Mean Curvature Flows in Higher Codimension

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Abstract

The mean curvature flow is an evolution process under which a submanifold deforms in the direction of its mean curvature vector. The hypersurface case has been much studied since the eighties. Recently, several theorems on regularity, global existence and convergence of the flow in various ambient spaces and codimensions were proved. We shall explain the results obtained as well as the techniques involved. The potential applications in symplectic topology and mirror symmetry will also be discussed.

1 Introduction

Let M be a Riemannian manifold and $F : \Sigma \mapsto M$ an isometric immersion of a smooth compact submanifold. The second fundamental form A of Σ is defined by

$$A : T\Sigma \times T\Sigma \mapsto N\Sigma,$$

$$A(X, Y) = (\nabla^M_X Y)^\perp \text{ where } X, Y \in T\Sigma$$

Here ∇^M is the Levi-Civita connection on M and $(\cdot)^\perp$ denotes the projection onto the normal bundle $N\Sigma$.

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The mean curvature vector H is defined by

$$H = Tr_g A \in N\Sigma$$

where the trace Tr_g is taken with respect to the induced metric g on Σ.

When $M = \mathbb{R}^N$, if $x^1, \cdots, x^n$ denote a local coordinate system on Σ then the second fundamental form can be represented by $(\frac{\partial^2 F}{\partial x^i \partial x^j})^\perp$ and the mean curvature vector is $H = (g^{ij}\frac{\partial^2 F}{\partial x^i \partial x^j})^\perp$, where g^{ij} is the inverse matrix to $g_{ij} = < \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} >$, the first fundamental form. Here $\frac{\partial^2 F}{\partial x^i \partial x^j}$ is considered as a vector in $\mathbb{R}^N$. In this case, it is not hard to check

$$H = \Delta_\Sigma F \tag{1.1}$$

where Δ_Σ is the Laplace operator of the induced metric on Σ.

The importance of the mean curvature vector lies in the first variation formula of area.

$$\delta_V(area(\Sigma)) = -\int_\Sigma H \cdot V$$

for any variation field V. Thus H is the normal vector field on Σ that points to the direction in which the area decreases most rapidly. Σ is called a minimal submanifold if H vanishes identically.

The mean curvature flow of $F : \Sigma \mapsto M$ is a family of immersions $F : \Sigma \times [0, \epsilon) \mapsto M$ parametrized by t that satisfies

$$\begin{aligned} &\frac{d}{dt}F_t(x) = H(x,t) \\ &F_0 = F \end{aligned} \tag{1.2}$$

where $H(x,t)$ is the mean curvature vector of $F_t(\Sigma)$ at $F_t(x)$.

This should be considered as the heat equation for submanifolds in view of equation (1.1). A submanifold tends to find its optimal shape inside the ambient manifold.

If we assume $M = \mathbb{R}^N$, in terms of coordinate $x^1, \cdots, x^n$ on Σ, the mean curvature flow is the solution to the following system of parabolic equations

$$F = F^A(x^1, \cdots, x^n, t), \; A = 1, \cdots N$$

$$\frac{\partial F^A}{\partial t} = \sum_{i,j,B} g^{ij}\, P^A_B\, \frac{\partial^2 F^B}{\partial x^i \partial x^j},\ A = 1, \cdots N$$

where $P^A_B = \delta^A_B - g^{kl} \frac{\partial F^A}{\partial x^k} \frac{\partial F^B}{\partial x^l}$ is the projection operator to the normal direction.

The equation (1.2) is a quasi-linear parabolic system and short time existence is guaranteed when the initial submanifold Σ is compact and smooth [18].

In general, the mean curvature flow fails to exist after a finite time. The singularity is completely characterized by the blow up of the second fundamental form. Namely, singularity at t_0 if and only if $\sup_{\Sigma_t} |A|^2 \to \infty$ as $t \to t_0$. See for example [14] for the hypersurface case.

The mean curvature flow has been studied by various approaches. In this article, we shall concentrated on the approaches of classical partial differential equations and geometric measure theory. For the level set approach and numerical methods, please see [3] and the reference therein.

There are many beautiful results in the hypersurface (codimension one) case.

Theorem 1.1 *(Huisken, 1984 ($N \geq 3$)[14], Gage-Hamilton, 1985 ($N = 2$)[10]) Any convex compact hypersurface in $\mathbb{R}^N$ contracts to a round point after finite time along the mean curvature flow.*

Theorem 1.2 *(Grayson, 1987 [11]) Any embedded closed curve in $\mathbb{R}^2$ contracts to a round point after finite time along the curvature flow.*

In codimension one case, H is essentially a scalar function and $H > 0$ is preserved along the flow [14]. As a contrast, in higher codimension H is a genuine vector and we do not know how to control the direction of H. There are relatively very few results in the higher codimension case, see [2], [3] and [21].

