

Long-time Existence and Convergence of Graphic Mean Curvature Flow in Arbitrary Codimension

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December 30, 2000, revised October 4, 2001

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Abstract

Let $f : \Sigma_1 \mapsto \Sigma_2$ be a map between compact Riemannian manifolds of constant curvature. This article considers the evolution of the graph of f in $\Sigma_1 \times \Sigma_2$ by the mean curvature flow. Under suitable conditions on the curvature of Σ_1 and Σ_2 and the differential of the initial map, we show that the flow exists smoothly for all time. At each instant t , the flow remains the graph of a map f_t and f_t converges to a constant map as t approaches infinity. This also provides a regularity estimate for Lipschitz initial data.

1 Introduction

The deformation of maps between Riemannian manifold has been studied for a long time. The idea is to find a natural process to deform a map to a "canonical" one. The harmonic heat flow is probably the most famous example. It is the gradient flow of the energy functional of maps. The classical work of Eells and Sampson [2] proves the flow converges to a harmonic map if the target manifold is of non-positive curvature. However, singularities do occur in the positive curvature case. Such example exists even for maps between two-spheres. In [8], [9], the author proposes the study of a new deformation process given by the mean curvature flow. The idea is to consider

the graph of the map as a submanifold in the product space and evolve it in the direction of its mean curvature vector. This is the gradient flow of the volume functional of graphs and a stationary point is a "minimal map" first introduced by Schoen in [6]. This proposal proves to be quite successful in the surface case. In fact, it provides an analytic proof of Smale's classical theorem of the diffeomorphism group of spheres. This article considers the arbitrary dimension and codimension case. We first prove that the graph condition is preserved and the solution exists for all time.

Theorem A. *Let (Σ_1, g) and (Σ_2, h) be Riemannian manifolds of constant curvature k_1 and k_2 respectively and f be a smooth map from Σ_1 to Σ_2 . Suppose $k_1 \geq |k_2|$. If $\det(g_{ij} + (f^*h)_{ij}) < 2$, the mean curvature flow of the graph of f remains a graph and exists for all time.*

The mean curvature flow for graphs appears to favor positively curved domain manifold. The convergence theorem is the following.

Theorem B. *Let (Σ_1, g) and (Σ_2, h) be Riemannian manifolds of constant curvature k_1 and k_2 respectively and f be a smooth map from Σ_1 to Σ_2 . Suppose $k_1 \geq |k_2|$ and $k_1 + k_2 > 0$. If $\det(g_{ij} + (f^*h)_{ij}) < 2$, then the mean curvature flow of the graph of f converges to the graph of a constant map at infinity.*

The condition $\det(g_{ij} + (f^*h)_{ij}) < 2$ is actually a geometric condition. When Σ_1 and Σ_2 are of the same dimension, it is closely related but slightly stronger than the condition that the Jacobian J_1 of the projection from the graph to Σ_1 be strictly greater than the absolute value of the Jacobian J_2 of the projection from the graph to Σ_2 . The geometric meaning is that we see more of the graph from Σ_1 than from Σ_2 .

An assumption of this type is clearly needed in general. In the two dimensional case, the condition $J_1 > |J_2|$ turns out to be the optimal one in [8]. But in higher dimension, it is not yet clear what would be the optimal condition for the global existence of the flow.

The stability of general gradient flow on Riemannian manifolds with analytic metric was proved by Simon in [7] where the smallness of the second derivative is assumed initially. The assumption in Theorem B is a condition on first derivatives and the proof relies on a curvature estimate which implies regularity for initial data with small Lipschitz norm. Such an estimate for codimension one graphic mean curvature flow was proved by Ecker and Huisken

[1]. A localized version in codimension one case indeed gives an alternative proof of the short time existence of the mean curvature flow.

Theorem A and B are proved in §4. The author would like to thank Professor R. Schoen, Professor L. Simon and Professor S.-T. Yau for their encouragement and advice.

2 Preliminaries

Let $f : \Sigma_1 \mapsto \Sigma_2$ be a smooth map between Riemannian manifolds. The graph of f is an embedded submanifold Σ in $M = \Sigma_1 \times \Sigma_2$. We denote the embedding by $F : \Sigma_1 \mapsto M$, $F = id \times f$. There are isomorphisms $T\Sigma_1 \mapsto T\Sigma$ by $X \mapsto X + df(X)$ and $T\Sigma_2 \mapsto N\Sigma$ by $Y \mapsto Y - (df)^T(Y)$ where $(df)^T : T\Sigma_2 \mapsto T\Sigma_1$ is the adjoint of df .

We assume the mean curvature flow of F can be written as a graph of f for $t \in [0, \epsilon)$ and derive the equation satisfied by f . It is given by a smooth family $F_t : \Sigma_1 \mapsto M$ which satisfies

$$\left(\frac{\partial F}{\partial t}\right)^\perp = H$$

where $(\cdot)^\perp$ denotes the projection onto $N\Sigma$ and H is the mean curvature vector of $F_t(\Sigma_1) = \Sigma_t$. By the definition of the mean curvature vector, this equation is equivalent to

$$\left(\frac{\partial F}{\partial t}\right)^\perp = \left(\Lambda^{ij} \nabla_{\frac{\partial F}{\partial x^i}}^M \frac{\partial F}{\partial x^j}\right)^\perp$$

where Λ^{ij} is the inverse to the induced metric Λ_{ij} on Σ .

$$\Lambda_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle$$

When Σ_1 and Σ_2 are both Euclidean space, $\frac{\partial F}{\partial t}$ and $\nabla_{\frac{\partial F}{\partial x^i}}^M \frac{\partial F}{\partial x^j}$ are both in $T\Sigma_2$. Since the projection to the normal part is an isomorphism when restricted to $T\Sigma_2$,

$$\frac{\partial F}{\partial t} = \Lambda^{ij} \nabla_{\frac{\partial F}{\partial x^i}}^M \frac{\partial F}{\partial x^j}$$

If we write $F_t = id \times f_t$, then $\Lambda_{ij} = g_{ij} + h_{\alpha\beta} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\beta}{\partial x^j} = g_{ij} + (f^*h)_{ij}$. f_t satisfies the following nonlinear partial differential equations.

$$\frac{\partial f^\alpha}{\partial t} = (g_{ij} + h_{\gamma\beta} \frac{\partial f^\gamma}{\partial x^i} \frac{\partial f^\beta}{\partial x^j})^{-1} \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}$$

3 Evolution equation for parallel form

In this section, we calculate the evolution equation of the restriction of a parallel n -form to an n -dimensional submanifold moving by the mean curvature flow.

We assume M is a Riemannian manifold with a parallel n form Ω . Let $F : \Sigma \mapsto M$ be an isometric immersion of an n -dimensional submanifold. We choose orthonormal frames $\{e_i\}_{i=1 \dots n}$ for $T\Sigma$ and $\{e_\alpha\}_{\alpha=n+1, \dots, n+m}$ for $N\Sigma$. The convention that i, j, k, \dots denote tangent indexes and $\alpha, \beta, \gamma \dots$ denote normal indexes is followed.

