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Gravitational energy seen by quasilocal observers

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Abstract

There have been many attempts to define quasilocal energy/mass for a spacelike 2-surface in a spacetime by the Hamilton–Jacobi method. The main difficulty in this approach is the subtle choice of the background configuration to be subtracted from the physical Hamiltonian. Quasilocal mass should be positive for general surfaces, but on the other hand should be zero for surfaces in the flat spacetime. In this paper, we survey the work in a series of papers [6, 25–27] in which a new notion of quasilocal mass/energy–momentum is proposed and investigated. In particular, the notion of energy observed by a 'quasilocal observer' will be discussed.

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(Some figures in this article are in color only in the electronic version)

1. Introduction

One of the greatest accomplishments of the theory of general relativity in the past century is the proof of the positive mass/energy theorem for asymptotically flat spacetime [22, 28]. This provides the theoretical foundation for stability of an isolated gravitating system. However, the concept of mass/energy remains a challenging problem because of the lack of a quasilocal description. Most observable physical models are finitely extended spatial regions and measurement of mass/energy on such a region is essential in many fundamental issues. In fact, among Penrose's list [19] of major unsolved problems in classical general relativity, the first one is 'Find a suitable quasilocal definition of energy–momentum in general relativity'.

In special relativity, quasilocal mass/energy is a well-defined notion. For a continuous matter distribution, the energy–momentum density is given by a symmetric (0, 2) tensor $T_{\mu\nu}$ which satisfies conservation law $T^{\mu}_{\nu,\mu} = 0$. Suppose t^{μ} is the 4-velocity of an observer, which is a unit future timelike translating killing vector field in the Minkowski space $\mathbb{R}^{3,1}$. Let Ω be a bounded spacelike region, then the energy seen by the observer t^{μ} and intercepted by Ω is

$$\int_{\Omega} T_{\mu\nu} t^{\mu} u^{\nu}, \qquad (1.1)$$

where u^{ν} is the future timelike unit normal of Ω.

By conservation law, $T_{\mu\nu}t^{\mu}$ is divergence free, and the dual 3-form is closed. A closed 3-form on $\mathbb{R}^{3,1}$ is exact, and thus there exists a 2-form ω such that (1.1) is expressed as $\int_{\Omega} d\omega$. By Stoke's theorem, this is equal to

$$\int_{\partial\Omega} \omega = E(\partial\Omega, t^{\mu}), \tag{1.2}$$

an integral over the boundary 2-surface $\partial\Omega$ which depends linearly on the observer t^{μ}. In relativity, energy depends on an observer and the rest mass can be obtained by minimizing among all energy seen by observers. Thus, minimizing $E(\partial\Omega, t^{\mu})$ among t^{μ} gives the quasilocal mass M of $\partial\Omega$. The quasilocal energy–momentum is then $M \cdot \bar{t}^{\mu}$ if the minimum energy is seen by the observer $\bar{t}^{\mu}$. This is the prototype of quasilocal mass and energy–momentum.

Can mass/energy be defined quasilocally when gravity is coupled? A general spacetime (N, g) is a four-dimensional manifold with the gravitation field represented by a Lorentzian metric g. Let T denote the energy–momentum tensor of matter density; Einstein's field equation then reads

$$Ric - \tfrac{1}{2}Rg = 8\pi T.$$

Several difficulties arise when we consider the previous formulation on a general spacetime N. First of all, a generic spacetime lacks the symmetry or Killing vector field to define conserved quantities. Even in the presence of a Killing field, the same integral (1.1) will only account for energy contribution from matter fields. As is well known, there are vacuum solutions (i.e. $T = 0$) for Einstein's field equation such as Schwarzchild's solutions, and there is nontrivial energy contribution from gravitation.

We nevertheless can pose the same question: suppose Ω is a bounded spacelike region in N, what is the quasilocal mass/energy of Ω, counting all contributions from matter fields and gravitation field? As no density exists for the gravitational field, mass/energy of the integral form (1.1) over Ω should not be expected. Instead, by energy conservation, we expect to read off the information from the boundary 2-surface $\partial\Omega$. For a fleet of observers along $\partial\Omega$, we may ask the quasilocal energy seen by such observers in the form of (1.2).