We shall discuss some new results about mean curvature flows in higher codimension in this article. The guideline is to identify positive quantities preserved along the flow.

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2 Applications in calibrated geometry

Let M be an N-dimensional Riemannian manifold and α an n-form on M. For any $p \in M$ and S any n-dimensional oriented subspace of T_pM. Represent $S = e_1 \wedge \cdots \wedge e_n$, where $e_1, \cdots e_n$ is an oriented orthonormal basis for S. Define the *comass* of α , $|\alpha|$ by

$$|\alpha|(p) = \sup_{S \subset T_pM} \alpha(e_1 \wedge \cdots \wedge e_n)$$

Following Harvey and Lawson [13], a closed form α with $|\alpha|(p) = 1$ at each $p \in M$ is called a *calibrating* form.

An oriented closed n-dimensional submanifold Σ of M is *calibrated* by α if $\alpha(T_p\Sigma) = 1$ for any $p \in \Sigma$, or $\alpha|_\Sigma = vol|_\Sigma$. A fundamental fact in calibrated geometry is: a calibrated submanifold minimizes area in its homology class. This follows from Stoke's Theorem:

Let Σ' be any submanifold with $[\Sigma'] = [\Sigma]$, then

$$\int_{\Sigma'} vol|_{\Sigma'} \geq \int_{\Sigma'} \alpha|_{\Sigma'} = \int_\Sigma \alpha|_\Sigma = \int_\Sigma vol|_\Sigma$$

We are interested in the following two classes of calibrated submanifolds.

(1) Let (M, ω) be a Kähler manifold and $\alpha = \frac{1}{n!}\omega^n$. Any $2n$ dimensional complex submanifold is calibrated by α.

(2) Let (M, ω, Ω) be Calabi-Yau of complex dimension m and Ω is the parallel holomorphic $(m, 0)$ form. $\alpha = Re\Omega$ is then a calibrating form. A Lagrangian submanifold calibrated by α is called a *special Lagrangian* submanifold.

Define the function $*\alpha(p) = \alpha(T_p\Sigma)$ on Σ, we may use $*\alpha$ to measure how far Σ is away from being calibrated. On a calibrated submanifold $*\alpha \equiv 1$. It turns out the condition $*\alpha > 0$ can be used to rule out a certain type of singularity.

There are two types of finite time singularity depending on the blow-up rate of $|A|$. Denote by t_0 the blow up time, then $\sup_{\Sigma_t} |A|^2 \to \infty$ as $t \to t_0$. The singularity is said to be *fast-forming* (type I) if there exists a $C > 0$ such that

$$\sup_t |A|^2 \leq \frac{C}{t_0 - t}$$

Otherwise, the singularity is called *slow-forming* (type II). For embedded curve on the plane, only type I singularity occurs.

Theorem 2.1 *[24] Let (M^4, ω) be a Kähler-Einstein four-manifold, then a symplectic surface, i.e. $*\omega > 0$ remains symplectic along the mean curvature flow and the flow does not develop any type I singularity.*

The results in Theorem 2.1 were obtained in the summer of 1999 and announced in February 2000 at Stanford's differential geometry seminar. Theorem 2.1 was also proved by Chen-Tian [5] and Chen-Li [4].

When M is a Calabi-Yau manifold of arbitrary dimension, we prove the following theorem.

Theorem 2.2 *[24] Let (M, ω, Ω) be a Calabi-Yau manifold, then a Lagrangian submanifold with $*Re\Omega > 0$ remains Lagrangian and $*Re\,\Omega > 0$ along the mean curvature flow and the flow does not develop any type I singularity.*

That being Lagrangian is preserved along the mean curvature flow in Kähler-Einstein manifolds was proved by Smoczyk in [21].