We first calculate the covariant derivative of the restriction of Ω on Σ .

$$\begin{aligned} & (\nabla_{e_k}^\Sigma \Omega)(e_{i_1}, \dots, e_{i_n}) \\ &= e_k(\Omega(e_{i_1}, \dots, e_{i_n})) - \Omega(\nabla_{e_k}^\Sigma e_{i_1}, \dots, e_{i_n}) - \dots - \Omega(e_{i_1}, \dots, \nabla_{e_k}^\Sigma e_{i_n}) \\ &= \Omega(\nabla_{e_k}^M e_{i_1} - \nabla_{e_k}^\Sigma e_{i_1}, \dots, e_{i_n}) + \dots + \Omega(e_{i_1}, \dots, \nabla_{e_k}^M e_{i_n} - \nabla_{e_k}^\Sigma e_{i_n}) \end{aligned}$$

where we have used $\nabla_{e_k}^M \Omega = 0$ because Ω is parallel. This equation can be abbreviated using the second fundamental form of F , $h_{\alpha ij} = \langle \nabla_{e_i}^M e_j, e_\alpha \rangle$.

$$\Omega_{i_1 \dots i_n, k} = \Omega_{\alpha i_2 \dots i_n} h_{\alpha i_1 k} + \dots + \Omega_{i_1 \dots i_{n-1} \alpha} h_{\alpha i_n k} \quad (3.1)$$

Likewise,

$$\Omega_{\alpha i_2 \dots i_n, k} = -\Omega_{j i_2 \dots i_n} h_{\alpha j k} + \Omega_{\alpha \beta i_3 \dots i_n} h_{\beta i_2 k} + \dots + \Omega_{\alpha i_2 \dots i_{n-1} \beta} h_{\beta i_n k} \quad (3.2)$$

Take the covariant derivative of equation (3.1) with respect to e_k again,

$$\begin{aligned} \Omega_{i_1 \dots i_n, k k} &= \Omega_{\alpha i_2 \dots i_n, k} h_{\alpha i_1 k} + \dots + \Omega_{i_1 \dots i_{n-1} \alpha, k} h_{\alpha i_n k} \\ &\quad + \Omega_{\alpha i_2 \dots i_n} h_{\alpha i_1 k, k} + \dots + \Omega_{i_1 \dots i_{n-1} \alpha} h_{\alpha i_n k, k} \end{aligned} \quad (3.3)$$

Plug equation (3.2) into (3.3) and apply the Codazzi equation $h_{\alpha ki,k} = h_{\alpha,i} + R_{\alpha kki}$ where R is the curvature operator of M . Now we specialize to $i_1 = 1, \dots, i_n = n$.

$$\begin{aligned} (\Delta^\Sigma \Omega)_{1\dots n} &= (-\Omega_{j2\dots n} h_{\alpha jk} + \Omega_{\alpha\beta3\dots n} h_{\beta 2k} + \dots + \Omega_{\alpha 2\dots(n-1)\beta} h_{\beta nk}) h_{\alpha 1k} \\ &\quad + \dots + (\Omega_{\beta 2\dots(n-1)\alpha} h_{\beta 1k} + \dots + \Omega_{1\dots(n-2)\beta\alpha} h_{\beta(n-1)k} - \Omega_{1\dots(n-1)j} h_{\alpha jk}) h_{\alpha nk} \\ &\quad + \Omega_{\alpha 2\dots n} h_{\alpha,1} + \dots + \Omega_{1\dots(n-1)\alpha} h_{\alpha,n} \\ &\quad + \Omega_{\alpha 2\dots n} R_{\alpha k k 1} + \dots + \Omega_{1\dots(n-1)\alpha} R_{\alpha k k n} \end{aligned}$$

where $\Delta^\Sigma \Omega$ is the rough Laplacian, i.e.

$$(\Delta^\Sigma \Omega)_{1\dots n} = (\nabla_{e_k}^\Sigma \nabla_{e_k}^\Sigma \Omega)(e_1, \dots, e_n)$$

Since Σ is of dimension n , after grouping terms we have

$$\begin{aligned} (\Delta^\Sigma \Omega)_{1\dots n} &= -\Omega_{12\dots n} \sum_{\alpha,k} (h_{\alpha 1k}^2 + \dots + h_{\alpha nk}^2) \\ &\quad + 2 \sum_{\alpha,\beta,k} [\Omega_{\alpha\beta3\dots n} h_{\alpha 1k} h_{\beta 2k} + \Omega_{\alpha 2\beta\dots n} h_{\alpha 1k} h_{\beta 3k} + \dots + \Omega_{1\dots(n-2)\alpha\beta} h_{\alpha(n-1)k} h_{\beta nk}] \\ &\quad + \sum_{\alpha,k} \Omega_{\alpha 2\dots n} h_{\alpha,1} + \dots + \Omega_{1\dots(n-1)\alpha} h_{\alpha,n} \\ &\quad + \sum_{\alpha,k} \Omega_{\alpha 2\dots n} R_{\alpha k k 1} + \dots + \Omega_{1\dots(n-1)\alpha} R_{\alpha k k n} \end{aligned} \tag{3.4}$$

We notice that $(\Delta^\Sigma \Omega)_{1\dots n} = \Delta(\Omega(e_1, \dots, e_n))$, where the Δ on the right hand side is the Laplacian of functions on Σ .

The terms in the bracket are formed in the following way. Choose two different indexes from 1 to n , replace the smaller one by α and the larger one by β . There are a total of $\frac{n(n-1)}{2}$ such terms.

Now we consider the mean curvature flow of Σ in M by $\frac{d}{dt} F_t = H_t$. In the following, we shall denote the image of F_t by Σ_t . Notice that here we require the velocity vector is in the normal direction. The evolution equation of $\Omega_{1\dots n}$ can be calculated as the following. We work in a local coordinate $\{\partial_i = \frac{\partial}{\partial x^i}\}$ on Σ , then

$$\begin{aligned}
& \frac{d}{dt} \Omega(\partial_1, \dots, \partial_n) \\
&= \Omega((\nabla_{\partial_1} H)^N, \partial_2, \dots, \partial_n) + \dots + \Omega(\partial_1, \partial_2, \dots, (\nabla_{\partial_n} H)^N) \\
&+ \Omega((\nabla_{\partial_1} H)^T, \partial_2, \dots, \partial_n) + \dots + \Omega(\partial_1, \partial_2, \dots, (\nabla_{\partial_n} H)^T)
\end{aligned}$$

Since $\frac{d}{dt}g_{ij} = \langle (\nabla_{\partial_i} H)^T, \partial_j \rangle$, if we choose a orthonormal frame and evolve the frame with respect to time so that it remains orthonormal, the terms in the last line vanish.

$$\frac{d}{dt} \Omega_{1\dots n} = \Omega_{\alpha 2\dots n} h_{\alpha,1} + \dots + \Omega_{1\dots(n-1)\alpha} h_{\alpha,n}$$

Combine this with equation (3.4) we get the parabolic equation satisfied by $\Omega_{1\dots n}$.

Proposition 3.1 *If Σ_t is an n -dimensional mean curvature flow in M and Ω is a parallel n -form on M . Then $\Omega_{1,\dots,n} = \Omega(e_1, \dots, e_n)$ satisfies*

$$\begin{aligned}
\frac{d}{dt} \Omega_{1\dots n} &= \Delta \Omega_{1\dots n} + \Omega_{1\dots n} \left(\sum_{\alpha,i,k} h_{\alpha ik}^2 \right) \\
&- 2 \sum_{\alpha,\beta,k} [\Omega_{\alpha\beta 3\dots n} h_{\alpha 1k} h_{\beta 2k} + \Omega_{\alpha 2\beta\dots n} h_{\alpha 1k} h_{\beta 3k} + \dots + \Omega_{1\dots(n-2)\alpha\beta} h_{\alpha(n-1)k} h_{\beta nk}] \\
&- \sum_{\alpha,k} [\Omega_{\alpha 2\dots n} R_{\alpha k k 1} + \dots + \Omega_{1\dots(n-1)\alpha} R_{\alpha k k n}]
\end{aligned} \tag{3.5}$$

where Δ denotes the time-dependent Laplacian on Σ_t .

When $M = \Sigma_1 \times \Sigma_2$ is product, the volume form Ω_i of each Σ_i is a parallel form on M . In fact, all the discussions in this paper apply to any locally Riemannian product manifold. At any point p on Σ , choose a oriented orthonormal basis e_1, \dots, e_n for $T_p \Sigma$. Then $\Omega_1(T\Sigma) = \Omega_1(e_1, \dots, e_n) = \Omega_1(\pi_1(e_1), \dots, \pi_1(e_n))$ is the Jacobian of the projection from $T\Sigma$ to $T\Sigma_1$. We shall use $*\Omega_1$ to denote this function as p varies along Σ , here $*$ is simply the Hodge operator with respect to the induced metric. By the implicit function theorem, $*\Omega_1 > 0$ near p if and only if Σ is locally a graph over Σ_1 near p .

