Subsets of Grassmannians Preserved by Mean Curvature Flows

Mu-Tao Wang

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Abstract

Let $M = \Sigma_1 \times \Sigma_2$ be the product of two compact Riemannian manifolds of dimension $n \geq 2$ and two, respectively. Let Σ be the graph of a smooth map $f : \Sigma_1 \mapsto \Sigma_2$, then Σ is an n-dimensional submanifold of M. Let $\mathfrak{G}$ be the Grassmannian bundle over M whose fiber at each point is the set of all n-dimensional subspaces of the tangent space of M. The Gauss map $\gamma : \Sigma \mapsto \mathfrak{G}$ assigns to each point $x \in \Sigma$ the tangent space of Σ at x. This article considers the mean curvature flow of Σ in M. When Σ_1 and Σ_2 are of the same non-negative curvature, we show a sub-bundle $\mathfrak{S}$ of the Grassmannian bundle is preserved along the flow, i.e. if the Gauss map of the initial submanifold Σ lies in $\mathfrak{S}$, then the Gauss map of Σ_t at any later time t remains in $\mathfrak{S}$. We also show that under this initial condition, the mean curvature flow remains a graph, exists for all time and converges to the graph of a constant map at infinity . As an application, we show that if f is any map from S^n to S^2 and if at each point, the restriction of df to any two dimensional subspace is area decreasing, then f is homotopic to a constant map.

1 Introduction

The maximum principle has proved to be a powerful tool in partial differential equations. In particular, the maximum principle of parabolic systems for tensors developed by R. Hamilton [4] plays an important role in the study of geometric evolution equations. The guiding principle is the following: an invariant convex subset in the space of curvature tensors preserved by the associated ordinary differential equations is preserved by the parabolic partial differential equations. This has been applied to the study of Ricci flow and curvature flow of hypersurfaces. In this article, we apply this idea to higher codimension mean curvature flow.

Let $M = \Sigma_1 \times \Sigma_2$ be the product of two compact Riemannian manifolds of dimension $n \geq 2$ and two, respectively. Let Σ be the graph of a map $f : \Sigma_1 \mapsto \Sigma_2$, then Σ is an n-dimensional submanifold of M. Let $\mathfrak{G}$ be the Grassmannian bundle over M whose fiber at each point is the set of all n-dimensional subspaces of the tangent space. The Gauss map $\gamma : \Sigma \mapsto \mathfrak{G}$ assigns to each point $x \in \Sigma$ the tangent space of Σ at x. The tangent space of M at x splits as $T_{\pi_1(x)}\Sigma_1 \times T_{\pi_2(x)}\Sigma_2$. Let $\mathfrak{G}' \subset \mathfrak{G}$ be the sub-bundle consisted of the graphs of linear transformations from $T_{\pi_1(x)}\Sigma_1$ to $T_{\pi_2(x)}\Sigma_2$. We show there exists a sub-bundle $\mathfrak{S} \subset \mathfrak{G}'$ that is preserved along the mean curvature flow.

Theorem A *Let $M = \Sigma_1 \times \Sigma_2$ be the product of two compact flat Riemannian manifolds and suppose Σ_2 is two-dimensional. If the gauss map of a compact oriented submanifold Σ of M lies in $\mathfrak{S}$, then along the mean curvature flow the gauss map of Σ_t remains in $\mathfrak{S}$. The flow of exists smoothly for all time and converges to a totally geodesic submanifold.*

This in particular implies Σ_t is the graph of a map f_t. The sub-bundle $\mathfrak{S}$ is best described in terms of f_t. In fact, if we denote the singular values of f_t by λ_1 and λ_2, then the Gauss map of Σ_t lies in $\mathfrak{S}$ if and only if $|\lambda_1\lambda_2| < 1$.

When Σ_1 is of positive curvature, we prove the following.

Theorem B *Let $M = S^n(k_1) \times \Sigma_2$ be the product of a sphere of curvature $k_1 > 0$ and a two-dimensional compact Riemannian manifold Σ_2 of constant curvature k_2 and $k_1 \geq |k_2|$. If the gauss map of a compact oriented submanifold Σ of M lies in $\mathfrak{S}$, then along the mean curvature flow the gauss map of Σ_t remains in $\mathfrak{S}$. The flow exists smoothly for all time and converges to a totally geodesic submanifold.*

Theorem A and B are proved by calculating the evolution equations of the Gauss map and applying maximum principle. The prototype is the following equation in the hypersurface case

$$\frac{d}{dt}N = \Delta N + |A|^2 N$$

where N denotes the unit normal vector and $|A|^2$ is the norm of the second fundamental form. If we take inner product of N with a constant vector ν, it is not hard to see that $\min_{\Sigma_t} < N, \nu >$ is non-decreasing in time. This is one of the key observation in [1] and [2] where the mean curvature flow of of entire graph of codimension one was studied. In codimension one case, N contains all the information of the Gauss map. While in higher codimension, a whole parabolic systems is needed in order to describe the evolution of the Gauss map.

The following is an application to higher homotopy groups of S^2.

Corollary *If f is any map from S^n to S^2, $n \geq 2$ and if at each point, the restriction of df to any two dimensional subspace is area decreasing, then f is homotopic to a constant map along the mean curvature flow.*

When $n = 2$, this is the same as saying the Jacobian of f is less than 1. In this case, f is of degree 0 and thus homotopic to a constant map. This homotopy can be realized through the mean curvature flow as was proved in [8]. As a contrast, the standard Hopf map from S^3 to S^2 has $|\lambda_1\lambda_2| = 4$.