Measuring gravitation energy in general relativity turns out to be a very subtle problem. Gravitation has no density by Einstein's equivalence principle. On the other hand, spacetime curvature distorts the underlying geometry and makes energy evaluation a nonlinear problem. Gravity is different from all other field theories in that it lacks a background configuration; there is no canonical identification between a curved spacetime and a flat one. An identification is only possible at the infinity of an asymptotically flat spacetime, and even in this case, there is ambiguity up to a choice of coordinate system.

In this paper, we survey the work in a series of papers [6, 25–27] in which a new notion of quasilocal mass/energy–momentum is proposed and investigated. In particular, the notion of energy $E(\partial\Omega, t^{\mu})$ observed by a 'quasilocal observer' t^{μ} along $\partial\Omega$ will be discussed.

2. Surface Hamiltonian of Brown–York

A promising approach to quasilocal mass is Brown–York's [1, 2] Hamilton–Jacobi analysis of the gravitational action. The analysis, when applied to the time history of a bounded spatial region, produces a surface Hamiltonian which we review in the following. Let Σ be a closed embedded spacelike 2-surface which bounds a spacelike region Ω in a spacetime N. Let u^{μ} denote the future timelike unit normal to Ω and v^{μ} denote the outward spacelike unit normal of Σ such that $u_{\mu}v^{\mu} = 0$. Denote by k the trace of the two-dimensional extrinsic curvature of

Σ in Ω in the direction of v^μ. Denote the Riemannian metric, the extrinsic curvature, and the trace of the extrinsic curvature on Ω by $g_{\mu\nu}$, $K_{\mu\nu} = \nabla_\mu u_\nu$, and $K = g^{\mu\nu}K_{\mu\nu}$, respectively. Let t^μ be a timelike vector field satisfying $t^\mu\nabla_\mu t = 1$. t^μ can be decomposed into the lapse function and shift vector $t^\nu = Nu^\nu + N^\nu$ along Ω. The calculation by Brown–York (see also Hawking–Horowitz [11]) leads to the surface Hamiltonian

$$\mathfrak{H} = -\frac{1}{8\pi}\int_\Sigma [Nk - N^\mu v^\nu (K_{\mu\nu} - Kg_{\mu\nu})] \tag{2.1}$$

on a solution M of the Einstein equation. We note that the Hamiltonian (2.1) is an integral on Σ that depends on the choices of a future timelike unit normal u^μ and a timelike vector field t^μ along Σ.

To define quasilocal energy, one needs to find a reference action that corresponds to fixing the metric on the timelike boundary, and compute the corresponding reference Hamiltonian $\mathfrak{H}_0$. The energy is then $E = \mathfrak{H} - \mathfrak{H}_0$.

Several ambiguities need to be anchored to make this well defined.

(1) What is the reference configuration? This should at least correspond to an isometric embedding of Σ into a flat spacetime.
(2) How do we choose u^ν, the timelike unit normal of Ω? Indeed, only a timelike unit normal along Σ is needed.
(3) How do we choose lapse N and shift N^ν in order to determine t^μ?

Brown and York proposed a prescription in [1, 2]:

(1) The reference is taken to be an isometric embedding of Σ into $\mathbb{R}^3$, considered as a flat three-dimensional slice with $K_{\mu\nu} = 0$ in a flat spacetime. References such as surfaces in the light cones [3, 14] have also been considered.
(2) The Brown–York energy seems to depend on an arbitrary choice of Ω and thus u^μ. In fact, different choices of u^μ were adopted in the calculation of large and small sphere limits [3, 4] in order to obtain the desired results.
(3) Brown–York chose $N = 1$ and $N^\mu = 0$, and the quasilocal energy is thus $\frac{1}{8\pi}\int_\Sigma (k_0 - k)$ where k_0 is the mean curvature of an isometric embedding of Σ into $\mathbb{R}^3$. When the intrinsic curvature is positive, this embedding is essentially unique.

