A STRONG STABILITY CONDITION ON MINIMAL SUBMANIFOLDS AND ITS IMPLICATIONS

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Abstract. We identify a strong stability condition on minimal submanifolds that implies uniqueness and dynamical stability properties. In particular, we prove a uniqueness theorem and a $\mathcal{C}^1$ dynamical stability theorem of the mean curvature flow for minimal submanifolds that satisfy this condition. The latter theorem states that the mean curvature flow of any other submanifold in a $\mathcal{C}^1$ neighborhood of such a minimal submanifold exists for all time, and converges exponentially to the minimal one. This extends our previous uniqueness and stability theorem [18] which applies only to calibrated submanifolds of special holonomy ambient manifolds.

1. Introduction

In our previous work [18], we study the uniqueness and $\mathcal{C}^1$ dynamical stability of calibrated submanifolds in manifolds of special holonomy with explicitly constructed Riemannian metrics. The result is extended to minimal submanifolds of general Riemannian manifolds in this paper. The assumption for the uniqueness and dynamical stability theorem is identified as a strongly stable condition which implies the stability of the minimal submanifold in the usual sense of the second variation of the volume functional. Recall that the mean curvature flow is the negative gradient flow of the volume functional. It is thus natural to ask whether a local minimizer (a stable minimal submanifold) of the volume functional is stable under the mean curvature flow. Such a question of great generality has been addressed in the celebrated work of L. Simon [14]: when is a local minimizer dynamically stable under the gradient flow, i.e. does the gradient flow of a small perturbation of a local minimizer still converge back to the local minimizer? The question in the context of [14] concerns a nonlinear parabolic system defined on a compact manifold, and it was proved that the analyticity of the functional and the smallness in $\mathcal{C}^2$ norm are sufficient for the validity of the dynamical stability. The question we addressed here corresponds to the specialization to the volume functional of compact submanifolds. A natural measurement of the distance between two submanifolds is the $\mathcal{C}^1$ (or Lipschitz) norm, which

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is essentially the weakest possible norm concerning the volume functional. Our results in this paper can be considered as such an optimal result.

As derived in [15, §3], the Jacobi operator of the second variation of the volume functional is $(\nabla^\perp)^*\nabla^\perp + \mathcal{R} - \mathcal{A}$, where $(\nabla^\perp)^*\nabla^\perp$ is the Bochner Laplacian of the normal bundle, $\mathcal{R}$ is an operator constructed from the restriction of the ambient Riemann curvature, and $\mathcal{A}$ is constructed from the second fundamental form. The precise definition can be found in §3.1. A minimal submanifold is said to be strongly stable if $\mathcal{R} - \mathcal{A}$ is a positive operator, see (3.2). Since $(\nabla^\perp)^*\nabla^\perp$ is a non-negative operator, strong stability implies stability in the sense of the second variation of the volume functional. In particular, the strong stability condition is satisfied by all the calibrated submanifolds considered in [18] which include (M denotes the ambient Riemannian manifold and Σ denotes the minimal submanifold):

(i) M is the total space of the cotangent bundle of a sphere, T^*S^n (for $n > 1$), with the Stenzel metric [17], and Σ is the zero section;
(ii) M is the total space of the cotangent bundle of a complex projective space, $T^*\mathbb{CP}^n$, with the Calabi metric [3] and Σ is the zero section;
(iii) M is the total space of one of the vector bundles $\mathbb{S}(S^3)$, $\Lambda^2_-(S^4)$, $\Lambda^2_-(\mathbb{CP}^2)$, and $\mathbb{S}_-(S^4)$ with the Ricci flat metric constructed by Bryant–Salamon [2], where $\mathbb{S}$ is the spinor bundle and $\mathbb{S}_-$ is the spinor bundle of negative chirality, and Σ is the zero section of the respective vector bundle.

These are essentially all metrics of special holonomy that are known to be written in a closed form. Note that in all these examples, the metrics of the total space are Ricci flat, and the zero sections are totally geodesic. Hence, the strongly stability in these examples is equivalent to the positivity of the operator $\mathcal{R}$. In [18], we proved uniqueness and dynamical stability theorems for the corresponding calibrated submanifolds and the proofs rely on the explicit knowledge of the ambient metric, whose coefficients are governed by solutions of ODE systems. A natural question was how general such rigidity phenomenon is. In this article, we discover that the strong stability condition is precisely the condition that makes everything work. Moreover, we identify more examples that satisfy the strong stability condition:

Proposition A. *Each of the following pairs (Σ, M) of minimal submanifolds Σ and their ambient Riemannian manifolds M satisfy the strong stability condition* (3.2) *:*

(i) *M is any Riemannian manifold of negative sectional curvature and Σ a totally geodesic submanifold;*
(ii) *M is any Kähler manifold and Σ is a complex submanifold whose normal bundle has positive holomorphic curvature.*
(iii) *M is any Calabi–Yau manifold and Σ is a special Lagrangian with positive Ricci curvature;*

(iv) *M is any G_2 manifold and Σ is a coassociative submanifold with positive definite $-2W_- + \frac{s}{3}$ on Λ^2_-.*

For example (i), the strong stability can be checked directly. The examples (ii), (iii), and (iv) will be explained in §3.2 and Appendix A.

We now state the main results of this paper. The first one says that a strongly stable minimal submanifold is rather unique.

Theorem A. *Let $\Sigma^n \subset (M, g)$ be a compact, oriented minimal submanifold which is strongly stable in the sense of* (3.2)*. Then there exists a tubular neighborhood U of Σ such that Σ is the only compact minimal submanifold in U with dimension no less than n.*

The second one is on the dynamical stability of a strongly stable minimal submanifold.

Theorem B. *Let $\Sigma \subset (M, g)$ be a compact, oriented minimal submanifold which is strongly stable in the sense of* (3.2)*. If Γ is a submanifold that is close to Σ in $\mathcal{C}^1$, the mean curvature flow Γ_t with $\Gamma_0 = \Gamma$ exists for all time, and Γ_t converges to Σ smoothly as $t \to \infty$.*

The precise statements can be found in Theorem 4.2 (Theorem A) and Theorem 6.2 (Theorem B), respectively. The $\mathcal{C}^1$ dynamical stability of the mean curvature flow for those calibrated submanifolds considered in [18] was proved in the same paper. In this regard, this theorem is a generalization of our previous result.

Here are some remarks on the strong stability condition. In the viewpoint of the second variational formula, the condition is natural, and is stronger than the positivity of the Jacobi operator. The main results of this paper are basically saying that the strong stability has nice geometric consequences. In particular, the minimal submanifold Σ needs not be a totally geodesic, while most known results about the convergence of higher codimensional mean curvature flow are under the totally geodesic assumption, e.g. [22].

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2. Local geometry near a submanifold

2.1. **Notations and basic properties.** Let (M, g) be a Riemannian manifold of dimension $n+m$, and $\Sigma \subset M$ be a compact, embedded, and oriented submanifold of dimension n. We use $\langle \cdot, \cdot \rangle$ to denote the evaluation of two tangent vectors by the metric tensor g. The notation $\langle \cdot, \cdot \rangle$ is also abused to denote the evaluation with respect to the induced metric on Σ. Denote by ∇ the Levi-Civita connection of (M, g), and by ∇^Σ the Levi-Civita connection of the induced metric on Σ.

Denote by $N\Sigma$ the normal bundle of Σ in M. The metric g and its Levi-Civita connection induce a bundle metric (also denoted by $\langle\cdot,\cdot\rangle$) and a metric connection for $N\Sigma$. The bundle connection on $N\Sigma$ will be denoted by $\nabla^\perp$.

In the following discussion, we are going to choose a local orthonormal frame $\{e_1,\cdots,e_n, e_{n+1},\cdots,e_{n+m}\}$ for TM near a point $p\in\Sigma$ such that the restriction of $\{e_1,\cdots,e_n\}$ on Σ is an oriented frame for $T\Sigma$ and the restrictions of $\{e_{n+1},\cdots,e_{n+m}\}$ is a frame for $N\Sigma$. The indexes i,j,k range from 1 to n, the indexes α,β,γ range from $n+1$ to $n+m$, the indexes A,B,C range from 1 to $n+m$, and repeated indexes are summed.

The convention of the Riemann curvature tensor is

$$\begin{aligned} R(e_C,e_D)e_B &= \nabla_{e_C}\nabla_{e_D}e_B - \nabla_{e_D}\nabla_{e_C}e_B - \nabla_{[e_C,e_D]}e_B\ , \\ R_{ABCD} &= R(e_A,e_B,e_C,e_D) = \langle R(e_C,e_D)e_B, e_A\rangle\ . \end{aligned}$$

What follows are some basic properties of the geometry of a submanifold. The details can be found in, for example [6, ch. 6].

(i) ∇^Σ is the projection of ∇ onto $T\Sigma\subset TM|_\Sigma$, and $\nabla^\perp$ is the projection of ∇ onto $N\Sigma\subset TM|_\Sigma$. Their curvatures are denoted by

$$\begin{aligned} R^\Sigma_{klij} &= \langle \nabla^\Sigma_{e_i}\nabla^\Sigma_{e_j}e_l - \nabla^\Sigma_{e_j}\nabla^\Sigma_{e_i}e_l - \nabla^\Sigma_{[e_i,e_j]}e_l, e_k\rangle\ , \\ R^\perp_{\alpha\beta ij} &= \langle \nabla^\perp_{e_i}\nabla^\perp_{e_j}e_\beta - \nabla^\perp_{e_j}\nabla^\perp_{e_i}e_\beta - \nabla^\perp_{[e_i,e_j]}e_\beta, e_\alpha\rangle\ , \end{aligned}$$

(ii) Given any two tangent vectors X,Y of Σ, the *second fundamental form* of Σ in M is defined by $\mathrm{I\!I}(X,Y) = (\nabla_X Y)^\perp$, where $(\cdot)^\perp : TM\to N\Sigma$ is the projection onto the normal bundle. The *mean curvature* of Σ is the normal vector field defined by $H = \mathrm{tr}_\Sigma\,\mathrm{I\!I}$. With a normal vector V, $\mathrm{I\!I}(X,Y,V)$ is defined to be $\langle\mathrm{I\!I}(X,Y),V\rangle = \langle\nabla_X Y, V\rangle$. In terms of the frame,

$$h_{\alpha ij} = \mathrm{I\!I}(e_i,e_j,e_\alpha) \quad\text{and}\quad H = h_{\alpha ii}\,e_\alpha\ .$$

(iii) For any tangent vectors X,Y,Z of Σ and a normal vector V, the Codazzi equation says that

$$\langle R(X,Y)Z,V\rangle = (\nabla_X\mathrm{I\!I})(Y,Z,V) - (\nabla_Y\mathrm{I\!I})(X,Z,V) \tag{2.1}$$

where

$$(\nabla_X\mathrm{I\!I})(Y,Z,V) = X\left(\mathrm{I\!I}(Y,Z,V)\right) - \mathrm{I\!I}(\nabla^\Sigma_X Y,Z,V) - \mathrm{I\!I}(Y,\nabla^\Sigma_X Z,V) - \mathrm{I\!I}(X,Y,\nabla^\perp_X V)\ . \tag{2.2}$$

In terms of the frame, denote $(\nabla_{e_i}\mathrm{I\!I})(e_j,e_k,e_\alpha)$ by $h_{\alpha jk;i}$, and (2.1) is equivalent to that $R_{\alpha kij} = h_{\alpha jk;i} - h_{\alpha ik;j}$.

2.2. **Geodesic coordinate and geodesic frame.** For any $p \in \Sigma$, we can construct a "partial" geodesic coordinate and a geodesic frame on a neighborhood of p in M as follows:

(i) Choose an oriented, orthonormal basis $\{e_1, \cdots, e_n\}$ for $T_p\Sigma$. The map

$$F_0 : \mathbf{x} = (x^1, \cdots, x^n) \mapsto \exp_p^\Sigma(x^j e_j)$$

parametrizes an open neighborhood of p in Σ, where $\exp^\Sigma$ is the exponential map of the induced metric on Σ. For any $\mathbf{x}$ of unit length, the curve $\gamma(t) = F_0(t\mathbf{x})$ is called a *radial geodesic on* Σ (*at* p). By using ∇^Σ to parallel transport $\{e_1, \cdots, e_n\}$ along these radial geodesics, we get a local orthonormal frame for $T\Sigma$ on a neighborhood of p in Σ. The frame is still denoted by $\{e_1, \cdots, e_n\}$.

(ii) Choose an orthonormal basis $\{e_{n+1}, \cdots, e_{n+m}\}$ for $N_p\Sigma$. By using $\nabla^\perp$ to parallel transport $\{e_{n+1}, \cdots, e_{n+m}\}$ along radial geodesics on Σ, we obtain a local orthonormal frame for $N\Sigma$ on a neighborhood of p in Σ. This frame is still denoted by $\{e_{n+1}, \cdots, e_{n+m}\}$. It is clear that $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}\}$ is a local orthonormal frame for $TM|_\Sigma$.

(iii) The map

$$F : (\mathbf{x}, \mathbf{y}) = \big((x^1, \cdots, x^n), (y^{n+1}, \cdots, y^{n+m})\big) \mapsto \exp_{F_0(\mathbf{x})}(y^\alpha e_\alpha)$$

parametrizes an open neighborhood of p in M. The map exp is the exponential map of (M, g). For any $\mathbf{y}$ of unit length, the curve $\sigma(t) = F(\mathbf{x}, t\mathbf{y}) = \exp_{F_0(\mathbf{x})}(t\mathbf{y})$ is called a *normal geodesic for* $\Sigma \subset M$.

(iv) For any $\mathbf{x}$, step (ii) gives an orthonormal basis $\{e_1, \cdots, e_{n+m}\}$ for $T_{F(\mathbf{x},0)}M$. By using ∇ to parallel transport it along normal geodesics, we have an orthonormal frame for TM on a neighborhood of p in M. This frame is again denoted by $\{e_1, \cdots, e_{n+m}\}$.

The freedom in the above construction is the choice of $\{e_1, \cdots, e_n\}$ and $\{e_{n+1}, \cdots, e_{n+m}\}$ at p, which is $\mathrm{SO}(n) \times \mathrm{O}(m)$. A particular choice will be made later on.

Remark 2.1. We will consider the curves $s \mapsto \exp_p^\Sigma(x^i e_i + s e_j)$ and $s \mapsto \exp_{F_0(\mathbf{x})}(y^\beta e_\beta + s e_\alpha)$ in the following discussion. They will be abbreviated as $F_0(\mathbf{x} + s e_j)$ and $F(\mathbf{x}, \mathbf{y} + s e_\alpha)$, respectively.

Remark 2.2. The frames $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}\}$ are constructed by parallel transport along radial geodesics on Σ and then normal geodesic for Σ. They are indeed smooth. We briefly explain the smoothness of $\{e_1, \cdots, e_n\}$ on a neighborhood of p in Σ. Write $e_i = S_{ij}(\mathbf{x}) \frac{\partial}{\partial x^j}$. The smoothness of the frame is equivalent to the smoothness of $S_{ij}(\mathbf{x})$. Let $\Gamma_{jk}^l(\mathbf{x})$ be the Christoffel symbols of ∇^Σ, i.e. $\nabla^\Sigma_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma_{jk}^l(\mathbf{x}) \frac{\partial}{\partial x^l}$. The Christoffel symbols $\Gamma_{jk}^l(\mathbf{x})$ are smooth functions. Since e_i is parallel along radial geodesics,

$$\nabla^\Sigma_{x^l \frac{\partial}{\partial x^l}} e_i = 0 = \left(x^l \frac{\partial\, S_{ij}(\mathbf{x})}{\partial x^l} + x^l S_{ik}(\mathbf{x}) \Gamma_{ik}^j(\mathbf{x}) \right) \frac{\partial}{\partial x^j} \ .$$

To avoid confusion, fix $\xi = (\xi^1, \cdots, \xi^n) \in \mathbb{R}^n$. Let $\gamma(t) = t\xi$ for $t \in [0,1]$. Since $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = \frac{1}{t}(x^l \frac{\partial}{\partial x^l} f(\mathbf{x}))|_{\gamma(t)}$,

$$\frac{\mathrm{d}S_{ij}(\gamma(t))}{\mathrm{d}t} = \xi^l \, S_{ik}(\gamma(t)) \, \Gamma^j_{ik}(t\xi) \ .$$

In other words, $[S_{ij}(\xi)]$ is the solution to the ODE system $\frac{\mathrm{d}S}{\mathrm{d}t} = F(S, t, \xi)$ at $t = 1$ with identity as the initial condition. Therefore, $S_{ij}(\xi)$ is smooth in ξ.

2.2.1. *The tubular neighborhood U_ε and the distance function.*

Definition 2.3. For any $\delta > 0$, let U_δ be the image of $\{V \in N\Sigma \mid |V| < \delta\}$ under the exponential map along Σ. By the implicit function theorem, there exists $\varepsilon > 0$, which is determined by the geometry of Σ and M, such that the following statements hold for U_ε:

(1) The map $\exp : \{V \in N\Sigma \mid |V| < 2\varepsilon\} \to U_{2\varepsilon}$ is a diffeomorphism.

(2) There exist the local coordinate system $(x^1, \cdots x^n, y^{n+1}, \cdots y^{n+m})$ and the frame $\{e_1, \cdots, e_{n+m}\}$ constructed in the last subsection.

(3) The function $\sum_\alpha (y^\alpha)^2$ is a well-defined smooth function on U_ε.

(4) On U_ε, the square root of $\sum_\alpha (y^\alpha)^2$ is the distance function to Σ.

(5) For any $q \in U_\varepsilon$, there exists a unique $p \in \Sigma$ such that there is a unique normal geodesic in U_ε connecting p and q.