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2 Analysis of Grassmannian bundle

Let us first describe the sub-bundle $\mathfrak{S}$. Let V_1 be an n-dimensional inner product space and V_2 a two-dimensional inner product space. Let $G(n, n+2)$ be the Grassmannian of all n-dimensional subspaces of $V_1 \times V_2$. Let $G' \subset G(n, n+2)$ be the set of all n-dimensional subspaces that can be written as graphs over V_1. For any $P \in G'$, P is the graph of a linear transformation $\mathfrak{p} : V_1 \mapsto V_2$. Then $(\mathfrak{p})^T \mathfrak{p}$ is self-adjoint and thus diagonalizable. The eigenvalues are denoted by $\{\lambda_1^2, \lambda_2^2\}$. λ_1 and λ_2 are the singular values of $\mathfrak{p}$. We now define S.

$$S = \{P \in G' |\, 1 - |\lambda_1\lambda_2| > 0\}$$

This is equivalent to saying $\mathfrak{p}$ is area decreasing on any two dimensional subspace of V_1.

Now let M be the product of two Riemannian manifolds $\Sigma_1 \times \Sigma_2$ of dimension n and 2 respectively. Let $\mathfrak{G}$ be the Grassmannian bundle on M whose fibers are isomorphic to $G(n, n+2)$. At each point x, $T_x M$ splits as the product of $T_{\pi_1(x)}\Sigma_1$ and $T_{\pi_2(x)}\Sigma_2$. The Riemannian structures on $T_{\pi_1(x)}\Sigma_1 = V_1$ and $T_{\pi_2(x)}\Sigma_2 = V_2$ defines the subset S of the fiber of $\mathfrak{G}$ at x.

Definition 2.1 $\mathfrak{S}$ *is the sub-bundle of the Grassmannian bundle* $\mathfrak{G}$ *whose fiber at each point consists of* S*.*

Let Σ be the graph of a smooth map $f : \Sigma_1 \mapsto \Sigma_2$. $T_x\Sigma$ is the graph of the differential of f at x, $df : T_{\pi_1(x)}\Sigma_1 \mapsto T_{\pi_2(x)}\Sigma_2$. Notice that we abuse the notation so that $T_x\Sigma, T_{\pi_1(x)}\Sigma_1$ and $T_{\pi_2(x)}\Sigma_2$ all denote subspaces of $T_x M$. At any point x, let λ_1,λ_2 be the singular values of df. They are well-defined up to a sign. Define

$$\eta(x) = \frac{1 - |\lambda_1\lambda_2|}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}$$

η is a function on Σ.

Proposition 2.1 $\eta > 0$ *on* Σ *if and only if the Gauss map of* Σ *lies in* $\mathfrak{S}$*.*

Later we shall give a characterization of η in terms of differential forms on M. Any differential form Ω on a Riemannian manifold can be considered as a function on the Grassmannian bundle $\mathfrak{G}$ of appropriate dimension. The *comass* of Ω at x is defined to be the supremum of Ω on $\mathfrak{G}_x$, the fiber of $\mathfrak{G}$ at x. This is an important concept in calibrated geometry, see Federer [3] or Harvey-Lawson [5]. Another description of $\mathfrak{S}$ can be given in terms of the comass.

Proposition 2.2 *If* Σ *is the graph of* $f : \Sigma_1 \mapsto \Sigma_2$*, then the Gauss map of* Σ *lies in* $\mathfrak{S}$ *if and only if the comass of* $f^*\Omega_2$ *is less than one.*

Here Ω_2 is the volume form on Σ_2 and $f^*\Omega_2$ is considered as a 2-form on Σ_1. Of course the comass is taken over all two-dimensional subspaces of the tangent space of Σ_1.

3 Evolution equation of n form

In this section, we calculate the evolution equation of the restriction of an n-form to an n-dimensional submanifold moving by the mean curvature flow. The case for a parallel form was calculated in [10]. Here we need to keep track of the terms that involve covariant derivatives of Ω.

We assume M is an $n+m$ dimensional Riemannian manifold with an n form Ω. Let $F:\Sigma\mapsto M$ be an isometric immersion of an n-dimensional submanifold. We shall compute near a point $p\in\Sigma$. We choose arbitrary orthonormal frames $\{e_i\}_{i=1\cdots n}$ for $T\Sigma$ and $\{e_\alpha\}_{\alpha=n+1,\cdots,n+m}$ for $N\Sigma$. ∇^M denotes the covariant derivative on M and ∇^Σ denotes the covariant derivative on Σ, which is simply the tangent part of ∇^M. $\nabla^M\Omega$ is the covariant derivative of Ω on M and $\nabla^\Sigma\Omega$ will denote the covariant derivative of the restriction of Ω to Σ.

We first calculate the covariant derivative of the restriction of Ω on Σ.

$$
\begin{aligned}
&(\nabla^\Sigma_{e_k}\Omega)(e_{i_1},\cdots,e_{i_n})\\
&=e_k(\Omega(e_{i_1},\cdots,e_{i_n}))-\Omega(\nabla^\Sigma_{e_k}e_{i_1},\cdots,e_{i_n})-\cdots-\Omega(e_{i_1},\cdots,\nabla^\Sigma_{e_k}e_{i_n})\\
&=(\nabla^M_{e_k}\Omega)(e_{i_1},\cdots,e_{i_n})+\Omega(\nabla^M_{e_k}e_{i_1}-\nabla^\Sigma_{e_k}e_{i_1},\cdots,e_{i_n})+\cdots+\Omega(e_{i_1},\cdots,\nabla^M_{e_k}e_{i_n}-\nabla^\Sigma_{e_k}e_{i_n})
\end{aligned}
$$

This equation can be abbreviated using the second fundamental form of F, $h_{\alpha ij}=<\nabla^M_{e_i}e_j,e_\alpha>$.