In the following, we assume Σ_t is locally a graph over Σ_1 so that $*\Omega_1 > 0$ on Σ_t and find orthonormal bases for the tangent and normal bundle of Σ_t so that we can represent the terms in equation (3.5) in a better form.

First, we need a simple linear algebra lemma. Let $V = V_1 \times V_2$ be a product of inner product spaces V_1 and V_2 of dimension n and m respectively. Let $D : V_1 \mapsto V_2$ be a linear transformation.

Lemma 3.1 *There exist orthonormal bases $\{a_i\}_{i=1\dots n}$ for V_1 and $\{a_\alpha\}_{\alpha=n+1\dots n+m}$ for V_2 such that $\lambda_{i\alpha} = \langle Da_i, a_\alpha \rangle \geq 0$ is diagonal.*

In fact, this is the Singular Value Decomposition and a proof is available in e.g. [5].

It is understood that if $n < m$, then $\lambda_{i\alpha} = 0$ for $\alpha > n$ and if $m < n$, then $\lambda_{i\alpha} = 0$ for $i > m$.

Now let T be the graph of D , i.e. $T = V_1 + D(V_1)$. N denotes the orthogonal complement of T . Let $\pi_i : V_1 \times V_2 \mapsto V_i$ be the projection map. We notice that there are isomorphism $\pi_1|_T : T \mapsto V_1$ and $\pi_2|_N : N \mapsto V_2$.

In the later application, $V_i = T_p \Sigma_i$ and D is given by the $df|_p$, the differential of f at the point p , where f is a locally defined map whose graph represents Σ near p .

Now $\{e_i = \frac{1}{\sqrt{1+\sum_\beta \lambda_{i\beta}^2}}(a_i + \sum_\beta \lambda_{i\beta} a_\beta)\}$ forms an orthonormal basis for T and $\{e_\alpha = \frac{1}{\sqrt{1+\sum_j \lambda_{j\alpha}^2}}(a_\alpha - \sum_j \lambda_{j\alpha} a_j)\}$ an orthonormal basis for N . It is not hard to check that $|\pi_1(e_i)| = \frac{1}{\sqrt{1+\sum_\beta \lambda_{i\beta}^2}}$ and

$$\begin{aligned}\pi_1(e_\alpha) &= -\sum_j \lambda_{j\alpha} \pi_1(e_j) \\ \pi_2(e_i) &= \sum_\beta \lambda_{i\beta} \pi_2(e_\beta)\end{aligned}\tag{3.6}$$

With these bases, we can calculate the terms in equation (3.5) for $\Omega = \Omega_1$. We first calculate the term

$$-2 \sum_{\alpha, \beta, k} [\Omega_{\alpha\beta 3\dots n} h_{\alpha 1k} h_{\beta 2k} + \Omega_{\alpha 2\beta\dots n} h_{\alpha 1k} h_{\beta 3k} + \dots + \Omega_{1\dots(n-2)\alpha\beta} h_{\alpha(n-1)k} h_{\beta nk}]$$

By equation (3.6),

$$\Omega_1(e_\alpha, e_\beta, e_3, \dots, e_n) = \Omega_1(\pi_1(e_\alpha), \pi_1(e_\beta), \pi_1(e_3), \dots, \pi_1(e_n)) = (\lambda_{1\alpha} \lambda_{2\beta} - \lambda_{2\alpha} \lambda_{1\beta}) * \Omega_1$$

Therefore, this term is

$$-2 \left[\sum_{\alpha, \beta, k, i < j} (\lambda_{i\alpha} \lambda_{j\beta} - \lambda_{j\alpha} \lambda_{i\beta}) h_{\alpha ik} h_{\beta jk} \right] * \Omega_1 \quad (3.7)$$

As for the curvature term,

$$\Omega_{\alpha, 2, \dots, n} R_{\alpha k k 1} = \Omega_1(e_\alpha, e_2, \dots, e_n) R(e_\alpha, e_k, e_k, e_1)$$

We assume Σ_i is of constant curvature k_i , therefore

$$\begin{aligned} & R(e_\alpha, e_k, e_k, e_1) \\ &= R_1(\pi_1(e_\alpha), \pi_1(e_k), \pi_1(e_k), \pi_1(e_1)) + R_2(\pi_2(e_\alpha), \pi_2(e_k), \pi_2(e_k), \pi_2(e_1)) \\ &= k_1 [\langle \pi_1(e_\alpha), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_1) \rangle - \langle \pi_1(e_\alpha), \pi_1(e_1) \rangle \langle \pi_1(e_k), \pi_1(e_k) \rangle] \\ &+ k_2 [\langle \pi_2(e_\alpha), \pi_2(e_k) \rangle \langle \pi_2(e_k), \pi_2(e_1) \rangle - \langle \pi_2(e_\alpha), \pi_2(e_1) \rangle \langle \pi_2(e_k), \pi_2(e_k) \rangle] \end{aligned}$$

Notice that

$$\langle X, Y \rangle = \langle \pi_1(X), \pi_1(Y) \rangle + \langle \pi_2(X), \pi_2(Y) \rangle$$

since $T\Sigma_1 \perp T\Sigma_2$.

Therefore the second term is

$$\begin{aligned} & k_2 \sum_k [(-\langle \pi_1(e_\alpha), \pi_1(e_k) \rangle)(\delta_{1k} - \langle \pi_1(e_k), \pi_1(e_1) \rangle) \\ & - (-\langle \pi_1(e_\alpha), \pi_1(e_1) \rangle)(1 - \langle \pi_1(e_k), \pi_1(e_k) \rangle)] \\ &= k_2 \sum_k [\langle \pi_1(e_\alpha), \pi_1(e_1) \rangle - \delta_{1k} \langle \pi_1(e_\alpha), \pi_1(e_k) \rangle \\ & + \langle \pi_1(e_\alpha), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_1) \rangle - \langle \pi_1(e_\alpha), \pi_1(e_1) \rangle \langle \pi_1(e_k), \pi_1(e_k) \rangle] \end{aligned}$$

Plug in $\pi_1(e_\alpha) = -\lambda_{j\alpha} \pi_1(e_j)$. We get

$$\begin{aligned} & \sum_{\alpha, k} \Omega_{\alpha 2 \dots n} R_{\alpha k k 1} \\ &= \sum_{\alpha, j, k} \lambda_{1\alpha} \lambda_{j\alpha} \{ k_1 [\langle \pi_1(e_j), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_1) \rangle - \langle \pi_1(e_j), \pi_1(e_1) \rangle |\pi_1(e_k)|^2] \\ & + k_2 [\langle \pi_1(e_j), \pi_1(e_1) \rangle - \delta_{1k} \langle \pi_1(e_j), \pi_1(e_k) \rangle \\ & + \langle \pi_1(e_j), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_1) \rangle - \langle \pi_1(e_j), \pi_1(e_1) \rangle |\pi_1(e_k)|^2] \} * \Omega_1 \end{aligned}$$

Similarly we can write down other terms and the curvature term becomes

$$\begin{aligned}
& - \sum_{\alpha,i,j,k} \lambda_{i\alpha} \lambda_{j\alpha} \{k_1[\langle \pi_1(e_j), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_i) \rangle - \langle \pi_1(e_j), \pi_1(e_i) \rangle |\pi_1(e_k)|^2] \\
& \quad + k_2[\langle \pi_1(e_j), \pi_1(e_i) \rangle - \delta_{ik} \langle \pi_1(e_j), \pi_1(e_k) \rangle \\
& \quad + \langle \pi_1(e_j), \pi_1(e_k) \rangle \langle \pi_1(e_k), \pi_1(e_i) \rangle - \langle \pi_1(e_j), \pi_1(e_i) \rangle |\pi_1(e_k)|^2]\} * \Omega_1
\end{aligned}$$

Notice that $\sum_{\alpha} \lambda_{i\alpha} \lambda_{j\alpha} \neq 0$ only if $i = j$. Let $\sum_{\alpha} \lambda_{i\alpha} \lambda_{j\alpha} = \delta_{ij} \lambda_i^2$ with $\lambda_i \geq 0$.

