A Bernstein Type Result for Special Lagrangian Submanifolds

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Abstract

Let Σ be a complete minimal Lagrangian submanifold of $\mathbb{C}^n$. We identify regions in the Grassmannian of Lagrangian subspaces so that whenever the image of the Gauss map of Σ lies in one of these regions, then Σ is an affine space.

1 Introduction

The well-known Bernstein theorem states any complete minimal surface that can be written as the graph of a function on $\mathbb{R}^2$ must be a plane. This type of result has been generalized in higher dimension and codimension under various conditions. See [2] and the reference therein for the codimension one case and [1], [3], and [6] for higher codimension case. In this note, we prove a Bernstein type result for complete minimal Lagrangian submanifolds of $\mathbb{C}^n$. We remark that Jost-Xin [7] obtained similar results from a somewhat different approach.

Recall a submanifold Σ of $\mathbb{C}^n$ is called Lagrangian if the Kähler form $\sum_{i=1}^n dx^i \wedge dy^i$ restricts to zero on Σ. If Σ happens to be the graph of a vector-valued function from a Lagrangian subspace L to its complement $L^\perp$ in $\mathbb{C}^n$. Rotating $\mathbb{C}^n$ by a element in $U(n)$, we may assume L is the x^i subspace and $L^\perp$ is the y^i subspace. In this case, there exists a smooth function $F : \mathbb{R}^n \to \mathbb{R}$ such that Σ is defined by the gradient of F, ∇F. The minimal Lagrangian equation can be written in terms of F.

$$Im(\det((I + i \text{ Hess } (F))) = \textit{ constant} \tag{1.1}$$

where I = identity matrix and Hess $F = (\frac{\partial^2 F}{\partial x^i \partial x^j})$.

Such minimal submanifolds were first studied by Harvey and Lawson [5] in the context of calibrated geometry. In fact, they are calibrated by n forms of the type $Re(e^{i\theta} dz^1 \wedge \cdots \wedge dz^n)$ for some constant θ. They are usually referred as special Lagrangian submanifold (SLg) in literature in a more general sense. Recently, Strominger-Yau-Zaslow [8] established a conjectural relation of fibrations by special Lagrangian tori with mirror symmetry.

In terms of (1.1), a Bernstein type question is to determine under what conditions an entire solution F becomes a quadratic polynomial.

The results in this paper imposes conditions on the image of the Gauss map of Σ. Recall the set of all Lagrangian subspaces of $\mathbb{C}^n$ is parametrized by the Lagrangian Grassmannian $U(n)/SO(n)$. The Gauss map of a Lagrangian submanifold $\gamma : \Sigma \mapsto U(n)/SO(n)$ assigns to each $x \in \Sigma$ the tangent space at x, $T_x\Sigma$.

A particular subset of the Lagrangian Grassmannian consists of the graphs of any symmetric linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. These can be considered as Lagrangians defined by the gradient of quadratic polynomials on $\mathbb{R}^n$.

For any $K > 0$, let $\mathfrak{B}_K$ to be the subset of the Lagrangian Grassmannian consisting of graphs of symmetric linear transformations $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ with eigenvalues $|\lambda_i| \leq K$ for each i. We remark that if the Gauss map of Σ lies in $\mathfrak{B}_K$ then Σ is the graph of $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ with uniformly bounded $|df|$.

Theorem A *Denote by Ξ the subset of the Lagrangian Grassmannian consisting of graphs of symmetric linear transformations $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ with eigenvalues $\lambda_i \lambda_j \geq -1$ for any i, j. Let Σ be a complete minimal Lagrangian submanifold of $\mathbb{C}^n$. Suppose there exists an element $g \in U(n)$ such that the image of the Gauss map of $g(\Sigma)$ lies in $\Xi \cap \mathfrak{B}_K$, then Σ is an affine space.*

We remark that the gradient of $g(\Sigma)$ is not necessarily bounded.

Indeed, the most general theorem of this type is the following.

