3 \geq\ & \alpha\sigma_k^{2\alpha}\lambda_1^{-1}\Big(\frac{2k\alpha-1}{k}\lambda_1^{-1}-2\frac{(k\alpha-1)}{k^2}\frac{\sigma_{k-1;1}}{\sigma_k}\Big)(\nabla_1\log\sigma_k)^2\\ &-\frac{\alpha}{k^2}\sigma_k^{2\alpha-1}\frac{\sigma_{k-1;1}}{\sigma_k}\lambda_1^{-1}(\nabla_1\log\sigma_k)^2\\ \geq\ & \alpha\frac{2k\alpha-1}{k^2}\sigma_k^{2\alpha}\lambda_1^{-2}\Big(k-\frac{\sigma_{k-1;1}\lambda_1}{\sigma_k}\Big)(\nabla_1\log\sigma_k)^2\\ \geq\ & \alpha\frac{(2k\alpha-1)(n-1)}{nk}\sigma_k^{2\alpha}\lambda_1^{-2}(\nabla_1\log\sigma_k)^2,\end{aligned}$$

thus $J_3\geq 0$. For $\mathcal{L}$ is an elliptic operator when $F=\sigma_k^\alpha$, at the maximum point of W, we have

$$0\geq\mathcal{L}W\geq J_1+J_2+J_3\geq 0.$$

Thus $J_1=0$, which implies $\lambda_1=\cdots=\lambda_n$. Since $\tilde{W}$ and W have the same maximum points, we finish the proof. □

By similar discussion, for $F=S_k^\alpha$, we have the following lemma.

Lemma 4.8. *For $F = S_k^\alpha$ and $C \le 0$, if $k \ge 1$ and $\alpha > \frac{1}{k}$, then at the maximum point of $\tilde{W}$, $\lambda_1 = \cdots = \lambda_n$.*

Proof. It is easy to check that $J_2 \ge 0$. We just show $J_3 \ge 0$ since the rest of the proof is similar to Lemma 4.7. Actually, for $F = S_k^\alpha$, we have

$$\begin{aligned}
F\lambda_1^{-2}\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}h_{ii1}h_{jj1} &= \alpha(\alpha-1)\lambda_1^{-2}S_k^{2\alpha}(\nabla_1 \log S_k)^2 + \alpha k(k-1)\lambda_1^{-2}S_k^{2\alpha-1}\lambda_i^{k-2}h_{ii1}^2 \\
&\ge \alpha(\alpha-1)\lambda_1^{-2}S_k^{2\alpha}(\nabla_1 \log S_k)^2 + \frac{(k-1)\alpha}{k}S_k^{2\alpha}\lambda_1^{-2}(\nabla_1 \log S_k)^2 \\
&= \frac{\alpha(k\alpha-1)}{k}S_k^{2\alpha}\lambda_1^{-2}(\nabla_1 \log S_k)^2,
\end{aligned}$$

where we use the Cauchy-Schwarz inequality for

$$(\nabla_1 \log S_k)^2 = k^2\Big(\frac{\lambda_i^{k-1}}{S_k}h_{ii1}\Big)^2 \le k^2\Big(\sum_i \frac{\lambda_i^k}{S_k}\Big)\Big(\sum_i \frac{\lambda_i^{k-2}}{S_k}h_{ii1}^2\Big) = k^2\sum_i \frac{\lambda_i^{k-2}}{S_k}h_{ii1}^2.$$

Therefore,

$$\begin{aligned}
J_3 \ge& \frac{\alpha(k\alpha-1)}{k}S_k^{2\alpha}\lambda_1^{-1}(\lambda_1^{-1} - \frac{2\lambda_1^{k-1}}{S_k})(\nabla_1 \log S_k)^2 \\
&+ \frac{\alpha(k\alpha-1)}{k}S_k^{2\alpha}\lambda_1^{-2}(\nabla_1 \log S_k)^2 \\
&= \frac{2\alpha(k\alpha-1)}{k}S_k^{2\alpha}\lambda_1^{-2}(1 - \frac{\lambda_1^k}{S_k})(\nabla_1 \log S_k)^2 \\
&\ge 0.
\end{aligned}$$

□

5. Proof of Theorem A

In this section, by considering the quantity

$$Z = F\mathrm{tr}b - \frac{n(\beta-1)}{2\beta}|X|^2,$$

we will prove Theorem A.

