Constraints on total conserved quantities in general relativity

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Abstract

This note discusses the recent work of Huang-Schoen-Wang [13] on specifying total conserved quantities of a vacuum initial data set.

1 Conserved quantities in special relativity

Conserved quantities in special relativity are associated with symmetry in the Minkowski space $\mathbb{R}^{3,1}$. The isometry group of $\mathbb{R}^{3,1}$ consists of spacetime translations and Lorentz rotations. An infinitesimal isometry corresponds to a Killing vector field of $\mathbb{R}^{3,1}$. In term of standard coordinates t, x^1, x^2, x^3 on $\mathbb{R}^{3,1}$, they are:

- Constant vector fields.
- Boost vector fields: $t\frac{\partial}{\partial x^i} + x^i\frac{\partial}{\partial t}, i = 1, 2, 3.$
- Rotation vector fields: $Y_i = \frac{\partial}{\partial x^i} \times \vec{x}, i = 1, 2, 3.$

For example, $Y_{(3)} = x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}$ corresponds to rotation about the x^3 axis.

A Killing vector field K satisfies the Killing equation

$$K_{a;b} + K_{b;a} = 0. \tag{1.1}$$

Given a timelike geodesic γ with 4-velocity V (thus $V^b V^a_{;b} = 0$) and energy-momentum 4-vector $p = mV$. By the geodesic equation and the Killing equation (1.1), we have $\langle p, K\rangle = p^a K_a$ is a constant along γ and thus conserved. In case when K is a future timelike unit constant vector field as the 4-velocity of an observer, $E = -\langle p, K\rangle$ is interpreted as the energy seen by the corresponding observer.

Each continuous distribution of matter field is attached with an energy-momentum stress tensor of matter density T_{ab}. For example, for electromagnetic field,

$$T_{ab} = \frac{1}{4\pi}(F_{ac}F_{bd}g^{cd} - \frac{1}{4}g_{ab}F_{cd}F_{ef}g^{ce}g^{df}).$$

T satisfies the conservation equation as a result of coordinate invariance of the associated Lagrangian:

$$T^a_{b;a} = 0.$$

For any spacelike hypersurface Ω in $\mathbb{R}^{3,1}$ which represents a time slice, the energy seen by an observer K and intercepted by Ω is the integral

$$\int_\Omega T(K, u) = \int_\Omega T_{ab}K^a u^b \tag{1.2}$$

where u^b is the future unit timelike normal of Ω. This is a conserved quantity by the Killing field equation for K, the conservation equation for T, and Stoke's theorem, i.e.

$$\int_{\Omega_1} T(K, u_1) = \int_{\Omega_2} T(K, u_2)$$

if $\partial\Omega_1 = \partial\Omega_2$.

Therefore, this expression depends only on the boundary 2-surface $\Sigma = \partial\Omega$ and is the quasi-local energy of Σ seen by the observer K if K is a future timelike Killing field.

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2 Energy in general relativity

In general relativity, spacetime is a 4-dimensional manifold with a Lorentz metric g, the gravitational field. Thus local causal structure is the same, and each tangent space is isometric to the Minkowski space. Gravitational force is represented by the spacetime curvature of g. Einstein's field equation relates gravitation field and matter fields by

$$Ric - \frac{1}{2}Rg = 8\pi T \tag{2.1}$$

where Ric is the Ricci curvature, and R is the scalar curvature of g, respectively. T represents the energy-momentum stress tensor of all matter fields. This is the Euler-Lagrange equation of the Hilbert-Einstein action.

It turns out first derivatives of g are all coordinate dependent, and thus there is no density for gravitational energy. This is manifestly Einstein's equivalence principle. If we try to form the same energy expression (1.2) by integrating T over a spacelike hypersurface as in special relativity, we encounter two difficulties:

Firstly, a generic spacetime does not have any Killing field and the expression is not conserved. Though T in (2.1) still satisfies the conservation equation $T^a_{b;a} = 0$. Secondly, the expression would give only the energy contribution from matters. There exists vacuum spacetime, i.e. $T = 0$, with nonzero energy such as the Kerr solution. There is gravitational energy by the sheer presence of spacetime curvature.