$$\Omega_{i_1\cdots i_n,k}=(\nabla^M_{e_k}\Omega)(e_{i_1},\cdots,e_{i_n})+\Omega_{\alpha i_2\cdots i_n}h_{\alpha i_1 k}+\cdots+\Omega_{i_1\cdots i_{n-1}\alpha}h_{\alpha i_n k} \quad (3.1)$$

Likewise, in $\Omega(e_\alpha,e_{i_2},\cdots,e_{i_n})$, Ω is considered as a section of $(N\Sigma)^*\wedge(\wedge(T\Sigma)^*)$.

$$\Omega_{\alpha i_2\cdots i_n,k}=(\nabla^M_{e_k}\Omega)(e_\alpha,e_{i_1},\cdots,e_{i_n})-\Omega_{j i_2\cdots i_n}h_{\alpha jk}+\Omega_{\alpha\beta i_3\cdots i_n}h_{\beta i_2 k}+\cdots+\Omega_{\alpha i_2\cdots i_{n-1}\beta}h_{\beta i_n k} \quad (3.2)$$

Now we calculate the second covariant derivative of the restriction of Ω on Σ.

$$
\begin{aligned}
&(\nabla^\Sigma_{e_k}\nabla^\Sigma_{e_k}\Omega)(e_1,\cdots e_n)\\
&=e_k((\nabla^\Sigma_{e_k}\Omega)(e_1,\cdots e_n))-(\nabla^\Sigma_{e_k}\Omega)(\nabla^\Sigma_{e_k}e_1,\cdots,e_n)-\cdots-(\nabla^\Sigma_{e_k}\Omega)(e_1,\cdots,\nabla^\Sigma_{e_k}e_n)
\end{aligned}
$$

The term $(\nabla^{\Sigma}_{e_k}\Omega)(\nabla^{\Sigma}_{e_k}e_1,\cdots,e_n)$ equals zero because $\nabla^{\Sigma}_{e_k}e_1$ is a tangent vector perpendicular to e_1 and thus a linear combination of $e_2,\cdots,e_n$. Likewise, other similar terms vanish.

$$
\begin{aligned}
&(\nabla^{\Sigma}_{e_k}\nabla^{\Sigma}_{e_k}\Omega)(e_1,\cdots e_n)\\
=&e_k[(\nabla^{M}_{e_k}\Omega)(e_1,\cdots,e_n)+\Omega_{\alpha 2\cdots n}h_{\alpha 1k}+\cdots+\Omega_{1\cdots n-1\alpha}h_{\alpha nk}]\\
=&(\nabla^{M}_{e_k}\nabla^{M}_{e_k}\Omega)(e_1,\cdots,e_n)+(\nabla^{M}_{e_k}\Omega)(\nabla^{M}_{e_k}e_1,\cdots,e_n)+\cdots+(\nabla^{M}_{e_k}\Omega)(e_1,\cdots,\nabla^{M}_{e_k}e_n)\\
&+\Omega_{\alpha 2\cdots n,k}h_{\alpha 1k}+\cdots+\Omega_{1\cdots n-1\alpha,k}h_{\alpha nk}\\
&+\Omega_{\alpha 2\cdots n}h_{\alpha 1k,k}+\cdots+\Omega_{1\cdots n-1\alpha}h_{\alpha nk,k}
\end{aligned}
\tag{3.3}
$$

Now $\nabla^{M}_{e_k}e_i=h_{\alpha ik}e_\alpha+\nabla^{\Sigma}_{e_k}e_i$ and $(\nabla^{M}_{e_k}\Omega)(\nabla^{\Sigma}_{e_k}e_1,\cdots,e_n)=0$
Therefore,

$$
\begin{aligned}
\Omega_{1\cdots n,kk}=&(\nabla^{M}_{e_k}\nabla^{M}_{e_k}\Omega)(e_1,\cdots,e_n)\\
&+(\nabla^{M}_{e_k}\Omega)(e_\alpha,\cdots,e_n)h_{\alpha 1k}+\cdots+(\nabla^{M}_{e_k}\Omega)(e_1,\cdots,e_\alpha)h_{\alpha nk}\\
&+\Omega_{\alpha 2\cdots n,k}h_{\alpha 1k}+\cdots+\Omega_{1\cdots n-1\alpha,k}h_{\alpha nk}\\
&+\Omega_{\alpha 2\cdots n}h_{\alpha 1k,k}+\cdots+\Omega_{1\cdots n-1\alpha}h_{\alpha nk,k}
\end{aligned}
\tag{3.4}
$$

Plug equation (3.2) into (3.4) and apply the Codazzi equation $h_{\alpha ki,k}=h_{\alpha,i}+R_{\alpha kki}$ where R is the curvature operator of M.

$$
\begin{aligned}
(\Delta^{\Sigma}\Omega)_{1\cdots n}=&-\Omega_{12\cdots n}\sum_{\alpha,k}(h^2_{\alpha 1k}+\cdots+h^2_{\alpha nk})\\
&+2\sum_{\alpha,\beta,k}[\Omega_{\alpha\beta 3\cdots n}h_{\alpha 1k}h_{\beta 2k}+\Omega_{\alpha 2\beta\cdots n}h_{\alpha 1k}h_{\beta 3k}+\cdots+\Omega_{1\cdots(n-2)\alpha\beta}h_{\alpha(n-1)k}h_{\beta nk}]\\
&+\sum_{\alpha,k}\Omega_{\alpha 2\cdots n}h_{\alpha,1}+\cdots+\Omega_{1\cdots(n-1)\alpha}h_{\alpha,n}\\
&+\sum_{\alpha,k}\Omega_{\alpha 2\cdots n}R_{\alpha kk1}+\cdots+\Omega_{1\cdots(n-1)\alpha}R_{\alpha kkn}\\
&+(\nabla^{M}_{e_k}\nabla^{M}_{e_k}\Omega)(e_1,\cdots,e_n)\\
&+2(\nabla^{M}_{e_k}\Omega)(e_\alpha,\cdots,e_n)h_{\alpha 1k}+\cdots+2(\nabla^{M}_{e_k}\Omega)(e_1,\cdots,e_\alpha)h_{\alpha nk}
\end{aligned}
\tag{3.5}
$$

We notice that $(\Delta^{\Sigma}\Omega)_{1\cdots n} = \Delta(\Omega(e_1, \cdots, e_n))$, where the Δ on the right hand side is the Laplacian of functions on Σ.