We now analyze the gradient of the function $\sum_\alpha (y^\alpha)^2$. To avoid confusion, let

$$\xi = (\xi^1, \cdots, \xi^n) \in \mathbb{R}^n \qquad \text{and} \qquad \eta = (\eta^{n+1}, \cdots, \eta^{n+m}) \in \mathbb{R}^m$$

be constant vectors. Consider the normal geodesic $\sigma(t) = F(\xi, t\eta)$; its tangent vector field is $\sigma'(t) = \eta^\alpha \frac{\partial}{\partial y^\alpha}$. On the other hand, $\sigma'(0)$ is also equal to $\eta^\alpha e_\alpha$, and $\eta^\alpha e_\alpha$ is defined and parallel along $\sigma(t)$. Thus, $\eta^\alpha \frac{\partial}{\partial y^\alpha} = \eta^\alpha e_\alpha$ on $\sigma(t)$. Since the y-coordinate of $\sigma(t)$ is $t\eta$, we find that

$$y^\alpha \frac{\partial}{\partial y^\alpha}\Big|_{\sigma(t)} = t\eta^\alpha \frac{\partial}{\partial y^\alpha}\Big|_{\sigma(t)} = t\eta^\alpha \, e_\alpha = t \, \sigma'(t) \ ; \tag{2.3}$$

at $t = 1$ it gives

$$y^\alpha \frac{\partial}{\partial y^\alpha} = y^\alpha e_\alpha \ . \tag{2.4}$$

By modifying the standard geodesic argument [4, p.4–9], the vector field $y^\alpha \frac{\partial}{\partial y^\alpha}|_{\sigma(t)}$ is half of the gradient vector field of $\sum_\alpha (y^\alpha)^2$. In addition, note that (2.4) implies that $\langle y^\alpha \frac{\partial}{\partial y^\alpha}, y^\alpha \frac{\partial}{\partial y^\alpha} \rangle = \sum_\alpha (y^\alpha)^2$. The Gauss lemma implies that $\langle y^\alpha \frac{\partial}{\partial y^\alpha}, s^\beta \frac{\partial}{\partial y^\beta} \rangle = 0$ if $\sum_\alpha y^\alpha s^\alpha = 0$. By considering the first variational formula of the one-parameter family of geodesics $\sigma(t, s) = \exp_{F_0(\xi + s e_j)}(t\eta)$,

one finds that $\langle y^\alpha \frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial x^j}\rangle = 0$. It follows from these relations that

$$\nabla\left(\sum_\alpha (y^\alpha)^2\right) = 2y^\alpha \frac{\partial}{\partial y^\alpha} \ . \tag{2.5}$$

For a locally defined smooth function near p, the following lemma establishes its expansion in terms of the coordinate system constructed above.

Lemma 2.4. *Let U_ε be a neighborhood of $p \in \Sigma$ in M as in Definition 2.3 with the coordinate system $(\mathbf{x}, \mathbf{y}) = (x^1, \cdots x^n, y^{n+1}, \cdots y^{n+m})$ and the frame $\{e_1, \cdots, e_n, e_{n+1}, \cdots e_{n+m}\}$. Then, any smooth function $f(\mathbf{x}, \mathbf{y})$ on U_ε has the following expansion:*

$$f(\mathbf{x}, \mathbf{y}) = f(\mathbf{0}, \mathbf{0}) + x^i\, e_i(f)|_p + y^\alpha\, e_\alpha(f)|_p + \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2) \ .$$

More precisely, it means that $\big|f(\mathbf{x}, \mathbf{y}) - f(\mathbf{0}, \mathbf{0}) - x^i\, e_i(f)|_p - y^\alpha\, e_\alpha(f)|_p\big| \le c(|\mathbf{x}|^2 + |\mathbf{y}|^2)$ for some constant c determined by the $\mathcal{C}^2$-norm of f and the geometry of M and Σ.

Proof. Let $q \in U_\varepsilon$ be any point. To avoid confusion, denote the coordinate of q by (ξ, η), where $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^m$ are regarded as constant vectors. Let $q_0 \in \Sigma$ be the point with normal coordinate $(\xi, \mathbf{0})$, and consider the radial geodesic on Σ joining q_0 and p, $\sigma_0(t) = F_0(t\xi)$. Applying Taylor's theorem on $f(\sigma_0(t))$ gives

$$f(\xi, \mathbf{0}) = f(\mathbf{0}, \mathbf{0}) + \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} f(\sigma_0(t)) + \int_0^1 (1-t^2) \frac{\mathrm{d}^2 f(\sigma_0(t))}{\mathrm{d}t^2} \ .$$

Since $\sigma_0'(t) = \xi^i\, e_i$, we find that

$$f(\xi, \mathbf{0}) = f(\mathbf{0}, \mathbf{0}) + \xi^i\, e_i(f)|_p + \xi^i\xi^j \int_0^1 (1-t)\, (e_j(e_i(f)))(\sigma_0(t))\, \mathrm{d}t \ . \tag{2.6}$$

Next, consider the normal geodesic joining q and q_0, $\sigma(t) = F(\xi, t\eta)$. Remember that $\sigma'(t) = \eta^\alpha e_\alpha$. By considering $f(\sigma(t))$,

$$f(\xi, \eta) = f(\xi, \mathbf{0}) + \eta^\alpha\, (e_\alpha(f))|_{q_0} + \eta^\alpha\eta^\beta \int_0^1 (1-t)\, (e_\beta(e_\alpha(f)))(\sigma(t))\, \mathrm{d}t \ . \tag{2.7}$$

Similar to (2.6), $(e_\alpha(f))|_{q_0} = (e_\alpha(f))|_p + \xi^j \int_0^1 e_j(e_\alpha(f))(\sigma_0(t))\, \mathrm{d}t$. Putting these together finishes the proof of this lemma. □

2.2.2. *The expansions of coordinate vector fields.*

Lemma 2.5. *Let U_ε be a neighborhood of $p \in \Sigma$ in M as in Definition 2.3 with the coordinate system $(\mathbf{x}, \mathbf{y}) = (x^1, \cdots x^n, y^{n+1}, \cdots y^{n+m})$ and the frame $\{e_1, \cdots, e_n, e_{n+1}, \cdots e_{n+m}\}$. Write*

$$\frac{\partial}{\partial x^i} = \langle \frac{\partial}{\partial x^i}, e_A\rangle e_A \quad \text{and} \quad \frac{\partial}{\partial y^\mu} = \langle \frac{\partial}{\partial y^\mu}, e_A\rangle e_A \ ,$$

then $\langle \frac{\partial}{\partial x^i}, e_A\rangle$ *and* $\langle \frac{\partial}{\partial y^\mu}, e_A\rangle$*, considered as locally defined multi-indexed functions, has the following expansions:*

$$\begin{aligned}\langle \frac{\partial}{\partial x^i}, e_j\rangle\big|_{(\mathbf{x},\mathbf{y})} &= \delta_{ij} - y^\alpha h_{\alpha ij}\big|_p + \mathcal{O}(|\mathbf{x}|^2+|\mathbf{y}|^2)\ ,\\ \langle \frac{\partial}{\partial y^\mu}, e_\beta\rangle\big|_{(\mathbf{x},\mathbf{y})} &= \delta_{\mu\beta} + \mathcal{O}(|\mathbf{x}|^2+|\mathbf{y}|^2)\ ,\end{aligned} \tag{2.8}$$

and both $\langle \frac{\partial}{\partial x^i}, e_\beta\rangle\big|_{(\mathbf{x},\mathbf{y})}$ *and* $\langle \frac{\partial}{\partial y^\mu}, e_j\rangle\big|_{(\mathbf{x},\mathbf{y})}$ *are of the order* $|\mathbf{x}|^2+|\mathbf{y}|^2$*. By inverting the matrices,*

$$e_i = \frac{\partial}{\partial x^i} + y^\alpha h_{\alpha ij}\frac{\partial}{\partial x^j} + \mathcal{O}(|\mathbf{x}|^2+|\mathbf{y}|^2) \qquad \textit{and} \qquad e_\alpha = \frac{\partial}{\partial y^\alpha} + \mathcal{O}(|\mathbf{x}|^2+|\mathbf{y}|^2)\ . \tag{2.9}$$

Proof. We apply Lemma 2.4 to these locally defined functions.

By construction, $\langle \frac{\partial}{\partial x^i}, e_j\rangle|_p = \delta_{ij}$. With a similar argument as that for (2.4), $x^i \frac{\partial}{\partial x^i} = x^i e_i$ on $\Sigma \cap U_\varepsilon$. It follows that

$$x^j = x^\ell \langle \frac{\partial}{\partial x^\ell}, e_j\rangle\ .$$

Differentiating the above equation first with respect to x^i and then with respect to x^k, and then evaluating at p which has $x^\ell = 0$ for all ℓ, we obtain

$$\left(\frac{\partial}{\partial x^k}\langle \frac{\partial}{\partial x^i}, e_j\rangle\right)\Big|_p + \left(\frac{\partial}{\partial x^i}\langle \frac{\partial}{\partial x^k}, e_j\rangle\right)\Big|_p = 0\ .$$

On the other hand, it follows from the construction that $(\nabla^\Sigma e_j)|_p = 0$, and

$$\left(\frac{\partial}{\partial x^k}\langle \frac{\partial}{\partial x^i}, e_j\rangle\right)\Big|_p = \langle \nabla^\Sigma_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}, e_j\rangle\Big|_p = \langle \nabla^\Sigma_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}, e_j\rangle\Big|_p = \left(\frac{\partial}{\partial x^i}\langle \frac{\partial}{\partial x^k}, e_j\rangle\right)\Big|_p\ .$$

Hence, $\frac{\partial}{\partial x^k}\langle \frac{\partial}{\partial x^i}, e_j\rangle$ is zero at p.

Since e_j is parallel with respect to ∇ along normal geodesics, $(\nabla_{e_\alpha} e_j)|_p = 0$. It follows that

$$\left(\frac{\partial}{\partial y^\alpha}\langle \frac{\partial}{\partial x^i}, e_j\rangle\right)\Big|_p = \langle \nabla_{\frac{\partial}{\partial y^\alpha}}\frac{\partial}{\partial x^i}, e_j\rangle\Big|_p = \langle \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial y^\alpha}, e_j\rangle\Big|_p = -\langle \frac{\partial}{\partial y^\alpha}, \nabla_{\frac{\partial}{\partial x^i}} e_j\rangle\Big|_p = -h_{\alpha ij}\big|_p$$

where the third equality follows from the fact that $\langle \frac{\partial}{\partial y^\alpha}, e_j\rangle \equiv 0$ on $\Sigma \cap U_\varepsilon$.

Note that $\langle \frac{\partial}{\partial x^i}, e_\beta\rangle$ vanishes on $\Sigma \cap U_\varepsilon$. Since e_β is parallel with respect to ∇ along normal geodesics, $(\nabla_{e_\alpha} e_\beta)|_p = 0$, and then

$$\left(\frac{\partial}{\partial y^\alpha}\langle \frac{\partial}{\partial x^i}, e_\beta\rangle\right)\Big|_p = \langle \nabla_{\frac{\partial}{\partial y^\alpha}}\frac{\partial}{\partial x^i}, e_\beta\rangle\Big|_p = \langle \nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial y^\alpha}, e_\beta\rangle\Big|_p\ .$$

By construction, $\frac{\partial}{\partial y^\alpha} = e_\alpha$ on $\Sigma \cap U_\varepsilon$ and $(\nabla^\perp e_\alpha)|_p = 0$. Therefore, $\frac{\partial}{\partial y^\alpha}\langle \frac{\partial}{\partial x^i}, e_\beta\rangle$ is zero at p.

The term $\langle \frac{\partial}{\partial y^\mu}, e_j\rangle$ also vanishes on $\Sigma \cap U_\varepsilon$. It follows from (2.4) that $y^\mu \langle \frac{\partial}{\partial y^\mu}, e_j\rangle = 0$. Differentiating the above equation first with respect to y^α and then with respect to y^β, we

obtain

$$\left(\frac{\partial}{\partial y^\alpha}\langle\frac{\partial}{\partial y^\beta}, e_j\rangle\right)\Big|_p + \left(\frac{\partial}{\partial y^\beta}\langle\frac{\partial}{\partial y^\alpha}, e_j\rangle\right)\Big|_p = 0 \ .$$

Since $\nabla_{e_\nu} e_j = 0$, the above two terms are always equal to each other, and thus both vanish.

For $\langle\frac{\partial}{\partial y^\mu}, e_\beta\rangle$, it follows from the construction that $\langle\frac{\partial}{\partial y^\mu}, e_\beta\rangle = \delta_{\mu\beta}$ on $\Sigma \cap U_\varepsilon$. According to (2.4), $y^\mu = y^\nu\langle\frac{\partial}{\partial y^\nu}, e_\mu\rangle$. By a similar argument as that for $\frac{\partial}{\partial x^k}\langle\frac{\partial}{\partial x^i}, e_j\rangle$, $\frac{\partial}{\partial y^\nu}\langle\frac{\partial}{\partial y^\mu}, e_\beta\rangle$ also vanishes at p. □

2.2.3. *The expansions of connection coefficients.*

Proposition 2.6. *Let U_ε be a neighborhood of $p \in \Sigma$ in M as in Definition 2.3 with the coordinate system $(\mathbf{x}, \mathbf{y}) = (x^1, \cdots x^n, y^{n+1}, \cdots y^{n+m})$ and the frame $\{e_1, \cdots, e_n, e_{n+1}, \cdots e_{n+m}\}$. Let*

$$\theta^B_A = \langle\nabla_{e_C} e_A, e_B\rangle\omega^C = \theta^B_A(e_C)\omega^C$$

be the connection 1-forms of the frame fields on U_ε, where $\{\omega^A\}_{A=1}^{n+m}$ is the dual coframe of $\{e_A\}_{A=1}^{n+m}$. Then, at a point $q \in U_\varepsilon$ with coordinates $(\mathbf{x}, \mathbf{y})$, $\theta^B_A(e_C)$, considered as locally defined multi-indexed functions, has the following expansions:

$$\begin{aligned}
\theta^j_i(e_k)|_{(\mathbf{x},\mathbf{y})} &= \frac{1}{2}x^l\, R^\Sigma_{jilk}\big|_p + y^\alpha\, R_{ji\alpha k}\big|_p + \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2) \ , \\
\theta^j_i(e_\beta)|_{(\mathbf{x},\mathbf{y})} &= \frac{1}{2}y^\alpha\, R_{ji\alpha\beta}\big|_p + \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2) \ ,
\end{aligned} \tag{2.10}$$

$$\theta^\alpha_i(e_j)|_{(\mathbf{x},\mathbf{y})} = h_{\alpha ij}\big|_p + x^k\, h_{\alpha ij;k}\big|_p + y^\beta\,(R_{\alpha i\beta j} + \sum_k h_{\alpha ik}h_{\beta jk})\big|_p + \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2) \ , \tag{2.11}$$

$$\begin{aligned}
\theta^\alpha_i(e_\beta)|_{(\mathbf{x},\mathbf{y})} &= \frac{1}{2}y^\gamma\, R_{\alpha i\gamma\beta}\big|_p + \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2) \ , \\
\theta^\alpha_\beta(e_i)|_{(\mathbf{x},\mathbf{y})} &= \frac{1}{2}x^j\, R^\perp_{\alpha\beta ji}\big|_p + y^\gamma\, R_{\alpha\beta\gamma i}\big|_p + \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2)
\end{aligned} \tag{2.12}$$

$$\theta^\alpha_\beta(e_\gamma)|_{(\mathbf{x},\mathbf{y})} = \frac{1}{2}y^\delta\, R_{\alpha\beta\delta\gamma}\big|_p + \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2) \ ,$$

where $R^\Sigma_{jilk}\big|_p$, $R_{ji\alpha k}\big|_p$, $R_{ji\alpha\beta}\big|_p$ $h_{\alpha ij}\big|_p$, $h_{\alpha ij;k}\big|_p$, $R_{\alpha i\beta j}\big|_p$, $R_{\alpha i\gamma\beta}\big|_p$,$R^\perp_{\alpha\beta ji}\big|_p$, $R_{\alpha\beta\gamma i}\big|_p$, $R_{\alpha\beta\delta\gamma}\big|_p$ all represent the evaluation of the corresponding tensors at p and with respect to the frame fields $\{e_i\}_{i=1}^n$ and $\{e_\alpha\}_{\alpha=n+1}^{n+m}$.

