# A Stability Criterion for Nonparametric Minimal Submanifolds

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**Abstract**

An $n$ dimensional minimal submanifold $\Sigma$ of $\mathbb{R}^{n+m}$ is called non-parametric if $\Sigma$ can be represented as the graph of a vector-valued function $f : D \subset \mathbb{R}^n \mapsto \mathbb{R}^m$. This note provides a sufficient condition for the stability of such $\Sigma$ in terms of the norm of the differential $df$.

## 1 Introduction

A minimal submanifold is stable if the second derivative of the volume functional with respect to any compact supported normal variational field is non-negative. A non-parametric minimal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ is always stable. This is no longer true when the codimension is greater than one. A non-stable non-parametric minimal surface in four dimension was constructed by Lawson and Osserman in [5]. It seems very little is known about the stability of higher codimension minimal submanifolds except for calibrated ones. Recall a submanifold $\Sigma$ is calibrated by a calibrating form $\Omega$ if $\Omega|_\Sigma$ is the volume form of $\Sigma$, or $*\Omega = 1$ where $*$ is the Hodge star operator. In particular, a non-parametric hypersurface in $\mathbb{R}^{n+1}$ is calibrated by the $n$ form $i(N)dx^1 \cdots dx^{n+1}$ where $N$ is a extension of the unit normal vector field of $\Sigma$.

In [8], the second author constructs solutions to the Dirichlet problem of minimal surface systems in higher dimension and codimension and the solutions satisfies $*\Omega > \frac{1}{2}$ for $\Omega$ the volume form of an $n$-dimensional subspace. When $\Sigma$ is the graph of $f : D \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ and $\Omega$ is the volume form of the domain $\mathbb{R}^n$ extending to the whole $\mathbb{R}^{n+m}$, we have the relation

$*\Omega = \frac{1}{\sqrt{\det(I+(df)^T df)}}$. $*\Omega$ is actually the Jacobian of the projection map $\pi : \Sigma \mapsto D$. In particular, a lower bound on $*\Omega$ implies an upper bound on the norm of $df$. In this paper, we discover a criterion for the stability of minimal submanifolds in terms of such condition.

**Theorem A.** *Let $\Sigma$ be the graph of $f : D \subset \mathbb{R}^n \mapsto \mathbb{R}^m$. If $\Sigma$ is minimal and $||df|| \leq \frac{\sqrt{1+c}-1}{\sqrt{c}}$, then $\Sigma$ is stable. Here the norm $||df||$ is defined to be $\sup_{|V|=1} |df(V)|$ and $c$ is a constant which is 1 if $m \leq 2$ or $n \leq 2$, and is $\min(m-1, n-1)$ in other cases.*

Since $*\Omega = \frac{1}{\sqrt{\det(I+(df)^T df)}} = \frac{1}{\sqrt{\prod(1+\lambda_i^2)}}$, it is not hard to see the following consequence.

**Corollary.** *Let $c$ be the constant as in Theorem A. If $\Sigma$ is minimal and $*\Omega \geq \frac{c}{2(c+1-\sqrt{1+c})}$, then $\Sigma$ is stable*

In particular, when $m \leq 2$ or $n \leq 2$, if $\Sigma$ is minimal and $*\Omega \geq \frac{2+\sqrt{2}}{4}$, then $\Sigma$ is stable. The condition $*\Omega > \frac{c}{2(c+1-\sqrt{1+c})}$ corresponds a region in the Grassmannian. For minimal surfaces in $\mathbb{R}^3$, Barbosa and Do Carmo [1],[2] proved if the area of the image of the Gauss map is less than $2\pi$ then $D$ is stable. Their result was also obtained by D. Fischer-Colbrie and R. Schoen in [4] by a different method. This raises the general question of how to characterize stability by the Gauss map of a minimal submanifold. We remark the condition is not sharp in view of the codimension one case.

The proof of Theorem A utilizes a second variation formula of R. Mclean [6] for calibrated submanifolds. The corresponding formula for complex submanifolds of Kähler manifolds was derived by J. Simons [7].