The terms in the bracket are formed in the following way. Choose two different indexes from 1 to n, replace the smaller one by α and the larger one by β. There are a total of $\frac{n(n-1)}{2}$ such terms.

Now we consider when $\Sigma = \Sigma_t$ is a time slice of a mean curvature flow in M by $\frac{d}{dt}F_t = H_t$. Notice that here we require the velocity vector is in the normal direction. We can extend $e_1, \cdots, e_n$ to a local coordinate $\{\partial_i = \frac{\partial}{\partial x^i}\}$ on Σ, then

$$
\begin{aligned}
&\frac{d}{dt}\Omega(\partial_1, \cdots, \partial_n) \\
&= (\nabla^M_H \Omega)(\partial_1, \cdots, \partial_n) + \Omega((\nabla_{\partial_1} H)^N, \partial_2, \cdots, \partial_n) + \cdots + \Omega(\partial_1, \partial_2, \cdots, (\nabla_{\partial_n} H)^N) \\
&+ \Omega((\nabla_{\partial_1} H)^T, \partial_2, \cdots, \partial_n) + \cdots + \Omega(\partial_1, \partial_2, \cdots, (\nabla_{\partial_n} H)^T)
\end{aligned}
$$

Since $\frac{d}{dt}g_{ij} =< (\nabla_{\partial_i} H)^T, \partial_j >$, if we choose a orthonormal frame and evolve the frame with respect to time so that it remains orthonormal, the terms in the last line vanish.

$$
\frac{d}{dt} * \Omega = *(\nabla^M_H \Omega) + \Omega_{\alpha 2\cdots n} h_{\alpha,1} + \cdots + \Omega_{1\cdots(n-1)\alpha} h_{\alpha,n}
$$

Combine this with equation (3.5) we get the parabolic equation satisfied by $*\Omega$.

Proposition 3.1 *If Σ_t is a time slice of an n-dimensional mean curvature flow in M and Ω is an n-form on M. For any point $p \in \Sigma_t$, let $\{e_1, \cdots, e_n\}$ be an orthonormal frame of $T\Sigma_t$ near p and $\{e_{n+1}, \cdots, e_{n+m}\}$ be an orthonormal* *frame of the normal bundle of Σ_t near p. Then $*\Omega = \Omega(e_1, \cdots, e_n)$ satisfies*

$$
\begin{aligned}
\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega(\sum_{\alpha,i,k} h_{\alpha ik}^2) \\
& - 2\sum_{\alpha,\beta,k}[\Omega_{\alpha\beta 3\cdots n}h_{\alpha 1k}h_{\beta 2k} + \Omega_{\alpha 2\beta\cdots n}h_{\alpha 1k}h_{\beta 3k} + \cdots + \Omega_{1\cdots(n-2)\alpha\beta}h_{\alpha(n-1)k}h_{\beta nk}] \\
& - \sum_{\alpha,k}[\Omega_{\alpha 2\cdots n}R_{\alpha kk1} + \cdots + \Omega_{1\cdots(n-1)\alpha}R_{\alpha kkn}] \\
& + *(\nabla_H^M\Omega) - (\nabla_{e_k}^M\nabla_{e_k}^M\Omega)(e_1, \cdots, e_n) \\
& - 2(\nabla_{e_k}^M\Omega)(e_\alpha, \cdots, e_n)h_{\alpha 1k} - \cdots - 2(\nabla_{e_k}^M\Omega)(e_1, \cdots, e_\alpha)h_{\alpha nk}
\end{aligned}
\tag{3.6}
$$

where Δ denotes the time-dependent Laplacian on Σ_t.

4 Proof of Theorem

Let us prove Theorem A now. We recall the statement.

Theorem A *Let $M = \Sigma_1 \times \Sigma_2$ be the product of two compact flat Riemannian manifolds of dimension n and 2 respectively. If the gauss map of a compact oriented submanifold Σ of M lies in $\mathfrak{S}$, then along the mean curvature flow the gauss map of Σ_t remains in $\mathfrak{S}$. The flow exists smoothly for all time and converges to a totally geodesic submanifold.*

Proof. Let Σ_t be the mean curvature flow of Σ given by a family of immersions $F : \Sigma \times [0, T) \mapsto M$. In the following calculation, it is useful to consider the total space of the mean curvature flow as $\Sigma \times [0, T)$. At each instant t, Σ is equipped with the induced metric by F_t. All geometric quantities defined on the image of $F(\cdot, t)$ are considered as defined on Σ.

Let Ω_1 and Ω_2 be the volume form of Σ_1 and Σ_2 respectively. They can be considered as parallel forms on M. Suppose initially the image of the Gauss map of Σ is in $\mathfrak{S}$. We may assume Σ is the graph of a map $f : \Sigma_1 \mapsto \Sigma_2$. This implies $\eta_1 = *\Omega_1 > 0$ and $\eta > 0$ on Σ at $t = 0$ by Proposition 2.1.

We shall characterize η in terms of differential forms. Consider Ξ the collection of n forms on M of the following type.