Proof. Since the restriction of the frame $\{e_i\}_{i=1}^n$ on Σ is parallel with respect to ∇^Σ along the radial geodesics, $x^k\theta^j_i(e_k)\big|_{(\mathbf{x},\mathbf{0})} = 0$ for any $i, j \in \{1, \ldots, n\}$. It follows that

$$\theta^j_i(e_k)\big|_{(\mathbf{x},\mathbf{0})} = -x^l\frac{\partial\,\theta^j_i(e_l)}{\partial x^k}\big|_{(\mathbf{x},\mathbf{0})} \quad \text{and thus} \quad \theta^j_i(e_k)\big|_p = 0 \ . \tag{2.13}$$

By taking the partial derivative in x^l and evaluating at $p = (\mathbf{0}, \mathbf{0})$, we find that

$$\frac{\partial\, \theta_i^j(e_k)}{\partial x^l}\Big|_p = -\frac{\partial\, \theta_i^j(e_l)}{\partial x^k}\Big|_p\ , \quad \text{or equivalently,} \quad e_l(\theta_i^j(e_k))|_p = -e_k(\theta_i^j(e_l))|_p \tag{2.14}$$

Similarly, since the restriction of $\{e_\mu\}_{\mu=n+1}^{n+m}$ on Σ is parallel with respect to $\nabla^\perp$ along radial geodesics, $x^k\theta_\mu^\nu(e_k)\big|_{(\mathbf{x},\mathbf{0})} = 0$ for any $\mu, \nu \in \{n+1, \ldots, n+m\}$. It follows that

$$\theta_\mu^\nu(e_k)\big|_p = 0 \qquad \text{and} \tag{2.15}$$

$$e_l(\theta_\mu^\nu(e_k))\big|_p = -e_k(\theta_\mu^\nu(e_l))\big|_p\ . \tag{2.16}$$

Since the frame $\{e_A\}_{i=1}^{n+m}$ is parallel with respect to ∇ along normal geodesics, $y^\mu\theta_A^B(e_\mu) = 0$ and it follows that

$$\theta_A^B(e_\mu) = -y^\nu\frac{\partial\, \theta_A^B(e_\nu)}{\partial y^\mu} \quad \Rightarrow \quad \theta_A^B(e_\mu)\big|_{(\mathbf{x},\mathbf{0})} = 0. \tag{2.17}$$

By taking partial derivatives,

$$\frac{\partial\, \theta_A^B(e_\mu)}{\partial x^k} = -y^\nu\frac{\partial^2\, \theta_A^B(e_\nu)}{\partial x^k\partial y^\mu} \quad \text{and} \quad \frac{\partial\, \theta_A^B(e_\mu)}{\partial y^\nu} = -\frac{\partial\, \theta_A^B(e_\nu)}{\partial y^\mu} - y^\delta\frac{\partial^2\, \theta_A^B(e_\delta)}{\partial y^\nu\partial y^\mu}\ . \tag{2.18}$$

Note that on Σ, $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ and $\{e_i\}_{i=1}^n$ are both bases for $T\Sigma$. Therefore,

$$e_k(\theta_A^B(e_\mu))\big|_{(\mathbf{x},\mathbf{0})} = 0\ . \tag{2.19}$$

By construction, $\frac{\partial}{\partial y^\mu} = e_\mu$ on Σ, and thus

$$e_\nu(\theta_A^B(e_\mu))\big|_{(\mathbf{x},\mathbf{0})} = -e_\mu(\theta_A^B(e_\nu))\big|_{(\mathbf{x},\mathbf{0})}\ . \tag{2.20}$$

In terms of the connection 1-forms, the components of the Riemann curvature tensor are

$$\begin{aligned}
&R_{ABCD}\\
&= \langle \nabla_{e_C}\nabla_{e_D}e_B - \nabla_{e_D}\nabla_{e_C}e_B - \nabla_{[e_C,e_D]}e_B, e_A\rangle\\
&= e_C(\theta_B^A(e_D)) - e_D(\theta_B^A(e_C)) - (\theta_B^E\wedge\theta_E^A)(e_C,e_D) - \theta_B^A(e_E)\theta_D^E(e_C) + \theta_B^A(e_E)\theta_C^E(e_D)\ .
\end{aligned} \tag{2.21}$$

With these preparations, we proceed to prove all the expansion formulae:

(The expansion of $\theta_i^j(e_k)$) It follows from (2.13) that the zeroth order term is zero. By (2.14), the coefficient of x^l in the expansion is

$$e_l(\theta_i^j(e_k))|_p = \frac{1}{2}\left[e_l(\theta_i^j(e_k)) - e_k(\theta_i^j(e_l))\right]\Big|_p = \frac{1}{2}\,R_{jilk}^\Sigma\big|_p\ .$$

Note that for $R_{\alpha\beta ji}^\Sigma$, all the indices of summation in (2.21) go from 1 to n. Due to (2.19), the coefficient of y^α in the expansion is

$$e_\alpha(\theta_i^j(e_k))\Big|_p = \left[e_\alpha(\theta_i^j(e_k)) - e_k(\theta_i^j(e_\alpha))\right]\Big|_p = R_{ji\alpha k}|_p\ .$$

(The expansion of $\theta_i^j(e_\beta)$) By (2.17), the zeroth order term is zero, and the coefficient of x^l in the expansion is zero. According to (2.20), the coefficient of y^α in the expansion is

$$e_\alpha(\theta_i^j(e_\beta))\Big|_p = \frac{1}{2}\left[e_\alpha(\theta_i^j(e_\beta)) - e_\beta(\theta_i^j(e_\alpha))\right]\Big|_p = R_{ji\alpha\beta}|_p \ .$$

(The expansion of $\theta_i^\alpha(e_j)$) On $\Sigma\cap U_\varepsilon$, $\theta_i^\alpha(e_j) = \langle\nabla_{e_j}e_i, e_\alpha\rangle = h_{\alpha ij}$. Its derivative along e_k is

$$e_k(\mathrm{I\!I}(e_i,e_j,e_\alpha)) = (\nabla_{e_k}\mathrm{I\!I})(e_i,e_j,e_\alpha) + \mathrm{I\!I}(\nabla^\Sigma_{e_k}e_i,e_j,e_\alpha) + \mathrm{I\!I}(e_i,\nabla^\Sigma_{e_k}e_j,e_\alpha) + \mathrm{I\!I}(e_i,e_j,\nabla^\perp_{e_k}e_\alpha) \ .$$

Due to (2.13) and (2.15), the last three terms vanish at p. It follows that $e_k(\mathrm{I\!I}(e_i,e_j,e_\alpha))|_p$ is equal to $h_{\alpha ij;k}|_p$.

The coefficient of y^β is $e_\beta(\theta_i^\alpha(e_j)|_p$. By (2.19) and (2.17),

$$R_{\alpha i\beta j}|_p = \left[e_\beta(\theta_i^\alpha(e_j)) + \theta_i^\alpha(e_k)\theta_\beta^k(e_j)\right]\Big|_p = [e_\beta(\theta_i^\alpha(e_j)) - h_{\alpha ik}h_{\beta kj}]|_p \ .$$

(The expansion of $\theta_i^\alpha(e_\beta)$) According to (2.17), the zeroth order term is zero, and the coefficient of x^l in the expansion is zero. By (2.20) and (2.17),

$$e_\gamma(\theta_i^\alpha(e_\beta))|_p = \frac{1}{2}\,[e_\gamma(\theta_i^\alpha(e_\beta)) - e_\beta(\theta_i^\alpha(e_\gamma))]|_p = \frac{1}{2}\,R_{\alpha i\gamma\beta}|_p \ .$$

(The expansion of $\theta_\beta^\alpha(e_i)$) By (2.15), the zeroth order term vanishes. With (2.16), (2.15) and (2.13),

$$e_j(\theta_\alpha^\beta(e_i))\Big|_p = \frac{1}{2}\left[e_j(\theta_\alpha^\beta(e_i)) - e_i(\theta_\alpha^\beta(e_j))\right]\Big|_p = \frac{1}{2}\,R^\perp_{\alpha\beta ji}\Big|_p \ .$$

Note that for $R^\perp_{\alpha\beta ji}$, the index of summation in the third term of (2.21) goes from $n+1$ to $n+m$, and the indices of summation in the last two terms of (2.21) go from 1 to n. By (2.19), (2.17) and (2.15),

$$e_\gamma(\theta_\alpha^\beta(e_i))\Big|_p = \left[e_\gamma(\theta_\alpha^\beta(e_i)) - e_i(\theta_\alpha^\beta(e_\gamma))\right]\Big|_p = R_{\alpha\beta\gamma i}|_p \ .$$

(The expansion of $\theta_\beta^\alpha(e_\gamma)$) Due to (2.17), $\theta_\beta^\alpha(e_\gamma)$ vanishes on $\Sigma\cap U_\varepsilon$. According to (2.20) and (2.17),

$$e_\delta(\theta_\beta^\alpha(e_\gamma))\big|_p = \frac{1}{2}\,\left[e_\delta(\theta_\beta^\alpha(e_\gamma)) - e_\gamma(\theta_\beta^\alpha(e_\delta))\right]\big|_p = \frac{1}{2}\,R_{\alpha\beta\delta\gamma}|_p \ .$$

This finishes the proof of this proposition. □

2.2.4. *Horizontal and vertical subspaces.* For any $q\in U_\varepsilon\subset M$, there exists a unique $p\in\Sigma$ such that there is a unique normal geodesic inside U_ε connecting q and p. Any tensor defined on Σ can be extended to U_ε by parallel transport of ∇ along normal geodesics. Here are some notions that will be used in this paper.

The parallel transport of $T\Sigma$ along normal geodesics defines an n-dimensional distribution of $TM|_{U_\varepsilon}$, which is called the *horizontal distribution*, and is denoted by $\mathcal{H}$. Its orthogonal

complement in TM is called the *vertical distribution*, and is denoted by $\mathcal{V}$. It is clear that $\mathcal{H} = \text{span}\{e_1, \cdots, e_n\}$ and $\mathcal{V} = \text{span}\{e_{n+1}, \cdots, e_{n+m}\}$. The parallel transport of the volume form of Σ along normal geodesics defines an n-form on U_ε, which is denoted by Ω. Let $\{\omega^1, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^{n+m}\}$ be the dual coframe of $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}\}$. In terms of the coframe,

$$\Omega = \omega^1 \wedge \cdots \wedge \omega^n \ . \tag{2.22}$$

For any $q \in U_\varepsilon$ and any oriented n-plane $L \subset T_qM$, consider the orthogonal projection onto $\mathcal{V}_q$, $\pi_\mathcal{V}$, and the evaluation of Ω on L. Suppose that $\Omega(L) > 0$. By the singular value decomposition, there exist oriented orthonormal basis $\{e_1, \cdots, e_n\}$ for $\mathcal{H}_q$, orthonormal basis $\{e_{n+1}, \cdots, e_{n+m}\}$ for $\mathcal{V}_q$ and angles $\phi_1, \cdots, \phi_n \in [0, \pi/2)$ such that

$$\{\tilde{e}_j = \cos\phi_j \, e_j + \sin\phi_j \, e_{n+j}\}_{j=1}^n \tag{2.23}$$

constitutes an oriented, orthonormal basis for L. If $n > m$, ϕ_j is set to be zero for $j > m$. It follows that

$$\Omega(L) = \cos\phi_1 \cdots \cos\phi_n \ , \tag{2.24}$$

and the operator norm of $\pi_\mathcal{V}$ is

$$\mathfrak{s}(L) := \left\| \pi_\mathcal{V}|_L \right\|_{\text{op}} = \max\{\sin\phi_1, \cdots, \sin\phi_n\} \ . \tag{2.25}$$

Remark 2.7. The construction (2.23) works for $\Omega(L) = 0$ as well, and some of the angles would be $\pi/2$. The formulae (2.24) and (2.25) remain valid. We briefly explain this linear-algebraic construction. Consider the orthogonal projection onto $\mathcal{H}_q$, $\pi_\mathcal{H}$. Let $L_\mathcal{V} = \ker(\pi_\mathcal{H} : L \to \mathcal{H}_q)$; it is a linear subspace of $\mathcal{V}_q$. Let L' be the orthogonal complement of $L_\mathcal{V}$ in L. Then, $L = L' \oplus L_\mathcal{V}$, and $\pi_\mathcal{H} : L' \to \mathcal{H}_q$ is injective. Note that $\pi_\mathcal{V}(L')$ is orthogonal to $L_\mathcal{V}$. The linear subspace L' is the graph of a linear map from $\pi_\mathcal{H}(L') \subset \mathcal{H}_q$ to $\mathcal{V}_q$. The basis (2.23) is constructed by applying the singular value decomposition to this linear map together with an orthonormal basis for $L_\mathcal{V}$.

It is easy to see that the orthogonal complement of L has the following orthonormal basis:

$$\{\tilde{e}_\alpha = -\sin\phi_\alpha \, e_{\alpha-n} + \cos\phi_\alpha \, e_\alpha\}_{\alpha=n+1}^m \tag{2.26}$$

where $\phi_\alpha = \phi_{\alpha-n}$. If $m > n$, ϕ_α is set to be zero for $\alpha > 2n$. The following estimates will be needed later, and are straightforward to come by:

$$\sum_{i=1}^n \left| (\omega^j \otimes \omega^k)(\tilde{e}_i, \tilde{e}_i) \right| \leq n \ , \quad \sum_{i=1}^n \left| (\omega^\alpha \otimes \omega^j)(\tilde{e}_i, \tilde{e}_i) \right| \leq n\mathfrak{s} \tag{2.27}$$

and

$$\begin{aligned}
&\Big|(\omega^1\wedge\cdots\wedge\omega^n)(\tilde e_\alpha,\tilde e_1,\cdots,\widehat{\tilde e_i},\cdots,\tilde e_n)\Big|\le\mathfrak{s}\ ,\\
&\Big|(\omega^1\wedge\cdots\wedge\omega^n)(\tilde e_\alpha,\tilde e_\beta,\tilde e_1,\cdots,\widehat{\tilde e_i},\cdots,\widehat{\tilde e_j}\cdots,\tilde e_n)\Big|\le\mathfrak{s}^2\ ,\\
&\Big|(\omega^\alpha\wedge\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^n)(\tilde e_1,\cdots,\tilde e_n)\Big|\le n\mathfrak{s}\ ,\\
&\Big|(\omega^\alpha\wedge\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^n)(\tilde e_\beta,\tilde e_1,\cdots,\widehat{\tilde e_j},\cdots,\tilde e_n)\Big|\le 1\ ,\\
&\Big|(\omega^\alpha\wedge\omega^\beta\wedge\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\widehat{\omega^j}\wedge\cdots\wedge\omega^n)(\tilde e_1,\cdots,\tilde e_n)\Big|\le n(n-1)\mathfrak{s}^2
\end{aligned}\tag{2.28}$$

for any $i,j,k\in\{1,\dots,n\}$ and $\alpha,\beta\in\{n+1,\dots,n+m\}$.

The above estimates are the zeroth order estimate. For the first and third inequalities of (2.28), a more refined version will also be needed. It follows from (2.23) and (2.26) that

$$(\omega^1\wedge\cdots\wedge\omega^n)(\tilde e_\alpha,\tilde e_1,\cdots,\widehat{\tilde e_i},\cdots,\tilde e_n)=(-1)^i\delta_{\alpha(n+i)}\,\frac{\sin\phi_i}{\cos\phi_i}\,\prod_{k=1}^n\cos\phi_k\ .\tag{2.29}$$

Let $\tilde\omega^1,\cdots,\tilde\omega^n,\tilde\omega^{n+1},\cdots,\tilde\omega^{n+m}$ be the dual basis of $\tilde e_1,\cdots,\tilde e_n,\tilde e_{n+1},\cdots,\tilde e_{n+m}$. According to (2.23) and (2.26),

$$\omega^j=\cos\phi_j\,\tilde\omega^j-\sin\phi_j\,\tilde\omega^{n+j}\quad\text{and}\quad\omega^\alpha=\sin\phi_\alpha\,\tilde\omega^{\alpha-n}+\cos\phi_\alpha\,\tilde\omega^\alpha\ .$$

Hence,

$$\begin{aligned}
&(\omega^\alpha\wedge\omega^1\wedge\cdots\wedge\widehat{\omega^i}\wedge\cdots\wedge\omega^n)(\tilde e_1,\cdots,\tilde e_n)\\
&=\Big((\sin\phi_\alpha\tilde\omega^{\alpha-n})\wedge(\cos\phi_1\tilde\omega^1)\wedge\cdots\wedge\widehat{(\cos\phi_i\tilde\omega^i)}\wedge\cdots\wedge(\cos\phi_n\tilde\omega^n)\Big)(\tilde e_1,\cdots,\tilde e_n)\\
&=(-1)^{i+1}\delta_{\alpha(n+i)}\,\frac{\sin\phi_i}{\cos\phi_i}\,\prod_{k=1}^n\cos\phi_k\ .
\end{aligned}\tag{2.30}$$

3. Minimal submanifolds and stability conditions

3.1. **The stability of a minimal submanifold.** A submanifold $\Sigma\subset(M,g)$ is said to be *minimal* if its mean curvature vanishes, $H=0$. It means that Σ is a critical point of the volume functional. A minimal submanifold Σ is said to be *stable* if the second variation of the volume functional is positive at Σ. We now recall the second variational formula of the volume functional. The detail can be found in [15, §3.2].

Suppose that V is a normal vector field on Σ. There are two linear operators on $N\Sigma$ in the second variation formula. The first one is the partial Ricci operator defined by

$$\mathcal{R}(V)=\mathrm{tr}_\Sigma\big(R(\,\cdot\,,V)\,\cdot\,\big)^\perp$$

where R is the Riemann curvature tensor of (M, g). The second one is basically the norm-square of the second fundamental form along V. The shape operator along V is a symmetric map from $T\Sigma$ to itself, and is defined by

$$\mathcal{S}_V(X) = -(\nabla_X V)^{T\Sigma} = -\nabla_X V + \nabla^{\perp}_X V \text{ , or equivalently } \langle \mathcal{S}_V(X), Y\rangle = \langle \mathrm{I\!I}(X, Y), V\rangle$$

for any tangent vectors X and Y of Σ. By regarding $\mathcal{S}$ as a map from $N\Sigma$ to $\mathrm{Sym}^2(T\Sigma)$, define

$$\mathcal{A}(V) = \mathcal{S}^t \circ \mathcal{S}(V)$$

where $\mathcal{S}^t : \mathrm{Sym}^2(T\Sigma) \to N\Sigma$ is the transpose map of $\mathcal{S}$.

With this understanding, the second variation of the volume functional in the direction of V is

$$\int_\Sigma |\nabla^{\perp} V|^2 + \langle \mathcal{R}(V), V\rangle - \langle \mathcal{A}(V), V\rangle \tag{3.1}$$

Therefore, Σ is stable if and only if $(\nabla^{\perp})^*\nabla^{\perp} + \mathcal{R} - \mathcal{A}$ is a positive operator. Note that $(\nabla^{\perp})^*\nabla^{\perp}$ is always non-negative definite, and $\mathcal{R} - \mathcal{A}$ is a linear map on $N\Sigma$. Hence, the positivity of $\mathcal{R} - \mathcal{A}$ is a condition easier to check, and implies the stability of Σ.

Definition 3.1. A minimal submanifold $\Sigma \subset (M, g)$ is said to be *strongly stable* if $\mathcal{R} - \mathcal{A}$ is a (pointwise) positive operator on $N\Sigma$.

In terms of the notations introduced in §2.1, Σ is strongly stable if there exists a constant $c_0 > 0$ such that

$$-\sum_{\alpha,\beta,i} R_{i\alpha i\beta} v^\alpha v^\beta - \sum_{\alpha,\beta,i,j} h_{\alpha ij} h_{\beta ij} v^\alpha v^\beta \geq c_0 \sum_\alpha (v^\alpha)^2 \tag{3.2}$$

for any $(v^{n+1}, \cdots, v^{n+m}) \in \mathbb{R}^m$.

In particular, for a hypersurface Σ, the condition is

$$-\mathrm{Ric}(\nu, \nu) - |A|^2 \geq c_0,$$

where ν is a unit normal and $|A|^2 = \sum_{i,j} h_{ij}^2$.

3.2. **Proof of Proposition A.** It is easy to see that (3.2) holds for a totally geodesic submanifold in a manifold with negative sectional curvature. When the geometry has special properties, the condition (3.2) is equivalent to some natural curvature condition on the minimal submanifold.