For other earlier attempts for quasilocal mass along the Hamilton–Jacobi analysis approach, we refer to [24, 25]. Among these proposals, the Brown–York mass and the Liu-Yau mass [13, 15] posses the important positivity property [15, 23]. However there exist surfaces in the Minkowski space with strictly positive Brown–York and Liu-Yau mass [17].

3. Surface Hamiltonian for spacelike 2-surfaces in the Minkowski space

Suppose Σ is a closed oriented spacelike 2-surface in $\mathbb{R}^{3,1}$. Given any constant timelike unit vector t^ν, we decompose it into the normal part and tangent part along Σ. Thus,

$$t^\nu = Nu^\mu + N^\mu \tag{3.1}$$

where u^ν is a future timelike unit normal vector field along Σ and N^μ is a tangent vector field along Σ. We note that u^μ is a normal vector field along Σ that is uniquely determined by this decomposition, which is independent of any spatial region Σ bounds.

Now suppose Σ bounds are spacelike hypersurface Ω in the Minkowski space. We assume Ω is connected. These are surfaces on which the quasilocal energy or mass is well-defined in special relativity. Let $\mathbb{R}^3$ be the totally geodesic spacelike 3-subspace of $\mathbb{R}^{3,1}$ that is orthogonal to t^μ. Let $\pi : \mathbb{R}^{3,1} \to \mathbb{R}^3$ be the projection and consider the restriction of π to Σ and Ω, which are maps of full rank. Since any spacelike hypersurface is achronal, the restriction of π to Ω is bijective and thus a diffeomorphism. In general, the image of Σ under the projection is a one-sided surface $\hat{\Sigma}$ in $\mathbb{R}^3$ and a continuous outward unit normal vector field is well defined. It turns out that the surface Hamiltonian on Σ can be expressed in terms of the geometry of $\hat{\Sigma}$. Let Σ be a closed oriented spacelike 2-surface Σ in the Minkowski space which bounds a spacelike region Ω. *Let t^μ be a unit future timelike translating Killing field in $\mathbb{R}^{3,1}$, u^μ be the unit future timelike normal defined by $t^\mu = Nu^\mu + N^\mu$ along Σ, and v^μ be the unit spacelike normal that is orthogonal to u^μ and outward pointing with respect to Ω. The surface Hamiltonian density on Σ with respect to t^μ and u^μ is given by*

$$Nk - N^\mu v^\nu (K_{\mu\nu} - K g_{\mu\nu}) = N\hat{k},$$

where $\hat{k}$ is the mean curvature of the projection of Σ onto the spacelike 3-subspace that is orthogonal to t^μ.

This formula was derived in [26] and a related one first appeared in [10]. In fact, suppose τ is the restriction of the time function (with respect to t^μ) to Σ, then the shift vector N^μ is $-\nabla\tau$, the gradient of τ with respect to the induced metric on Σ, and the lapse is $N = \sqrt{1 + |\nabla\tau|^2}$. We thus have

$$\int_\Sigma [Nk - N^\mu v^\nu (K_{\mu\nu} - K g_{\mu\nu})] = \int_{\hat{\Sigma}} \hat{k}.$$

That t^μ is Killing implies

$$\nabla_\mu t_\nu + \nabla_\nu t_\mu = 0. \tag{3.2}$$

Let $\Pi_{\mu\nu}$ be the induced metric on Σ. Tracing the equation on Σ yields

$$\Pi^{\mu\nu}(\nabla_\mu t_\nu) = 0. \tag{3.3}$$

With the decomposition $t^\mu = Nu^\mu + N^\mu$, (3.3) implies

$$-N\Pi^{\mu\nu}(\nabla_\nu u_\mu) = \Pi^{\mu\nu}(\nabla_\nu N_\mu). \tag{3.4}$$

On the left-hand side, the term $\Pi^{\mu\nu}(\nabla_\nu u_\mu)$ is the expansion in the direction of u^μ, while the right-hand side is a divergence expression that is equal to $\Delta\tau$ where Δ is the Laplace operator with respect to the induced metric on Σ.