Lemma 5.1.

$$\mathcal{L}Z + R(\nabla Z) = L_1 + L_2 + L_3,$$

where $R(\nabla Z)$ denote the terms containing ∇Z,

$$L_1 = (\beta-1)F\mathrm{tr}b - \frac{n(\beta-1)}{\beta}\sum_i \frac{\partial F}{\partial\lambda_i} + C(n\beta F - \mathrm{tr}b\frac{\partial F}{\partial\lambda_i}\lambda_i^2),$$

$$L_2 = \Big(\frac{n(\beta-1)}{\beta}\lambda_i^{-1}(2F^{-1}\frac{\partial F}{\partial\lambda_i} + \lambda_i^{-1}) - 2F^{-1}\frac{\partial F}{\partial\lambda_i}\mathrm{tr}b\Big)(\nabla_i F)^2$$

and

$$\begin{aligned}
L_3 =& 2F\frac{\partial F}{\partial\lambda_i}\lambda_p^{-2}\lambda_q^{-1}h_{pqi}^2 + F\lambda_p^{-2}\frac{\partial^2 F}{\partial\lambda_i\partial\lambda_j}h_{iip}h_{jjp} \\
&+ F\lambda_p^{-2}\sum_{i\ne j}(\frac{\partial F}{\partial\lambda_i} - \frac{\partial F}{\partial\lambda_j})(\lambda_i-\lambda_j)^{-1}h_{ijp}^2.
\end{aligned}$$

Proof. By Proposition 3.1, we have

$$\begin{aligned}\mathcal{L}Z =& \langle X, \nabla(F\mathrm{tr}b)\rangle + (\beta-1)F\mathrm{tr}b - \frac{n(\beta-1)}{\beta}\sum_i \frac{\partial F}{\partial h_{ii}} \\ &+ C(n\beta F - \mathrm{tr}b\frac{\partial F}{\partial h_{ij}}h_{jl}h_{li}) + 2\frac{\partial F}{\partial h_{ij}}\nabla_i F\nabla_j \mathrm{tr}b \\ &+ Fb^{kp}b^{qk}\frac{\partial^2 F}{\partial h_{ij}\partial h_{st}}h_{ijp}h_{stq} + 2Fb^{ks}b^{pt}b^{kq}\frac{\partial F}{\partial h_{ij}}h_{sti}h_{pqj}.\end{aligned}$$

From

$$\nabla_j Z = \mathrm{tr}b\nabla_j F + F\nabla_j \mathrm{tr}b - \frac{n(\beta-1)}{\beta}\langle X, e_j\rangle,$$

we have

$$\begin{aligned}\nabla_l(F\mathrm{tr}b)\langle X, e_l\rangle &= \nabla_l Z\langle X, e_l\rangle + \frac{n(\beta-1)}{\beta}\sum_l \langle X, e_l\rangle^2 \\ &= \nabla_l Z\langle X, e_l\rangle + \frac{n(\beta-1)}{\beta}\lambda_l^{-2}(\nabla_l F)^2\end{aligned}$$

and

$$\begin{aligned}\nabla_i \mathrm{tr}b &= F^{-1}(\nabla_i Z - \mathrm{tr}b\nabla_i F + \frac{n(\beta-1)}{\beta}\lambda_i^{-1}\nabla_i F) \\ &= F^{-1}\nabla_i Z + F^{-1}(-\mathrm{tr}b + \frac{n(\beta-1)}{\beta}\lambda_i^{-1})\nabla_i F.\end{aligned} \tag{5.1}$$