3.2.1. *Complex submanifolds in Kähler manifolds.* Let (M^{2n}, g, J, ω) be a Kähler manifold, and $\Sigma^{2p} \subset M$ be a complex submanifold. The submanifold Σ is automatically minimal. In fact, the second variation (3.1) is always non-negative. In this case, the operator $\mathcal{R} - \mathcal{A}$ was studied by Simons in the famous paper [15, §3.5]. We briefly summarize his results. The condition is equivalent to that

$$\left\langle -J\left(\sum_{i=1}^{p} R^{\perp}(e_i, f_i)(V)\right), V\right\rangle \geq c_0 \, |V|^2$$

where $\{e_1, \cdots, e_p, f_1, \cdots, f_p\}$ is an orthonormal frame for $T\Sigma$ with $f_i = Je_i$. In other words, the normal bundle curvature contracting with ω_Σ is positive definite. It implies that the normal bundle of Σ admits no non-trivial holomorphic cross section.

3.2.2. *Minimal Lagrangians in Kähler–Einstein manifolds.* Let (M^{2n}, g, ω) be a Kähler–Einstein manifold, where ω is the Kähler form. Denote the Einstein constant by c, i.e. $\mathrm{Ric} = c\, g$. A half-dimensional submanifold $L^n \subset M$ is said to be Lagrangian if $\omega|_L$ vanishes. Suppose that L is both minimal and Lagrangian. Then, (3.2) is equivalent to the condition that

$$\mathrm{Ric}^L - c \ \text{ is a positive definite operator on } \ TL \ , \tag{3.3}$$

where Ric^L is the Ricci curvature of $g|_L$. For completeness, the derivation is included in Appendix A.1. We remark that when $c < 0$, a minimal Lagrangian is always stable. That is to say, the second variation (3.1) is strictly positive for any non-identically zero V; see [5, 13].

A case of particular interest is *special Lagrangians* in a Calabi–Yau manifold; see [8, §III]. The constant $c = 0$ for a Calabi–Yau manifold, and the strong stability condition (3.2) is equivalent to the positivity of Ric^L. By the Bochner formula, it implies that the first Betti number of L is zero. According to the result of McLean [12, Corollary 3.8], L is infinitesimally rigid as a special Lagrangian submanifold.

3.2.3. *Coassociatives in G_2 manifolds.* A G_2 manifold (M, g) is a 7-dimensional Riemannian manifold with G_2 holonomy. A *coassociative* submanifold is a special class of minimal, 4-dimensional submanifold in M. A complete story can be found in [8, §IV] and [10, ch.11–12], and a brief summary is included in Appendix A.2.

Suppose that $\Sigma^4 \subset M$ is coassociative. The strong stability condition (3.2) is equivalent to that

$$-2W_- + \frac{s}{3} \ \text{ is a positive definite operator on } \ \Lambda^2_- \ , \tag{3.4}$$

where W_- anti-self-dual part of the Weyl curvature of $g|_\Sigma$, and s is the scalar curvature of $g|_\Sigma$. The computation bears its own interest in G_2 geometry, and is included in Appendix A.2. According to the Weitzenböck formula for anti-self-dual 2-forms [7, Appendix C], (3.4)

implies that Σ has no non-trivial anti-self-dual harmonic 2-forms. Due to [12, Corollary 4.6], Σ is infinitesimally rigid as a coassociative submanifold.

3.3. **The Codazzi equation on a minimal submanifold.** Suppose that Σ is a minimal submanifold. Choose a local orthonormal frame $\{e_1, \cdots, e_{n+m}\}$ such that the restriction of $\{e_1, \cdots, e_n\}$ on Σ are tangent to Σ and the restriction of $\{e_{n+1}, \cdots, e_{n+m}\}$ to Σ are normal to Σ. Consider the following equation on Σ:

$$h_{\alpha ii;k} = e_k(h_{\alpha ii}) - 2\langle \nabla^{\Sigma}_{e_j} e_i, e_k\rangle h_{\alpha ki} - \langle \nabla^{\perp}_{e_j} e_\alpha, e_\beta\rangle h_{\beta ii} \ .$$

Since the mean curvature vanishes, the first and third terms are zero. For the second term, $\langle \nabla^{\Sigma}_{e_j} e_i, e_k\rangle$ is skew-symmetric in i and k, and $h_{\alpha ki}$ is symmetric in i and k. Hence, the second term is also zero. By combining it with the Codazzi equation (2.1),

$$R_{\alpha iij} = h_{\alpha ji;i} \ . \tag{3.5}$$

4. The convexity of ψ and a local uniqueness theorem of minimal submanifolds

Suppose that Σ is a minimal submanifold in (M, g) and consider the function $\psi = \sum_\alpha (y^\alpha)^2$ on the tubular neighborhood U_ε of Σ as in §2.2.1. Similar to [18], the strong stability of Σ is closely related to the positivity of the trace of $\mathrm{Hess}(\psi)$ over an n-dimensional subspace.

Proposition 4.1. *Let $\Sigma^n \subset (M, g)$ be a compact, oriented minimal submanifold that is strongly stable in the sense of* (3.2)*. There exist positive constants ε_1 and c which depend on the geometry of M and Σ and which have the following property. For any $q \in U_{\varepsilon_1}$ and any oriented n-plane $L \subset T_qM$,*

$$\mathrm{tr}_L \,\mathrm{Hess}(\psi) \geq c\left(\big(\mathfrak{s}(L)\big)^2 + \psi(q)\right) \tag{4.1}$$

where $\mathfrak{s}(L)$ is defined by (2.25)*.*

Proof. Let $p \in \Sigma$ be the point such that there is a normal geodesic in U_ε connecting p and q. To calculate $\mathrm{Hess}(\psi)$, take the frame $\{e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+m}\}$ constructed in §2.2. Let $\{\omega^1, \cdots, \omega^n, \omega^{n+1}, \cdots, \omega^{n+m}\}$ be the dual coframe. According to (2.4) and (2.5), $\mathrm{d}\psi = 2y^\alpha \omega^\alpha$, and thus $e_j(\psi) \equiv 0$. By (2.11),

$$\begin{aligned}
&\mathrm{Hess}(\psi)(e_i, e_j)\\
&= e_i(e_j(\psi)) - (\nabla_{e_i} e_j)(\psi) = -2y^\alpha\, \theta^\alpha_j(e_i)\\
&= -2y^\alpha\, h_{\alpha ij}\big|_p - 2y^\alpha x^k\, h_{\alpha ij;k}\big|_p - 2y^\alpha y^\beta\, R_{\alpha j\beta i}\big|_p - 2y^\alpha y^\beta\, (h_{\alpha jk} h_{\beta ik})\big|_p + \mathcal{O}((|\mathbf{x}|^2 + |\mathbf{y}|^2)^{\frac{3}{2}}) \ .
\end{aligned}$$

By (2.9), (2.11) and (2.12),

$$\begin{aligned}
\mathrm{Hess}(\psi)(e_\alpha, e_i) &= e_\alpha(e_i(\psi)) - (\nabla_{e_\alpha} e_i)(\psi) = -2y^\beta\, \theta_i^\beta(e_\alpha) \\
&= \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2)\ , \\
\mathrm{Hess}(\psi)(e_\alpha, e_\beta) &= e_\alpha(e_\beta(\psi)) - (\nabla_{e_\alpha} e_\beta)(\psi) = 2e_\alpha(y^\beta) - 2y^\gamma\, \theta_\beta^\gamma(e_\alpha) \\
&= 2\delta_{\alpha\beta} + \mathcal{O}(|\mathbf{x}|^2 + |\mathbf{y}|^2)\ .
\end{aligned} \tag{4.2}$$

We choose the frame so that L has an oriented, orthonormal basis of the form (2.23) and evaluate $\mathrm{tr}_L \mathrm{Hess}(\psi)$; note that all the x^j-coordinate of q are zero:

$$\begin{aligned}
\mathrm{tr}_L \mathrm{Hess}(\psi) &= \sum_j \mathrm{Hess}(\psi)(\cos\phi_j\, e_j + \sin\phi_j\, e_{n+j}, \cos\phi_j\, e_j + \sin\phi_j\, e_{n+j}) \\
&= \sum_j \Big[2\cos^2\phi_j \left(-y^\alpha\, h_{\alpha jj}\big|_p - y^\alpha y^\beta\, R_{\alpha j\beta j}\big|_p - y^\alpha y^\beta\, (h_{\alpha jk} h_{\beta jk})\big|_p \right) + 2\sin^2\phi_j \Big] \\
&\quad + \mathcal{O}(|\mathbf{y}|^3) + \mathfrak{s}(L)\cdot\mathcal{O}(|\mathbf{y}|^2) \\
&\geq 2\sum_{j,\alpha,\beta} \Big[-y^\alpha y^\beta\, R_{\alpha j\beta j}\big|_p - y^\alpha y^\beta\, (h_{\alpha jk} h_{\beta jk})\big|_p \Big] + 2\sum_j \sin^2\phi_j \\
&\quad + 2\sum_{j,\alpha,\beta} \sin^2\phi_j \Big[y^\alpha y^\beta\, R_{\alpha j\beta j}\big|_p + y^\alpha y^\beta\, (h_{\alpha jk} h_{\beta jk})\big|_p \Big] - \mathfrak{s}^2(L) - c'|\mathbf{y}|^3
\end{aligned}$$

for some $c' > 0$. By using the strong stability condition (3.2), this finishes the proof of the proposition. □

By the same argument as in [18], the convexity of ψ implies the following local uniqueness theorem of minimal submanifolds near Σ.

Theorem 4.2. *(Theorem A) Let $\Sigma^n \subset (M, g)$ be a compact minimal submanifold which is strongly stable in the sense of* (3.2)*. Then, there exists a tubular neighborhood U of Σ such that any compact minimal submanifold Γ in U with $\dim\Gamma \geq n$ must be contained in Σ. In other words, Σ is the only compact minimal submanifold in U with dimension no less than n.*

Proof. It basically follows from [18, Lemma 5.1] and Proposition 4.1. The only point to check is that the estimate of Proposition 4.1 holds for dimension greater than n. Namely, it remains to show that for any $q \in U_{\varepsilon_1}$ and any $\bar{n}$-plane $\bar{L} \subset T_qM$ with $\bar{n} > n$,

$$\mathrm{tr}_{\bar{L}} \mathrm{Hess}(\psi) \geq c_0$$

for some positive constant c_0.

The argument is similar to Remark 2.7. Pick an $(\bar{n} - n)$-subspace of $\ker(\pi_{\mathcal{H}} : \bar{L} \to \mathcal{H}_q)$. Denote it by $L_{\mathcal{V}}$. Note that $L_{\mathcal{V}}$ belong to $\mathcal{V}_q$. Let L be the orthogonal complement of $L_{\mathcal{V}}$ in

$\bar{L}$. The dimension of L is n. By Proposition 4.1 and (4.2), the trace of the Hessian of ψ over $\bar{L}$ has the following lower bound:

$$\begin{aligned}\operatorname{tr}_{\bar{L}} \operatorname{Hess}(\psi) &= \operatorname{tr}_L \operatorname{Hess}(\psi) + \operatorname{tr}_{L_\nu} \operatorname{Hess}(\psi) \\ &\geq c\,\psi(q) + \big(2(\bar{n}-n) - c'\psi(q)\big) \ .\end{aligned}$$

Thus, the quantity is positive when $\psi(q)$ is sufficiently small. □

Remark 4.3. In this rigidity theorem, it is not hard to see that the minimal submanifold Σ needs not to be orientable. However, in order to have ψ to be well-defined on a tubular neighborhood, Σ has to be embedded.

5. Further estimates needed for the stability theorem

From now on, Σ is taken to be a strongly stable minimal submanifold and we see in the last section that the distance function ψ to Σ defined on U_ε satisfies a convexity condition.

To study the dynamical stability of mean curvature flows near Σ, we need to measure how close a nearby submanifold is to Σ. The distance function ψ gives such a measurement in C^0. In order to obtain measurements in higher derivatives, we extend the volume form and the second fundamental form of Σ to the tubular neighborhood U_ε. In particular, in §2.2.4, the volume form of Σ is extended to an n-form Ω on U_ε. The restriction of Ω to another n-dimensional submanifold Γ, which is denoted by $*\Omega$, measures how close Γ is to Σ in C^1. The evolution equation of $*\Omega$ along the mean curvature flow plays an essential role for the estimates. The equation naturally involves the restriction of the covariant derivatives/second covariant derivatives of Ω on Γ. In this section, we derive estimates of these quantities in preparation for the proof of the stability theorem.

5.1. Extension of auxiliary tensors to U_ε.

We adopt the frame and coordinate constructed in §2.

The second fundamental form of Σ can also be extended to U_ε by parallel transport along normal geodesics, as explained in §2.2.4. Denote the extension by $\mathrm{I\!I}^\Sigma$, which, in terms of the frames, is given by

$$\mathrm{I\!I}^\Sigma = h_{\alpha ij}\,\omega^i \otimes \omega^j \otimes e_\alpha \ . \tag{5.1}$$

In other words, for any $q \in U_\varepsilon$, $h_{\alpha ij}(q) = h_{\alpha ij}(p)$ where $p \in \Sigma$ is the unique point such that there is a normal geodesic in U_ε connecting p and q, see Definition 2.3. To avoid introducing more notations, we use the metric g to lower the indices of $\mathrm{I\!I}^\Sigma$, and then $\mathrm{I\!I}^\Sigma = h_{\alpha ij}\, e_i \otimes e_j \otimes e_\alpha$.

Suppose that Γ is an oriented, n-dimensional submanifold in $U_\varepsilon \subset M$ with $\Omega(\Gamma) > 0$. With the above extension, we can compare the second fundamental form of Γ with that of Σ. For any $q \in \Gamma$, choose a local orthonormal frame $\{\tilde{e}_1, \cdots, \tilde{e}_n, \tilde{e}_{n+1}, \cdots, \tilde{e}_{n+m}\}$ on a neighborhood

of q in M such that the restriction of $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$ on Γ form an oriented frame for $T\Gamma$, and the restriction of $\{\tilde{e}_{n+1}, \cdots, \tilde{e}_{n+m}\}$ on Γ form a frame for $N\Gamma$. With this, the second fundamental form of Γ is

$$\mathrm{I\!I}^{\Gamma} = \tilde{h}_{\alpha ij}\, \tilde{e}_i \otimes \tilde{e}_j \otimes \tilde{e}_\alpha \quad \text{where} \quad \tilde{h}_{\alpha ij} = \langle \nabla_{\tilde{e}_i} \tilde{e}_j, \tilde{e}_\alpha \rangle \ . \tag{5.2}$$

As explained in §2.2.4, we may assume that these frames are of the form (2.23) and (2.26) at q. The inverse transform reads

$$e_j = \cos\phi_j\, \tilde{e}_j - \sin\phi_j\, \tilde{e}_{n+j} \quad \text{and} \quad e_\alpha = \sin\phi_\alpha\, \tilde{e}_{\alpha-n} + \cos\phi_\alpha\, \tilde{e}_\alpha \ . \tag{5.3}$$

It follows that

$$\mathrm{I\!I}^{\Sigma}\big|_q = h_{\alpha ij}(p)\, (\cos\phi_i\, \tilde{e}_i - \sin\phi_i\, \tilde{e}_{n+i}) \otimes (\cos\phi_j\, \tilde{e}_j - \sin\phi_j\, \tilde{e}_{n+j}) \otimes (\sin\phi_\alpha\, \tilde{e}_{\alpha-n} + \cos\phi_\alpha\, \tilde{e}_\alpha). \tag{5.4}$$

Hence,

$$\langle \mathrm{I\!I}^{\Gamma}, \mathrm{I\!I}^{\Sigma} \rangle\big|_q = \sum_{\alpha,i,j} \Big(\cos\phi_i \cos\phi_j \cos\phi_\alpha\, \tilde{h}_{\alpha ij}\, h_{\alpha ij}(p) \Big) \ . \tag{5.5}$$

In the above expression, $h_{\alpha ij}(p)$ depends only on $p \in \Sigma$, while ϕ_i, ϕ_α, and $\tilde{h}_{\alpha ij}$ all depend on $q \in \Gamma$.

We extend another tensor which is related to the strong stability condition (3.2). Consider the parallel transport of the following tensor on Σ along normal geodesics:

$$(R_{\alpha i\beta j} + h_{\alpha ik} h_{\beta jk})\ (\omega^i \otimes \omega^j) \otimes (e^\alpha \otimes e^\beta) \ ,$$

which is considered to be defined on U_ε. Pairing the last component with $\nabla\psi/2$ produces a tensor of the same type as $\mathrm{I\!I}^{\Sigma}$, which is denoted by S^{Σ}:

$$S^{\Sigma}|_q = y^\beta\, (R_{\alpha i\beta j}(p) + (h_{\alpha ik} h_{\beta jk})(p))\ \omega^i \otimes \omega^j \otimes e_\alpha \tag{5.6}$$

where $p \in \Sigma$ is the point such that there is a unique normal geodesic in U_ε connecting p and q. Similarly,

$$\langle \mathrm{I\!I}^{\Gamma}, S^{\Sigma} \rangle\big|_q = \sum_{\alpha,\beta,i,j} \left(\cos\phi_i \cos\phi_j \cos\phi_\alpha\, \tilde{h}_{\alpha ij}\, y^\beta \big(R_{\alpha i\beta j}(p) + \sum_k (h_{\alpha ik} h_{\beta jk})(p) \big) \right) \ . \tag{5.7}$$

Again in the above expression, $R_{\alpha i\beta j}(p) + \sum_k (h_{\alpha ik} h_{\beta jk})(p)$ depends only on $p \in \Sigma$, while $\phi_i, \phi_\alpha, y^\beta$, and $\tilde{h}_{\alpha ij}$ all depend on $q \in \Gamma$.