4. New quasilocal energy

Now we review the definition of quasilocal energy in [25]. Suppose Σ is a spacelike 2-surface which bounds a spacelike hypersurface Ω in a spacetime M. The mean curvature vector field h^ν is the unique normal vector field along Σ that is characterized by 'the expansion $\Pi^\mu_\nu \nabla_\mu u^\nu$ along any normal vector field u^ν is $-h_\nu u^\nu$'.

Consider a reference isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ of the induced metric on Σ. Suppose $i(\Sigma)$ bounds a spacelike hypersurface Ω_0 in $\mathbb{R}^{3,1}$. Let t_0^ν be a unit future timelike translating Killing field in $\mathbb{R}^{3,1}$. Choose $({u_0}^\nu, {v_0}^\nu)$ as in the previous section, i.e. u_0^ν is the unit future timelike normal defined by $t_0^\nu = Nu_0^\mu + N^\nu$ along Σ, and v_0^ν is the unit spacelike normal that is orthogonal to u_0^ν and outward pointing with respect to Ω_0. $({u_0}^\nu, {v_0}^\nu)$ along $i(\Sigma)$ in $\mathbb{R}^{3,1}$ is the reference normal gauge we shall fix, and it depends on the choice of the pair $\left(i, t_0^\nu\right)$.

When the mean curvature vector h^ν of Σ in M is spacelike, $\left(i, t_0^\mu\right)$ determine a canonical future timelike normal vector field $\bar{u}^\nu$ in M along Σ. Indeed, there is a unique $\bar{u}^\nu$ that satisfies

$$h_\nu \bar{u}^\nu = (h_0)_\nu {u_0}^\nu \tag{4.1}$$

where h_0^ν is the mean curvature vector of $i(\Sigma)$ in $\mathbb{R}^{3,1}$. Physically, (4.1) means the expansions of $\Sigma \subset M$ and $i(\Sigma) \subset \mathbb{R}^{3,1}$ along the respective directions $\bar{u}^\nu$ and ${u_0}^\nu$ are the same.

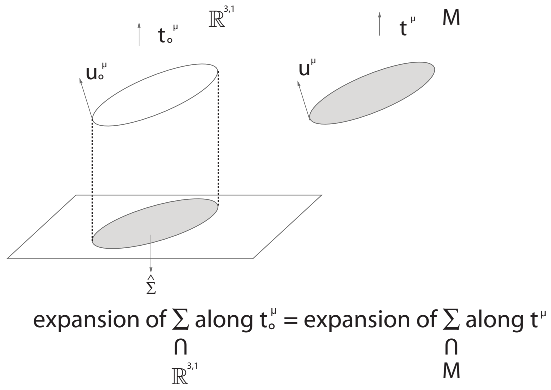

Figure 1. Canonical gauge.

Take $\bar{v}^\nu$ to be the spacelike normal vector that is orthogonal to $\bar{u}^\nu$ and outward pointing with respect to the spacelike hypersurface Ω. $(\bar{u}^\nu, \bar{v}^\nu)$ is called the *canonical gauge* (figure 1) with respect to the pair $\left(i, t_0^\nu\right)$.

4-vectors in $\mathbb{R}^{3,1}$ and M, along $i(\Sigma)$ and Σ, respectively, can be identified through

$$u_0{}^\nu \to \bar{u}^\nu, \qquad v_0{}^\nu \to \bar{v}^\nu, \tag{4.2}$$

and the identification of tangent vectors on $i(\Sigma)$ and Σ. Therefore, we obtain a quasilocal observer $\bar{t}^\mu = N\bar{u}^\mu + N^\mu$ along Σ with the same lapse and shift as $t_0^\mu = Nu_0^\mu + N^\mu$ along $i(\Sigma)$.

The *quasilocal energy* $E\left(i, t_0^\mu\right)$ *of* Σ *in the canonical gauge with respect to* $\left(i, t_0^\nu\right)$ is then the difference of the two surface Hamiltonian $\mathfrak{H}(\bar{t}^\mu, \bar{u}^\mu) - \left(t_0^\mu, u_0^\mu\right)$.