$\Xi = \{\, \Omega_1 - \Omega_2 \wedge \omega \,|\, \omega$ is any parallel simple $(n-2)$ form of comass one on $\Sigma_1\}$

At any point x, by Singular Value Decomposition we can take an orthonormal basis $\{a_i\}_{i=1\cdots n}$ for $T_{\pi_1(x)}\Sigma_1$ and $\{a_\alpha\}_{\alpha=n+1,n+2}$ for $T_{\pi_2(x)}\Sigma_2$ so that $df(a_i) = \lambda_i a_{n+i}$, $a_1^* \wedge \cdots \wedge a_n^*$ is the volume form of $T_{\pi_1(x)}\Sigma_1$ and $a_{n+1}^* \wedge a_{n+2}^*$ is the volume form for $T_{\pi_2(x)}\Sigma_2$. Therefore,

$$\{e_1 = \frac{1}{\sqrt{1+\lambda_1^2}}(a_1 + \lambda_1 a_{n+1}), e_2 = \frac{1}{\sqrt{1+\lambda_2^2}}(a_2 + \lambda_2 a_{n+2}), e_3 = a_3, \cdots, e_n = a_n\} \tag{4.1}$$

forms an orthonormal basis for $T_x\Sigma$ and

$$\{e_{n+1} = \frac{1}{\sqrt{1+\lambda_1^2}}(a_{n+1} - \lambda_1 a_1), e_{n+2} = \frac{1}{\sqrt{1+\lambda_2^2}}(a_{n+2} - \lambda_2 a_2)\} \tag{4.2}$$

an orthonormal basis for $N_x\Sigma$. Thus,

$$\begin{aligned} *(\Omega_1 - \Omega_2 \wedge \omega) &= (\Omega_1 - \Omega_2 \wedge \omega)(e_1, \cdots, e_n) \\ &= \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}(1 - \lambda_1\lambda_2\omega(a_3, \cdots, a_n)) \end{aligned} \tag{4.3}$$

On the other hand

$$\eta_1 = \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}$$

Recall

$$\eta = \frac{1 - |\lambda_1\lambda_2|}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}$$

It is not hard to see

$$\eta(x) = \min_{\Omega \in \Xi} *\Omega(x)$$

Suppose at $t = t_0$, the image of the Gauss map hits the boundary of $\mathfrak{S}$ for the first time. Therefore each Σ_t, $t < t_0$ can be written as the graph of $f_t : \Sigma_1 \mapsto \Sigma_2$ and the singular values of f_t satisfy $|\lambda_1\lambda_2| < 1$.

We claim Σ_{t_0} remains a graph. Indeed, since Ω_1 is a parallel form, η_1 satisfies the following equation by equation (3.6).

$$\frac{d}{dt}\eta_1 = \Delta\eta_1 + \eta_1[\sum_{\alpha,i,k} h^2_{\alpha ik} - 2\sum_k \lambda_1\lambda_2(h_{n+1,1k}h_{n+2,2k} - h_{n+2,1k}h_{n+1,2k})] \tag{4.4}$$

where we use

$$\Omega_1(e_{n+1}, e_{n+2}, e_3, \cdots, e_n) = \frac{\Omega_1(a_{n+1} - \lambda_1 a_1, a_{n+2} - \lambda_2 a_2, a_3, \cdots, a_n)}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}} = \lambda_1\lambda_2\eta_1$$

Notice this equation is valid at any point x. Since $|\lambda_1\lambda_2| < 1$ for $0 \leq t < t_0$, applying maximum principle to equation (4.4) implies $\min_{\Sigma_t}\eta_1$ is non-decreasing in t and thus $\eta_1 > 0$ at t_0.

Now η is well-defined at t_0. Take any p so that $\eta(p, t_0) = 0$, we shall show that $\frac{d}{dt}|_{t=t_0}\eta \geq 0$ at p.

It is clear that $\lambda_1\lambda_2 \neq 0$ at p. Otherwise, $\eta_1 = \eta = 0$, a contradiction.

By the previous characterization of η and Hamilton's maximum principle [4] , we only need to show $\frac{d}{dt}|_{t=t_0} * \Omega \geq 0$ at the point p for any $\Omega \in \Xi$ such that $*\Omega(p) = \eta(p)$. At p, we apply Singular Value Decomposition to get an orthonormal basis $\{a_i\}_{i=1,\cdots n}$ for $T_{\pi_1(p)}\Sigma_1$ as before. Such Ω is of the form $\Omega_1 - \Omega_2 \wedge \omega$ with $\omega(a_3, \cdots, a_n) = 1$ or $\omega = a_3^* \wedge \cdots \wedge a_n^*$ by equation (4.3).

$*\Omega$ satisfies

$$\begin{aligned}\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega(\sum_{\alpha,i,k} h^2_{\alpha ik})\\ - 2\sum_{\alpha,\beta,k}[\Omega_{\alpha\beta 3\cdots n}h_{\alpha 1k}h_{\beta 2k} + \Omega_{\alpha 2\beta\cdots n}h_{\alpha 1k}h_{\beta 3k} + \cdots + \Omega_{1\cdots(n-2)\alpha\beta}h_{\alpha(n-1)k}h_{\beta nk}]\end{aligned} \tag{4.5}$$

At this point p,

$$(\Omega_1 - \Omega_2\wedge\omega)(e_{n+1}, e_{n+2}, e_{i_1}, \cdots e_{i_{n-2}}) = \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}(\lambda_1\lambda_2 - 1)\omega(e_{i_1}, \cdots, e_{i_{n-2}})$$

Thus

$$\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega[|A|^2 + 2(h_{n+1,1k}h_{n+2,2k} - h_{n+1,2k}h_{n+2,1k})]$$

This can be completed square and we get

$$\frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega[\sum_{\alpha,2<i\leq n,k} h^2_{\alpha ik} + \sum_k (h_{n+1,1k} + h_{n+2,2k})^2 + \sum_k (h_{n+1,2k} - h_{n+2,1k})^2]$$

Therefore $\frac{d}{dt} *\Omega \geq 0$ at (p, t_0). Since this is true for any Ω that achieves the minimum of $*\Omega$ in Ξ, we have $\frac{d}{dt}\eta \geq 0$. Thus the sub-bundle $\mathfrak{S}$ is preserved along the mean curvature flow.