In the rest of this subsection, we assume $\Omega(T_q\Gamma) > \frac{1}{2}$ and estimate $\langle \mathrm{I\!I}^{\Gamma}, \mathrm{I\!I}^{\Sigma} \rangle\big|_q$ and $\langle \mathrm{I\!I}^{\Gamma}, S^{\Sigma} \rangle\big|_q$. We assume that $T_q\Gamma$ has an oriented frame of the form (2.23), and $N_q\Gamma$ has a frame of the form (2.26). Since $\Omega(T_q\Gamma) > \frac{1}{2}$, it follows from (2.24) that

$$\cos\phi_j \geq \cos\phi_1 \cdots \cos\phi_n > \frac{1}{2} \quad \text{for } j \in \{1, \ldots, n\} \ .$$

A direct computation shows that

$$\begin{aligned} &0 < 1 - \cos\phi_j = 1 - \sqrt{1-\sin^2\phi_j} \le 2\left(\mathfrak{s}(T_q\Gamma)\right)^2 \ , \\ &\left|\frac{1}{\cos\phi_j} - \cos\phi_j\right| = \frac{\sin^2\phi_j}{\cos\phi_j} < 2\left(\mathfrak{s}(T_q\Gamma)\right)^2 . \end{aligned} \tag{5.8}$$

Suppose that $\mathfrak{s}$ in (2.25) is achieved at ϕ_1, and then

$$\begin{aligned} &\Omega(T_q\Gamma) \le \cos\phi_1 = \sqrt{1-\sin^2\phi_1} \le 1 - \frac{1}{2}\sin^2\phi_1 \\ \Rightarrow\ &1 - \Omega(T_q\Gamma) \ge \frac{1}{2}\left(\mathfrak{s}(T_q\Gamma)\right)^2 \ . \end{aligned} \tag{5.9}$$

On the other hand,

$$1 - \Omega(T_q\Gamma) \le 1 - \left(\Omega(T_q\Gamma)\right)^2 = 1 - \prod_{j=1}^{n}(1-\sin^2\phi_j) \le c(n)\left(\mathfrak{s}(T_q\Gamma)\right)^2 \tag{5.10}$$

for some dimensional constant $c(n)$.

Applying the estimate (5.8) to (5.5) and (5.7), we obtain

$$\begin{aligned} &\left|\langle \mathrm{I\!I}^\Gamma, \mathrm{I\!I}^\Sigma\rangle\big|_q - \sum_{\alpha,i,j}\tilde{h}_{\alpha ij}\, h_{\alpha ij}(p)\right| \le c\left(\mathfrak{s}(q)\right)^2 |\mathrm{I\!I}^\Gamma| \ , \\ &\left|\langle \mathrm{I\!I}^\Gamma, S^\Sigma\rangle\big|_q - \sum_{\alpha,\beta,i,j}\left(\tilde{h}_{\alpha ij}\, y^\beta\big(R_{\alpha i\beta j}(p) + \sum_k (h_{\alpha ik}h_{\beta jk})(p)\big)\right)\right| \le c\left(\mathfrak{s}(q)\right)^2\sqrt{\psi(q)}\,|\mathrm{I\!I}^\Gamma| \end{aligned} \tag{5.11}$$

for some constant c depending on the geometry of Σ and M.

5.2. **Estimates involving the derivatives of Ω.** In this subsection, we derive estimates that involve derivatives of Ω, which are needed in the proof of Theorem B. In the following three lemmas, we estimate quantities that appear naturally in the evolution equation of $*\Omega$ (6.1).

Let Γ be an n-dimensional submanifold in the tubular neighborhood of Σ. The function $*\Omega$ is the Hodge star of $\Omega|_\Gamma$ with respect to the induced metric on Γ, and is the same as $\Omega(T_q\Gamma)$. We assume throughout this subsection that $*\Omega(q) > \frac{1}{2}$ for any $q \in \Gamma$. For each $q \in \Gamma$, let $p \in \Sigma$ be the point such that there is a unique normal geodesic in U_ε connecting p and q; see Definition 2.3. We use the coordinate and frame constructed in §2.1 to carry out the computation. Moreover, we assume that $T_q\Gamma$ has an oriented frame of the form (2.23), and $N_q\Gamma$ has a frame of the form (2.26). For $q \in \Gamma$, $\psi(q)$ is a $\mathcal{C}^0$ order quantity. $*\Omega(q)$ and $\mathfrak{s}(q)$ are both $\mathcal{C}^1$ order quantities that depend on the tangent space $T_q\Gamma$ at q, where $\mathfrak{s}(q) = \mathfrak{s}(T_q\Gamma)$ is defined in (2.25).

5.2.1. *The restriction of the derivative of* Ω *to* Γ. To compute $\nabla\Omega$, it is convenient to introduce the following shorthand notations:

$$\Omega^j = \iota(e_j)\Omega = (-1)^{j+1}\omega^1 \wedge \cdots \wedge \widehat{\omega^j} \wedge \cdots \wedge \omega^n \ , \tag{5.12}$$

$$\Omega^{jk} = \iota(e_k)\iota(e_j)\Omega = \begin{cases} (-1)^{j+k}\,\omega^1 \wedge \cdots \wedge \widehat{\omega^k} \wedge \cdots \wedge \widehat{\omega^j} \wedge \cdots \wedge \omega^n & \text{if } k < j \ , \\ (-1)^{j+k+1}\,\omega^1 \wedge \cdots \wedge \widehat{\omega^j} \wedge \cdots \wedge \widehat{\omega^k} \wedge \cdots \wedge \omega^n & \text{if } k > j \ . \end{cases} \tag{5.13}$$

The covariant derivative of Ω is

$$\nabla\Omega = (\nabla\omega^j) \wedge \Omega^j = \theta_j^\alpha \otimes (\omega^\alpha \wedge \Omega^j) \ . \tag{5.14}$$

Lemma 5.1. *Let $\Sigma^n \subset (M, g)$ be a compact, oriented minimal submanifold. Then, there exist a positive constant c which depends on the geometry of M and Σ and which has the following property. Suppose that $\Gamma \subset U_\varepsilon$ is an oriented n-dimensional submanifold with $*\Omega(q) > \frac{1}{2}$ for any $q \in \Gamma$. Then,*

$$\begin{aligned} &\left| \sum_{\alpha,j,k} \left[(-1)^j\, \tilde{h}_{\alpha jk}\, (\nabla_{\tilde{e}_k}\Omega)(\tilde{e}_\alpha, \tilde{e}_1, \cdots, \widehat{\tilde{e}_j}, \cdots, \tilde{e}_n) \right] + (*\Omega)\, \langle \mathrm{I\!I}^\Gamma, \mathrm{I\!I}^\Sigma + S^\Sigma \rangle \right| \\ &\leq c \left(\big(\mathfrak{s}(q)\big)^2 + \psi(q) \right) \left| \mathrm{I\!I}^\Gamma \right| \end{aligned}$$

at any $q \in \Gamma$. The summation is indeed a contraction between $\mathrm{I\!I}^\Gamma$ and $\nabla\Omega$, and is independent of the choice of the orthonormal frame.

Proof. By (5.14),

$$\sum_{\alpha,j,k} \left[(-1)^j\, \tilde{h}_{\alpha jk}\, (\nabla_{\tilde{e}_k}\Omega)(\tilde{e}_\alpha, \tilde{e}_1, \cdots, \widehat{\tilde{e}_j}, \cdots, \tilde{e}_n) \right] \Big|_q = - \sum_{\alpha,j,k} \left[\tilde{h}_{\alpha jk} \theta_j^\alpha(\tilde{e}_k) \frac{\cos\phi_\alpha}{\cos\phi_j} (*\Omega) \right] \ .$$

According to (2.11), (2.23) and the fact that the x^j-coordinates of q are all zero,

$$\left| \theta_j^\alpha(\tilde{e}_k)\big|_q - \cos\phi_k\, h_{\alpha jk}(p) - \cos\phi_k\, y^\beta\, (R_{\alpha j\beta k} + h_{\alpha jl} h_{\beta kl})(p) \right| \leq c_1 \left(\big(\mathfrak{s}(q)\big)^2 + \psi(q) \right) \tag{5.15}$$

at q. Combining this with (5.8) and (5.11) finishes the proof of this lemma. □

5.2.2. *The restriction of the second derivative of* Ω *to* Γ. Since

$$\begin{aligned} \nabla\omega^\alpha &= -\theta_i^\alpha \otimes \omega^i - \theta_\beta^\alpha \otimes \omega^\beta \quad \text{and} \\ \nabla\Omega^j &= (\nabla\omega^k) \wedge \Omega^{jk} = \theta_j^k \otimes \Omega^k + \theta_k^\alpha \otimes (\omega^\alpha \wedge \Omega^{jk}) \ , \end{aligned}$$

the covariant derivative of (5.14) is

$$\begin{aligned} \nabla^2\Omega = &-(\theta_i^\alpha \otimes \theta_i^\alpha) \otimes \Omega + (\theta_k^\alpha \otimes \theta_j^\beta) \otimes (\omega^\beta \wedge \omega^\alpha \wedge \Omega^{jk}) \\ &+ \left(\nabla\theta_i^\alpha + \theta_\beta^\alpha \otimes \theta_i^\beta + \theta_k^i \otimes \theta_k^\alpha \right) \otimes (\omega^\alpha \wedge \Omega^i) \end{aligned} \tag{5.16}$$

where $\nabla\theta_i^\alpha$ is the covariant derivative of a local section of T^*M.

Lemma 5.2. *Let $\Sigma^n \subset (M,g)$ be a compact, oriented minimal submanifold. Then there exists a positive constant c which depends on the geometry of M and Σ and which has the following property. Suppose that $\Gamma \subset U_\varepsilon$ is an oriented n-dimensional submanifold with $*\Omega(q) > \frac{1}{2}$ for any $q \in \Gamma$. Then,*

$$\left| \sum_k \left[(\nabla^2_{\tilde{e}_k,\tilde{e}_k} \Omega)(\tilde{e}_1, \cdots, \tilde{e}_n) \right] - \sum_{\alpha,i,k} \left[(-1)^i \Omega(\tilde{e}_\alpha, \tilde{e}_1, \cdots, \widehat{\tilde{e}_i}, \cdots, \tilde{e}_n) R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{i}} \right] + (*\Omega) \left| \mathrm{I\!I}^\Sigma + S^\Sigma \right|^2 \right|$$
$$\leq c \left(\mathfrak{s}^2(q) + \psi(q) \right)$$

at any $q \in \Gamma$, where $R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{j}} = \langle R(\tilde{e}_k, \tilde{e}_j)\tilde{e}_k, \tilde{e}_\alpha \rangle$ are components of the restriction of the curvature tensor of M along Γ. Note that the two summations are independent of the choice of the orthonormal frame.

Proof. We examine the components on the right hand side of (5.16). Due to (2.11) and (2.12),

$$\begin{aligned} |\theta^j_i(\tilde{e}_k)| &\leq c_2 \left(\mathfrak{s}^2(q) + \sqrt{\psi(q)} \right) , \\ |\theta^\beta_\alpha(\tilde{e}_k)| &\leq c_2 \left(\mathfrak{s}^2(q) + \sqrt{\psi(q)} \right) \end{aligned} \tag{5.17}$$

for any $i,j,k \in \{1,\ldots,n\}$ and $\alpha,\beta \in \{n+1,\ldots,n+m\}$. With (5.15) and the third and fifth line of (2.28),

$$\begin{aligned} \left| \left(\theta^\alpha_j(\tilde{e}_k)\right) \left(\theta^\beta_i(\tilde{e}_k)\right) \left((\omega^\beta \wedge \omega^\alpha \wedge \Omega^{ij})(\tilde{e}_1, \cdots, \tilde{e}_n) \right) \right| &\leq c_3 \big(\mathfrak{s}(q)\big)^2 , \\ \left| \left(\theta^\beta_\alpha(\tilde{e}_k)\right) \left(\theta^\beta_j(\tilde{e}_k)\right) \left((\omega^\alpha \wedge \Omega^j)(\tilde{e}_1, \cdots, \tilde{e}_n) \right) \right| &\leq c_3 \left(\big(\mathfrak{s}(q)\big)^2 + \psi(q) \right) , \\ \left| \left(\theta^j_i(\tilde{e}_k)\right) \left(\theta^\alpha_i(\tilde{e}_k)\right) \left((\omega^\alpha \wedge \Omega^j)(\tilde{e}_1, \cdots, \tilde{e}_n) \right) \right| &\leq c_3 \left(\big(\mathfrak{s}(q)\big)^2 + \psi(q) \right) . \end{aligned} \tag{5.18}$$

According to (5.15) and (5.8),

$$\left| \sum_{\alpha,j,k} \big(\theta^\alpha_j(\tilde{e}_k)\big)^2 - |\mathrm{I\!I}^\Sigma + S^\Sigma|^2 \right| \leq c_4 \left((\mathfrak{s}(q))^2 + \psi(q) \right) . \tag{5.19}$$

By (5.16), (5.19) and (5.18),

$$\begin{aligned} &\left| \left[(\nabla^2_{\tilde{e}_k,\tilde{e}_k} \Omega)(\tilde{e}_1, \cdots, \tilde{e}_n) \right] - \left((\nabla \theta^\alpha_i)(\tilde{e}_k, \tilde{e}_k) \right) \left((\omega^\alpha \wedge \Omega^i)(\tilde{e}_1, \cdots, \tilde{e}_n) \right) + (*\Omega) \left| \mathrm{I\!I}^\Sigma + S^\Sigma \right|^2 \right| \\ &\leq c_5 \left((\mathfrak{s}(q))^2 + \psi(q) \right) . \end{aligned} \tag{5.20}$$

The next step is to compute $\nabla \theta^\alpha_i$:

$$\begin{aligned} &\theta^\alpha_i = \theta^\alpha_i(e_j)\, \omega^j + \theta^\alpha_i(e_\beta)\, \omega^\beta \\ \Rightarrow\ &\nabla \theta^\alpha_i = \mathrm{d}(\theta^\alpha_i(e_j)) \otimes \omega^j + \mathrm{d}(\theta^\alpha_i(e_\beta)) \otimes \omega^\beta + \theta^\alpha_i(e_j)\, \nabla \omega^j + \theta^\alpha_i(e_\beta)\, \nabla \omega^\beta . \end{aligned} \tag{5.21}$$

By (5.17) and (5.15), we have the following estimate at q:

$$\begin{aligned}\left|(\nabla\omega^j)(\tilde{e}_k,\tilde{e}_k)\right| &= \left|\theta_i^j(\tilde{e}_k)\,\omega^i(\tilde{e}_k)-\theta_j^\alpha(\tilde{e}_k)\,\omega^\alpha(\tilde{e}_k)\right| \le c_6\left(\mathfrak{s}(q)+\sqrt{\psi(q)}\right)\ ,\\ \left|(\nabla\omega^\beta)(\tilde{e}_k,\tilde{e}_k)\right| &= \left|\theta_j^\beta(\tilde{e}_k)\,\omega^j(\tilde{e}_k)+\theta_\gamma^\beta(\tilde{e}_k)\,\omega^\gamma(\tilde{e}_k)\right| \le c_6\ .\end{aligned}$$

Together with (2.11),

$$\left|\left(\theta_i^\alpha(e_j)\,\nabla\omega^j+\theta_i^\alpha(e_\beta)\,\nabla\omega^\beta\right)(\tilde{e}_k,\tilde{e}_k)\right| \le c_7\left(\mathfrak{s}(q)+\sqrt{\psi(q)}\right)\ . \tag{5.22}$$

It follows from (2.11) and (2.8) that

$$\begin{aligned}\left|\mathrm{d}(\theta_i^\alpha(e_j))-(h_{\alpha ij;k})\,(p)\omega^k+(R_{\alpha i\beta j}+h_{\alpha ik}h_{\beta jk})\,(p)\omega^\beta\right| &\le c_8\sqrt{\psi(q)}\ ,\\ \left|\mathrm{d}(\theta_i^\alpha(e_\beta))-\frac{1}{2}R_{\alpha i\gamma\beta}(p)\omega^\gamma\right| &\le c_8\end{aligned} \tag{5.23}$$

where the norm on the left hand side is induced by the Riemannian metric g. By combining (5.21), (5.22) an (5.23),

$$\left|(\nabla\theta_i^\alpha)(\tilde{e}_k,\tilde{e}_k)-\cos^2\phi_k\,h_{\alpha ik;k}(p)\right| \le c_9\left(\mathfrak{s}(q)+\sqrt{\psi(q)}\right)\ .$$

It together with (2.30) and (5.8) gives that

$$\begin{aligned}&\left|-\big((\nabla\theta_i^\alpha)(\tilde{e}_k,\tilde{e}_k)\,(\omega^\alpha\wedge\Omega^i)(\tilde{e}_1,\cdots,\tilde{e}_n)\big)+(*\Omega)\sin\phi_i\,h_{(n+i)ik;k}(p)\right|\\ &\le c_{10}\left((\mathfrak{s}(q))^2+\psi(q)\right)\ .\end{aligned} \tag{5.24}$$

It remains to calculate the second term in the asserted inequality of the lemma. By (2.29),

$$\sum_{\alpha,i,k}(-1)^i\Omega(\tilde{e}_\alpha,\tilde{e}_1,\cdots,\widehat{\tilde{e}_i},\cdots,\tilde{e}_n)R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{i}} = (*\Omega)\sum_{i,k}\frac{\sin\phi_i}{\cos\phi_i}R(\tilde{e}_{n+i},\tilde{e}_k,\tilde{e}_k,\tilde{e}_i)\ .$$

With (2.23), (2.26) and (5.8),

$$\left|\sum_{\alpha,i,k}(-1)^i\Omega(\tilde{e}_\alpha,\tilde{e}_1,\cdots,\widehat{\tilde{e}_i},\cdots,\tilde{e}_n)R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{i}}+(*\Omega)\sum_{i,k}\sin\phi_i R_{(n+i)kki}\right| \le c_{11}\,(\mathfrak{s}(q))^2\ .$$

Since $\left|R_{(n+i)kki}|_q-R_{(n+i)kki}|_p\right|\le c_{12}\sqrt{\psi(q)}$, we have

$$\left|\sum_{\alpha,i,k}(-1)^i\Omega(\tilde{e}_\alpha,\tilde{e}_1,\cdots,\widehat{\tilde{e}_i},\cdots,\tilde{e}_n)R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{i}}+(*\Omega)\sum_{i,k}\sin\phi_i R_{(n+i)kki}(p)\right| \le c_{13}\left((\mathfrak{s}(q))^2+\psi(q)\right)\ . \tag{5.25}$$

By (5.20), (5.24), (5.25) and the Codazzi equation (3.5), it finishes the proof of the lemma. □

Remark 5.3. The tensor S^Σ is needed for Lemma 5.2; otherwise the error term would be bigger. However, S^Σ will only be used in some intermediate steps in the proof of Theorem B.