We recall the Kerr metric in Boyer-Lindquist coordinates is of the form:

$$\begin{aligned} ds^2 = -\frac{\Delta}{U}\left(dt - a\,\sin^2\theta\, d\phi\right)^2 + U\left(\frac{dr^2}{\Delta} + d\theta^2\right) \\ + \frac{\sin^2\theta}{U}\left(a\,dt - (r^2 + a^2)\, d\phi^2\right)^2 \end{aligned}$$

$$\begin{aligned} U &= r^2 + a^2\,\cos^2\theta \\ \Delta &= r^2 - 2\,m\,r + a^2 \end{aligned}$$

This is a stationary vacuum solution which is axially symmetric solution and $\frac{\partial}{\partial\phi}$ is the corresponding ration Killing field. The total mass is m and the total angular momentum with respect to $\frac{\partial}{\partial\phi}$ is a. When $a = 0$, this reduces to the Schwarzschild solution which is a static vacuum solution that is also spherically symmetric.

Einstein's field equation is derived from variation of the Einstein-Hilbert action on a spacetime domain M:

$$\frac{1}{16\pi}\int_M R + \frac{1}{8\pi}\int_{\partial M} \mathfrak{K} + \int_M L(g, \Phi)$$

where $\mathfrak{K}$ is the trace of the second fundamental form of ∂M and Φ represents all the matter fields. The variation of the last term with respect to g gives T. The second term is indeed a divergence term $\int_M \partial^a I_a$, where I_a consists of first derivatives of g. Formally applying Hamilton-Jacobi analysis to this action, one obtains ${T^*}_{ab}$, the so called Einstein pseudo tensor, which is expressed in terms of first derivatives of g and satisfies the conservation equation ${T^*}^a_{;ba} = 0$.

Here is Weyl's [17] comment on T^*_{ab} (see chapter 3 of [6] for the English translation quoted here)

Nevertheless it seems to be physically meaningless to introduce the T^*_{ab} as energy components of the gravitational field; for, these quantities are neither a tensor nor are they symmetric. In fact by choosing an appropriate coordinate system all the T^*_{ab} can be made to vanish at any given point; for this purpose one only needs to choose a geodesic (normal) coordinate system. And on the other hand one gets $T^*_{ab} \neq 0$ in a 'Euclidean' completely gravitationless world when using a curved coordinate system, but where no gravitational energy exists. Although the differential relations ($\nabla^a {T^*}_{ab} = 0$) are without a physical meaning, nevertheless by integrating them over an isolated system one gets invariant conserved quantities.

3 Isolated systems and total conserved quantities

An isolated system is modeled on an asymptotically flat spacetime where gravitation is weak at spatial infinity. Arnowitt-Deser-Misner [1] applied Hamilton-Jacobi analysis of the Einstein-Hilbert action to such a system, and obtained total conserved quantities. It turns out the application of Noether's theorem to general relativity requires a reference system which is taken to be $\mathbb{R}^{3,1}$. These include the total energy and linear momentum, as well as the angular momentum and center of mass which altogether correspond to the 10-dimensional Killing vector fields of the Minkowski space.

Let (M, g, p) be an unbounded spacelike hypersurface in spacetime , where g is the induced Riemannian metric and p is the second fundamental form, we usually use the canonical momentum $\pi = p - (tr_g p)g$ instead of p. (M, g, π) is called an initial data set as it represents a Cauchy data for the Einstein equation as a hyperbolic PDE system. We say (M, g, π) is asymptotically flat if outside a compact set there exists an asymptotically flat coordinate system $\{x^i\}_{i=1,2,3}$ so that $g_{ij} - \delta_{ij} = O(|x|^{-1})$ and $\pi_{ij} = O(|x|^{-2})$ and derivatives of g_{ij} and p_{ij} satisfy appropriate decay conditions.

Let (M, g, π) be an asymptotically flat initial data set. Let $E, \mathbf{C}, \mathbf{P}, \mathbf{J}$ denote the energy, center of mass, linear momentum, and angular momentum of (g, π). They are defined as limits of flux integrals over coordinate spheres S_ρ of radius ρ with respect to the asymptotic flat coordinate system.

$$
\begin{aligned}
E &= \frac{1}{16\pi} \lim_{\rho\to\infty} \int_{S_\rho} \sum_{i,j} (g_{ij,i} - g_{ii,j})\nu^j \\
\mathbf{P}_i &= \frac{1}{8\pi} \lim_{\rho\to\infty} \int_{S_\rho} \sum_{j} \pi_{ij}\nu^j \\
\mathbf{J}_i &= \frac{1}{8\pi E} \lim_{\rho\to\infty} \int_{S_\rho} \sum_{j,k} \pi_{jk} Y^j_{(i)}\nu^k \\
\mathbf{C}^k &= \frac{1}{16\pi E} \lim_{\rho\to\infty} \int_{S_\rho} \left[x^k \sum_{i,j} (g_{ij,i} - g_{ii,j})\nu^j - \sum_i (g_{ik}\nu^i - g_{ii}\nu^k) \right]
\end{aligned}
$$

Here ν^i is the outward unit normal of S_ρ and $Y_{(i)}, i = 1, 2, 3$ is the rotation Killing field with respect to the x^i axis.