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## 2 Second Variation Formula

We first recall the second variation formula for minimal submanifolds. Let $F_0 : \Sigma \mapsto \mathbb{R}^{n+m}$ be a minimal submanifold and let $F : \Sigma \times [0,1) \mapsto \mathbb{R}^{n+m}$ be a one-parameter family of immersions with $F(\cdot, 0) = F_0$. We may assume the variation field $V = F_*(\frac{\partial}{\partial s})$ is normal and of compact support. For simplicity, we will identify $F_0(\Sigma)$ with $\Sigma$ and denote $F(\cdot, s)$ by $F_s$. A coordinate system $\{x^i\}$ in a neighborhood of $p \in \Sigma$ is fixed. Let $g_{ij}(s)$ be the induced metric and $dv_s = \sqrt{\det g_{ij}}(s)\, dX$ be the volume form on $F_s(\Sigma)$. At $s = 0$, the volume form will be written as $dv$ instead.

We recall the second variation formula from [3]:

$$\frac{d^2}{ds^2}|_{s=0} \int_\Sigma dv_s = \int_\Sigma (||\nabla^N V||^2 - B(V,V))\, dv$$

where $\nabla^N V$ is the covariant derivative of $V$ as a section of the normal bundle and $B(V,V) = \sum_{ijkl} g^{ik} g^{jl} \langle \frac{\partial^2 F}{\partial x^i \partial x^j}, V\rangle \langle \frac{\partial^2 F}{\partial x^k \partial x^l}, V\rangle$. The minimal submanifold $\Sigma$ is stable if and only if

$$\int_\Sigma ||\nabla^N V||^2\, dv \geq \int_\Sigma B(V,V)\, dv$$

for any normal vector field with compact support. Since the second variation formula does not depend on $\frac{\partial^2 F}{\partial s^2}$, we may consider only the case $\frac{\partial^2 F}{\partial s^2}$ is zero at $s = 0$. In this case, the following equation holds at every point.

$$\frac{\partial^2}{\partial s^2}|_{s=0} \sqrt{\det g_{ij}}(s) = (||\nabla^N V||^2 - B(V,V)) \sqrt{\det g_{ij}} \tag{2.1}$$

In [6], a different second variation formula is derived in the presence of a calibrating form $\Omega$. In the following, we derive the formula for completeness. We shall assume $\Omega$ is locally an exact form.

Now $\int_{F_s(\Sigma)} \Omega = \int_\Sigma F_s^* \Omega$ is a constant . Write $F_s^*\Omega = *\Omega(s) \sqrt{\det g_{ij}}(s)\, dX$, where $*\Omega = \frac{1}{\sqrt{\det g_{ij}}} \Omega(\frac{\partial F}{\partial x^1}, \cdots \frac{\partial F}{\partial x^n})$.

Since

$$\begin{aligned} 0 &= \frac{d^2}{ds^2} \int_\Sigma F_s^* \Omega \\ &= \int_\Sigma [(\frac{\partial^2}{\partial s^2} * \Omega) \sqrt{\det g_{ij}}(s) + 2(\frac{\partial}{\partial s} * \Omega)(\frac{\partial}{\partial s} \sqrt{\det g_{ij}}(s)) \\ &+ *\Omega \frac{\partial^2}{\partial s^2} \sqrt{\det g_{ij}}(s)]\, dX \end{aligned}$$

and $\frac{\partial}{\partial s}|_{s=0}\sqrt{\det g_{ij}}(s)=0$ by the minimal condition, at $s=0$ we have

$$\int_\Sigma *\Omega\frac{\partial^2}{\partial s^2}\sqrt{\det g_{ij}}(s)\,dX=-\int_\Sigma(\frac{\partial^2}{\partial s^2}*\Omega)\,dv$$

That is,

$$\int_\Sigma *\Omega(||\nabla^N V||^2-B(V,V))\,dv=-\int_\Sigma(\frac{\partial^2}{\partial s^2}|_{s=0}*\Omega)\,dv \tag{2.2}$$