We can rearrange the basis $\{a_{\alpha}\}$ so that

$$\lambda_{i\alpha} = \delta_{\alpha, n+i} \lambda_i$$

If we represent Σ_t locally as the graph of a locally defined map f_t , then λ_i 's are in fact the eigenvalues of $\sqrt{(df_t)^T df_t}$.

$$\begin{aligned}
& \sum_{\alpha,i,k} \lambda_{i\alpha}^2 \{k_1[|\pi_1(e_i)|^2 |\pi_1(e_k)|^2 - \langle \pi_1(e_i), \pi_1(e_k) \rangle^2] \\
& + k_2[|\pi_1(e_i)|^2 |\pi_1(e_k)|^2 - \langle \pi_1(e_i), \pi_1(e_k) \rangle^2 + \delta_{ik} \langle \pi_1(e_i), \pi_1(e_k) \rangle - |\pi_1(e_i)|^2]\} * \Omega_1
\end{aligned}$$

By the choice of $\{e_i\}$, $\langle \pi_1(e_i), \pi_1(e_k) \rangle = 0$ unless $i = k$, and if we write $\sum_i |\pi_1(e_i)|^2 = |\pi_1|^2$, then the term is

$$\sum_{\alpha,i} \lambda_{i\alpha}^2 |\pi_1(e_i)|^2 \{k_1(|\pi_1|^2 - |\pi_1(e_i)|^2) + k_2(|\pi_1|^2 - |\pi_1(e_i)|^2 + 1 - n)\} * \Omega_1$$

Now we shall write the equation for $*\Omega$ in terms of λ_i and the second fundamental form. Notice that $|\pi_1(e_i)|^2 = \frac{1}{1+\lambda_i^2}$.

Proposition 3.2 *Suppose $M = \Sigma_1 \times \Sigma_2$ and Σ_i is a Riemannian manifold of constant curvature k_i , $i = 1, 2$. The volume form of Σ_i is denoted by Ω_i . Let $F_0 : \Sigma \hookrightarrow M$ be an embedding such that Σ is locally a graph over Σ_1 . If each Σ_t is locally a graph over Σ_1 along the mean curvature flow of F_0 for $t \in [0, \epsilon)$, then $*\Omega = *\Omega_1$ satisfies the following equation.*

$$\begin{aligned}
& \frac{d}{dt} * \Omega = \Delta * \Omega + *\Omega \left\{ \sum_{\alpha,i,k} h_{\alpha ik}^2 - 2 \sum_{k,i < j} \lambda_i \lambda_j h_{n+i,ik} h_{n+j,jk} + 2 \sum_{k,i < j} \lambda_i \lambda_j h_{n+j,ik} h_{n+i,jk} \right. \\
& \left. + \sum_i \frac{\lambda_i^2}{1+\lambda_i^2} \left[k_1 \left(\sum_{j \neq i} \frac{1}{1+\lambda_j^2} \right) + k_2 \left(1 - n + \sum_{j \neq i} \frac{1}{1+\lambda_j^2} \right) \right] \right\}
\end{aligned} \tag{3.8}$$

where λ_i 's are the eigenvalues of $\sqrt{(df_t)^T df_t}$ and f_t is a locally defined map whose graph represents Σ_t locally.

It is understood that in case $n > m$, we pretend $h_{n+i,jk} = 0$ for $m < i \leq n$.

If Σ_t is indeed the graph of a map f_t , then λ_i 's are the eigenvalues of $\sqrt{(df_t)^T df_t}$. As a comparison with the harmonic heat flow, $*\Omega = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}}$ and the energy density of f is $|df|^2 = \sum_{i=1}^n \lambda_i^2$. A lower bound of $*\Omega$ implies an upper bound for $|df|^2$.

When $k_1 = k_2 = c$, the equation becomes.

$$\begin{aligned} \frac{d}{dt} * \Omega &= \Delta * \Omega + * \Omega \left\{ \sum_{\alpha,i,k} h_{\alpha ik}^2 - 2 \sum_{k,i < j} \lambda_i \lambda_j h_{n+i,ik} h_{n+j,jk} + 2 \sum_{k,i < j} \lambda_i \lambda_j h_{n+j,ik} h_{n+i,jk} \right. \\ &+ c \sum_i \frac{\lambda_i^2}{1 + \lambda_i^2} \left[\left(\sum_{j \neq i} \frac{2}{1 + \lambda_j^2} \right) + 1 - n \right] \left. \right\} \end{aligned} \quad (3.9)$$

4 Long time existence and Convergence

In this section, we consider the mean curvature flow of Σ in $M = \Sigma_1 \times \Sigma_2$ in the case when Σ is the graph of $f : \Sigma_1 \mapsto \Sigma_2$. In particular, we prove the long time existence and convergence, assuming that $\det(g_{ij} + (f^*h)_{ij})$ is less than 2 initially. In our notation, $\det(g_{ij} + (f^*h)_{ij}) = \det(\delta_{ij} + \langle f_*(a_i), f_*(a_j) \rangle_h)$, where $\{a_i\}$ is any orthonormal basis for (Σ_1, g) .

Let us explain the hypothesis $\det(g_{ij} + (f^*h)_{ij}) < 2$ in more detail when Σ_1 and Σ_2 are both two-dimensional surfaces. As remarked in §1, this condition is equivalent to the Jacobian J_1 of the projection from Σ onto Σ_1 is greater than $\frac{1}{\sqrt{2}}$ and is slightly stronger than $J_1 > |J_2|$.

By the Singular Value Decomposition at any point $p \in \Sigma_1$, we can choose orthonormal bases $\{a_1, a_2\}$ for $T_p \Sigma_1$ and $\{a_3, a_4\}$ for $T_{f(p)} \Sigma_2$ such that $df|_p(a_1) = \lambda_1 a_3$ and $df|_p(a_2) = \lambda_2 a_4$. Then $\det(g_{ij} + (f^*h)_{ij}) = (1 + \lambda_1^2)(1 + \lambda_2^2)$. Let Ω_1 and Ω_2 be the volume form of Σ_1 and Σ_2 respectively. They can be extended as parallel forms on $\Sigma_1 \times \Sigma_2$. We also have the projections $\pi_1 : \Sigma_1 \times \Sigma_2 \mapsto \Sigma_1$ and $\pi_2 : \Sigma_1 \times \Sigma_2 \mapsto \Sigma_2$. At any point $(p, f(p)) \in \Sigma$ and any orthonormal basis $\{e_1, e_2\}$ for $T_{(p,f(p))} \Sigma$, $\Omega_1(e_1, e_2)$ is the Jacobian of $\pi_1|_\Sigma$, the restriction of π_1 to Σ , and $\Omega_2(e_1, e_2)$ is the Jacobian of $\pi_2|_\Sigma$. Now

we can take the orthonormal basis for $T_{(p,f(p))}\Sigma$ to consist of

$$e_1 = \frac{1}{\sqrt{1+\lambda_1^2}}(a_1 + \lambda_1 a_3), \quad e_2 = \frac{1}{\sqrt{1+\lambda_2^2}}(a_2 + \lambda_2 a_4)$$

Since $\Omega_1 = a_1^* \wedge a_2^*$ and $\Omega_2 = a_3^* \wedge a_4^*$, we have $\Omega_1(e_1, e_2) = \frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}$ and $\Omega_2(e_1, e_2) = \frac{\lambda_1 \lambda_2}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}}$. The assumption $(1 + \lambda_1^2)(1 + \lambda_2^2) < 2$ is equivalent to

$$\Omega_1(e_1, e_2) > \frac{1}{\sqrt{2}} \quad (4.1)$$

Taking into account of the fact that $(\frac{1}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}})^2 + (\frac{\lambda_1 \lambda_2}{\sqrt{(1+\lambda_1^2)(1+\lambda_2^2)}})^2 \leq 1$, (4.1) implies the weaker condition

$$\Omega_1(e_1, e_2) > |\Omega_2(e_1, e_2)|$$

In fact the condition $\Omega_1(e_1, e_2) > |\Omega_2(e_1, e_2)|$ is equivalent to Σ being symplectic with respect to both symplectic forms $\Omega_1 + \Omega_2$ and $\Omega_1 - \Omega_2$. This is exactly the assumption in [8] where we prove the global existence and convergence in the two-dimensional case. First we prove Theorem A.