Then, by Lemma 3.2, we have

$$\begin{aligned}\mathcal{L}Z + R(\nabla Z) =& (\beta-1)F\mathrm{tr}b - \frac{n(\beta-1)}{\beta}\sum_i \frac{\partial F}{\partial \lambda_i} + C(n\beta F - \mathrm{tr}b\frac{\partial F}{\partial \lambda_i}\lambda_i^2) \\ &+ \Big(\frac{n(\beta-1)}{\beta}\lambda_i^{-1}(2F^{-1}\frac{\partial F}{\partial \lambda_i} + \lambda_i^{-1}) - 2F^{-1}\frac{\partial F}{\partial \lambda_i}\mathrm{tr}b\Big)(\nabla_i F)^2 \\ &+ 2F\frac{\partial F}{\partial \lambda_i}\lambda_p^{-2}\lambda_q^{-1}h_{pqi}^2 + F\lambda_p^{-2}\frac{\partial^2 F}{\partial \lambda_i\partial \lambda_j}h_{iip}h_{jjp} \\ &+ F\lambda_p^{-2}\sum_{i\neq j}(\frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j})(\lambda_i - \lambda_j)^{-1}h_{ijp}^2.\end{aligned}$$

□

Lemma 5.2. *For $F = \sigma_k^\alpha$, if $k\alpha \geq 1$ and $C \leq 0$, then $L_1 \geq 0$. In particular, for $F = \sigma_n^\alpha$, if $C \leq 0$ and $\alpha \geq 0$, then $L_1 \geq 0$.*

Proof. For $F = \sigma_k^\alpha$ and $C \leq 0$, by Newton-Maclaurin inequality and $\mathrm{tr}b = \frac{\sigma_{n-1}}{\sigma_n}$, we have

$$\begin{aligned}L_1 &= (k\alpha-1)\sigma_k^\alpha(\mathrm{tr}b - \frac{n(n-k+1)\sigma_{k-1}}{k\sigma_k}) - \alpha C\sigma_k^\alpha\Big(-nk + \mathrm{tr}b\big(\sigma_1 - (k+1)\frac{\sigma_{k+1}}{\sigma_k}\big)\Big) \\ &\geq (k\alpha-1)\sigma_k^\alpha(\frac{\sigma_{n-1}}{\sigma_n} - \frac{n(n-k+1)\sigma_{k-1}}{k\sigma_k}) - \frac{k\alpha}{n}C\sigma_k^\alpha\Big(-n^2 + \mathrm{tr}b\sigma_1\Big) \\ &\geq 0.\end{aligned}$$

□

Theorem 5.3. *For $F=\sigma_k^\alpha$ and $C\le 0$, if $\alpha>\frac{1}{k}$, then a strictly convex closed solution of (1.5) is a round sphere.*

Proof. For $F=\sigma_k^\alpha$,

$$
\begin{aligned}
L_2 &= \Big(\frac{n(k\alpha-1)}{k\alpha}\lambda_i^{-1}(\frac{2\alpha\sigma_{k-1;i}}{\sigma_k}+\lambda_i^{-1})-\frac{2\alpha\sigma_{k-1;i}}{\sigma_k}\mathrm{tr}b\Big)(\nabla_i\sigma_k^\alpha)^2\\
&=\alpha\sigma_k^{2\alpha}\Big(\frac{n(k\alpha-1)}{k}\lambda_i^{-2}(\frac{2\alpha\sigma_{k-1;i}\lambda_i}{\sigma_k}+1)-\frac{2\alpha^2\sigma_{k-1;i}}{\sigma_k}\mathrm{tr}b\Big)(\nabla_i\log\sigma_k)^2
\end{aligned}
$$

and

$$
\begin{aligned}
L_3 =& 2\alpha\sigma_k^{2\alpha-1}\sigma_{k-1;i}\lambda_p^{-2}\lambda_q^{-1}h_{pqi}^2+\alpha(\alpha-1)\sigma_k^{2\alpha-2}\lambda_p^{-2}(\nabla_p\sigma_k)^2\\
&+\alpha\sigma_k^{2\alpha-1}\lambda_p^{-2}\sigma_{k-2;ij}(h_{iip}h_{jjp}-h_{ijp}^2)\\
=&\alpha(\alpha-1)\sigma_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log\sigma_k)^2+\alpha\sigma_k^{2\alpha-1}\lambda_p^{-2}(\sigma_{k-1;i}\lambda_i^{-1}h_{iip}^2+\sigma_{k-2;ij}h_{iip}h_{jjp})\\
&+\alpha\sigma_k^{2\alpha-1}\lambda_p^{-2}\sum_{i\ne j}(2\sigma_{k-1;i}\lambda_j^{-1}-\sigma_{k-2;ij})h_{ijp}^2+\alpha\sigma_k^{2\alpha-1}\lambda_p^{-2}\sigma_{k-1;i}\lambda_i^{-1}h_{iip}^2.
\end{aligned}
$$