We shall compute $\frac{\partial^2}{\partial s^2}*\Omega$ using the formula $*\Omega=\frac{1}{\sqrt{\det g_{ij}}}\Omega(\frac{\partial F}{\partial x^1},\cdots\frac{\partial F}{\partial x^n})$. Thus

$$\begin{aligned}\frac{\partial^2}{\partial s^2}*\Omega &= \frac{\partial^2}{\partial s^2}(\frac{1}{\sqrt{\det g_{ij}}})\Omega(\frac{\partial F}{\partial x^1},\cdots\frac{\partial F}{\partial x^n})\\ &+2\frac{\partial}{\partial s}(\frac{1}{\sqrt{\det g_{ij}}})\frac{\partial}{\partial s}\Omega(\frac{\partial F}{\partial x^1},\cdots\frac{\partial F}{\partial x^n})\\ &+(\frac{1}{\sqrt{\det g_{ij}}})\frac{\partial^2}{\partial s^2}\Omega(\frac{\partial F}{\partial x^1},\cdots\frac{\partial F}{\partial x^n})\end{aligned} \tag{2.3}$$

At $s=0$, the minimal condition implies the second term vanishes and the first term becomes

$$\frac{\partial^2}{\partial s^2}|_{s=0}(\frac{1}{\sqrt{\det g_{ij}}})=-(\det g_{ij})^{-1}\frac{\partial^2}{\partial s^2}|_{s=0}(\sqrt{\det g_{ij}})$$

Because $\frac{\partial^2 F}{\partial s^2}$ is zero at $s=0$, the third term is

$$\frac{\partial^2}{\partial s^2}\Omega(\frac{\partial F}{\partial x^1},\cdots\frac{\partial F}{\partial x^n})=2[\Omega(\frac{\partial^2 F}{\partial s\partial x^1},\frac{\partial^2 F}{\partial s\partial x^2}\cdots\frac{\partial F}{\partial x^n})+\cdots] \tag{2.4}$$

Denote $\frac{\partial F}{\partial x^i}$ by $\partial_i$, then $\frac{\partial^2 F}{\partial s\partial x^i}=\frac{\partial^2 F}{\partial x^i\partial s}=\nabla_{\partial_i}V$ where $\nabla$ is the connection on $\mathbb{R}^{n+m}$ and $V=\frac{\partial F}{\partial s}$ is the variation field . In the following computation $(\cdot)^T$ and $(\cdot)^N$ denote the tangent and normal part of vector respectively.

$$\begin{aligned}&\Omega(\frac{\partial^2 F}{\partial s\partial x^1},\frac{\partial^2 F}{\partial s\partial x^2}\cdots\frac{\partial F}{\partial x^n})\\ &=\Omega((\nabla_{\partial_1}V)^T+(\nabla_{\partial_1}V)^N,(\nabla_{\partial_2}V)^T+(\nabla_{\partial_2}V)^N,\cdots\partial_n)\\ &=\Omega((\nabla_{\partial_1}V)^T,(\nabla_{\partial_2}V)^T,\cdots\partial_n)+\Omega((\nabla_{\partial_1}V)^T,(\nabla_{\partial_2}V)^N,\cdots\partial_n)\\ &+\Omega((\nabla_{\partial_1}V)^N,(\nabla_{\partial_2}V)^T,\cdots\partial_n)+\Omega((\nabla_{\partial_1}V)^N,(\nabla_{\partial_2}V)^N,\cdots\partial_n)\end{aligned}$$

We can assume $\{x^i\}$ is a normal coordinate system in a neighborhood of $p$ with respect to the induced metric on $\Sigma$. Hence $g_{ij}(0) = \delta_{ij}$ at $p$. We do the computation at point $p$ and get

$$\Omega((\nabla_{\partial_1} V)^T, (\nabla_{\partial_2} V)^T, \cdots \partial_n) = *\Omega(\langle V, \nabla_{\partial_1}\partial_1\rangle\langle V, \nabla_{\partial_2}\partial_2\rangle - \langle V, \nabla_{\partial_1}\partial_2\rangle^2)$$