Proof. In [24](see page 324, remark 5.1), we sketch a proof of this theorem. For a Lagrangian submanifold of Calabi-Yau manifolds, the fundamental equations are, (see for example [22] or [23])

$$\Omega|_\Sigma = e^{i\theta} vol|_\Sigma \tag{2.1}$$

$$H = J(\nabla\theta)$$

Once we know being Lagrangian is preserved, then θ satisfies the heat equation

$$\frac{d}{dt}\theta = \Delta\theta$$

By equation (2.1), $*Re\Omega = \cos\theta$. A straightforward calculation using $|H|^2 = |\nabla\theta|^2$ shows

$$\frac{d}{dt} * Re\Omega = \Delta * Re\Omega + |H|^2 * Re\Omega$$

The same argument in Proposition 5.2 of [24] shows on a type I blow up limit Σ_∞, we will have $|H|^2 = 0$. Since any type I blow up limit is smooth and satisfies $F^\perp = H$ [15], Σ_∞ must be a flat space. White's regularity theorem [27] shows there is no type I singularity.

□

3 Applications in mapping deformations

Let $f : \Sigma_1 \mapsto \Sigma_2$ be a smooth map between compact Riemannian manifolds. The volume form ω_1 of Σ_1 extends to a parallel calibrating form on the product space $\Sigma_1 \times \Sigma_2$. Let Σ be the graph of f as a submanifold of $\Sigma_1 \times \Sigma_2$. On Σ, $*\omega_1 = Jac(\pi_1|_\Sigma)$ is the Jacobian of the projection $\pi_1 : \Sigma_1 \times \Sigma_2 \mapsto \Sigma_1$ restricting to Σ. Any submanifold Σ' of $\Sigma_1 \times \Sigma_2$ is a locally a graph over Σ_1 if $*\omega_1 > 0$ on Σ' by the inverse function theorem.

We shall evolve Σ by the mean curvature flow in $\Sigma_1 \times \Sigma_2$. If $*\omega_1 > 0$ is preserved along the flow then each Σ_t is a graph over Σ_1 and thus the flow gives a deformation f_t of the original map f. It turns out a stronger inequality is preserved.

Theorem 3.1 *[24] If a smooth map $f : S^2 \mapsto S^2$ satisfies $*\omega_1 > |*\omega_2|$ on the graph of f, then the inequality remains true along the mean curvature flow, the flow exists smoothly for all time and f_t converges to a constant map.*

The assumption is the same as $Jac(\pi_1|_\Sigma) > |Jac(\pi_2|_\Sigma)|$. In other words, if we see more of Σ from Σ_1 than from Σ_2, then Σ converges to some $\Sigma_1 \times \{p\}$ eventually. This is a natural geometric assumption and we believe such assumption is necessary for higher codimension mean curvature flow.

This theorem is generalized to arbitrary dimension and codimension in [26] under a slightly stronger assumption.

Theorem 3.2 *[26] Let $f : S^n \mapsto S^m$ be a smooth map. If $*\omega_1 > \frac{1}{\sqrt{2}}$ on the graph of f, then the mean curvature flow of the graph of f in $S^n \times S^m$ exists for all time, remains a graph, and converges smoothly to the graph of a constant map at infinity.*

This theorem is true under various curvature assumptions, please see [26] for the more general version. $*\omega_1$ should be considered as the inner product of the tangent space of Σ and the tangent space of S_n. The condition $*\omega_1 > \frac{1}{\sqrt{2}}$ guarantees $T\Sigma$ is closer to TS^n than to any other competing directions.

When $*\omega_1 = *\omega_2$, we proved the following theorem.

Theorem 3.3 *[25] Let $f : S^2 \mapsto S^2$ be a smooth map such that $*\omega_1 = *\omega_2 > 0$ on the graph of f, then the equality is preserved along the mean curvature flow and f_t converges to an isometry of S^2.*

The condition translates to $f^*\omega_2 = \omega_1$, or f is an area-preserving diffeomorphism (or symplectomorphism). Recall the harmonic heat flow of Eells-Sampson considers the deformation of a map $f : M \mapsto N$ along the gradient flow of the energy functional. When the sectional curvature of the target N is non-positive, the flow exists for all time and f_t converges to a harmonic map as $t \mapsto \infty$. For maps of nonzero degree between two-spheres $f : S^2 \mapsto S^2$, singularities do occur in the harmonic heat flow even after finite time. It is quite surprising that the mean curvature deformation exists for all time and converges.

f_t indeed provides a path in the diffeomorphism (symplectomorphism) group of S^2.

Theorem 3.4 *[25] Any area preserving diffeomorphism of two-sphere deforms to an isometry through area preserving diffeomorphisms along the mean curvature flow.*

For a Riemann surface Σ with positive genus, the same result holds [25] when Σ has hyperbolic metric and the map $f : \Sigma \mapsto \Sigma$ is homotopic to identity.

4 Proof of Theorem 3.1

We shall explain the techniques involved in the proof of Theorem 3.1 as it is the first complete solution to a higher codimension mean curvature flow.