By using Lemma 2.7, $\sigma_{k-1;i}\lambda_j^{-1}\ge\sigma_{k-2;ij}$ and

$$
k\sum_i\frac{\sigma_{k-1;i}}{\lambda_i\sigma_k}h_{iip}^2=\Big(\sum_i\frac{\lambda_i\sigma_{k-1;i}}{\sigma_k}\Big)\Big(\sum_i\frac{\sigma_{k-1;i}}{\lambda_i\sigma_k}h_{iip}^2\Big)\ge\Big(\sum_i\frac{\sigma_{k-1;i}}{\sigma_k}h_{iip}\Big)^2.
$$

on the second, the third and the last terms, respectively, we obtain

$$
\begin{aligned}
L_3 &\ge \alpha(\alpha-1)\sigma_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log\sigma_k)^2+\alpha\sigma_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log\sigma_k)^2+\frac{\alpha}{k}\sigma_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log\sigma_k)^2\\
&=\frac{\alpha(k\alpha+1)}{k}\sigma_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log\sigma_k)^2.
\end{aligned}
$$

Then,

$$
L_2+L_3\ge\alpha\sigma_k^{2\alpha}\Big(\lambda_i^{-2}(\frac{n(k\alpha-1)}{k}\frac{2\alpha\sigma_{k-1;i}\lambda_i}{\sigma_k}+\frac{(n+1)k\alpha-n+1}{k})-\frac{2\alpha^2\sigma_{k-1;i}}{\sigma_k}\mathrm{tr}b\Big)(\nabla_i\log\sigma_k)^2.
$$

Assume that x_0 is a maximum point of $\tilde{W}$. Then it is follows from Lemma 4.7 that x_0 is an umbilic point. At x_0, thus we have

$$
\begin{aligned}
&\lambda_i^{-2}(\frac{n(k\alpha-1)}{k}\frac{2\alpha\sigma_{k-1;i}\lambda_i}{\sigma_k}+\frac{(n+1)k\alpha-n+1}{k})-\frac{2\alpha^2\sigma_{k-1;i}}{\sigma_k}\mathrm{tr}b\\
=&\lambda_i^{-2}(\frac{n(k\alpha-1)}{k}\frac{2k\alpha}{n}+\frac{(n+1)k\alpha-n+1}{k}-\frac{2\alpha^2k}{n}n)\\
=&\lambda_i^{-2}(n-1)(\alpha-\frac{1}{k})>0.
\end{aligned}
$$

Since $Z\le n\tilde{W}\le nW(x_0)=Z(x_0)$, Z attains its maximum at x_0. Hence, there exists a neighborhood of x_0, denoted by U, such that in U, $\mathcal{L}Z+R(\nabla Z)\ge 0$. By the strong maximum principle, we know $Z=Z(x_0)$ is constant in U, which implies $\tilde{W}$ is also constant in U. Then the set of points where $\tilde{W}$ attains its maximum is an open set. Hence $\tilde{W}$ is constant on M, which implies that M is totally umbilic. □

In order to discuss $F=\sigma_k^\alpha$ further, we need the following lemma.

Lemma 5.4. *Suppose $y_i \in \mathbb{R}$ and $t_i = \frac{\sigma_k}{\lambda_i \sigma_{k-1;i}}$ for $1 \le i \le n$. For any $1 \le m \le n$, the following inequality holds*

$$\sum_i t_i y_i^2 - 4\alpha y_m (\sum_i y_i) \ge \Big(\frac{1}{k}(\frac{2\alpha}{t_m} - 1)^2 - \frac{4\alpha^2}{t_m}\Big)(\sum_i y_i)^2.$$

Proof. If $\sum_i y_i = 0$, the inequality is trivial. If $\sum_i y_i \neq 0$, we may assume $\sum_i y_i = 1$. In fact, we will estimate the minimum of

$$f(y_1, ..., y_n) = \sum_i t_i y_i^2 - 4\alpha y_m$$

under the condition $\sum_i y_i = 1$. Using Lagrangian multiplier technique, we solve the following equations for $\tilde{f} = f + \tau(\sum_i y_i - 1)$,

$$0 = \frac{\partial}{\partial y_i}\tilde{f} = 2t_i y_i - 4\alpha\delta_{im} + \tau,$$
$$0 = \frac{\partial}{\partial \tau}\tilde{f} = \sum_i y_i - 1.$$