Let $\mathfrak{M}$ be the set of graphs of all symmetric linear transformation $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ whose eigenvalues (λ_i) satisfy the following two conditions:

1. $$F(h_{ijk}) = \sum_{i,j,k} h_{ijk}^2 + \sum_{k,i} \lambda_i^2 h_{iik}^2 + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{ijk}^2 \geq 0$$

   for any trace-free symmetric three tensor h_{ijk}.

2. $$F(h_{ijk}) = 0$$

   if and only if $h_{ijk} = 0$ for all i, j, k.

Here h_{ijk} is any element in $\otimes^3 \mathbb{R}^n$ that is symmetric in i, j and k. h_{ijk} being trace-free means $\sum_{i=1}^n h_{iik} = 0$ for any k. In fact, h_{ijk} corresponds to the second fundamental form of a Lagrangian submanifold. The trace-free condition corresponds to vanishing mean curvature vector. It is clear that Ξ is a subset of $\mathfrak{M}$.

Theorem B *The conclusion for Theorem A holds for* $\mathfrak{M}_K$ *, the subset of the Lagrangian Grassmannian consisting of graphs of symmetric linear transformations in* $\mathfrak{M} \cap \mathfrak{B}_K$.

These theorems are proved by maximum principle. When Σ is the graph over a Lagrangian subspace L, we calculate the Laplacian of $\ln *\Omega$ where $*\Omega$ is the Jacobian of the projection from Σ to L. This is a positive function and when the Gauss map of Σ satisfies the above conditions it is indeed superharmonic. The parabolic version of this equation was first derived in [9] in the study of higher co-dimension mean curvature flow.

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2 Proof of Theorem

Let Σ be a complete submanifold of $\mathbb{R}^{2n}$. Around any point $p \in \Sigma$, we choose orthonormal frames $\{e_i\}_{i=1 \cdots n}$ for $T\Sigma$ and $\{e_\alpha\}_{\alpha=n+1,\cdots,2n}$ for $N\Sigma$, the normal bundle of Σ . The convention that $i, j, k, \cdots$ denote tangent indexes and $\alpha, \beta, \gamma \cdots$ denote normal indexes is followed.

The second fundamental form of Σ is denoted by $h_{\alpha ij} =< \nabla_{e_i} e_j, e_\alpha >$.

The following formula was essentially derived in [9]. To apply to the current situation, we note that minimal submanifold corresponds to stationary phase of mean curvature flow.

Proposition 2.1 *Let Σ be a the graph of $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ and (λ_i) be the eigenvalues of $\sqrt{(df)^T df}$. If Σ is a minimal submanifold, then $*\Omega = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}}$ satisfies the following equation.*

$$\Delta * \Omega = - * \Omega \{ \sum_{\alpha,i,k} h_{\alpha ik}^2 - 2 \sum_{k,i<j} \lambda_i \lambda_j h_{n+i,ik} h_{n+j,jk} + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{n+j,ik} h_{n+i,jk} \} \tag{2.1}$$

where Δ is the Laplace operator of the induced metric on Σ.

Geometrically, $*\Omega$ is the Jacobian of the projection from Σ to the domain $\mathbb{R}^n$.

Proof of Theorem A. First we show if the Gauss map of Σ lies in $\Xi \cap \mathfrak{B}_K$, then Σ is an affine space. The general case follows from the following observation: if $g \in U(n)$ then $g(\Sigma)$ is again a minimal Lagrangian submanifold.

We rewrite equation (2.1) in the Lagrangian case. Hence the tangent bundle is canonically isomorphic to the normal bundle by the complex structure J. We define

$$h_{ijk} =< \nabla_{e_i} e_j, J(e_k) >$$

then h_{ijk} is symmetric in i, j and k.