4.1 Maximum principle

The maximum principle of parabolic systems developed by R. Hamilton [12] plays an important role in the study of geometric evolution equations. The first step is to use maximum principle to show the inequalities $*\omega_1 + *\omega_2 > 0$ and $*\omega_1 - *\omega_2 > 0$ are preserved along the flow.

In fact, if we denote the singular values of df by λ_1 and λ_2, then

$$*\omega_1 = \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}$$

and

$$*\omega_2 = \frac{\lambda_1\lambda_2}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}$$

Let $\eta_1 = *\omega_1 + *\omega_2$ and $\eta_2 = *\omega_1 - *\omega_2$, then $0 < \eta_1, \eta_2 \leq 1$. The following equations are derived in [24] (see [26] for general parallel forms).

$$\frac{d}{dt}\eta_1 = \Delta\eta_1 + \eta_1|A_1|^2 + \eta_1(1-\eta_1^2) \tag{4.1}$$

$$\frac{d}{dt}\eta_2 = \Delta\eta_2 + \eta_2|A_2|^2 + \eta_2(1-\eta_2^2) \tag{4.2}$$

where A_1 and A_2 are part of the second fundamental form with $|A_1|^2 + |A_2|^2 = 2|A|^2$.

The assumption of Theorem 3.1 implies $\eta_1, \eta_2 > 0$ initially. By maximum principle of parabolic equations, $\min_{\Sigma_t} \eta_i$ is nondecreasing. This guarantees $*\omega_1 > |*\omega_2|$ is preserved. Adding equations (4.1) and (4.1), we get for $\mu = *\omega_1$,

$$\frac{d}{dt}\mu \geq \Delta\mu + c|A|^2 \tag{4.3}$$

where $c > 0$ is $\min\{\eta_1, \eta_2\}$ at $t = 0$.

4.2 Blow-up analysis

The blow-up analysis is used in proving long time existence of the flow. First let us recall the blow-up analysis for minimal surfaces. To study a possible singularity x_0, we blow up the minimal surface Σ^n at x_0 by $B_\lambda : x \mapsto \lambda(x - x_0)$,

$\lambda > 0$. Any limit as $\lambda \to \infty$ is still minimal since the minimal surface equation is invariant under the scaling. It must be a cone as a consequence of the monotonicity formula. A minimal cone is rigid in the following sense: if it is close enough to a plane, then it must be a plane. The closeness is measured by the density function.

$$\Theta(x_0) = \lim_{r\to 0} \Theta(x_0, r) = \lim_{r\to 0} \frac{area(B(x_0, r) \cap \Sigma)}{\omega^n r^n}$$

where ω^n is the area of an n-dimensional unit ball. The monotonicity formula in minimal surface theory says $\Theta(x_0, r)$ is non-increasing as r approaches 0, in particular the limit exists. Allard's regularity theorem [1] then asserts there exists an $\epsilon > 0$ such that if $\Theta(x_0) < 1 + \epsilon$, then x_0 is a regular point. We refer to Simon's book [20] for minimal surface theory.

For the mean curvature flow, we consider the total space time as a submanifold in $M \times \mathbb{R}$ and use parabolic blow up at a space time point (x_0, t_0). The limit is still a mean curvature flow and the monotonicity formula of Huisken implies a time slice satisfies $F^\perp = H$, or the limit flow is self-similar. For a smooth point, we obtain the stationary flow of a plane.

The density function is now replaced by the integral of the backward heat kernel. To be more precise, we isometrically embed M into $\mathbb{R}^N$. For any $\lambda > 1$, the parabolic blow up D_λ at (x_0, t_0) is defined by

$$\begin{aligned} D_\lambda : \mathbb{R}^N \times [0, t_0) &\to \mathbb{R}^N \times [-\lambda^2 t_0, 0) \\ (x, t) &\to (\lambda(x - x_0), \lambda^2(t - t_0)) \end{aligned} \tag{4.4}$$

The (n-dimensional) backward heat kernel ρ_{x_0,t_0} at (x_0, t_0) is

$$\rho_{x_0,t_0}(x, t) = \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} \exp\left(\frac{-|x - x_0|^2}{4(t_0 - t)}\right) \tag{4.5}$$

Notice that the integral $\int \rho_{x_0,t_0} d\mu_t$ is invariant under the parabolic blow up, where $d\mu_t = \sqrt{\det g_{ij}(F_t)} d\mu$ is the pull back volume form by F_t.