It turns out the well-definedness of $\mathbf{J}_i$ and $\mathbf{C}_k$ rely on extra assumptions at spatial infinity. We impose the Regge–Teitelboim [14] condition that

$$g_{ij}(x) - g_{ij}(-x) = O(|x|^{-2}) \text{ and } \pi_{ij}(x) + \pi_{ij}(-x) = O(|x|^{-3})$$

and similar parity conditions on ∂g_{ij} and $\partial \pi_{ij}$. These quantities are conserved under the evolution of Einstein's equation of a maximal slice with appropriate assumptions on the decay rate of (g, π), see chapter 3 of Christodoulou [6]. There are other different conditions (see for example, Ashtekar-Hansen [2]) to ensure the finiteness of angular momentum and center of mass. An important property of the Regge-Teitelboim condition is, by a theorem of Corvino-Schoen [10], that initial data sets satisfying this condition are dense, and thus a generic initial data set can be approximated by these sets.

(M, g, π) as a hypersurface in spacetime satisfies the constraint equation

$$\frac{1}{2}(R(g) + \frac{1}{2}(Tr_g\pi)^2 - |\pi|^2) = \mu, \text{ and } div_g(\pi) = \mathfrak{J} \tag{3.1}$$

where μ, $\mathfrak{J}$ are from the energy-momentum stress tensor of matter fields which is assumed to satisfy the dominant energy condition $\mu \geq |\mathfrak{J}|$. We shall assume both μ and $\mathfrak{J}$ vanish and thus (M, g, π) is a vacuum initial data set.

Question: Given a vacuum initial data set which is asymptotically flat, is there any constraint on the conserved quantities $E, \mathbf{C}, \mathbf{P}, \mathbf{J}$?

Schoen-Yau's positive mass theorem [15, 16] [3] imposes the most important constraint on these quantities, namely that the energy-momentum vector is a future timelike vector. In particular, this says that the magnitude of the linear momentum vector is bounded above by the energy:

$$E \geq |\mathbf{P}|$$

and thus the total mass $m = \sqrt{E^2 - |\mathbf{P}|^\mathbf{2}}$ is always non-negative.

For the Kerr solutions which describe rotating stationary axially symmetric vacuum black holes, we have the mass-angular momentum inequality

$$m \geq |\mathbf{J}|.$$

It has been shown by Dain [11, 12] and Chruściel et al.[7, 8] that such an inequality is also satisfied by general axially symmetric black hole solutions of the Einstein equations (see also work of Zhang [18]).

4 Specifying total conserved quantities

In [13], we show that the mass-angular inequality does not hold true in general.

Theorem There are no constraints on the angular momentum and center of mass in terms of the energy-momentum vector for general vacuum solutions of the Einstein equations.

In fact, given any 10 real numbers $E, \mathbf{C}, \mathbf{P}, \mathbf{J}$ with $E \geq |\mathbf{P}|$, we can construct a vacuum initial data set that has them as conserved quantities.

An effective procedure was given for adding a specified amount of angular momentum to a solution of the vacuum Einstein equations, producing a new solution with specified angular momentum but with only slightly perturbed energy-momentum vector. A similar result was obtained for the center of mass. Then, by considering a family of initial data set near the given one, and by doing the construction continuously and a degree argument, we obtain a perturbation with arbitrarily specified angular momentum and center of mass, while leaving the energy-momentum vector unchanged.

From a technical point of view the reason it is possible to make these constructions is that the angular momentum and center of mass are determined by terms in the expansion of the solution which are of lower order than those which determine the energy and linear momentum.

The idea then is to make perturbations near infinity which affect only the lower order terms in the expansion. We do this by explicitly constructing linear perturbations supported in a shell near infinity which impose the required change in angular momentum (or center of mass), and then by finding a solution of the vacuum constraint equations which is sufficiently close to the perturbed system so that the change in angular momentum (or center of mass) persists. This is an application of the Corvino-Schoen [9, 10] gluing construction of initial data set.

The added term in the perturbation vanishes in the axially symmetric case, and we cannot increase the angular momentum while keeping the axially symmetric condition. Thus the result is consistent with the mass-angular momentum inequality of Dain and Chruściel et al.

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