The Lagrangian condition also implies $< df(X), J(Y) >$ is symmetric in X, Y. We can find an orthonormal basis $\{e_i\}_{i=1\cdots n}$ for $T_p\Sigma$ so that $df(e_i) = \lambda_i J(e_i)$ and $\{J(e_i)\}_{i=1\cdots n}$ becomes an orthonormal basis for the normal bundle. Equation (2.1) becomes

$$\Delta * \Omega = - * \Omega \{ \sum_{i,j,k} h_{ijk}^2 - 2 \sum_{k,i<j} \lambda_i \lambda_j h_{iik} h_{jjk} + 2 \sum_{k,i<j} \lambda_i \lambda_j h_{jik} h_{ijk} \} \tag{2.2}$$

We shall calculate

$$\Delta(\ln *\Omega) = \frac{*\Omega\Delta(*\Omega) - |\nabla * \Omega|^2}{| * \Omega|^2} \tag{2.3}$$

The covariant derivative of $*\Omega$ can be calculated as in equation (3.1) in [9].

$$(*\Omega)_k = - * \Omega(\sum_i \lambda_i h_{iik})$$

Plug this and equation (2.2) into equation (2.3) and we obtain

$$\Delta(\ln *\Omega) = -\{\sum_{i,j,k} h_{ijk}^2 + \sum_{k,i} \lambda_i^2 h_{iik}^2 + 2 \sum_{k,i<j} \lambda_i\lambda_j h_{ijk}^2 \} \tag{2.4}$$

If the Gauss map of Σ lies in Ξ, then it is obvious that $\Delta(\ln *\Omega) \leq 0$. The condition $|\lambda_i| \leq K$ means Σ is the graph of a vector-valued function with bounded gradient. $\sum_{i,j,k} h_{ijk}^2 + \sum_{k,i} \lambda_i^2 h_{iik}^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{ijk}^2 = 0$ forces $h_{ijk} = 0$ for any i, j, k by symmetry consideration. This immediately implies any minimal Lagrangian cone satisfies the assumption of the theorem is flat by maximum principle. For the general case, we can apply the standard blow-down and Allard regularity theorem to conclude Σ is totally geodesic and thus an affine space.

If Σ is minimal Lagrangian, so is any $g(\Sigma)$ for $g \in U(n)$. This is because $U(n)$ is contained in the isometry group of $\mathbb{C}^n$ and it preserves the standard Kähler form. This completes the proof of Theorem A.

□

Proof of Theorem B.

This follows immediately from the definition of the set Ξ'_K and equation (2.4). Because Σ is minimal, we only need to consider trace-free h_{ijk}.

□

In the rest of the paper, we identify another region of the Lagrangian Grassmannian where the Bernstein-type theorem also applies. This is not as general as the region in Theorem B. However, we expect it will provide a better estimate in future application.

Theorem C *The conclusion for Theorem A holds for $\Xi' \cap \mathfrak{B}_K$ where Ξ' is the subset of Lagrangian Grassmannian consisting of graphs of symmetric linear transformations $L : \mathbb{R}^n \mapsto \mathbb{R}^n$ with eigenvalues $\lambda_i\lambda_j + \lambda_i\lambda_k + \lambda_j\lambda_k \geq 0$ for any pairwise distinct i, j, k.*

Proof of Theorem C.

We rewrite the right hand side of equation (2.4).

$$\begin{aligned}
&-\{\sum_{i,j,k} h_{ijk}^2 + \sum_{k,i} \lambda_i^2 h_{iik}^2 + 2\sum_{k,i<j} \lambda_i\lambda_j h_{ijk}^2\ \} \\
=&-\{\sum_{i,j,k} h_{ijk}^2 + \sum_{i} \lambda_i^2 h_{iii}^2 + \sum_{i<k}(\lambda_i^2 + 2\lambda_i\lambda_k)h_{iik}^2 + \sum_{i>k}(\lambda_i^2 + 2\lambda_i\lambda_k)h_{iik}^2 \\
&+ 2\sum_{i<j<k}(\lambda_i\lambda_j + \lambda_j\lambda_k + \lambda_k\lambda_i))h_{ijk}^2\}
\end{aligned} \tag{2.5}$$