In comparison to Brown–York's definition, we have the following.

(1) The image of the reference isometric embedding is in $\mathbb{R}^{3,1}$ and it may not be contained in any totally geodesic $\mathbb{R}^3$.

(2) The timelike normal vector field $\bar{u}^\mu$ is fixed by the canonical gauge condition (4.1).

(3) The shift vector $N^\mu = -\nabla\tau$ is the gradient vector of a function defined on Σ and $N = \sqrt{1 + N^\mu N_\mu}$. This follows by comparing (4.1) with (3.4).

The canonical gauge is characterized by the property that it gives the maximal energy among all possible gauges (see Proposition 2.1. in [26]). When the mean curvature vector h^ν is spacelike, this maximum is achieved at the unique $(\bar{u}^\mu, \bar{v}^\mu)$. In general, we can still define the quasilocal energy $E\left(i, t_0^\mu\right)$ to be the maximal value.

5. 2-surface geometry and mean curvature vector

It turns out, the physical data that is needed to determine the quasilocal mass is best represented by intrinsic tensors on the surface instead of spacetime coordinates. This is especially useful when the mean curvature vector is spacelike. The spacetime metric g induces a Riemannian metric σ on Σ. We shall denote the mean curvature vector by H without any indication of spacetime coordinates. The mean curvature vector, as a section of the normal bundle of Σ, also defines a connection one-form α_H of the normal bundle which we describe in the following.

Take a tetrad frame (e_1, e_2, l, n) where e_1 and e_2 form an orthonormal basis of the tangent space of Σ, and l and n are future null normal vector fields normalized so that $g(l, n) = -1$. Consider the following Newman–Penrose coefficients:

$$\rho = -\frac{1}{2}\sum_{a=1}^{2} g(\nabla_{e_a} l, e_a) \qquad \text{and} \qquad \mu = \frac{1}{2}\sum_{a=1}^{2} g(\nabla_{e_a} n, e_a).$$

In terms of these, the mean curvature vector is $H = -2\rho n + 2\mu l$. Note that the definition of ρ and μ depends on the choice of l and n, but the definition of H does not. Suppose H is spacelike and thus $\rho < 0$ and $\mu < 0$, we denote

$$|H| = \sqrt{8\rho\mu}. \tag{5.1}$$

In this case, the mean curvature vector determines the following orthonormal basis in the normal bundle:

$$e_3 = -\frac{H}{|H|} = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\rho}{\mu}}n - \sqrt{\frac{\mu}{\rho}}l\right)$$

and

$$e_4 = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\rho}{\mu}}n + \sqrt{\frac{\mu}{\rho}}l\right).$$

This is chosen so that on the standard configuration of a round sphere in the Minkowski space, e_3, is the outward spacelike normal and e_4 is the future timelike normal. As in [26], we consider the connection one-form on Σ defined by

$$\alpha_H(V) = g(\nabla_V e_3, e_4),$$

for any tangent vector V of Σ. In terms of ρ and μ,

$$\alpha_H(V) = \frac{1}{2}V\left(\log\frac{\mu}{\rho}\right) - \frac{1}{2}g(\nabla_V l, n).$$

Again this is an expression that is independent of the choice of l and n.

Another expression $\alpha_H(V) = g\left(\nabla_V \frac{J}{|H|}, \frac{H}{|H|}\right)$ is used in [26], in which $J = 2\rho n + 2\mu l$ is the future timelike normal J that is dual to the mean curvature vector along the light cone in the normal bundle.

6. Properties of quasilocal mass

Let Σ be a spacelike 2-surface which bounds a spacelike region in a spacetime. We describe properties of quasilocal energy $E\left(i, t_0^\nu\right)$ with respect to a quasilocal observer $\left(i, t_0^\nu\right)$. *Rigidity—if Σ is a spacelike 2-surface which bounds a spacelike region in $\mathbb{R}^{3,1}$, then there exists a quasilocal observer $\left(i, t_0^\mu\right)$ such that the quasilocal energy $E\left(i, t_0^\nu\right) = 0$.*

In fact, there exists an isometric embedding $i : \Sigma \to \mathbb{R}^{3,1}$ such that $E\left(i, t_0'^\nu\right) = 0$ for any constant future timelike unit vector $t_0'^\nu$ in $\mathbb{R}^{3,1}$. The latter property can be interpreted as the vanishing of quasilocal energy–momentum.