Theorem A. *Let (Σ_1, g) and (Σ_2, h) be Riemannian manifolds of constant curvature k_1 and k_2 respectively and f be a smooth map from Σ_1 to Σ_2 . Suppose $k_1 \geq |k_2|$. If $\det(g_{ij} + (f^*h)_{ij}) < 2$, the mean curvature flow of the graph of f remains a graph and exists for all time.*

Proof. Following the notation in the previous section with $\Omega = \Omega_1$. It is not hard to see that $*\Omega = \frac{1}{\sqrt{\det(g_{ij} + (f^*h)_{ij})}} = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}}$ and the assumption implies $\prod_{i=1}^n (1+\lambda_i^2) \leq 2 - \delta$ for some $\delta > 0$, and in particular $\sum_{i=1}^n \lambda_i^2 \leq 1 - \delta$.

Now we take a look at the quadratic terms of the second fundamental form in equation (3.8). First we assume $n \leq m$, so $n + m \geq 2n$. The tangent indices i, j, k run from 1 to n and the normal index α runs from $n + 1$ to $n + m$ unless they are specified otherwise. We divide $\sum h_{\alpha ik}^2$ into two parts:

$$\sum_{\alpha, i, k} h_{\alpha ik}^2 = \sum_{n+1 \leq \alpha \leq 2n, i, k} h_{\alpha ik}^2 + \sum_{2n < \alpha, i, k} h_{\alpha ik}^2$$

In the first summand, write $\alpha = n + j$ then j runs from 1 to n , therefore

$$\sum_{n+1 \leq \alpha \leq 2n, i, k} h_{\alpha ik}^2 = \sum_{j, i, k} h_{n+j, i, k}^2 = \sum_{i < j, k} (h_{n+i, j, k}^2 + h_{n+j, i, k}^2) + \sum_{i, k} h_{n+i, i, k}^2$$

The quadratic terms of the second fundamental form in equation (3.8) become.

$$\begin{aligned} & \sum_{\alpha, i, k} h_{\alpha ik}^2 - 2 \sum_{i < j, k} \lambda_i \lambda_j h_{n+i, ik} h_{n+j, jk} + 2 \sum_{i < j, k} \lambda_j \lambda_i h_{n+j, ik} h_{n+i, jk} \\ &= \delta |A|^2 + (1 - \delta) \sum_{\alpha > 2n, i, k} h_{\alpha, ik}^2 + (1 - \delta) \sum_{i, k} h_{n+i, ik}^2 + (1 - \delta) \sum_{i < j, k} (h_{n+i, jk}^2 + h_{n+j, ik}^2) \\ & - 2 \sum_{i < j, k} \lambda_i \lambda_j h_{n+i, ik} h_{n+j, jk} + 2 \sum_{i < j, k} \lambda_j \lambda_i h_{n+j, ik} h_{n+i, jk} \\ & \geq \delta |A|^2 + (1 - \delta) \sum_{\alpha > 2n, i, k} h_{\alpha, ik}^2 + \left(\sum_{i, k} h_{n+i, ik}^2 \right) \left(\sum_i \lambda_i^2 \right) + (1 - \delta) \sum_{i < j, k} (h_{n+i, jk}^2 + h_{n+j, ik}^2) \\ & - 2 \sum_{i < j, k} \lambda_i \lambda_j h_{n+i, ik} h_{n+j, jk} - 2(1 - \delta) \sum_{i < j, k} |h_{n+j, ik} h_{n+i, jk}| \end{aligned}$$

where we have used $\sum_i \lambda_i^2 \leq 1 - \delta$ and $|\lambda_i \lambda_j| \leq 1 - \delta$.

Drop the non-negative term $(1 - \delta) \sum_{\alpha > 2n, i, k} h_{\alpha, ik}^2$ and the last expression is no less than

$$\begin{aligned} & \delta |A|^2 + \left(\sum_i \lambda_i h_{n+i, ik} \right)^2 - 2 \sum_{i < j, k} \lambda_i \lambda_j h_{n+i, ik} h_{n+j, jk} + (1 - \delta) \sum_{i < j, k} (|h_{n+i, jk}| - |h_{n+j, ik}|)^2 \\ & \geq \delta |A|^2 + \sum_i \lambda_i^2 h_{n+i, ik}^2 + (1 - \delta) \sum_{i < j, k} (|h_{n+i, jk}| - |h_{n+j, ik}|)^2 \end{aligned}$$

which is non-negative.

If $n > m$, since $h_{n+i,jk} = 0$ for $m < i \leq n$, the quadratic terms become

$$\begin{aligned}
& \sum_{\alpha,i,k} h_{\alpha ik}^2 - 2 \sum_{1 \leq i < j \leq m,k} \lambda_i \lambda_j h_{n+i,ik} h_{n+j,jk} + 2 \sum_{1 \leq i < j \leq m,k} \lambda_j \lambda_i h_{n+j,ik} h_{n+i,jk} \\
&= \sum_{\alpha, m < i \leq n, k} h_{\alpha, ik}^2 + \sum_{\alpha, 1 \leq i \leq m, k} h_{\alpha, ik}^2 \\
&\quad - 2 \sum_{1 \leq i < j \leq m, k} \lambda_i \lambda_j h_{n+i, ik} h_{n+j, jk} + 2 \sum_{1 \leq i < j \leq m, k} \lambda_j \lambda_i h_{n+j, ik} h_{n+i, jk} \\
&= \sum_{\alpha, m < i \leq n, k} h_{\alpha, ik}^2 + \sum_{1 \leq i \leq m, k} h_{n+i, ik}^2 + \sum_{1 \leq i < j \leq m, k} h_{n+i, jk}^2 + \sum_{1 \leq i < j \leq m, k} h_{n+j, ik}^2 \\
&\quad - 2 \sum_{1 \leq i < j \leq m, k} \lambda_i \lambda_j h_{n+i, ik} h_{n+j, jk} + 2 \sum_{1 \leq i < j \leq m, k} \lambda_j \lambda_i h_{n+j, ik} h_{n+i, jk}
\end{aligned}$$

By a similar argument, this term is non-negative and bounded below by $\delta|A|^2$.

As for the curvature term, for each i we have

$$\begin{aligned}
& k_1 \left(\sum_{j \neq i} \frac{1}{1 + \lambda_j^2} \right) + k_2 \left(1 - n + \sum_{j \neq i} \frac{1}{1 + \lambda_j^2} \right) \\
&= (k_1 + k_2) \left(\sum_{j \neq i} \frac{1}{1 + \lambda_j^2} \right) + k_2 (1 - n) \\
&\geq \frac{(k_1 - k_2)}{2} (n - 1) + (k_1 + k_2) \left[\left(\sum_{j \neq i} \frac{1}{1 + \lambda_j^2} \right) - \frac{n - 1}{2} \right]
\end{aligned} \tag{4.2}$$

Because each λ_j^2 is less than 1 and $k_1 \geq |k_2|$, this term is nonnegative. When $k_1 + k_2 > 0$, this term is indeed strictly positive.