The monotonicity formula of Huisken [15] says for $t < t_0$

$$\frac{d}{dt} \int \rho_{x_0,t_0} d\mu_t \le 0$$

so the limit as $t \to t_0$ exists. This formula holds only for mean curvature flows in Euclidean spaces. For a general ambient manifold, a modification to take care of curvature terms is necessary, see [28] or [16]. The analogue of Allard's regularity theorem in mean curvature flow is the following theorem of White.

Theorem 4.1 *[27] There is an $\epsilon > 0$ such that if*

$$\lim_{t\to t_0} \int \rho_{x_0,t_0} d\mu_t < 1+\epsilon$$

then (x_0, t_0) is a regular point.

In the proof of Theorem 3.1, the equation (4.3) helps us find a subsequence $t_i \to t_0$ and blow up rate $\lambda_i \to \infty$ such that the L^2 norm of the second fundamental form $\int |A|^2$ of $\lambda_i \Sigma_{t_i}$ approaches zero. The limit is thus a plane and $\int \rho_{x_0,t_0} d\mu_{t_i} \to 1$ as $t_i \to \infty$. Monotonicity formula implies $\lim_{t\to t_0} \int \rho_{x_0,t_0} d\mu_t = 1$. By White's theorem again it can be concluded that (x_0, t_0) is a regular point.

4.3 Curvature estimate

From the calculation in [24], the norm of the second fundamental form $|A|^2$ satisfies

$$\frac{d}{dt}|A|^2 \le \Delta|A|^2 - 2|\nabla A|^2 + K_1|A|^4 + K_2|A|^2$$

where K_1, K_2 are constants depending on the curvature and the covariant derivatives of curvature of the ambient space .

In general, $|A|^2$ blows up in finite time because of the $|A|^4$ term. However, the equation of μ (4.3) helps to control $|A|^2$ in the proof of Theorem 3.1.

By equation (4.3) for any k, $0 < k < 1$, we have

$$\frac{d}{dt}(\mu - k) \ge \Delta(\mu - k) + c\frac{\mu}{\mu - k}(\mu - k)|A|^2$$

where we use $0 < \mu \le 1$.

If $\min_{\Sigma_t} \mu$ is very close to one when t is large, we may choose k close to 1 so that $\mu - k > 0$ is preserved after some t_1 and $\frac{\mu}{\mu-k}$ is large. The quantity $g = \frac{|A|^2}{\mu-k}$ after time t_1 then satisfies

$$\frac{d}{dt} g \leq \Delta g + V \cdot \nabla g - K_3 g^2 + K_4 g$$

with $K_3 > 0$. The maximum principle shows g is uniformly bounded for $t > t_1$.

By equations (4.1) and (4.2) and a comparison argument, we see $\min_{\Sigma_t} \eta_1 \to 1$ and $\min_{\Sigma_t} \eta_2 \to 1$ as $t \to \infty$. In particular , $\min_{\Sigma_t} \mu \to 1$. The assumption on μ is true when t is large enough and thus $|A|^2$ is uniformly bounded. Integrate equation (4.3) over space and time shows $\int_{\Sigma_t} |A|^2 \to 0$ and the sub mean value inequality in [16] shows $\sup |A|^2 \to 0$. The last step is to apply Simon's [19] general convergence theorem for gradient flows.

5 Related problems

Recall [21] that being Lagrangian is preserved along the mean curvature flow in Kähler-Einstein manifolds. The graph of a symplectomorphism of a Kähler-Einstein manifold M is a Lagrangian submanifold in the product space $(M \times M, \omega_1 - \omega_2)$. The following question is thus a natural generalization of Theorem 3.3.

Question 1 *Can one prove the long time existence and convergence of mean curvature flows of symplectomorphisms of Kähler-Einstein manifolds?*

Theorem 3.3 and the corresponding theorems for higher genus Riemann surfaces implies any Lagrangian graph is Lagrangian isotopic to a minimal Lagrangian graph along the mean curvature flow. This is related to the following conjecture due to Thomas and Yau [23].

Question 2 *Can one prove the long time existence and convergence of mean curvature flow of a stable graded Lagrangian submanifold in a Calabi-Yau manifold?*

A notion of stability for Lagrangian submanifolds was formulated in [23] in terms of the range of the phase function θ.

$$\Omega|_\Sigma = e^{i\theta} vol|_\Sigma$$

on any Lagrangian submanifold of Calabi-Yau manifold. Theorem 2.2 implies if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ on a Lagrangian submanifold, then the mean curvature flow does not develop any type I singularity. How to exclude or perturb away type II singularities seems a very interesting yet hard problem.

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