Continue from equation (2.4), we derive

$$\begin{aligned}
&2[\Omega(\frac{\partial^2 F}{\partial s \partial x^1}, \frac{\partial^2 F}{\partial s \partial x^2} \cdots \frac{\partial F}{\partial x^n}) + \cdots] \\
&= 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^T, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n) \\
&+ 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^N, \cdots (\nabla_{\partial_j} V)^T, \cdots \partial_n) \\
&+ 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^N, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n) \\
&+ 2 * \Omega(\sum_{i<j} \langle V, \nabla_{\partial_i}\partial_i\rangle\langle V, \nabla_{\partial_j}\partial_j\rangle - \langle V, \nabla_{\partial_i}\partial_j\rangle^2)
\end{aligned}$$

It thus follows from (2.3) that

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}|_{s=0} * \Omega = &- * \Omega ||\nabla^N V||^2 + *\Omega \sum_{i,j} \langle V, \nabla_{\partial_i}\partial_j\rangle^2 \\
&+ 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^T, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n) \\
&+ 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^N, \cdots (\nabla_{\partial_j} V)^T, \cdots \partial_n) \\
&+ 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^N, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n) \\
&+ 2 * \Omega(\sum_{i<j} \langle V, \nabla_{\partial_i}\partial_i\rangle\langle V, \nabla_{\partial_j}\partial_j\rangle - \langle V, \nabla_{\partial_i}\partial_j\rangle^2)
\end{aligned}$$

However,

$$\begin{aligned}
&* \Omega \sum_{i,j} \langle V, \nabla_{\partial_i}\partial_j\rangle^2 + 2 * \Omega(\sum_{i<j} \langle V, \nabla_{\partial_i}\partial_i\rangle\langle V, \nabla_{\partial_j}\partial_j\rangle - \langle V, \nabla_{\partial_i}\partial_j\rangle^2) \\
&= *\Omega(\sum_i \langle V, \nabla_{\partial_i}\partial_i\rangle)^2 = 0
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial^2}{\partial s^2}|_{s=0} * \Omega = &- * \Omega||\nabla^N V||^2 + 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^T, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n) \\ &+ 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^N, \cdots (\nabla_{\partial_j} V)^T, \cdots \partial_n) \\ &+ 2\sum_{i<j} \Omega(\partial_1, \cdots (\nabla_{\partial_i} V)^N, \cdots (\nabla_{\partial_j} V)^N, \cdots \partial_n)\end{aligned}$$

Combine equations (2.1) and (2.2), we obtain

**Proposition 2.1** *Let $\Omega$ be an exact parallel n-form and $\Sigma$ be an n dimensional minimal submanifold in $\mathbb{R}^{n+m}$. Assume that $V$ is a normal variation field and $\nabla_V V = 0$ along $\Sigma$. Then one has*

$$\begin{aligned}&\int_\Sigma *\Omega(||\nabla^N V||^2 - B(V,V))\, dv \\ &= \int_\Sigma [*\Omega||\nabla^N V||^2 \\ &- 2\sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^T, \cdots, (\nabla_{\partial_j} V)^N, \cdots, \partial_n) \\ &- 2\sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^N, \cdots, (\nabla_{\partial_j} V)^N, \cdots, \partial_n) \\ &- 2\sum_{i<j} \Omega(\partial_1, \cdots, (\nabla_{\partial_i} V)^T, \cdots, (\nabla_{\partial_j} V)^N, \cdots, \partial_n)]\, dv\end{aligned} \tag{2.5}$$

# 3 Proof of Theorem A

The idea now is to show the right hand side of equation (2.5) is greater than or equal to

$$\delta \int_\Sigma *\Omega(||\nabla^N V||^2 - B(V,V))\, dv$$

for some $\delta < 1$. We shall express the integrand in the right hand side of equation (2.5) in terms of a particular orthonormal basis. At any point $p$, we consider the singular value decomposition of $df : \mathbb{R}^n \mapsto \mathbb{R}^m$. We have