5.2.3. *The derivative of* $*\Omega$ *along* Γ. The following lemma relates the derivative of $*\Omega$ along Γ and the second fundamental form of Γ.

Lemma 5.4. *Let $\Sigma^n \subset (M, g)$ be a compact, oriented minimal submanifold. Then, there exist a positive constant c which depends on the geometry of M and Σ and which has the following property. Suppose that $\Gamma \subset U_\varepsilon$ is an oriented n-dimensional submanifold with $*\Omega(q) > \frac{1}{2}$ for any $q \in \Gamma$. Then,*

$$|\nabla^\Gamma(*\Omega)|^2 \leq c\,(\mathfrak{s}(q)(*\Omega))^2\,|\mathrm{I\!I}^\Gamma - \mathrm{I\!I}^\Sigma|^2 + c\left((\mathfrak{s}(q))^2 + \psi(q)\right)^2$$

for any $q \in \Gamma$.

Proof. We compute

$$\begin{aligned}
\nabla^\Gamma(*\Omega) &= [\tilde{e}_j(\Omega(\tilde{e}_1, \cdots, \tilde{e}_n))]\,\tilde{\omega}^j \\
&= \left[(\nabla_{\tilde{e}_j}\Omega)(\tilde{e}_1, \cdots, \tilde{e}_n) + \sum_{i=1}^n \Omega(\tilde{e}_1, \cdots, \tilde{e}_{i-1}, \nabla_{\tilde{e}_j}\tilde{e}_i, \tilde{e}_{i+1}, \cdots, \tilde{e}_n)\right]\tilde{\omega}^j \\
&= \left[(\nabla_{\tilde{e}_j}\Omega)(\tilde{e}_1, \cdots, \tilde{e}_n) + \sum_{i=1}^n \tilde{h}_{\alpha ij}\Omega(\tilde{e}_1, \cdots, \tilde{e}_{i-1}, \tilde{e}_\alpha, \tilde{e}_{i+1}, \cdots, \tilde{e}_n)\right]\tilde{\omega}^j\ .
\end{aligned}$$

Note that the expression is tensorial, and we use the frame (2.23) and (2.26) to proceed.

Due to (5.14) and (2.30),

$$\begin{aligned}
(\nabla_{\tilde{e}_j}\Omega)(\tilde{e}_1, \cdots, \tilde{e}_n) &= \theta_i^\alpha(\tilde{e}_j)\,(\omega^\alpha \wedge \Omega^i)(\tilde{e}_1, \cdots, \tilde{e}_n) \\
&= \left(\cos\phi_j\,\theta_i^{n+i}(e_j) + \sin\phi_j\,\theta_i^{n+i}(e_{n+j})\right)\frac{\sin\phi_i}{\cos\phi_i}(*\Omega)\ .
\end{aligned}$$

By (2.11) and (5.8), at q,

$$\begin{aligned}
&\left|(\nabla_{\tilde{e}_j}\Omega)(\tilde{e}_1, \cdots, \tilde{e}_n) - (*\Omega)\sum_{i=1}^n\left[\frac{\sin\phi_i}{\cos\phi_i}\cos\phi_{n+i}\cos\phi_i\cos\phi_j h_{(n+i)ij}(p)\right]\right| \\
&\leq c_1\left((\mathfrak{s}(q))^2 + \psi(q)\right)\ .
\end{aligned}$$

According to (2.29),

$$\sum_{i=1}^n \tilde{h}_{\alpha ij}\Omega(\tilde{e}_1, \cdots, \tilde{e}_{i-1}, \tilde{e}_\alpha, \tilde{e}_{i+1}, \cdots, \tilde{e}_n) = -(*\Omega)\sum_{i=1}^n\left[\frac{\sin\phi_i}{\cos\phi_i}\tilde{h}_{(n+i)ij}\right]\ .$$

To sum up,

$$\begin{aligned}
\left|\nabla^{\Gamma}(*\Omega)\right|^2 &= \sum_{j=1}^{n}\left[\tilde{e}_j(\Omega(\tilde{e}_1,\cdots,\tilde{e}_n))\right]^2 \\
&\le 2\sum_{j=1}^{n}\left|(\nabla_{\tilde{e}_j}\Omega)(\tilde{e}_1,\cdots,\tilde{e}_n) - (*\Omega)\sum_{i=1}^{n}\left[\frac{\sin\phi_i}{\cos\phi_i}\cos\phi_{n+i}\cos\phi_i\cos\phi_j h_{(n+i)ij}(p)\right]\right|^2 \\
&\quad + 2(*\Omega)^2\sum_{j=1}^{n}\left|\sum_{i=1}^{n}\frac{\sin\phi_i}{\cos\phi_i}\left(\cos\phi_{n+i}\cos\phi_i\cos\phi_j h_{(n+i)ij}(p) - \tilde{h}_{(n+i)ij}\right)\right|^2 \\
&\le 4nc_1^2\left((\mathfrak{s}(q))^2+\psi(q)\right)^2 \\
&\quad + 8n\left(\mathfrak{s}(q)\right)^2(*\Omega)^2\sum_{i,j}\left|\tilde{h}_{(n+i)ij} - \cos\phi_{n+i}\cos\phi_i\cos\phi_j h_{(n+i)ij}(p)\right|^2
\end{aligned}$$

By (5.2) and (5.4),

$$|\mathrm{I\!I}^{\Gamma} - \mathrm{I\!I}^{\Sigma}|^2 \ge \sum_{\alpha,i,j}\left|\tilde{h}_{\alpha ij} - \cos\phi_i\cos\phi_j\cos\phi_\alpha h_{\alpha ij}(p)\right|^2$$

This completes the proof of this lemma. □

6. Stability of the mean curvature flow

After the preparation in the last sections, we consider the mean curvature flow. We first recall the following proposition from [23, Proposition 3.1].

Proposition 6.1. *Along the mean curvature flow Γ_t in M, $*\Omega = \Omega(\tilde{e}_1,\cdots,\tilde{e}_n)$ satisfies*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}*\Omega &= \Delta^{\Gamma_t}*\Omega + *\Omega\Big(\sum_{\alpha,i,k}\tilde{h}_{\alpha ik}^2\Big) \\
&\quad - 2\sum_{\alpha,\beta,k}[\tilde{\Omega}_{\alpha\beta 3\cdots n}\tilde{h}_{\alpha 1k}\tilde{h}_{\beta 2k} + \tilde{\Omega}_{\alpha 2\beta\cdots n}\tilde{h}_{\alpha 1k}\tilde{h}_{\beta 3k} + \cdots + \tilde{\Omega}_{1\cdots(n-2)\alpha\beta}\tilde{h}_{\alpha(n-1)k}\tilde{h}_{\beta nk}] \\
&\quad - 2(\nabla_{\tilde{e}_k}\Omega)(\tilde{e}_\alpha,\cdots,\tilde{e}_n)\tilde{h}_{\alpha 1k} - \cdots - 2(\nabla_{\tilde{e}_k}\Omega)(\tilde{e}_1,\cdots,\tilde{e}_\alpha)\tilde{h}_{\alpha nk} \\
&\quad - \sum_{\alpha,k}[\tilde{\Omega}_{\alpha 2\cdots n}R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{1}} + \cdots + \tilde{\Omega}_{1\cdots(n-1)\alpha}R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{n}}] - (\nabla^2_{\tilde{e}_k,\tilde{e}_k}\Omega)(\tilde{e}_1,\cdots,\tilde{e}_n)
\end{aligned} \tag{6.1}$$

where Δ^{Γ_t} denotes the time-dependent Laplacian on Γ_t, $\tilde{\Omega}_{\alpha\beta 3\cdots n} = \Omega(\tilde{e}_\alpha,\tilde{e}_\beta,\tilde{e}_3,\cdots,\tilde{e}_n)$ etc., and $R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{1}} = \langle R(\tilde{e}_k,\tilde{e}_1)\tilde{e}_k,\tilde{e}_\alpha\rangle$, etc. are the coefficients of the curvature operators of M.

When Ω is a parallel form in M, $\nabla\Omega\equiv 0$, this recovers an important formula in proving the long time existence result of the graphical mean curvature flow in [22].

6.1. **Proof of Theorem B.** A finite time singularity of the mean curvature flow happens exactly when the second fundamental becomes unbounded; see Huisken [9], also [23]. The following theorem shows that if we start with a submanifold which is $\mathcal{C}^1$ close to a strongly stable minimal submanifold Σ, then the mean curvature flow exists for all time, and converges smoothly to Σ.

Theorem 6.2. *(Theorem B) Let $\Sigma^n \subset (M, g)$ be a compact, oriented, strongly stable minimal submanifold. Then, there exist positive constants $\kappa \ll 1$ and c which depend on the geometry of M and Σ and which have the following significance. Suppose that $\Gamma \subset U_\varepsilon$ is an oriented n-dimensional submanifold satisfying*

$$\sup_{q\in\Gamma} (1 - (*\Omega) + \psi) < \kappa \ . \tag{6.2}$$

Then, the mean curvature flow Γ_t with $\Gamma_0 = \Gamma$ exists for all $t > 0$. Moreover, $\sup_{q\in\Gamma_t} |\mathrm{I\!I}^t| \le c$ for any $t > 0$, where $\mathrm{I\!I}^t$ the second fundamental form of Γ_t, and Γ_t converges smoothly to Σ as $t \to \infty$.

Proof. The constant κ will be chosen to be smaller than ε^2 and $\frac{1}{2}$; its precise value will be determined later. Suppose that the condition (6.2) holds for all $\{\Gamma_t\}_{0\le t<T}$.

Denote by H_t the mean curvature vector of Γ_t. According to Proposition 4.1

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi = H_t(\psi) = \Delta^{\Gamma_t}\psi - \mathrm{tr}_{\Gamma_t} \mathrm{Hess}\, \psi \le \Delta^{\Gamma_t}\psi - c_1(\psi + \mathfrak{s}^2) \ . \tag{6.3}$$

By applying Lemma 5.1, Lemma 5.2 and the second line of (2.28) to (6.1),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(*\Omega) &\ge \Delta^{\Gamma_t}(*\Omega) + (*\Omega)|\mathrm{I\!I}^t|^2 - 2c_2\mathfrak{s}^2|\mathrm{I\!I}^t|^2 \\
&\quad - 2(*\Omega)\langle \mathrm{I\!I}^t, \mathrm{I\!I}^\Sigma + S^\Sigma\rangle - c_2(\mathfrak{s}^2 + \psi)|\mathrm{I\!I}^t| \\
&\quad + (*\Omega)|\mathrm{I\!I}^\Sigma + S^\Sigma|^2 - c_2(\mathfrak{s}^2 + \psi) \\
&\ge \Delta^{\Gamma_t}(*\Omega) + (*\Omega)|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma - S^\Sigma|^2 - c_3(\mathfrak{s}^2 + \psi)|\mathrm{I\!I}^t|^2 - c_3(\mathfrak{s}^2 + \psi) \\
&\ge \Delta^{\Gamma_t}(*\Omega) + \frac{1}{2}(*\Omega)|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 - (*\Omega)|S^\Sigma|^2 \\
&\quad - 2c_3(\mathfrak{s}^2 + \psi)|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 - c_3(\mathfrak{s}^2 + \psi)(1 + 2|\mathrm{I\!I}^\Sigma|^2) \ .
\end{aligned}$$

If $\kappa \le 1/(48c_3)$, it follows from (5.9) that $2c_3(\mathfrak{s}^2 + \psi) \le (*\Omega)/6$. Since $|S^\Sigma|^2 \le c_4\psi$ and $|\mathrm{I\!I}^\Sigma|^2 \le c_4$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(*\Omega) \ge \Delta^{\Gamma_t}(*\Omega) + \frac{1}{3}(*\Omega)|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 - c_5(\mathfrak{s}^2 + \psi) \ . \tag{6.4}$$

By combining it with (6.3), (5.9) and (5.10), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(1-(*\Omega)+c_6\psi\right) &\leq \Delta^{\Gamma_t}\left(1-(*\Omega)+c_6\psi\right)-\frac{1}{3}(*\Omega)|\mathrm{I\!I}^t-\mathrm{I\!I}^\Sigma|^2-c_7\left(1-(*\Omega)+\psi\right)\\ &\leq \Delta^{\Gamma_t}\left(1-(*\Omega)+c_6\psi\right)-\frac{1}{3}(*\Omega)|\mathrm{I\!I}^t-\mathrm{I\!I}^\Sigma|^2-\frac{c_7}{c_6}\left(1-(*\Omega)+c_6\psi\right)\end{aligned} \tag{6.5}$$

where $c_6 = 1 + c_5/c_1$. By the maximum principle, $\max_{\Gamma_t}\left(1-(*\Omega)+c_6\psi\right)$ is non-increasing.

The evolution equation for the norm of the second fundamental form for a mean curvature flow is derived in [20, Proposition 7.1]. In particular, $|\mathrm{I\!I}^t|^2 = \sum_{\alpha,i,k}\tilde{h}^2_{\alpha ik}$ satisfies the following equation along the flow:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{I\!I}^t|^2 &= \Delta^{\Gamma_t}|\mathrm{I\!I}^t|^2-2|\nabla^{\Gamma_t}\mathrm{I\!I}^t|^2+2\left[\left(\nabla_{\tilde{e}_k}R\right)_{\tilde{\alpha}\tilde{i}\tilde{j}\tilde{k}}+\left(\nabla_{\tilde{e}_j}R\right)_{\tilde{\alpha}\tilde{k}\tilde{i}\tilde{k}}\right]\tilde{h}_{\alpha ij}\\ &\quad-4R_{\tilde{l}\tilde{i}\tilde{j}\tilde{k}}\tilde{h}_{\alpha lk}\tilde{h}_{\alpha ij}+8R_{\tilde{\alpha}\tilde{\beta}\tilde{j}\tilde{k}}\tilde{h}_{\beta ik}\tilde{h}_{\alpha ij}-4R_{\tilde{l}\tilde{k}\tilde{i}\tilde{k}}\tilde{h}_{\alpha lj}\tilde{h}_{\alpha ij}+2R_{\tilde{\alpha}\tilde{k}\tilde{\beta}\tilde{k}}\tilde{h}_{\beta ij}\tilde{h}_{\alpha ij}\\ &\quad+2\sum_{\alpha,\gamma,i,j}\left(\sum_k\tilde{h}_{\alpha ik}\tilde{h}_{\gamma jk}-\tilde{h}_{\alpha jk}\tilde{h}_{\gamma ik}\right)^2+2\sum_{i,j,k,l}\left(\sum_\alpha\tilde{h}_{\alpha ij}\tilde{h}_{\alpha kl}\right)^2.\end{aligned} \tag{6.6}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{I\!I}^t|^2 \leq \Delta^{\Gamma_t}|\mathrm{I\!I}^t|^2-2|\nabla^{\Gamma_t}\mathrm{I\!I}^t|^2+c_8\left(|\mathrm{I\!I}^t|^4+|\mathrm{I\!I}^t|^2+1\right). \tag{6.7}$$

The quartic term $|\mathrm{I\!I}^t|^4$ could potentially lead to the finite time blow-up of $|\mathrm{I\!I}^t|$. We apply the same method in [22]: use the evolution equation of $(*\Omega)^p$ to help. Let p be a constant no less than 1, whose precise value will be determined later. According to (6.4),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(*\Omega)^p &= p(*\Omega)^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}(*\Omega)\\ &\geq p(*\Omega)^{p-1}\Delta^{\Gamma_t}(*\Omega)+\frac{p}{3}(*\Omega)^p|\mathrm{I\!I}^t-\mathrm{I\!I}^\Sigma|^2-c_5p(\mathfrak{s}^2+\psi)\\ &= \Delta^{\Gamma_t}(*\Omega)^p-p(p-1)(*\Omega)^{p-2}|\nabla^{\Gamma_t}\mathrm{I\!I}^t|^2+\frac{p}{3}(*\Omega)^p|\mathrm{I\!I}^t-\mathrm{I\!I}^\Sigma|^2-c_5p(\mathfrak{s}^2+\psi).\end{aligned}$$

After an appeal to Lemma 5.4,

$$\frac{\mathrm{d}}{\mathrm{d}t}(*\Omega)^p \geq \Delta^{\Gamma_t}(*\Omega)^p+\frac{p}{3}\left(1-c_9\,p\,\mathfrak{s}^2\right)(*\Omega)^p|\mathrm{I\!I}^t-\mathrm{I\!I}^\Sigma|^2-c_9\,p^2(\mathfrak{s}^2+\psi).$$

If $\kappa \leq 1/(24c_9p)$, it follows from (5.9) that $c_9\,p\,\mathfrak{s}^2 \leq 1/12$. It together with (6.3) gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((*\Omega)^p-Kp^2\psi\right) \geq \Delta^{\Gamma_t}\left((*\Omega)^p-Kp^2\psi\right)+\frac{p}{4}\left((*\Omega)^p-Kp^2\psi\right)|\mathrm{I\!I}^t-\mathrm{I\!I}^\Sigma|^2 \tag{6.8}$$

where $K = c_9/c_1$. The maximum principle implies that if $(*\Omega)^p - Kp^2\psi > 0$ on Γ, then $\min_{\Gamma_t}\left((*\Omega)^p-Kp^2\psi\right)$ is non-decreasing. Moreover, for any $p \geq 1$, we may choose κ such that (6.2) implies that $(*\Omega)^p - Kp^2\psi > 1/2$.