By Proposition 3.2 and the previous paragraph, $*\Omega$ satisfies the differential inequality.

$$\frac{d}{dt} *\Omega \geq \Delta * \Omega + \delta|A|^2 \tag{4.3}$$

According to the maximum principle for parabolic equations, $\min_{\Sigma_t} *\Omega$ is nondecreasing in time. In particular, $*\Omega$ has a positive lower bound. Since $*\Omega$ is the Jacobian of the projection map from Σ_t to Σ_1 , by the implicit function theorem, this implies Σ_t remains the graph of a map $f_t : \Sigma_1 \mapsto \Sigma_2$ whenever the flow exists.

Now we isometrically embed $M = \Sigma_1 \times \Sigma_2$ into \mathbb{R}^N . The mean curvature flow equation in terms of the coordinate function $F(x, t)$ in \mathbb{R}^N becomes

$$\frac{d}{dt}F(x, t) = H = \overline{H} + E$$

where $H \in TM/T\Sigma$ is the mean curvature vector of Σ_t in M and $\overline{H} \in T\mathbb{R}^N/T\Sigma$ is the mean curvature vector of Σ_t in \mathbb{R}^N .

To detect a possible singularity at (y_0, t_0) , recall the (n-dimensional) backward heat kernel ρ_{y_0, t_0} at (y_0, t_0) introduced by Huisken [3].

$$\rho_{y_0, t_0}(y, t) = \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} \exp\left(\frac{-|y - y_0|^2}{4(t_0 - t)}\right)$$

The monotonicity formula of Huisken asserts $\lim_{t \rightarrow t_0} \int \rho_{y_0, t_0} d\mu_t$ exists. ρ_{y_0, t_0} satisfies the following backward heat equation derived in [8] along the mean curvature flow. Here ∇ and Δ represent the covariant derivative and the Laplace operator of the induced metric on Σ_t respectively.

$$\frac{d}{dt}\rho_{y_0, t_0} = -\Delta\rho_{y_0, t_0} - \rho_{y_0, t_0}\left(\frac{|F^\perp|^2}{4(t_0 - t)^2} + \frac{F^\perp \cdot \overline{H}}{t_0 - t} + \frac{F^\perp \cdot E}{2(t_0 - t)}\right) \quad (4.4)$$

where F^\perp is the component of $F \in T\mathbb{R}^N$ in $T\mathbb{R}^N/T\Sigma_t$.

Recall that

$$\frac{d}{dt}d\mu_t = -|H|^2 d\mu_t = -\overline{H} \cdot (\overline{H} + E) d\mu_t$$

Combine this equation with equations (4.3) and (4.4), we get

$$\begin{aligned} & \frac{d}{dt} \int (1 - *\Omega)\rho_{y_0, t_0} d\mu_t \\ & \leq \int [\Delta(1 - *\Omega) - \delta|A|^2]\rho_{y_0, t_0} d\mu_t \\ & \quad - \int (1 - *\Omega)\left[\Delta\rho_{y_0, t_0} + \rho_{y_0, t_0}\left(\frac{|F^\perp|^2}{4(t_0 - t)^2} + \frac{F^\perp \cdot \overline{H}}{t_0 - t} + \frac{F^\perp \cdot E}{2(t_0 - t)}\right)\right] d\mu_t \\ & \quad - \int (1 - *\Omega)[\overline{H}(\overline{H} + E)]\rho_{y_0, t_0} d\mu_t \end{aligned} \quad (4.5)$$

By rearranging terms, the right hand side can be written as

$$\begin{aligned} & \int [\Delta(1 - *\Omega)\rho_{y_0, t_0} - (1 - *\Omega)\Delta\rho_{y_0, t_0}] d\mu_t - \delta \int |A|^2 \rho_{y_0, t_0} d\mu_t \\ & - \int (1 - *\Omega)\rho_{y_0, t_0} \left[\frac{|F^\perp|^2}{4(t_0 - t)^2} + \frac{F^\perp \cdot \overline{H}}{t_0 - t} + \frac{F^\perp \cdot E}{2(t_0 - t)} + |\overline{H}|^2 + \overline{H} \cdot E \right] d\mu_t \end{aligned}$$

The first term vanishes by integration by parts and the third term can be completed square. Therefore

$$\begin{aligned} & \frac{d}{dt} \int (1 - *\Omega)\rho_{y_0, t_0} d\mu_t \\ & \leq -\delta \int |A|^2 \rho_{y_0, t_0} d\mu_t - \int (1 - *\Omega)\rho_{y_0, t_0} \left| \frac{|F^\perp|}{2(t_0 - t)} + \overline{H} + \frac{E}{2} \right|^2 d\mu_t \\ & \quad + \int (1 - *\Omega)\rho_{y_0, t_0} \left| \frac{E}{2} \right|^2 d\mu_t \end{aligned}$$

Since E is bounded and $\int (1 - *\Omega)\rho_{y_0, t_0} d\mu_t \leq \int \rho_{y_0, t_0} d\mu_t$ is finite, this implies

$$\begin{aligned} & \frac{d}{dt} \int (1 - *\Omega)\rho_{y_0, t_0} d\mu_t \\ & \leq C - \delta \int |A|^2 \rho_{y_0, t_0} d\mu_t \end{aligned}$$

for some constant C . From this we see that $\lim_{t \rightarrow t_0} \int (1 - *\Omega)\rho_{y_0, t_0} d\mu_t$ exists.

For $\lambda > 1$, the parabolic dilation D_λ at (y_0, t_0) is defined by

$$\begin{aligned} D_\lambda : \mathbb{R}^N \times [0, t_0] & \rightarrow \mathbb{R}^N \times [-\lambda^2 t_0, 0] \\ (y, t) & \rightarrow (\lambda(y - y_0), \lambda^2(t - t_0)) \end{aligned} \tag{4.6}$$

Let $\mathcal{S} \subset \mathbb{R}^N \times [0, t_0]$ be the total space of the mean curvature flow, we shall study the flow $\mathcal{S}^\lambda = D^\lambda(\mathcal{S}) \subset \mathbb{R}^N \times [-\lambda^2 t_0, 0]$. Denote the new time parameter by s , then $t = t_0 + \frac{s}{\lambda^2}$. Let $d\mu_s^\lambda$ denote the induced volume form on Σ by $F_s^\lambda = \lambda F_{t_0 + \frac{s}{\lambda^2}}$. The image of F_s^λ is the s -slice of \mathcal{S}^λ and is denoted by Σ_s^λ . Therefore,

$$\begin{aligned} & \frac{d}{ds} \int (1 - *\Omega)\rho_{0,0} d\mu_s^\lambda \\ & = \frac{1}{\lambda^2} \frac{d}{dt} \int (1 - *\Omega)\rho_{y_0, t_0} d\mu_t \\ & \leq \frac{C}{\lambda^2} - \frac{\delta}{\lambda^2} \int \rho_{y_0, t_0} |A|^2 d\mu_t \end{aligned}$$

Notice that $*\Omega$ is a invariant under the parabolic dilation. It is not hard to check that

$$\frac{1}{\lambda^2} \int \rho_{y_0, t_0} |A|^2 d\mu_t = \int \rho_{0,0} |A|^2 d\mu_s^\lambda$$

This is because $\rho_{y_0, t_0} d\mu_t$ is invariant under the parabolic scaling and the norm of second fundamental form scales like the inverse of the distance.

Therefore

$$\begin{aligned} & \frac{d}{ds} \int (1 - *\Omega) \rho_{0,0} d\mu_s^\lambda \\ & \leq \frac{C}{\lambda^2} - \delta \int \rho_{0,0} |A|^2 d\mu_s^\lambda \end{aligned}$$

This reflects the correct scaling for the parabolic blow-up.