Since Σ is minimal, the mean curvature vector $\sum_{i=1}^n h_{iik} = 0$ for each k , we have

$$\begin{aligned}
&\sum_{i} \lambda_i^2 h_{iii}^2 \\
=&\sum_{i<j} \lambda_i^2 h_{ijj}^2 + \sum_{i>j} \lambda_i^2 h_{ijj}^2 + 2\sum_{i\neq j, i\neq l, j<l} \lambda_i^2 h_{ijj}h_{ill} \\
=&\sum_{k<i} \lambda_k^2 h_{kii}^2 + \sum_{k>i} \lambda_k^2 h_{kii}^2 + 2\sum_{i\neq j, i\neq l, j<l} \lambda_i^2 h_{ijj}h_{ill}
\end{aligned} \tag{2.6}$$

Plug equations (2.6) into (2.5),

$$\begin{aligned}
&\Delta(\ln *\Omega) \\
=&-\{\sum_{i,j,k} h_{ijk}^2 + 2\sum_{i\neq j, i\neq l, j<l} \lambda_i^2 h_{ijj}h_{ill} + \sum_{i<k}(\lambda_i^2 + 2\lambda_i\lambda_k + \lambda_k^2)h_{iik}^2 + \\
&\sum_{i>k}(\lambda_i^2 + 2\lambda_i\lambda_k + \lambda_k^2)h_{iik}^2 + 2\sum_{i<j<k}(\lambda_i\lambda_j + \lambda_j\lambda_k + \lambda_k\lambda_i))h_{ijk}^2\} \\
=&-\{\sum_{i,j,k} h_{ijk}^2 + 2\sum_{i\neq j, i\neq l, j<l} \lambda_i^2 h_{ijj}h_{ill} + \sum_{p\neq q}(\lambda_p + \lambda_q)^2 h_{pqq}^2 + \\
&+ 2\sum_{i<j<k}(\lambda_i\lambda_j + \lambda_j\lambda_k + \lambda_k\lambda_i)h_{ijk}^2\ \}
\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&2\sum_{i\neq j,i\neq l,j<l}\lambda_i^2 h_{ijj}h_{ill}+\sum_{p\neq q}(\lambda_p+\lambda_q)^2h_{pqq}^2\\
&\geq \sum_{i\neq j,i\neq l,j<l} 2\lambda_i^2 h_{ijj}h_{ill}+(\lambda_i+\lambda_l)^2h_{ill}^2+(\lambda_i+\lambda_j)^2h_{ijj}
\end{aligned}
\tag{2.8}
$$

If $j\neq i\neq l$ and $\lambda_i\lambda_j+\lambda_j\lambda_l+\lambda_l\lambda_i\geq 0$ then

$$2\lambda_i^2 h_{ijj}h_{ill}+(\lambda_i+\lambda_l)^2h_{ill}^2+(\lambda_i+\lambda_j)^2h_{ijj}\geq 0.$$

Thus $\Delta(\ln *\Omega)\leq -|A|^2$. The rest is identical to that of Theorem A and B.

□

We remark the condition $\lambda_i\lambda_j+\lambda_i\lambda_k+\lambda_j\lambda_k\geq 0$ for any pairwise distinct i,j,k is void in two-dimension. Indeed, this is true even without the Lagrangian assumption and hence rediscover the results of [1] (see also [3]) that an non-parametric minimal cone of dimension three must be flat. [1]

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[1]After this paper was finished, we were informed by Yu Yuan that he has also derived formula (2.4) from a different point of view. In his paper "A Bernstein problem for special Lagrangian equations", Yu Yuan had the following interesting observation: the linear transformation $(x^i,y^i)\mapsto(\frac{x^i+y^i}{\sqrt{2}},\frac{-x^i+y^i}{\sqrt{2}})$ takes a convex function F to a function $\overline{F}$ with $-I\leq D^2\overline{F}\leq I$. Since this transformation (so called Lewy transformation) is an element of $U(n)$, our Theorem A implies a convex entire solution to equation (1.1) is a quadratic polynomial. This result was also proved in Yu Yuan's paper. We would like to thank Yu Yuan for sending us his preprint before publication .

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