Denote $(*\Omega)^p - Kp^2\psi$ by η. Due to (6.7) and (6.8),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\eta^{-1}|\mathrm{I\!I}^t|^2) \leq \eta^{-1}\Delta^{\Gamma_t}|\mathrm{I\!I}^t|^2 - 2\eta^{-1}|\nabla^{\Gamma_t}\mathrm{I\!I}^t|^2 + c_8\eta^{-1}\left(|\mathrm{I\!I}^t|^4 + |\mathrm{I\!I}^t|^2 + 1\right)\\ - \eta^{-2}|\mathrm{I\!I}^t|^2\left(\Delta^{\Gamma_t}\eta + \frac{p}{4}\eta|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2\right) .\end{aligned}$$

Since $\Delta^{\Gamma_t}(\eta^{-1}|\mathrm{I\!I}^t|^2) = \eta^{-1}\Delta^{\Gamma_t}|\mathrm{I\!I}^t|^2 - \eta^{-2}|\mathrm{I\!I}^t|^2\Delta^{\Gamma_t}\eta - 2\eta^{-1}\langle\nabla^{\Gamma_t}\eta, \nabla^{\Gamma_t}(\eta^{-1}|\mathrm{I\!I}^t|^2)\rangle$ and $|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 \geq \frac{1}{2}|\mathrm{I\!I}^t|^2 - c_{10}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\eta^{-1}|\mathrm{I\!I}^t|^2) &\leq \Delta^{\Gamma_t}(\eta^{-1}|\mathrm{I\!I}^t|^2) + 2\eta^{-1}\langle\nabla^{\Gamma_t}\eta, \nabla^{\Gamma_t}(\eta^{-1}|\mathrm{I\!I}^t|^2)\rangle\\ &\quad - \frac{p}{8}\eta^{-1}|\mathrm{I\!I}^t|^4 + c_{11}\eta^{-1}\left(|\mathrm{I\!I}^t|^4 + (p+1)|\mathrm{I\!I}^t|^2 + 1\right)\\ &\leq \Delta^{\Gamma_t}(\eta^{-1}|\mathrm{I\!I}^t|^2) + 2\eta^{-1}\langle\nabla^{\Gamma_t}\eta, \nabla^{\Gamma_t}(\eta^{-1}|\mathrm{I\!I}^t|^2)\rangle\\ &\quad - \left(\frac{p}{8} - 2c_{10}\right)(\eta^{-1}|\mathrm{I\!I}^t|^2)^2 + c_{11}(\eta^{-1}|\mathrm{I\!I}^t|^2) + 2c_{10} .\end{aligned} \tag{6.9}$$

Choose $p \geq 16c_{11}+1$. It follows from the maximum principle that $\eta^{-1}|\mathrm{I\!I}^t|^2$ is uniformly bounded, and hence there is no finite time singularity.

The $\mathcal{C}^0$ convergence is easy to come by. The differential inequality (6.3) implies that ψ converges to zero exponentially. Similarly, it follows from (6.10) that $1 - (*\Omega) + c_6\psi$ converges to zero exponentially. Therefore, $*\Omega$ converges to 1 exponentially, and we conclude the $\mathcal{C}^1$ convergence.

For the $\mathcal{C}^2$ and smooth convergence, consider $\eta = (*\Omega)^p - Kp^2\psi$. It follows from the above discussion that η has a positive lower bound. It is clear that $\eta \leq 1$. Moreover, η converges to 1 as $t \to \infty$. Integrating (6.8) gives

$$\int_{\Gamma_t} |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 \,\mathrm{d}\mu_t \leq c_{12}\int_{\Gamma_t}\frac{\mathrm{d}\eta}{\mathrm{d}t}\,\mathrm{d}\mu_t .$$

Recall that the Lie derivative of $\mathrm{d}\mu_t$ in H_t is $-|H_t^2|\,\mathrm{d}\mu_t$; see [20, §2]. It follows that

$$\frac{1}{c_{12}}\int_{\Gamma_t}|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2\,\mathrm{d}\mu_t \leq \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}\eta\,\mathrm{d}\mu_t + \int_{\Gamma_t}\eta\,|H_t|^2\,\mathrm{d}\mu_t \tag{6.10}$$

We claim that the improper integral of the right hand side for $0 \leq t < \infty$ converges. To start, note that $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_t}\mathrm{d}\mu_t = -\int_{\Gamma_t}|H_t|^2\,\mathrm{d}\mu_t \leq 0$. Thus, $\mathrm{vol}(\Gamma_t) = \int_{\Gamma_t}\mathrm{d}\mu_t$ is positive and non-increasing, and must converge as $t \to \infty$. For the first term on right hand side of (6.10),

$$\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}s}\int_{\Gamma_s}\eta\,\mathrm{d}\mu_s\right)\mathrm{d}s = \int_{\Gamma_t}\eta\,\mathrm{d}\mu_t - \int_{\Gamma_0}\eta\,\mathrm{d}\mu_0$$

Since η converges to 1 (uniformly) and $\mathrm{vol}(\Gamma_t)$ converges as $t \to \infty$, $\int_{\Gamma_t} \eta \, \mathrm{d}\mu_t$ converges as $t \to \infty$. For the second term on the right hand side of (6.10),

$$\begin{aligned}\int_0^t \left(\int_{\Gamma_s} \eta \, |H_s|^2 \, \mathrm{d}\mu_s \right) \mathrm{d}s &\leq \int_0^t \left(\int_{\Gamma_s} |H_s|^2 \, \mathrm{d}\mu_s \right) \mathrm{d}s = \int_0^t \left(-\frac{\mathrm{d}}{\mathrm{d}s} \int_{\Gamma_s} \mathrm{d}\mu_s \right) \mathrm{d}s \\ &= \mathrm{vol}(\Gamma_0) - \mathrm{vol}(\Gamma_t) \leq \mathrm{vol}(\Gamma_0) \ .\end{aligned}$$

It is bounded from above, and is clearly non-decreasing in t. Therefore, it converges as $t \to \infty$.

It follows from the claim and (6.10) that

$$\int_0^\infty \left(\int_{\Gamma_t} |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 \, \mathrm{d}\mu_t \right) \mathrm{d}t < \infty \ . \tag{6.11}$$

On the other hand, $|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2$ obeys a differential inequality of the same form as (6.7):

$$\frac{\mathrm{d}}{\mathrm{d}t} |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 \leq \Delta^{\Gamma_t} |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 + c_{13} \left(|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^4 + |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 + 1 \right) \ . \tag{6.12}$$

The derivation for this inequality is in Appendix B. By (6.12) and the uniform boundedness of $|\mathrm{I\!I}^t|$, $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_t} |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 \, \mathrm{d}\mu_t$ is bounded from above uniformly. Due to Lemma 6.3, which is proved at the end of this subsection, we find that

$$\lim_{t \to 0} \int_{\Gamma_t} |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 \, \mathrm{d}\mu_t = 0 \ . \tag{6.13}$$

Since $|\mathrm{I\!I}^t|$ is uniformly bounded, in view of (6.12), we find that $\frac{\mathrm{d}}{\mathrm{d}t} |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 - \Delta^{\Gamma_t} |\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2$ is bounded from above, independent of t. This together with (6.13) implies that $|\mathrm{I\!I}^t - \mathrm{I\!I}^\Sigma|^2 \to 0$ as $t \to \infty$ in the sup norm.

We can then write Γ_t as a graph (in the geodesic coordinate defined in §2.2) over Σ defined by $y^\alpha = f_t^\alpha, \alpha = n+1, \cdots n+m$ for f_t^α functions on Σ. The above estimates imply that f_t^α converges to 0 in $\mathcal{C}^2$ as $t \to \infty$. As a mean curvature flow, f_t^α satisfies a second order quasilinear parabolic system, and standard arguments lead to the smooth convergence of f_t^α. □

Lemma 6.3. *Let $a > 0$ and $f(t)$ be a smooth function for $t \in (a, \infty)$. Suppose that $f(t) \geq 0$, $\int_a^\infty f(t) \, \mathrm{d}t$ converges, and $f'(t) \leq C$ for some constant $C > 0$. Then, $f(t) \to 0$ as $t \to \infty$.*

Proof. It follows from $f'(t) \leq C$ that $f(t) \geq f(t_1) - C(t_1 - t)$ for any $t_1 > t > a$. Since $f(t) \geq 0$ and $\int_a^\infty f(t) \, \mathrm{d}t < \infty$, given any $\epsilon \in (0,1)$, there exists an $A_\epsilon > a$ such that $\int_{A_\epsilon}^\infty f(t) \, \mathrm{d}t < \epsilon$. Thus, for any $t_1 > A_\epsilon + 1 > A_\epsilon + \sqrt{\epsilon}$,

$$\begin{aligned}\epsilon > \int_{t_1 - \sqrt{\epsilon}}^{t_1} f(t) \, \mathrm{d}t &\geq \int_{t_1 - \sqrt{\epsilon}}^{t_1} (f(t_1) - C(t_1 - t)) \, \mathrm{d}t \\ &= \sqrt{\epsilon} \, (f(t_1) - C \, t_1) + C \left(\sqrt{\epsilon} \, t_1 - \frac{1}{2} \epsilon \right) \ .\end{aligned}$$

It follows that $f(t) < (1 + \frac{1}{2} C) \sqrt{\epsilon}$ for any $t > A_\epsilon + 1$. □

Appendix A. Computations related to strong stability

For minimal Lagrangians in a Kähler–Einstein manifold and coassociatives in a G_2 manifold, the condition (3.2) can be rewritten as a curvature condition on the submanifold. One ingredient is the geometric properties of $\mathrm{U}(n)$ and G_2 holonomy. Another ingredient is the Gauss equation:

$$R_{ijk\ell} - R^{\Sigma}_{ijk\ell} = h_{\alpha i\ell} h_{\alpha jk} - h_{\alpha ik} h_{\alpha j\ell} \ . \tag{A.1}$$

A.1. **Minimal Lagrangians in Kähler–Einstein manifolds.** Let (M^{2n}, g, J, ω) be a Kähler–Einstein manifold, where J is the complex structure and ω is the Kähler form. Denote the Einstein constant by c; namely,

$$\sum_C R_{ACBC} = \mathrm{Ric}_{AB} = c\, g_{AB} \ .$$

A submanifold $L^n \subset M^{2n}$ is Lagrangian if $\omega|_L$ vanishes. It implies that J induces an isomorphism between its tangent bundle TL and normal bundle NL. In terms of the notations introduced in §2.1, the correspondence is

$$v^i e_i \longleftrightarrow v^i\, Je_i \ . \tag{A.2}$$

In particular, if $\{e_1, \cdots, e_n\}$ is an orthonormal frame for TL, $\{Je_1, \cdots, Je_n\}$ is an orthonormal frame for NL. Denote Je_k by $e_{J(k)}$, and let

$$C_{kij} = h_{J(k)ij} = \langle \nabla_{e_i} e_j, Je_k \rangle \ .$$

Since J is parallel, it is easy to verify that C_{kij} is totally symmetric.

Now, suppose that L is also minimal. By using the correspondence (A.2), the strong stability condition (3.2) can be rewritten as follows.

$$\begin{aligned}
-R_{iJ(k)iJ(\ell)}\, v^k\, v^\ell - C_{kij}\, C_{\ell ij}\, v^k\, v^\ell &= -c\, g_{k\ell}\, v^k\, v^\ell + R_{J(i)J(k)J(i)J(\ell)}\, v^k\, v^\ell - C_{kij}\, C_{\ell ij}\, v^k\, v^\ell \\
&= -c\, |v|^2 + R_{ik i\ell}\, v^k\, v^\ell - C_{kij}\, C_{\ell ij}\, v^k\, v^\ell \\
&= -c\, |v|^2 + R^L_{ik i\ell}\, v^k\, v^\ell + C_{jki}\, C_{j\ell i}\, v^k\, v^\ell - C_{kij}\, C_{\ell ij}\, v^k\, v^\ell \\
&= -c\, |v|^2 + \mathrm{Ric}^L(v, v) \ .
\end{aligned}$$

The first equality uses the Kähler–Einstein condition. The second equality follows from the parallelity of J. The third equality uses the Gauss equation and the minimal condition. The last equality relies on the fact that C_{kij} is totally symmetric. This computation says that (3.2) is equivalent to the condition that $\mathrm{Ric}^L - c$ is a positive definite operator on TL.

A.2. **Coassociative submanifolds in G_2 manifolds.** In this case, the ambient space is 7-dimensional, and the submanifold is 4-dimensional.

A.2.1. *Four dimensional Riemannian geometry.* The Riemann curvature tensor has a nice decomposition in 4 dimensions. What follows is a brief summary of the decomposition; readers are directed to [1] for more.

Let Σ be an oriented, 4-dimensional Riemannian manifold. The Riemann curvature tensor in general defines a self-adjoint transform on Λ^2 by

$$\mathfrak{R}(e_i \wedge e_j) = \frac{1}{2} R^{\Sigma}_{k\ell ij}\, e_k \wedge e_\ell \ .$$

In 4 dimensions, Λ^2 decomposes into self-dual, Λ^2_+, and anti-self-dual part, Λ^2_-. In terms of the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, the curvature map $\mathfrak{R}$ has the form

$$\mathfrak{R} = \begin{bmatrix} W_+ + \frac{s}{12}\mathbf{I} & B \\ B^T & W_- + \frac{s}{12}\mathbf{I} \end{bmatrix} \ .$$

Here, $s = R^{\Sigma}_{ijij}$ is the scalar curvature, $W_\pm$ is the self-dual and anti-self-dual part of the Weyl tensor, B is the traceless Ricci tensor, and $\mathbf{I}$ is the identity homomorphism.

With respect to the basis $\{e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3\}$, the lower-right block $W_- + \frac{s}{12}\mathbf{I}$ is

$$\frac{1}{2}\begin{bmatrix} R^{\Sigma}_{1212} + R^{\Sigma}_{3434} - 2R^{\Sigma}_{1234} & R^{\Sigma}_{1213} + R^{\Sigma}_{1224} - R^{\Sigma}_{3413} - R^{\Sigma}_{3424} & R^{\Sigma}_{1214} - R^{\Sigma}_{1223} - R^{\Sigma}_{3414} + R^{\Sigma}_{3423} \\ R^{\Sigma}_{1312} - R^{\Sigma}_{1334} + R^{\Sigma}_{2412} - R^{\Sigma}_{2434} & R^{\Sigma}_{1313} + R^{\Sigma}_{2424} + 2R^{\Sigma}_{1324} & R^{\Sigma}_{1314} - R^{\Sigma}_{1323} + R^{\Sigma}_{2414} - R^{\Sigma}_{2423} \\ R^{\Sigma}_{1412} - R^{\Sigma}_{1434} - R^{\Sigma}_{2312} + R^{\Sigma}_{2334} & R^{\Sigma}_{1413} + R^{\Sigma}_{1424} - R^{\Sigma}_{2313} - R^{\Sigma}_{2324} & R^{\Sigma}_{1414} + R^{\Sigma}_{2323} - 2R^{\Sigma}_{1423} \end{bmatrix} . \tag{A.3}$$

The operator will be needed is $W_- - \frac{s}{6}\mathbf{I} = (W_- + \frac{s}{12}\mathbf{I}) - \frac{s}{4}\mathbf{I}$. One-fourth of the scalar curvature is

$$\frac{s}{4} = \frac{1}{2}\left(R^{\Sigma}_{1212} + R^{\Sigma}_{3434} + R^{\Sigma}_{1313} + R^{\Sigma}_{2424} + R^{\Sigma}_{1313} + R^{\Sigma}_{2424}\right) \ . \tag{A.4}$$

A.2.2. *G_2 geometry.* A 7-dimensional Riemannian manifold M with G_2 holonomy can be characterized by the existence of a parallel, positive 3-form φ. A complete story can be found in [10, ch.11]. In terms of a local orthonormal coframe, the 3-form and its Hodge star are

$$\begin{aligned} \varphi &= \omega^{567} + \omega^{125} - \omega^{345} + \omega^{136} + \omega^{246} + \omega^{147} - \omega^{237} \ , \\ *\varphi &= \omega^{1234} - \omega^{1267} + \omega^{3467} + \omega^{1357} + \omega^{3457} - \omega^{1456} + \omega^{2356} \end{aligned} \tag{A.5}$$

where ω^{123} is short for $\omega^1 \wedge \omega^2 \wedge \omega^3$. It is known that the holonomy is G_2 if and only if $\nabla\varphi = 0$, which is also equivalent to $\mathrm{d}\varphi = 0 = \mathrm{d} * \varphi$.

Remark A.1. There are two commonly used conventions for the 3-form; see [11] for instance. The convention here is the same as that in [12]; the deformation of coassociatives will then be determined by anti-self-dual harmonic forms. If one use the convention in [10], the deformation of coassociatives will be determined by self-dual harmonic forms.

The 3-form φ determines a product map $\times$ for tangent vectors of M. For any two tangent vectors X and Y,

$$X \times Y = \big(\varphi(X, Y, \cdot)\big)^\sharp \ .$$

For instance, $e_1 \times e_2 = e_5$. Since φ and the metric tensor are both parallel, $\times$ is parallel as well.

As a consequence,

$$R(e_A, e_B)(e_1 \times e_2) = \big(R(e_A, e_B)e_1\big) \times e_2 + e_1 \times \big(R(e_A, e_B)e_2\big) \ ,$$

and its e_3-component gives $R_{53AB} - R_{62AB} - R_{71AB} = 0$ for any $A, B \in \{1, \ldots, 7\}$. In total, the parallelity of $\times$ leads to following seven identities:

$$\begin{aligned} R_{52AB} + R_{63AB} + R_{74AB} &= 0 \ , \\ R_{51AB} - R_{64AB} + R_{73AB} &= 0 \ , \\ R_{54AB} + R_{61AB} - R_{72AB} &= 0 \ , \\ -R_{53AB} + R_{62AB} + R_{71AB} &= 0 \ , \end{aligned} \qquad \begin{aligned} R_{67AB} + R_{12AB} - R_{34AB} &= 0 \ , \\ -R_{57AB} + R_{13AB} + R_{24AB} &= 0 \ , \\ R_{56AB} - R_{14AB} - R_{23AB} &= 0 \ . \end{aligned} \tag{A.6}$$

These identities imply that a G_2 manifold is always Ricci flat.

A.2.3. *Coassociative geometry.* According to [8, §IV], an oriented, 4-dimensional submanifold Σ of a G_2 manifold is said to be coassociative if $*\varphi|_\Sigma$ coincides with the volume form of the induced metric. Harvey and Lawson also proved that being coassociative is equivalent to that $\varphi|_\Sigma$ vanishes. Similar to the Lagrangian case, the normal bundle of a coassociative submanifold is canonically isomorphic to an intrinsic bundle. The following discussion is basically borrowed from [12, §4].