And, using $\sum_i \lambda_i \sigma_{k-1;i} = k\sigma_k$, we have $y_i = \frac{2\alpha\delta_{im}}{ti} - \frac{1}{2t_i}\tau$ and $\tau = \frac{4\alpha}{kt_m} - \frac{2}{k}$. Thus, $y_i = \frac{1}{t_i}(2\alpha\delta_{im} - \frac{2\alpha}{kt_m} + \frac{1}{k})$. Because $t_i > 0$, we know

$$\begin{aligned} f_{\min} &= \sum_i \frac{1}{t_i}(2\alpha\delta_{im} - \frac{2\alpha}{kt_m} + \frac{1}{k})^2 - \frac{4\alpha}{t_m}(2\alpha - \frac{2\alpha}{kt_m} + \frac{1}{k}) \\ &= \sum_{i\neq m} \frac{1}{k^2 t_i}(\frac{2\alpha}{t_m} - 1)^2 + \frac{1}{t_m}(-2\alpha + \frac{2\alpha}{kt_m} - \frac{1}{k})(2\alpha + \frac{2\alpha}{kt_m} - \frac{1}{k}) \\ &= \sum_i \frac{1}{k^2 t_i}(\frac{2\alpha}{t_m} - 1)^2 - \frac{4\alpha^2}{t_m} \\ &= \frac{1}{k}(\frac{2\alpha}{t_m} - 1)^2 - \frac{4\alpha^2}{t_m}. \end{aligned}$$

□

Now, we use (5.1) to estimate L_2 and L_3 in a different way.

Theorem 5.5. *For $F = \sigma_k^\alpha$ and $C \le 0$, if $2 \le k \le n-1$ and $\frac{1}{k} \le \alpha \le \frac{1}{2}$ or $k = \alpha = 1$, the strictly convex closed solution of (1.5) is a round sphere. For $F = \sigma_n^\alpha$ and $C < 0$, if $\frac{1}{n+2} \le \alpha \le \frac{1}{2}$, the strictly convex closed solution of (1.5) is a round sphere.*

Proof. Using Lemma 2.7 and $\sigma_{k-1;i}\lambda_j^{-1} - \sigma_{k-2;ij} > 0$, we have

$$\begin{aligned} \frac{1}{\alpha\sigma_k^{2\alpha}} L_3 &= (\alpha - 1)\lambda_p^{-2}(\nabla_p \log\sigma_k)^2 + \sigma_k^{-1}\lambda_p^{-2}(\sigma_{k-1;i}\lambda_i^{-1}h_{iip}^2 + \sigma_{k-2;ij}h_{iip}h_{jjp}) \\ &\quad + \frac{\sigma_{k-1;i}}{\sigma_k}\lambda_p^{-2}\lambda_i^{-1}h_{iip}^2 + \sigma_k^{-1}\sum_{\neq}\lambda_p^{-2}(2\sigma_{k-1;i}\lambda_j^{-1} - \sigma_{k-2;ij})h_{ijp}^2 \\ &\quad + 2\sigma_k^{-1}\sum_{i\neq j}\lambda_i^{-2}(\sigma_{k-1;i}\lambda_j^{-1} - \sigma_{k-2;ij})h_{iij}^2 + 2\sum_{i\neq j}\frac{\sigma_{k-1;i}}{\sigma_k}\lambda_j^{-3}h_{ijj}^2 \\ &\ge \alpha\lambda_p^{-2}(\nabla_p\log\sigma_k)^2 + \frac{\sigma_{k-1;i}}{\sigma_k}\lambda_p^{-2}\lambda_i^{-1}h_{iip}^2 + 2\sum_{i\neq j}\frac{\sigma_{k-1;i}}{\sigma_k}\lambda_j^{-3}h_{ijj}^2. \end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{1}{\alpha\sigma_k^{2\alpha}}(L_2+L_3)\geq{}&\lambda_i^{-1}\Big(\frac{2\alpha n(k\alpha-1)}{k}\frac{\sigma_{k-1;i}}{\sigma_k}+\frac{(n+1)k\alpha-n}{k}\lambda_i^{-1}\Big)(\nabla_i\log\sigma_k)^2\\&-\frac{2\alpha^2\sigma_{k-1;i}}{\sigma_k}\mathrm{tr}b(\nabla_i\log\sigma_k)^2+2\sum_{i\neq j}\frac{\sigma_{k-1;i}}{\sigma_k}\lambda_j^{-3}h_{ijj}^2\\&+\frac{\sigma_{k-1;i}}{\sigma_k}\lambda_p^{-2}\lambda_i^{-1}h_{iip}^2.\end{aligned}$$