Now we prove long time existence and convergence. By a similar argument, we can show if $\min \eta = \delta > 0$ at $t = 0$, then this is preserved along the flow. This implies in particular,

$$|\lambda_1 \lambda_2| \leq 1 - \delta, \tag{4.6}$$

and

$$\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)} \leq \frac{1}{\delta} \tag{4.7}$$

Since $|\lambda_1\lambda_2| \leq 1 - \delta$, we have

$$\frac{d}{dt} * \Omega_1 \geq \Delta * \Omega_1 + \delta * \Omega_1 |A|^2 \tag{4.8}$$

In particular, $*\Omega_1$ has a uniform lower bound, each Σ_t can be written as the graph of a map $f_t : \Sigma_1 \mapsto \Sigma_2$, and f_t has uniform gradient bound.

Integrating $\frac{d}{dt} * \Omega_1 \geq \Delta * \Omega_1 + \delta * \Omega_1 |A|^2$ over space and time from $t = 0$ to $t = \infty$ we get

$$\delta \int_0^\infty \int_{\Sigma_t} *\Omega_1 |A|^2 \leq \int_0^\infty \int_{\Sigma_t} *\Omega_1 |H|^2$$

For a mean curvature flow, $\int_0^\infty \int_{\Sigma_t} |H|^2 < \infty$, thus $\int_0^\infty \int_{\Sigma_t} |A|^2 < \infty$. We can extract a subsequence $t_i \to \infty$ such that $\int_{\Sigma_{t_i}} |A|^2 \to 0$. Because each f_i has bounded gradient, this is the same as $\int_{\Sigma_1} |\nabla df_i|^2 \to 0$. Therefore df_i

is in $W^{1,2}$ which is compactly imbedded in $L^{\frac{2n}{n-2}}$. We can further extract a convergent subsequence which converges in $C^\alpha \cap W^{1,\frac{2n}{n-2}}$ norm. Apply the Sobolev inequality shows df_i converges to a constant and the limit of f_i is a totally geodesic submanifold. The uniform convergence of f_t follows as the proof of Theorem C in [9], which uses the property that distance function to any totally geodesic submanifold in a Riemannian manifold of non-positive sectional curvature is convex. We remark that in this case, the limit is actually the graph of a linear map.

□

Theorem B *Let $M = S^n(k_1) \times \Sigma_2^m$ be the product of a sphere of curvature $k_1 > 0$ and a compact Riemannian manifold Σ_2 of constant curvature k_2 and $k_1 \geq |k_2|$. If the gauss map of a compact oriented submanifold Σ^n of M lies in $\mathfrak{S}$, then along the mean curvature flow the gauss map of Σ_t^n remains in $\mathfrak{S}$. The flow exists smoothly for all time and converges to an $S^n(k_1)$ slice in M.*

Proof.

The proof follows the same strategy as that of Theorem A. We actually show the image of the Gauss map never hits the boundary of $\mathfrak{S}$. Suppose the contrary happens at $t = t_0$. Again, we look at the equation of $*\Omega_1$. Using $|\lambda_1\lambda_2| \leq 1$ for $0 \leq t < t_0$ and Proposition 3.2 in [10], we see

$$\begin{aligned}&\frac{d}{dt} * \Omega_1 \geq \Delta * \Omega_1\\&+ *\Omega_1 \sum_i \frac{\lambda_i^2}{1+\lambda_i^2}[k_1(\sum_{j\neq i}\frac{2}{1+\lambda_j^2}) + k_2(1-n+\sum_{j\neq i}\frac{2}{1+\lambda_j^2})]\end{aligned} \tag{4.9}$$

The last term comes from the curvature of M. Rewrite

$$k_1(\sum_{j\neq i}\frac{2}{1+\lambda_j^2})+k_2(1-n+\sum_{j\neq i}\frac{2}{1+\lambda_j^2}) = \frac{k_1-k_2}{2}(n-1)+\frac{k_1+k_2}{2}(\sum_{j\neq i}\frac{2}{1+\lambda_j^2}+1-n)$$

We claim the curvature term is always non-negative under our assumption. Since $k_1 - k_2 \geq 0, k_1 + k_2 \geq 0$, we only need to show

$$\sum_{i=1}^n \frac{\lambda_i^2}{1+\lambda_i^2}(1-n+\sum_{k\neq i}^n \frac{2}{1+\lambda_k^2}) \geq 0$$

This is indeed

$$\frac{\lambda_1^2}{1+\lambda_1^2}(n-3+\frac{2}{1+\lambda_2^2})+\frac{\lambda_2^2}{1+\lambda_2^2}(n-3+\frac{2}{1+\lambda_1^2})$$

This can be rewritten as

$$(n-2)\frac{\lambda_1^2+\lambda_2^2+2\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}+\frac{\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}$$

which is non-negative under the assumption $|\lambda_1\lambda_2|\leq 1$.

Therefore, at t_0, Σ_{t_0} remains a graph and $\min_{\Sigma_{t_0}}\eta=0$. Take any p so that $\eta(p,t_0)=0$. We may assume $\lambda_1>0,\lambda_2>0$ at p.

As before, we choose orthonormal basis at p that corresponds to the singular value decomposition of df. We can extend the orthonormal basis $\{a_i\}_{i=1\cdots n}$ at $T_{\pi_1(p)}\Sigma_1$ to an orthonormal frame field in a neighborhood $U_1\subset\Sigma_1$ such that at this point p, $\nabla^M a_i=0$, $i=1\cdots n$. This is possible because the Riemannian structure is a product and each Σ_1 slice is totally geodesic in M. Take

$$\Omega=\Omega_1-\Omega_2\wedge\omega$$

where $\omega=a_3^*\wedge\cdots\wedge a_n^*$.