Take any $\tau > 0$ and integrate from $-1 - \tau$ to -1 .

$$\begin{aligned} & \delta \int_{-1-\tau}^{-1} \int \rho_{0,0} |A|^2 d\mu_s^\lambda ds \\ & \leq \int (1 - *\Omega) \rho_{0,0} d\mu_{-1}^\lambda - \int (1 - *\Omega) \rho_{0,0} d\mu_{-1-\tau}^\lambda + \frac{C}{\lambda^2} \end{aligned} \tag{4.7}$$

Notice that

$$\int (1 - *\Omega) \rho_{0,0} d\mu_s^\lambda = \int (1 - *\Omega) \rho_{y_0, t_0} d\mu_{t_0 + \frac{s}{\lambda^2}}$$

By the fact that $\lim_{t \rightarrow t_0} \int (1 - *\Omega) \rho_{y_0, t_0} d\mu_t$ exists, the right hand side of equation (4.7) approaches zero as $\lambda \rightarrow \infty$. Take a sequence $\lambda_j \rightarrow \infty$, for a fixed $\tau > 0$,

$$\int_{-1-\tau}^{-1} \int \rho_{0,0} |A|^2 d\mu_s^{\lambda_j} ds \leq C(j)$$

where $C(j) \rightarrow 0$ as $\lambda_j \rightarrow \infty$.

Choose $\tau_j \rightarrow 0$ such that $\frac{C(j)}{\tau_j} \rightarrow 0$ and $s_j \in [-1 - \tau_j, -1]$ so that

$$\int \rho_{0,0} |A|^2 d\mu_{s_j}^{\lambda_j} \leq \frac{C(j)}{\tau_j}$$

We investigate this inequality more carefully. Notice that

$$\rho_{0,0}(F_{s_j}^{\lambda_j}) = \frac{1}{4\pi(-s_j)} \exp\left(\frac{-|F_{s_j}^{\lambda_j}|^2}{4(-s_j)}\right)$$

where $F_{s_j}^{\lambda_j} = \lambda_j F_{t_0 + \frac{s_j}{\lambda_j^2}}$.

If we consider for any $R > 0$, the ball of radius R , $B_R(0) \subset \mathbb{R}^N$, when j is large enough, we may assume $-1 < s_j < -\frac{1}{2}$, then

$$\int \rho_{0,0} |A|^2 d\mu_{s_j}^{\lambda_j} \geq \frac{1}{2\pi} \exp\left(\frac{-R^2}{2}\right) \int_{\Sigma_{s_j}^{\lambda_j} \cap B_R(0)} |A|^2 d\mu_{s_j}^{\lambda_j}$$

This implies for any compact set $K \subset \mathbb{R}^N$,

$$\int_{\Sigma_{s_j}^{\lambda_j} \cap K} |A|^2 d\mu_{s_j}^{\lambda_j} \rightarrow 0 \text{ as } j \rightarrow \infty \quad (4.8)$$

Now we claim in the rest of the proof this together with the fact that $*\Omega$ has a positive lower bound imply $\lim_{j \rightarrow \infty} \int \rho_{y_0, t_0} d\mu_{t_0 + \frac{s_j}{\lambda_j^2}} = \lim_{j \rightarrow \infty} \int \rho_{0,0} d\mu_{s_j}^{\lambda_j} \leq$

1. We may assume the origin is a limit point of $\Sigma_{s_j}^{\lambda_j}$, otherwise the limit is zero and there is nothing to be proved.

$*\Omega$ is in fact the Jacobian of the projection $\pi_1 : \Sigma_t \mapsto \Sigma_1$. Each Σ_t can be written as the graph of a map $f_t : \Sigma_1 \mapsto \Sigma_2$ with uniformly bounded $|df_t|$. This is because $\det(g_{ij} + (f_t^* h)_{ij}) = \prod_{i=1}^n (1 + \lambda_i^2)$ is bounded and $\prod_{i=1}^n (1 + \lambda_i^2) \geq 1 + \sum_{i=1}^n \lambda_i^2 = 1 + |df_t|^2$. Denote $f_{t_0 + \frac{s_j}{\lambda_j^2}}$ by f_j . Now we

consider the blow up of the graph of f_j in \mathbb{R}^N by λ_j . This is the graph of the function \tilde{f}_j defined on $\lambda_j \Sigma_1 \subset \mathbb{R}^N$ which corresponds to a part of $\Sigma_{s_j}^{\lambda_j}$. Now $|d\tilde{f}_j|$ is also uniformly bounded and our assumption on $\Sigma_{s_j}^{\lambda_j}$ implies $\lim_{j \rightarrow \infty} \tilde{f}_j(0) = 0$. Therefore we may assume $\tilde{f}_j \rightarrow \tilde{f}_\infty$ in C^α on compact sets. \tilde{f}_∞ is an entire graph defined on \mathbb{R}^n .

Other the other hand,

$$|A|_j \leq |\nabla d\tilde{f}_j| \leq (1 + |d\tilde{f}_j|^2)^{\frac{3}{2}} |A|_j \quad (4.9)$$

where $|A|_j$ is the norm of the second fundamental form of $\Sigma_{s_j}^{\lambda_j}$ and $|\nabla d\tilde{f}_j|$ is the norm of the covariant derivatives of $d\tilde{f}_j$. Inequalities (4.9) can be derived as equations (29) on page 31 of [4].

Use the equation (4.8), we can show $\tilde{f}_j \rightarrow \tilde{f}_\infty$ in $C^\alpha \cap W_{loc}^{1,2}$ and \tilde{f}_∞ has vanishing second derivatives. This implies $\Sigma_{s_j}^{\lambda_j} \rightarrow \Sigma_{-1}^\infty$ as Radon measures and Σ_{-1}^∞ is the graph of a linear function. Therefore

$$\lim_{j \rightarrow \infty} \int \rho_{0,0} d\mu_{s_j}^{\lambda_j} = \int \rho_{0,0} d\mu_{-1}^\infty = 1$$

This implies

$$\lim_{j \rightarrow \infty} \int \rho_{y_0, t_0} d\mu_{t_0 + \frac{s_j}{\lambda_j^2}} = \lim_{t \rightarrow t_0} \int \rho_{y_0, t_0} d\mu_t = 1$$

The regularity now follows from White's theorem [10] which asserts (y_0, t_0) is a regular point whenever $\lim_{t \rightarrow t_0} \int \rho_{y_0, t_0} d\mu_t \leq 1 + \epsilon$. □

Theorem B. *Suppose $k_1 \geq |k_2|$ and $k_1 + k_2 > 0$. If $\det(g_{ij} + (f^*h)_{ij}) < 2$, then the flow exists for all time and the corresponding map converges to a constant map at infinity.*

Proof. Long time existence is already proved in Theorem A. Since $*\Omega = \frac{1}{\sqrt{\det(g_{ij} + (f^*h)_{ij})}} = \frac{1}{\sqrt{\prod_{i=1}^n (1 + \lambda_i^2)}}$, the assumption is equivalent to $*\Omega > \frac{1}{\sqrt{2}}$.

By equation (4.2), we have

$$k_1 \left(\sum_{j \neq i} \frac{1}{1 + \lambda_j^2} \right) + k_2 (1 - n + \sum_{j \neq i} \frac{1}{1 + \lambda_j^2}) \geq 2c_1$$

for any i , where c_1 is a constant that depends on the initial condition. By Proposition 3.2, $*\Omega$ satisfies

$$\frac{d}{dt} * \Omega \geq \Delta * \Omega + 2c_1 \sum_{i=1}^n \frac{\lambda_i^2}{1 + \lambda_i^2}$$

That $\lambda_i^2 < 1$ for each i implies

$$\frac{d}{dt} * \Omega \geq \Delta * \Omega + c_1 \sum_{i=1}^n \lambda_i^2$$

Since $\lambda_i^2 < 1$ for each i , we have

$$1 + c_2 \sum_{i=1}^n \lambda_i^2 \geq \prod_{i=1}^n (1 + \lambda_i^2) \geq 1 + \sum_{i=1}^n \lambda_i^2$$

where c_2 is a constant that depends on n . Therefore $\sum_{i=1}^n \lambda_i^2 \geq \frac{1}{c_2} (\frac{1}{(*\Omega)^2} - 1)$. and $*\Omega$ satisfies

$$\frac{d}{dt} * \Omega \geq \Delta * \Omega + c_3 (\frac{1}{(*\Omega)^2} - 1)$$

By the comparison theorem for parabolic equations, $\min_{\Sigma_t} *\Omega$ is non-decreasing in t and $\min_{\Sigma_t} *\Omega \rightarrow 1$ as $t \rightarrow \infty$.