It follows from (5.1) that

$$-\frac{n(k\alpha-1)}{k}\lambda_i^{-1}\frac{\sigma_{k-1;p}}{\sigma_k}h_{ppi}=R(\nabla Z)+\sum_p\lambda_p^{-1}(\lambda_p^{-1}h_{ppi}-\alpha\frac{\sigma_{k-1;q}}{\sigma_k}h_{qqi}). \tag{5.2}$$

By using (5.2), we can estimate the following two terms

$$\begin{aligned}&-\frac{2\alpha^2\sigma_{k-1;i}}{\sigma_k}\mathrm{tr}b(\nabla_i\log\sigma_k)^2+2\sum_{i,j}\frac{\sigma_{k-1;i}}{\sigma_k}\lambda_j^{-3}h_{ijj}^2\\=&2\sum_i\frac{\sigma_{k-1;i}}{\sigma_k}\sum_j\lambda_j^{-1}(\lambda_j^{-2}h_{ijj}^2-\alpha^2(\nabla_i\log\sigma_k)^2)\\=&2\sum_i\frac{\sigma_{k-1;i}}{\sigma_k}\sum_j\lambda_j^{-1}(\lambda_j^{-1}h_{ijj}-\alpha\nabla_i\log\sigma_k)^2-\frac{4\alpha n(k\alpha-1)}{k}\frac{\sigma_{k-1;i}}{\sigma_k}\lambda_i^{-1}(\nabla_i\log\sigma_k)^2+R(\nabla Z).\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{\alpha\sigma_k^{2\alpha}}(L_2+L_3)+R(\nabla Z)\\\geq{}&\lambda_i^{-1}\Big(-\frac{2\alpha((n-1)k\alpha-n)}{k}\frac{\sigma_{k-1;i}}{\sigma_k}+\frac{(n+1)k\alpha-n}{k}\lambda_i^{-1}\Big)(\nabla_i\log\sigma_k)^2\\&+2\sum_i\frac{\sigma_{k-1;i}}{\sigma_k}\sum_{j\neq i}\lambda_j^{-1}(\lambda_j^{-1}h_{ijj}-\alpha\nabla_i\log\sigma_k)^2+\lambda_i^{-2}\frac{\sigma_{k-1;p}}{\sigma_k}\lambda_p^{-1}h_{ppi}^2\\&-4\alpha\frac{\sigma_{k-1;i}}{\sigma_k}\lambda_i^{-2}h_{iii}\nabla_i\log\sigma_k.\end{aligned}$$

Let $t_i=\frac{\sigma_k}{\lambda_i\sigma_{k-1;i}}$ and using Lemma 5.4, we have

$$\begin{aligned}&\lambda_i^{-2}\frac{\sigma_{k-1;p}}{\sigma_k}\lambda_p^{-1}h_{ppi}^2-4\alpha\frac{\sigma_{k-1;i}}{\sigma_k}\lambda_i^{-2}h_{iii}\nabla_i\log\sigma_k\\={}&\sum_i\lambda_i^{-2}\Big\{\sum_p t_p\Big(\frac{\sigma_{k-1;p}}{\sigma_k}h_{ppi}\Big)^2-4\alpha\frac{\sigma_{k-1;i}}{\sigma_k}h_{iii}\Big(\sum_p\frac{\sigma_{k-1;p}}{\sigma_k}h_{ppi}\Big)\Big\}\\\geq{}&\sum_i\lambda_i^{-2}\Big(\frac{1}{k}(\frac{2\alpha}{t_i}-1)^2-\frac{4\alpha^2}{t_i}\Big)(\nabla_i\log\sigma_k)^2.\end{aligned}$$