Positivity result holds if the quasilocal observer satisfies the admissible pair condition defined in [25] (definition 2).

Positivity—suppose the spacetime satisfies the dominant energy condition and Σ is a spacelike 2-surface which bounds a spacelike region. We also assume that Σ has spacelike mean curvature and $\left(i, t_0^\nu\right)$ satisfies the admissible pair condition, then $E\left(i, t_0^\nu\right) \geqslant 0$.

We can define the quasilocal mass M to be the minimum of $E\left(i, t_0^\nu\right)$ among all admissible pairs and the quasilocal energy–momentum to be $M \cdot \bar{t}_0^\nu$ if the minimum is achieved at $\bar{t}_0^\nu$.

A challenging problem is to characterize those surfaces in the rigidity case. Namely, when the quasilocal energy–momentum is 0, the surface should be sitting in the Minkowski space. This is a stronger assumption than the vanishing of quasilocal mass. Those surfaces with zero quasilocal mass but nonzero quasilocal energy–momentum should correspond to purely radiative configurations.

In [6, 27], we prove the quasilocal mass approaches the ADM mass and the Bondi mass at spatial and null infinity, respectively. In the spatial infinity case, we take Σ_r to be coordinate spheres with respect to an asymptotically flat coordinates on an end. In the null infinity case, we take Σ_r to be the r-slice at a fixed retarded time $w = c$ with respect to a Bondi coordinate at null infinity. In either case, the 2-surface Σ_r has a positive Gauss curvature when r is large enough and can take the isometric embedding i_r to be the unique one into $\mathbb{R}^3 \subset \mathbb{R}^{3,1}$ by the Weyl embedding theorem (Nirenberg [18] and Pogorelov [20]).

Asymptotic behavior—for a fixed $t_0^\nu \in \mathbb{R}^{3,1}$, in either the spatial infinity or null infinity case, the limit of quasilocal energy $E(\Sigma_r, i_r, t_0^\nu)$ is a linear expression in t_0^μ and defines an energy–momentum 4-vector which coincides with the ADM energy–momentum and the Bondi–Sachs energy–momentum [5, 21], respectively.

It is not hard to see from the expression of $E\big(i, t_0^\mu\big)$ that E is rather nonlinear in t_0^μ; it nevertheless gets linearized and acquires the Lorentzian symmetry at infinity.

We write a general future timelike unit vector t_0^ν in $\mathbb{R}^{3,1}$ in the form $(\sqrt{1+|a|^2}, a^1, a^2, a^3)$ for $a^1, a^2, a^3 \in \mathbb{R}$. It is shown that

$$\lim_{r\to\infty} E(\Sigma_r, X_r, t_0) = \sqrt{1+|a|^2}E + a^k P_k$$

where (E, P_1, P_2, P_3) is equal to the ADM energy–momentum 4-vector in the spatial infinity case and the Bondi–Sachs energy–momentum 4-vector in the null infinity case, respectively.

In terms of the data on Σ_r, we have

$$E = \lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} (|H_0| - |H|),$$

and

$$P_k = \lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} \alpha_H(\nabla x^k) = -\lim_{r\to\infty} \frac{1}{8\pi} \int_{\Sigma_r} (\mathrm{div}_{\Sigma_r} \alpha_H) x^k \tag{6.1}$$

where $X_r = (x^1, x^2, x^3) : \Sigma_r \to \mathbb{R}^3$ is the isometric embedding into $\mathbb{R}^3$ and H_0 is the mean curvature of the image of X_r. We consider each x^k as a function on Σ_r and ∇x^k is the gradient of x^k with respect to the induced metric on Σ_r as a tangent vector field. This gives a uniform expression for the ADM and Bondi–Sachs energy–momentum 4-vectors that is independent of the asymptotically flat coordinates at infinity.