$$df(a_i) = \lambda_i a_{n+i}$$

where $\lambda_i \geq 0$ are the singular values of $df$, or eigenvalues of $\sqrt{(df)^T df}$. $\{a_i\}_{i=1,\cdots,n}$ is an orthonormal basis of eigenvectors of $\sqrt{(df)^T df}$. The set $\{a_{n+i}\}$ can be completed to form an orthonormal basis $\{a_\alpha\}_{\alpha=n+1,\cdots,n+m}$ for $\mathbb{R}^m$. (In case $m < n$, we will have $\lambda_i = 0$ for $i > m$ and $\{a_{n+i}\}_{i=1,\cdots,m}$ forms an orthonormal basis for $\mathbb{R}^m$.) Now $\{e_i = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i a_{n+i})\}$ and $\{e_{n+i} = \frac{1}{\sqrt{1+\lambda_i^2}}(a_{n+i} - \lambda_i a_i)\}$ can be completed to give orthonormal basis of the tangent and normal space. The orthonormal basis for the normal space is denoted by $\{e_\alpha\}_{\alpha=n+1,\cdots,n+m}$. In these bases we denote $(\nabla_{e_i} V)^N = \sum_\alpha V_i^\alpha e_\alpha$ and $(\nabla_{e_i} V)^T = -\sum_{\alpha,j} V^\alpha h_{\alpha ij} e_j$. We shall assume $m \geq n$ in the following calculation, the other case can be treated similarly.

At the point $p$ in these bases, the integrand of the right hand side of equation (2.5) can be written as

$$\begin{aligned}
&* \Omega[\sum_{i,\alpha}(V_i^\alpha)^2 - 2\sum_{i<j}\lambda_i\lambda_j V_i^{n+i}V_j^{n+j} + 2\sum_{i<j}\lambda_i\lambda_j V_i^{n+j}V_j^{n+i} \\
&+ 2\sum_\alpha\sum_{i<j} V^\alpha h_{\alpha ii} V_j^{n+j}\lambda_j - 2\sum_\alpha\sum_{i<j} V^\alpha h_{\alpha ij} V_j^{n+i}\lambda_i \\
&+ 2\sum_\alpha\sum_{i<j} V^\alpha h_{\alpha jj} V_i^{n+i}\lambda_i - 2\sum_\alpha\sum_{i<j} V^\alpha h_{\alpha ij} V_i^{n+j}\lambda_j]
\end{aligned}$$

which is the same as

$$\begin{aligned}
&* \Omega[\sum_{i,\alpha}(V_i^\alpha)^2 - \sum_{i\neq j}\lambda_i\lambda_j V_i^{n+i}V_j^{n+j} + \sum_{i\neq j}\lambda_i\lambda_j V_i^{n+j}V_j^{n+i} \\
&+ 2\sum_\alpha\sum_{i\neq j} V^\alpha h_{\alpha ii} V_j^{n+j}\lambda_j - 2\sum_\alpha\sum_{i\neq j} V^\alpha h_{\alpha ij} V_j^{n+i}\lambda_i]
\end{aligned}$$

By minimality, we have

$$2\sum_{i\neq j} V^\alpha h_{\alpha ii} V_j^{n+j}\lambda_j = -2\sum_j V^\alpha h_{\alpha jj} V_j^{n+j}\lambda_j$$

Define $\Xi$ by

$$\begin{aligned}
\Xi = &\sum_{i,\alpha}(V_i^\alpha)^2 - \sum_{i\neq j}\lambda_i\lambda_j V_i^{n+i}V_j^{n+j} + \sum_{i\neq j}\lambda_i\lambda_j V_i^{n+j}V_j^{n+i} \\
&- 2\sum_\alpha\sum_{i,j} V^\alpha h_{\alpha ij} V_j^{n+i}\lambda_i
\end{aligned}$$

If we can show

$$\Xi \geq \delta[\sum_{i,\alpha}(V_i^\alpha)^2 - \sum_{ij}(\sum_\alpha V^\alpha h_{\alpha ij})^2]$$

for some $\delta < 1$, then we are done in review of equation (2.5).