Then, we have

$$
\begin{aligned}
&\frac{1}{\alpha\sigma_k^{2\alpha}}(L_2+L_3)+R(\nabla Z)\\
\geq &\sum_i \lambda_i^{-2}\Big\{\frac{2\alpha^2}{t_i}(\frac{2}{kt_i}-n-1)+\alpha(\frac{2(n-2)}{kt_i}+n+1)-\frac{n-1}{k}\Big\}(\nabla_i\log\sigma_k)^2\\
=&\sum_i \lambda_i^{-2}\Big\{(\frac{2\alpha}{t_i}-1)((\frac{2}{kt_i}-n-1)\alpha+\frac{n-1}{k})\Big\}(\nabla_i\log\sigma_k)^2.
\end{aligned}
$$

Since $t_i\geq 1$, if $k\geq 2$ and $\alpha\in[\frac{n-1}{k(n+1)-2},\frac{1}{2}]$ or $k=\alpha=1$, then $L_2+L_3\geq 0$. Since $L_1\geq 0$, by the strong maximum principle, we know Z is constant. Hence, $L_1=L_2+L_3=0$. In case $C<0$ or $\alpha>\frac{1}{k}$, $L_1=0$ implies M is totally umbilic; in other cases, $L_2+L_3=0$ implies that the second fundamental form is parallel. Either of these implies the solution is a round sphere. □

Proof of Theorem A. Combining Theorem 5.3 with Theorem 5.5, we complete the proof of Theorem A. □

6. Proof of Theorem B

For $F=S_k^\alpha$, by similar discussion, we have

Lemma 6.1. *For $F=S_k^\alpha$ and $C\leq 0$, if $k\geq 1$ and $k\alpha\geq 1$, then $L_1\geq 0$.*

Proof. For $F=S_k^\alpha$ and $C\leq 0$, by $\frac{S_{k+1}}{S_k}\geq\frac{S_1}{n}$, we have

$$
L_1=(k\alpha-1)S_k^\alpha(\mathrm{tr}b-\frac{nS_{k-1}}{S_k})-k\alpha CS_k^{\alpha-1}\Big(\mathrm{tr}bS_{k+1}-nS_k\Big)\geq 0.
$$

□

Theorem 6.2. *For $F=S_k^\alpha$ and $C\leq 0$, if $k\geq 1$ and $\alpha>\frac{1}{k}$, solution of* (1.5) *is a round sphere.*

Proof. For $F=S_k^\alpha$,

$$
L_2=\alpha S_k^{2\alpha}\Big(\frac{n(k\alpha-1)}{k}\lambda_p^{-2}(\frac{2k\alpha}{S_k}\lambda_p^k+1)-2k\alpha^2\frac{\lambda_p^{k-1}}{S_k}\mathrm{tr}B\Big)(\nabla_p\log S_k)^2
$$

and

$$
\begin{aligned}
L_3\geq\ & 2k\alpha S_k^{2\alpha-1}\lambda_i^{k-1}\lambda_p^{-2}\lambda_q^{-1}h_{pqi}^2+\alpha(\alpha-1)S_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log S_k)^2\\
&+\alpha k(k-1)S_k^{2\alpha-1}\lambda_p^{-2}\lambda_i^{k-2}h_{iip}^2\\
\geq\ &\alpha k(k+1)S_k^{2\alpha-1}\lambda_p^{-2}\lambda_i^{k-2}h_{iip}^2+\alpha(\alpha-1)S_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log S_k)^2\\
\geq\ &\frac{\alpha(k+1)}{k}S_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log S_k)^2+\alpha(\alpha-1)S_k^{2\alpha}\lambda_p^{-2}(\nabla_p\log S_k)^2,
\end{aligned}
$$

where the last inequality is from Cauchy-Schwarz inequality for

$$
(\nabla_p\log S_k)^2=k^2\Big(\frac{\lambda_i^{k-1}}{S_k}h_{iip}\Big)^2\leq k^2\Big(\sum_i\frac{\lambda_i^k}{S_k}\Big)\Big(\sum_i\frac{\lambda_i^{k-2}}{S_k}h_{iip}^2\Big)=k^2\sum_i\frac{\lambda_i^{k-2}}{S_k}h_{iip}^2.
$$