7. Spherically symmetric case and monotonicity property

Among earlier proposals for quasilocal mass, the Hawking mass and the Bartnik mass enjoy a nice monotonicity property in the outward spatial direction. This is consistent with other matter field theory, as the mass is represented as the integral of the density function which is pointwise positive. In particular, the monotonicity of Hawking mass along the inverse mean curvature flow is critical in Huisken–Ilmanen's [12] proof of the Riemannian Penrose inequality. On the other hand, it is well known that the energy method is the most power tool in studying the hyperbolic equation. Energy estimates will be extremely useful in controlling the dynamics of Einstein's equation in spacetime evolution. In fact, in the proof of stability of the Minkowski space [9] and black hole formation theorem [8], the integral of the Bel-Robinson tensor, as an approximation of the gravitational energy, is essential in deriving curvature estimates. In this section, we test the proposed quasilocal mass in a spherically symmetric spacetime and its monotonicity, and speculate on the general case. In a spherically symmetric spacetime, the metric is of the form:

$$g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b + r^2\tilde{\sigma}$$

where $\tilde{\sigma}$ is the standard metric on the unit 2-sphere S^2. The areal radius r of an $SO(3)$ orbit, is a function on the quotient manifold Q with Lorentz metric $g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$ of signature (1, 1). Each point $p \in Q$ represents a round 2-sphere $\Sigma(p)$ with radius $r(p)$. The energy–momentum tensor is also spherically symmetric and is thus of the form

$$T_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b + r^2 S\tilde{\sigma}$$

for a function S on Q. The mean curvature vector of a sphere $\Sigma(p)$ is computed

$$H(p) = -\frac{2}{r}\partial^a r\frac{\partial}{\partial x^a}.$$

Suppose the mean curvature vector of $\Sigma(p)$ is spacelike, or equivalently $\partial^a r\frac{\partial}{\partial x^a}$ is spacelike. Our mass, which coincides with the Liu-Yau mass, is computed as

$$M(p) = r(1 - \sqrt{\partial^a r\partial_a r}),$$

while the Misner–Sharpe mass [16] is

$$m(p) = \frac{r}{2}(1 - \partial^a r\partial_a r).$$

It is understood that if the mean curvature vector is non-spacelike, $\sqrt{\partial^a r\partial_a r}$ is replaced by $-\sqrt{-\partial^a r\partial_a r}$. The relationship between M and m is given exactly as

$$m = M - \frac{M^2}{2r}. \tag{7.1}$$

In particular, as long as M is bounded, M and m approach to the same limit as r goes to infinity. On the other hand, at horizon where $\nabla r = 0$, we have $M(r_0) = r_0$ and $m(r_0) = \frac{r_0}{2}$. Thus, the two notions of quasilocal mass are equivalent in the sense that $\frac{M}{2} \leqslant m \leqslant M$. On a Schwarzchild's solution, M is monotone decreasing along $\frac{\partial}{\partial r}$ and $M(\infty) = \frac{1}{2}M(r_0)$, while m is a constant in r with $m(\infty) = m(r_0)$. It was shown in [7] (p 362, (3.18)) that m, as a function on the quotient manifold Q, satisfies

$$\partial_a m = 4\pi r^2(T_{ab} - g_{ab}\mathrm{tr}T)\partial^b r,$$

where $\mathrm{tr}T = g^{ab}T_{ab}$. Thus, m is monotone non-decreasing along $\frac{\partial}{\partial r}$ (assuming spacelike) in a spherically symmetric spacetime that satisfies the dominant energy condition, in particular, by (7.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(M - \frac{M^2}{2r}\right) \geqslant 0. \tag{7.2}$$

Contrary to some expectation for the monotonicity of quasilocal mass, we believe that a straightforward monotonicity may not hold true in a general spacelike slice for the following reasons: the gravitational energy has no density and the gravitation binding energy may be negative. However, an inequality such as (7.2) may still hold for our quasilocal mass. This will be discussed in a forthcoming work.

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