To prove convergence at infinity, we first show that $\max_{x \in \Sigma_t} |A|^2(x) \rightarrow 0$ as $t \rightarrow \infty$. We need to take a look at the quadratic term of the second fundamental form in equation (3.8).

Let $\Lambda = (\lambda_{i\alpha})$ be a matrix and

$$Q(x) = \sum_{i,\alpha} x_{i\alpha}^2 - 2 \sum_{\alpha,\beta,i < j} (\lambda_{i\alpha} \lambda_{j\beta} - \lambda_{j\alpha} \lambda_{i\beta}) x_{i\alpha} x_{j\beta}$$

be the quadratic form defined for $x = (x_{i\alpha}) \in \mathbb{R}^n \times \mathbb{R}^m$. Choose ϵ small enough such that $Q(x) > \frac{1}{2}|x|^2$ when $|\Lambda|^2 \leq \epsilon$.

The quadratic term of the second fundamental form in equation (3.8) in the original index (see also equation (3.7)) is

$$[\sum_{\alpha,i,k} h_{\alpha ik}^2 - 2 \sum_{\alpha,\beta,i < j,k} (\lambda_{i\alpha} \lambda_{j\beta} - \lambda_{j\alpha} \lambda_{i\beta}) h_{\alpha ik} h_{\beta jk}] * \Omega_1 \quad (4.10)$$

Now take $\epsilon_2 < \epsilon$. There exists a time T such that $*\Omega > \frac{1}{\sqrt{1+\epsilon_2}}$ and $\sum \lambda_i^2 < \epsilon_2$ for $t > T$. Therefore we have

$$\frac{d}{dt} *\Omega \geq \Delta *\Omega + \frac{1}{2} *\Omega |A|^2$$

Let $\eta = *\Omega$, then by equation (3.1)

$$\begin{aligned} |\nabla \eta|^2 &= \sum_k \left(\sum_{\alpha} \Omega_{\alpha 2 \dots n} h_{\alpha 1 k} + \dots + \Omega_{1 \dots, n-1 \alpha} h_{\alpha n k} \right)^2 \\ &\leq n \sum_k \left[\sum_{\alpha} (\Omega_{\alpha 2 \dots n} h_{\alpha 1 k})^2 + \dots + (\Omega_{1 \dots, n-1 \alpha} h_{\alpha n k})^2 \right] \\ &\leq n \sum_k \left[\sum_{\alpha} (\lambda_{\alpha 1} h_{\alpha 1 k})^2 + \dots + (\lambda_{\alpha n} h_{\alpha n k})^2 \right] (*\Omega)^2 \end{aligned}$$

Therefore

$$|\nabla \eta|^2 \leq n \epsilon_2 \eta^2 |A|^2 \quad (4.11)$$

Let p be a positive number to be determined, η^p satisfies

$$\begin{aligned} \frac{d}{dt} \eta^p &= p \eta^{p-1} \frac{d}{dt} \eta \\ &\geq p \eta^{p-1} (\Delta \eta + \frac{1}{2} \eta |A|^2) \\ &= \Delta \eta^p - p(p-1) \eta^{p-2} |\nabla \eta|^2 + \frac{p}{2} \eta^p |A|^2 \end{aligned}$$

Use the inequality (4.11), we get

$$\frac{d}{dt} \eta^p \geq \Delta \eta^p + [\frac{p}{2} - p(p-1)n\epsilon_2] \eta^p |A|^2$$

Recall from [8] that $|A|^2$ satisfies the following equation along the mean curvature flow.

$$\begin{aligned} \frac{d}{dt} |A|^2 &= \Delta |A|^2 - 2|\nabla A|^2 + [(\nabla_{\partial_k}^M R)_{\alpha i j k} + (\nabla_{\partial_j}^M R)_{\alpha k i k}] h_{\alpha i j} \\ &\quad - 2R_{l i j k} h_{\alpha l k} h_{\alpha i j} + 4R_{\alpha \beta j k} h_{\beta i k} h_{\alpha i j} - 2R_{l k i k} h_{\alpha l j} h_{\alpha i j} + R_{\alpha k \beta k} h_{\beta i j} h_{\alpha i j} \\ &\quad + \sum_{\alpha, \gamma, i, m} \left(\sum_k h_{\alpha i k} h_{\gamma m k} - h_{\alpha m k} h_{\gamma i k} \right)^2 + \sum_{i, j, m, k} \left(\sum_{\alpha} h_{\alpha i j} h_{\alpha m k} \right)^2 \end{aligned} \quad (4.12)$$

where R_{ABCD} is the curvature tensor and ∇^M is the covariant derivative of M .

In our case, the curvature operator is parallel and it follows that $|A|^2$ satisfies

$$\frac{d}{dt}|A|^2 \leq \Delta|A|^2 - 2|\nabla A|^2 + K_1|A|^4 + K_2|A|^2$$

for K_1, K_2 constants that depend on the dimensions of Σ_1 and Σ_2 . Applying the same technique in [9] to calculate $\eta^{-2p}|A|^2$ we get

$$\begin{aligned} \frac{d}{dt}(\eta^{-2p}|A|^2) &\leq \Delta(\eta^{-2p}|A|^2) - \eta^{2p}\nabla(\eta^{-2p}) \cdot \nabla(\eta^{-2p}|A|^2) \\ &\quad + \eta^{-2p}[K_1|A|^4 + K_2|A|^2 - 2|A|^4(\frac{p}{2} - p(p-1)n\epsilon_2)] \end{aligned}$$

We may further assume ϵ_2 is small so that $K_1 + 1 - \frac{1}{\sqrt{2n\epsilon_2}} < 0$. Choose p so that $2p(p-1)n\epsilon_2 = 1$, so $2np^2 \geq \frac{1}{\epsilon_2}$. Therefore $K_1 - p + 2p(p-1)n\epsilon_2 \leq K_1 + 1 - \frac{1}{\sqrt{2n\epsilon_2}} < 0$.

Denote $\eta^{-2p}|A|^2$ by g , then g satisfies

$$\frac{d}{dt}g \leq \Delta g - \eta^{2p}\nabla(\eta^{-2p}) \cdot \nabla g + \eta^{2p}(K_1 + 1 - \frac{1}{\sqrt{2n\epsilon_2}})g^2 + K_2g$$

By the maximal principle and comparison theorem for parabolic equations and notice that $0 < \eta < 1$ is bounded away from zero, $\max_{\Sigma_t}|A|^2 \leq \frac{c_3K_2}{\frac{1}{\sqrt{2n\epsilon_2}} - (K_1+1)}$ if t is large enough. Since ϵ_2 can be arbitrarily small, this implies $\max_{\Sigma_t}|A|^2 \rightarrow 0$ as $t \rightarrow \infty$. Since the mean curvature flow is a gradient flow, the metrics are analytic, by Simon's [7] theorem, the flow converges to a unique limit at infinity. That the limiting map is a constant map follows from $*\Omega = \frac{1}{\sqrt{\prod_{i=1}^n(1+\lambda_i^2)}} \rightarrow 1$ as $t \rightarrow \infty$, thus $\lambda_i \rightarrow 0$ and $|df_t| \rightarrow 0$. \square

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