By Cauchy-Schwarz inequality, for any $\epsilon > 0$ to be determined,

$$-2\sum_{i,j,\alpha} V^\alpha h_{\alpha ij} V_j^{n+i}\lambda_i \geq -\epsilon\sum_{i,j}(\sum_\alpha V^\alpha h_{\alpha ij})^2 - \frac{1}{\epsilon}\sum_{i,j}(V_j^{n+i}\lambda_i)^2$$

Therefore

$$\begin{aligned}\Xi \geq &\sum_{i\alpha}(V_i^\alpha)^2 - \sum_{i\neq j}\lambda_i\lambda_j V_i^{n+i}V_j^{n+j} + \sum_{i\neq j}\lambda_i\lambda_j V_i^{n+j}V_j^{n+i} - \frac{1}{\epsilon}\sum_{i,j}(V_j^{n+i}\lambda_i)^2 \\ &- \epsilon\sum_{i,j}(\sum_\alpha V^\alpha h_{\alpha ij})^2\end{aligned}$$

Now assume each $\lambda_i \leq \eta$, then

$$\begin{aligned}\Xi \geq &\sum_{i,\alpha}(V_i^\alpha)^2 - \eta^2\sum_{i\neq j}|V_i^{n+i}||V_j^{n+j}| - \eta^2\sum_{i\neq j}|V_i^{n+j}||V_j^{n+i}| - \frac{\eta^2}{\epsilon}\sum_{i,j}(V_j^{n+i})^2 \\ &- \epsilon\sum_{i,j}(\sum_\alpha V^\alpha h_{\alpha ij})^2\end{aligned} \tag{3.1}$$

By Cauchy-Schwarz inequality

$$\sum_{i\neq j}|V_i^{n+i}||V_j^{n+j}| \leq (n-1)\sum_i(V_i^{n+i})^2$$

and

$$\sum_{i\neq j}|V_i^{n+j}||V_j^{n+i}| \leq \sum_{i\neq j}(V_i^{n+j})^2$$

In a general case, the coefficient in the right hand side of the first inequality should be $\min(m-1, n-1)$. Let $c = 1$ if $m \leq 2$ or $n \leq 2$ and $c = \min(m-1, n-1)$ in other cases. Plug into equation (3.1), we obtain,

$$\Xi \geq (1 - \frac{\eta^2}{\epsilon} - c\eta^2)\sum_{i,\alpha}(V_i^\alpha)^2 - \epsilon\sum_{i,j}(\sum_\alpha V^\alpha h_{\alpha ij})^2 \tag{3.2}$$

We need to have $1-\frac{\eta^2}{\epsilon}-c\eta^2 \geq \epsilon$ which is $\eta^2 \leq \frac{\epsilon(1-\epsilon)}{1+c\epsilon}$. It is not hard to see $\max_{0<\epsilon<1}\frac{\epsilon(1-\epsilon)}{1+c\epsilon} = \frac{(\sqrt{c+1}-1)^2}{c}$ when $\epsilon = \frac{\sqrt{1+c}-1}{c}$. Thus if we assume $||df|| \leq \frac{\sqrt{1+c}-1}{\sqrt{c}}$, then each $|\lambda_i| \leq \frac{\sqrt{1+c}-1}{\sqrt{c}}$ and

$$\Xi \geq \frac{\sqrt{1+c}-1}{c}[\sum_{i,\alpha}(V_i^\alpha)^2-\sum_{i,j}(\sum_\alpha V^\alpha h_{\alpha ij})^2]$$

Theorem A is proved.

# 4 Examples

The construction in [8] supplies examples for such stable minimal submanifolds. Given any $\phi : D \subset \mathbb{R}^n \mapsto \mathbb{R}^m$ defined on a convex domain $D$, we can scale $\phi$ so that on the graph of $\phi$, $*\Omega \geq \frac{c}{2(c+1-\sqrt{1+c})}$ and the derivative of $\phi$ satisfies the requirement in [8]. It was proved in [8] that the Cauchy-Dirichlet problem of the mean curvature flow for initial data $\phi$ is solvable and $*\Omega \geq \frac{c}{2(c+1-\sqrt{1+c})}$ is preserved along the flow. The flows converges to a minimal submanifold which is stable by the Corollary.

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