Ω is an n form defined on $U=U_1\times\Sigma_2$ that satisfies $\nabla^M\Omega=0$ at p.

Now we extend Ω to a global form on M. Take a cut-off function ϕ such that $\phi\equiv 1$ in a neighborhood of p and ϕ has compact support. Then

$$\Omega=\Omega_1-\phi\Omega_2\wedge\omega$$

is such a global extension. Now $*\Omega(p,t_0)=0$ and (p,t_0), $*\Omega>0$ for $0\leq t<t_0$ and $*\Omega\geq 0$ at t_0. Therefore $\frac{d}{dt}*\Omega\leq 0$ and $\Delta*\Omega\geq 0$ at(p,t_0).

We recall the evolution equation from equation (3.6).

$$\begin{aligned}\frac{d}{dt} * \Omega &= \Delta * \Omega + *\Omega(\sum_{\alpha,i,k} h_{\alpha ik}^2) \\ &- 2\sum_{\alpha,\beta,k}[\Omega_{\alpha\beta 3\cdots n}h_{\alpha 1k}h_{\beta 2k} + \Omega_{\alpha 2\beta\cdots n}h_{\alpha 1k}h_{\beta 3k} + \cdots + \Omega_{1\cdots(n-2)\alpha\beta}h_{\alpha(n-1)k}h_{\beta nk}] \\ &- \sum_{\alpha,k}[\Omega_{\alpha 2\cdots n}R_{\alpha kk1} + \cdots + \Omega_{1\cdots(n-1)\alpha}R_{\alpha kkn}] \\ &+ *(\nabla_H^M\Omega) - (\nabla_{e_k}^M\nabla_{e_k}^M\Omega)(e_1,\cdots,e_n) \\ &- 2(\nabla_{e_k}^M\Omega)(e_\alpha,\cdots,e_n)h_{\alpha i_1 k} - \cdots - 2(\nabla_{e_k}^M\Omega)(e_1,\cdots,e_\alpha)h_{\alpha i_n k}\end{aligned} \tag{4.10}$$

By the way Ω is constructed, $\nabla^M\Omega = 0$ at p. We claim the term $(\nabla_{e_k}^M\nabla_{e_k}^M\Omega)(e_1,\cdots,e_n)$ also vanishes at p.

In fact, consider

$$\begin{aligned}&\nabla_{e_k}^M\nabla_{e_k}^M(\Omega_2\wedge\omega)(e_1,\cdots,e_n) \\ &= (\Omega_2\wedge\nabla_{e_k}^M\nabla_{e_k}^M\omega)(e_1,\cdots,e_n) \\ &= \frac{\lambda_1\lambda_2}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}(\nabla_{e_k}^M\nabla_{e_k}^M\omega)(a_3,\cdots,a_n)\end{aligned}$$

For $i = 3,\cdots,n$, $(\nabla_X^M\nabla_Y^M a_i^*)(a_i) = X(\nabla_Y^M a_i^*(a_i)) - (\nabla_Y^M a_i^*)(\nabla_X^M a_i)$. $(\nabla_Y^M a_i^*)(a_i) =< \nabla_Y^M a_i, a_i >=< \nabla_Y^{\Sigma_1} a_i, a_i >$ is zero, so $X(\nabla_Y^M a_i^*(a_i)) = 0$. Therefore $\nabla_X^M\nabla_Y^M a_i^*(a_i) = 0$ at p. Since $\omega = a_3^*\wedge\cdots\wedge a_n^*$, we get $(\nabla_{e_k}^M\nabla_{e_k}^M\omega)(a_3,\cdots,a_n) = 0$.

Now the left hand side of equation (4.10) is non-positive, we shall show the curvature term

$$-\sum_{\alpha,k}[\Omega_{\alpha 2\cdots n}R_{\alpha kk1} + \cdots + \Omega_{1\cdots(n-1)\alpha}R_{\alpha kkn}]$$

is strictly positive, thus achieves contradiction because all other terms on the right hand side are non-negative.

We calculate the curvature term.

$$\Omega(e_\alpha, e_2,\cdots,e_n) = -\delta_{\alpha,n+1}\frac{\lambda_1+\lambda_2}{\sqrt{(1+\lambda^2)(1+\lambda_2^2)}}$$

Likewise,

$$\Omega(e_1, e_\alpha, e_3 \cdots, e_n) = -\delta_{\alpha,n+2} \frac{\lambda_1 + \lambda_2}{\sqrt{(1+\lambda^2)(1+\lambda_2^2)}}$$

and

$$\Omega(e_1, e_2, e_\alpha, e_4 \cdots, e_n) = 0$$

We assume Σ_1 and Σ_2 are of constant curvature k_1 and k_2 respectively, by the calculation in [10] we have

$$\begin{aligned}
&\sum_k R(e_\alpha, e_k, e_k, e_i) \\
&= k_1[\sum_k < \pi_1(e_\alpha), \pi_1(e_k) >< \pi_1(e_k), \pi_1(e_1) > - < \pi_1(e_\alpha), \pi_1(e_1) > \sum_k < \pi_1(e_k), \pi_1(e_k) >] \\
&+ k_2[\sum_k < \pi_2(e_\alpha), \pi_2(e_k) >< \pi_2(e_k), \pi_2(e_1) > - < \pi_2(e_\alpha), \pi_2(e_1) > \sum_k < \pi_2(e_k), \pi_2(e_k) >] \\
=&k_1[\sum_k < \pi_1(e_\alpha), \pi_1(e_k) >< \pi_1(e_k), \pi_1(e_i) > - < \pi_1(e_\alpha), \pi_1(e_i) > \sum_k |\pi_1(e_k)|^2] \\
&+ k_2[(n-1) < \pi_1(e_\alpha), \pi_1(e_i) > \\
&+ \sum_k < \pi_1(e_\alpha), \pi_1(e_k) >< \pi_1(e_k), \pi_1(e_1) > - < \pi_1(e_{n+1}), \pi_1(e_1) > \sum_k |\pi_1(e_k)|^2]
\end{aligned}$$