Then,

$$
L_2+L_3\geq\alpha S_k^{2\alpha}\Big\{2n\alpha(k\alpha-1)\frac{\lambda_p^{k-2}}{S_k}+\frac{(n+1)k\alpha-n+1}{k}\lambda_p^{-2}-2k\alpha^2\frac{\lambda_p^{k-1}\mathrm{tr}b}{S_k}\Big\}(\nabla_p\log S_k)^2.
$$

At an umbilic point, we have

$$
\begin{aligned}
&2n\alpha(k\alpha-1)\frac{\lambda_p^{k-2}}{S_k}+\frac{(n+1)k\alpha-n+1}{k}\lambda_p^{-2}-2k\alpha^2\frac{\lambda_p^{k-1}\mathrm{tr}b}{S_k}\\
=&\lambda_p^{-2}(2\alpha(k\alpha-1)+\frac{(n+1)k\alpha-n+1}{k}-2k\alpha^2)\\
=&\frac{(n-1)(k\alpha-1)}{k}\lambda_p^{-2}\\
>&0.
\end{aligned}
$$

The rest of the proof is similar to Theorem 5.3.

□

Theorem 6.3. *For $F=S_k^\alpha$ and $C\le 0$, if $k\ge 1$ and $\alpha=\frac{1}{k}$, the solution of* (1.5) *is a round sphere.*

Proof. In fact, for $F=S_k^\alpha$ and $\alpha=\frac{1}{k}$, we have

$$L_2=-\frac{2}{k^2}S_k^{\frac{2-k}{k}}\mathrm{tr}b\lambda_i^{k-1}(\nabla_i\log S_k)^2$$

and

$$
\begin{aligned}
L_3\ge\;&2S_k^{\frac{2-k}{k}}\lambda_i^{k-1}\lambda_p^{-3}h_{ppi}^2+\frac{1-k}{k^2}\lambda_p^{-2}(\nabla_p\log S_k)^2\\
&+(k-1)S_k^{\frac{2-k}{k}}\lambda_p^{-2}\lambda_i^{k-2}h_{iip}^2\\
\ge\;&2S_k^{\frac{2-k}{k}}\lambda_i^{k-1}\lambda_p^{-3}h_{ppi}^2,
\end{aligned}
$$

where the last inequality is from Cauchy-Schwarz inequality for

$$(\nabla_p\log S_k)^2=k^2\Big(\frac{\lambda_i^{k-1}}{S_k}h_{iip}\Big)^2\le k^2\Big(\sum_i\frac{\lambda_i^k}{S_k}\Big)\Big(\sum_i\frac{\lambda_i^{k-2}}{S_k}h_{iip}^2\Big)=k^2\sum_i\frac{\lambda_i^{k-2}}{S_k}h_{iip}^2.$$

By (5.1) for $\alpha=\frac{1}{k}$, we have

$$R(\nabla Z)+\sum_p\lambda_p^{-1}(\lambda_p^{-1}h_{ppi}-\frac{1}{k}\nabla_i\log S_k)=0. \tag{6.1}$$

Using (6.1), we have

$$
\begin{aligned}
L_2+L_3+R(\nabla Z)&=2S_k^{\frac{2-k}{k}}\lambda_i^{k-1}\sum_p\lambda_p^{-1}(\lambda_p^{-2}h_{ppi}^2-\frac{1}{k^2}(\nabla_i\log S_k)^2)\\
&=2S_k^{\frac{2-k}{k}}\lambda_i^{k-1}\sum_p\lambda_p^{-1}(\lambda_p^{-1}h_{ppi}-\frac{1}{k}\nabla_i\log S_k)^2\\
&\ge 0.
\end{aligned}
$$

Since $L_1\ge 0$, by the strong maximum principle, we know Z is constant. Hence, $L_1=L_2+L_3=0$. This implies that the solution is a sphere.

□

Proof of Theorem B. Combining Theorem 6.2 with Theorem 6.3, we complete the proof of Theorem B. □

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