Because $e_{n+i} = \frac{1}{\sqrt{1+\lambda_i^2}}(a_{n+i} - \lambda_i a_i)$, $\pi_1(e_{n+i}) = \frac{-\lambda_1}{\sqrt{1+\lambda_i^2}} a_i$. Likewise, $\pi_1(e_k) = \frac{1}{\sqrt{1+\lambda_k^2}} a_k$.

Therefore,

$$R_{n+i,kki} = \frac{\lambda_i}{1+\lambda_i^2}[k_1(\sum_{k \neq i} \frac{1}{1+\lambda_k^2}) + k_2(1 - n + \sum_{k \neq i} \frac{1}{1+\lambda_k^2})]$$

Therefore the curvature term $-\sum_{\alpha,k}[\Omega_{\alpha 2 \cdots n} R_{\alpha kk1} + \cdots + \Omega_{1 \cdots (n-1)\alpha} R_{\alpha kkn}]$ in equation (3.6) becomes

$$\frac{(\lambda_1 + \lambda_2)}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}} \sum_{i=1}^2 \frac{\lambda_i}{1+\lambda_i^2}[k_1(\sum_{j \neq i}^n \frac{1}{1+\lambda_j^2}) + k_2(1 - n + \sum_{j \neq i}^n \frac{1}{1+\lambda_j^2})]$$

$$k_1(\sum_{j\neq i}\frac{1}{1+\lambda_j^2})+k_2(1-n+\sum_{j\neq i}\frac{1}{1+\lambda_j^2}) = \frac{k_1-k_2}{2}(n-1)+\frac{k_1+k_2}{2}(\sum_{j\neq i}\frac{2}{1+\lambda_j^2}+1-n)$$

Since $\lambda_1+\lambda_2>0$ and $k_1-k_2\geq 0, k_1+k_2\geq 0$, we only need to show

$$\sum_{i=1}^n \frac{\lambda_i}{1+\lambda_i^2}(1-n+\sum_{k\neq i}^n\frac{2}{1+\lambda_k^2})\geq 0$$

This is indeed

$$\frac{\lambda_1}{1+\lambda_1^2}(n-3+\frac{2}{1+\lambda_2^2})+\frac{\lambda_2}{1+\lambda_2^2}(n-3+\frac{2}{1+\lambda_1^2})$$

This can be rewritten as

$$(n-2)\frac{(\lambda_1+\lambda_2)(1+\lambda_1\lambda_2)}{(1+\lambda_1^2)(1+\lambda_2^2)}+\frac{(\lambda_1+\lambda_2)(1-\lambda_1\lambda_2)}{(1+\lambda_1^2)(1+\lambda_2^2)}$$

which is strictly positive under the assumption $|\lambda_1\lambda_2|<1$.

Now we turn to long-time existence and convergence. As in Theorem A, we can show

$$|\lambda_1\lambda_2|\leq 1-\delta, \tag{4.11}$$

Recall the equation satisfied by $*\Omega_1$,

$$\begin{aligned}&\frac{d}{dt}*\Omega_1\geq\Delta*\Omega_1+\delta*\Omega_1|A|^2\\&+*\Omega_1\{\frac{k_1-k_2}{2}(n-1)[\sum_i\frac{\lambda_i^2}{1+\lambda_i^2}]+\frac{k_1+k_2}{2}[(n-2)\frac{\lambda_1^2+\lambda_2^2+2\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}+\frac{\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}]\}\end{aligned} \tag{4.12}$$

In fact, $\sum_i\frac{\lambda_i^2}{1+\lambda_i^2}=\frac{\lambda_1^2+\lambda_2^2+2\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}$. Since $*\Omega_1=\frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}$, we have $1-*\Omega_1^2=\frac{\lambda_1^2+\lambda_2^2+\lambda_1^2\lambda_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}$. It is not hard to see there exists a constant c' such that $\lambda_1^2+\lambda_2^2-2\lambda_1^2\lambda_2^2\geq c'(\lambda_1^2+\lambda_2^2+\lambda_1^2\lambda_2^2)$ if $|\lambda_1\lambda_2|\leq 1-\delta$.

Therefore,

$$\frac{d}{dt}*\Omega_1\geq\Delta*\Omega_1+\delta*\Omega_1|A|^2+c*\Omega_1(1-*\Omega_1^2) \tag{4.13}$$

for some constant $c > 0$.

As in the proof of Theorem A in [10], this equation implies long time existence by blowing up argument and White's regularity Theorem [11].

By maximum principle, $\min_{\Sigma_t} *\Omega_1 \to 1$ as $t \to \infty$, then we can use the estimate in the proof of Theorem B in [10] to show $\max_{\Sigma_t} |A|^2 \to 0$ and apply Simon's Theorem. We get smooth convergence in this case. In the limit, $*\Omega_1 = 1$ and thus $\lambda_1 = \lambda_2 = 0$.

$\square$

Corollary *If f is any smooth map from S^n to S^2 and if at each point, the restriction of df to any two dimensional subspace is area decreasing, then f is homotopic to a constant map along the mean curvature flow.*

Again, we remark the condition is equivalent to the comass of $f^*\Omega_2$ is less than one, where Ω_2 is the area form on S^2.

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