Orthonormal frame. Suppose that $\Sigma \subset M$ is coassociative. One can find a local orthonormal frame $\{e_1, \cdots, e_7\}$ such that $\{e_1, e_2, e_3, e_4\}$ are tangent to Σ, $\{e_5, e_6, e_7\}$ are normal to Σ, and φ takes the form (A.5) in this frame. Here is a sketch of the construction. Start with a unit normal vector, e_5, and a unit tangent vector, e_1, of Σ. Let $e_2 = e_5 \times e_1$. Then, set e_3 to be a unit vector tangent to Σ and orthogonal to $\{e_1, e_2\}$. Finally, let $e_4 = e_3 \times e_5$, $e_6 = e_1 \times e_3$ and $e_7 = e_3 \times e_2$.

Normal bundle and second fundamental form. The normal bundle of Σ is isomorphic to the bundle of anti-self-dual 2-forms of Σ via the following map:

$$V \mapsto (V \lrcorner \varphi)|_\Sigma \ . \tag{A.7}$$

In terms of the above frame, e_5 corresponds to $\omega^{12} - \omega^{34}$, e_6 corresponds to $\omega^{13} + \omega^{24}$, and e_7 corresponds to $\omega^{14} - \omega^{23}$.

As shown in [8], a coassociative submanifold must be minimal. In fact, its second fundamental form has certain symmetry. For instance,

$$\begin{aligned} h_{51i} &= \langle \nabla_{e_i} e_1, e_5 \rangle = -\langle e_1, \nabla_{e_i}(e_6 \times e_7) \rangle \\ &= -\langle e_1, (\nabla_{e_i} e_6) \times e_7 \rangle - \langle e_1, e_6 \times (\nabla_{e_i} e_7) \rangle \\ &= -\langle e_4, \nabla_{e_i} e_6 \rangle + \langle e_3, \nabla_{e_i} e_7 \rangle = h_{64i} - h_{73i} \ . \end{aligned}$$

What follows are all the relations:

$$\begin{aligned} h_{52i} + h_{63i} + h_{74i} &= 0 \ , & h_{54i} + h_{61i} - h_{72i} &= 0 \ , \\ h_{51i} - h_{64i} + h_{73i} &= 0 \ , & -h_{53i} + h_{62i} + h_{71i} &= 0 \end{aligned} \tag{A.8}$$

for any $i \in \{1, 2, 3, 4\}$. These relations imply that the mean curvature vanishes. They can be encapsulated as $\sum_j e_j \times \mathrm{I\!I}(e_i, e_j) = 0$.

A.2.4. *Strong stability for coassociatives.* For any sections of $N\Sigma$, $\mathbf{v}$, denote the symmetric bilinear form on the left hand side of (3.2) by $Q(\mathbf{v}, \mathbf{v})$. Under the identification (A.7), $\tilde{Q}(\mathbf{v}, \mathbf{v}) = -2\,\mathbf{v}^T W_- \mathbf{v} + \frac{s}{3}|\mathbf{v}|^2$ is also a symmetric bilinear form.

We now check that $Q(\mathbf{v}, \mathbf{v}) = \tilde{Q}(\mathbf{v}, \mathbf{v})$ for any unit vector $\mathbf{v} \in N_p\Sigma$ at any $p \in \Sigma$. As explained above, we may take $e_5 = \mathbf{v}$ and construct the other orthonormal vectors. With respect such a frame, it follows from (A.3) and (A.4) that

$$\tilde{Q}(\mathbf{v}, \mathbf{v}) = R^{\Sigma}_{1313} + R^{\Sigma}_{2424} + R^{\Sigma}_{1313} + R^{\Sigma}_{2424} + 2R^{\Sigma}_{1234} \ .$$

The quantity $Q(\mathbf{v}, \mathbf{v})$ can be rewritten as follows.

$$\begin{aligned} Q(\mathbf{v}, \mathbf{v}) &= -\sum_i R_{i5i5} - \sum_{i,j} (h_{5ij})^2 \\ &= R_{6565} + R_{7575} - \sum_{i,j} (h_{5ij})^2 \\ &= R_{1313} + R_{2424} + R_{1313} + R_{2424} - 2R_{1423} + 2R_{1324} - \sum_{i,j} (h_{5ij})^2 \\ &= R_{1313} + R_{2424} + R_{1313} + R_{2424} + 2R_{1234} - \sum_{i,j} (h_{5ij})^2 \ . \end{aligned}$$

The second equality follows from Ricci flatness. The third equality uses (A.6). The last equality is the first Bianchi identity. With the Gauss equation and some simple manipulation,

$$\begin{aligned} & Q(\mathbf{v}, \mathbf{v}) - \tilde{Q}(\mathbf{v}, \mathbf{v}) \\ &= \sum_{\alpha} \big((h_{\alpha 14} + h_{\alpha 23})^2 + (h_{\alpha 13} - h_{\alpha 24})^2 - (h_{\alpha 11} + h_{\alpha 22})(h_{\alpha 33} + h_{\alpha 44}) \big) - \sum_{i,j} (h_{5ij})^2 \ . \end{aligned} \tag{A.9}$$

By appealing to (A.8),

$$h_{614}+h_{623}=-h_{522}-h_{544}=h_{511}+h_{533}\ ,\quad h_{613}-h_{624}=-h_{512}-h_{534}\ ,$$
$$h_{714}+h_{723}=-h_{512}+h_{534}\ ,\qquad\qquad h_{713}-h_{724}=-h_{522}+h_{533}=-h_{511}-h_{544}\ ,$$

and

$$h_{611}+h_{622}=-h_{633}-h_{644}=-h_{514}+h_{523}\ ,$$
$$h_{711}+h_{722}=-h_{733}-h_{744}=h_{513}+h_{524}\ .$$

By using these relations, it is not hard to verify that (A.9) vanishes. Therefore, the strong stability condition (3.2) is equivalent to the positivity of $-2W_-+\frac{s}{3}$.

As a final remark, this equivalence can also be seen by combining [12, Theorem 4.9] and the Weitzenböck formula [7, Appendix C]. Nevertheless it is nice to derive the equivalence directly by highlighting the geometry of G_2.

Appendix B. Evolution equation for tensors

Suppose that Ψ be a tensor defined on M of type $(0,3)$. The main purpose of this section is to calculate its evolution equation along the mean curvature flow. Since there will be some different connections, we denote the Levi-Civita connection of (M,g) by $\overline{\nabla}$ to avoid confusions.

Let Γ_t be the mean curvature flow at time t. The tensor Ψ is a section of $(T^*M\otimes T^*M\otimes T^*M)|_{\Gamma^t}$. The connection $\overline{\nabla}$ naturally induces a connection $\tilde{\nabla}$ on this bundle. The only difference between $\overline{\nabla}$ and $\tilde{\nabla}$ is that the direction vector in $\tilde{\nabla}$ must be tangent to Γ_t.

connection	bundle and base
$\overline{\nabla}$	Levi-Civita connection of (M,g)
∇^{Γ_t}	Levi-Civita connection of Γ_t with the induced metric
$\nabla^\perp$	connection of the normal bundle of Γ_t
$\tilde{\nabla}$	connection of $(T^*M\otimes T^*M\otimes T^*M)|_{\Gamma^t}$
∇	connection of $T^*\Gamma_t\otimes T^*\Gamma_t\otimes N^*\Gamma_t$ defined by (2.2)

From the construction, ∇ is the composition of $\tilde{\nabla}$ with the orthogonal projection.

Proposition B.1. *Let Ψ be a tensor of type $(0,3)$ defined on the ambient manifold M. Along the mean curvature flow Γ_t in M,*

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{I\!I}^t-\Psi|^2\le\Delta^{\Gamma_t}|\mathrm{I\!I}^t-\Psi|^2-|\tilde{\nabla}(\mathrm{I\!I}^t-\Psi)|^2+c(|\mathrm{I\!I}^t-\Psi|^4+|\mathrm{I\!I}^t-\Psi|^2+1)\qquad\text{(B.1)}$$

where $c>0$ is determined by the Riemann curvature tensor of M and the sup-norm of Ψ, $\overline{\nabla}\Psi$, $\overline{\nabla}^2\Psi$.

Proof. The mean curvature flow can be regarded as a map from $\Gamma_0\times[0,\epsilon)\to M$. For any $p\in\Gamma_0$ and $t_0\in[0,\epsilon)$, choose a geodesic coordinate for Γ_0 at p: $\{\tilde{x}^1,\cdots,\tilde{x}^n\}$. We also choose

a local orthonormal frame $\{\tilde{e}_\alpha\}$ for $N\Gamma_t$. The following computations on derivatives are *always evaluated at the point* (p, t_0).

Let $H = \tilde{h}_\alpha \tilde{e}_\alpha$ be the mean curvature vector of Γ_t. The components of the second fundamental form and its covariant derivative are denoted by

$$\begin{aligned}
\tilde{h}_{\alpha ij} &= \langle \overline{\nabla}_{\partial_{\tilde{i}}} \partial_{\tilde{j}}, \tilde{e}_\alpha \rangle = \mathrm{I\!I}(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) \ , \\
\tilde{h}_{\alpha ij,k} &= (\nabla_{\partial_{\tilde{k}}} \mathrm{I\!I})(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) \ , \\
\tilde{h}_{\alpha,k} &= \langle \nabla^\perp_{\partial_{\tilde{k}}} H, \tilde{e}_\alpha \rangle \ .
\end{aligned}$$

At (p, t_0), $\tilde{h}_\alpha = \tilde{h}_{\alpha kk}$ and $\tilde{h}_{\alpha,i} = \tilde{h}_{\alpha kk,i}$.

Note that on $\Gamma_0 \times [0, \epsilon)$, H is ∂_t, and thus commutes with $\partial_{\tilde{i}}$. It follows that the evolution of the metric is:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \tilde{g}_{ij} &= H \langle \partial_{\tilde{i}}, \partial_{\tilde{j}} \rangle = \langle \overline{\nabla}_H \partial_{\tilde{i}}, \partial_{\tilde{j}} \rangle + \langle \partial_{\tilde{i}}, \overline{\nabla}_H \partial_{\tilde{j}} \rangle \\
&= -\langle H, \overline{\nabla}_{\partial_{\tilde{i}}} \partial_{\tilde{j}} \rangle - \langle \overline{\nabla}_{\partial_{\tilde{j}}} \partial_{\tilde{i}}, H \rangle = -2 \tilde{h}_\alpha \tilde{h}_{\alpha ij} \ , \qquad \text{(B.2)} \\
\frac{\mathrm{d}}{\mathrm{d}t} \tilde{g}^{ij} &= 2 \tilde{h}_\alpha \tilde{h}_{\alpha ij} \ . \qquad \text{(B.3)}
\end{aligned}$$

The covariant derivative of $\partial_{\tilde{i}}$ and $\tilde{e}_\alpha$ along H can be expressed as follows:

$$\begin{aligned}
\overline{\nabla}_H \partial_{\tilde{i}} &= \langle \overline{\nabla}_H \partial_{\tilde{i}}, \partial_{\tilde{j}} \rangle \partial_{\tilde{j}} + \langle \overline{\nabla}_H \partial_{\tilde{i}}, \tilde{e}_\alpha \rangle \tilde{e}_\alpha \\
&= -\tilde{h}_\alpha \tilde{h}_{\alpha ij} \, \partial_{\tilde{j}} + \tilde{h}_{\alpha,i} \, \tilde{e}_\alpha \ , \qquad \text{(B.4)} \\
\overline{\nabla}_H \tilde{e}_\alpha &= \langle \overline{\nabla}_H \tilde{e}_\alpha, \partial_{\tilde{i}} \rangle \partial_{\tilde{i}} + \langle \overline{\nabla}_H \tilde{e}_\alpha, \tilde{e}_\beta \rangle \tilde{e}_\beta \\
&= -\tilde{h}_{\alpha,i} \, \partial_{\tilde{i}} + \langle \overline{\nabla}_H \tilde{e}_\alpha, \tilde{e}_\beta \rangle \tilde{e}_\beta \ . \qquad \text{(B.5)}
\end{aligned}$$

The last part of the preparation is to relate the covariant derivative of Ψ in H to its Bochner–Laplacian in the ambient manifold M.

$$\begin{aligned}
\overline{\nabla}_H \Psi &= \overline{\nabla}_{(\overline{\nabla}_{\partial_{\tilde{j}}} \partial_{\tilde{j}})^\perp} \Psi = -\overline{\nabla}_{\nabla^\Sigma_{\partial_{\tilde{j}}} \partial_{\tilde{j}}} \Psi + \overline{\nabla}_{\overline{\nabla}_{\partial_{\tilde{j}}} \partial_{\tilde{j}}} \Psi \\
&= \left(\tilde{\nabla}_{\partial_{\tilde{j}}} \tilde{\nabla}_{\partial_{\tilde{j}}} \Psi - \tilde{\nabla}_{\nabla^\Sigma_{\partial_{\tilde{j}}} \partial_{\tilde{j}}} \Psi \right) + \left(-\overline{\nabla}_{\partial_{\tilde{j}}} \overline{\nabla}_{\partial_{\tilde{j}}} \Psi + \overline{\nabla}_{\overline{\nabla}_{\partial_{\tilde{j}}} \partial_{\tilde{j}}} \Psi \right) \\
&= -\tilde{\nabla}^* \tilde{\nabla} \Psi + \mathrm{tr}_{\Gamma_t} (\overline{\nabla}^2 \Psi) \ . \qquad \text{(B.6)}
\end{aligned}$$

Indeed, $\nabla^\Sigma_{\partial_{\tilde{j}}} \partial_{\tilde{j}}$ is zero at (p, t_0). The tensor $\overline{\nabla}^* \overline{\nabla} \Psi$ is defined in the ambient space, and has nothing to do with the submanifold Γ_t. It follows from (B.6) that the evolution of of $|\Psi|^2$ is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} |\Psi|^2 &= H\big(\langle \Psi, \Psi \rangle\big) = 2 \langle \overline{\nabla}_H \Psi, \Psi \rangle \\
&= -2 \langle \tilde{\nabla}^* \tilde{\nabla} \Psi, \Psi \rangle + 2 \langle \mathrm{tr}_{\Gamma_t} (\overline{\nabla}^2 \Psi), \Psi \rangle \\
&= \Delta^{\Gamma_t} |\Psi|^2 - 2 |\tilde{\nabla} \Psi|^2 + 2 \langle \mathrm{tr}_{\Gamma_t} (\overline{\nabla}^2 \Psi), \Psi \rangle \qquad \text{(B.7)}
\end{aligned}$$

The next task is to calculate the evolution equation for $\langle \mathrm{I\!I}^t, \Psi\rangle = \tilde{g}^{ik}\tilde{g}^{jl}\tilde{h}_{\alpha kl}\tilde{\Psi}_{\alpha ij}$ where $\tilde{\Psi}_{\alpha ij} = \Psi(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha)$. According to (B.4) and (B.5),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Psi}_{\alpha ij} &= H\big(\Psi(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha)\big) \\
&= (\overline{\nabla}_H\Psi)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) + \Psi(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \overline{\nabla}_H\tilde{e}_\alpha) + \Psi(\overline{\nabla}_H\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) + \Psi(\partial_{\tilde{i}}, \overline{\nabla}_H\partial_{\tilde{j}}, \tilde{e}_\alpha) \\
&= (\overline{\nabla}_H\Psi)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) - \tilde{h}_{\alpha,k}\tilde{\Psi}_{kij} + \langle\overline{\nabla}_H\tilde{e}_\alpha, \tilde{e}_\beta\rangle\tilde{\Psi}_{\beta ij} \\
&\quad - \tilde{h}_\gamma\tilde{h}_{\gamma ik}\tilde{\Psi}_{\alpha kj} + \tilde{h}_{\gamma,i}\tilde{\Psi}_{\alpha\gamma j} - \tilde{h}_\gamma\tilde{h}_{\gamma jk}\tilde{\Psi}_{\alpha ik} + \tilde{h}_{\gamma,j}\tilde{\Psi}_{\alpha i\gamma}
\end{aligned} \tag{B.8}$$

The difference between $\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t$ and $\nabla^*\nabla\mathrm{I\!I}^t$ is:

$$\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t - \nabla^*\nabla\mathrm{I\!I}^t = \tilde{\nabla}_{\partial_{\tilde{k}}}\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t - \nabla_{\partial_{\tilde{k}}}\nabla_{\partial_{\tilde{k}}}\mathrm{I\!I}^t \tag{B.9}$$

Since

$$\begin{aligned}
(\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\cdot,\cdot,\cdot) &= (\overline{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\cdot,\cdot,\cdot) \\
&= \partial_{\tilde{k}}\big(\mathrm{I\!I}^t(\cdot,\cdot,\cdot)\big) - \mathrm{I\!I}^t\big((\overline{\nabla}_{\partial_{\tilde{k}}}\cdot)^T,\cdot,\cdot\big) - \mathrm{I\!I}^t\big(\cdot,(\overline{\nabla}_{\partial_{\tilde{k}}}\cdot)^T,\cdot\big) - \mathrm{I\!I}^t\big(\cdot,\cdot,(\overline{\nabla}_{\partial_{\tilde{k}}}\cdot)^\perp\big)\ ,
\end{aligned}$$

the tensor $\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t$ has only the following components:

$$\begin{aligned}
(\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) &= (\nabla_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) = \tilde{h}_{\alpha ij,k}\ , \\
(\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \partial_{\tilde{l}}) &= -\tilde{h}_{\alpha ij}\tilde{h}_{\alpha kl}\ , \\
(\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\tilde{e}_\beta, \partial_{\tilde{j}}, \tilde{e}_\alpha) &= \tilde{h}_{\beta ki}\tilde{h}_{\alpha ij}\ , \\
(\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \tilde{e}_\beta, \tilde{e}_\alpha) &= \tilde{h}_{\beta kj}\tilde{h}_{\alpha ij}\ .
\end{aligned} \tag{B.10}$$

The above four equations hold everywhere, but not only at (p, t_0). It follows that

$$\begin{aligned}
(\tilde{\nabla}_{\partial_{\tilde{k}}}\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) &= \partial_{\tilde{k}}\Big((\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha)\Big) - (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)\big(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \overline{\nabla}_{\partial_{\tilde{k}}}\tilde{e}_\alpha\big) \\
&\quad - (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)\big(\overline{\nabla}_{\partial_{\tilde{k}}}\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha\big) - (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)\big(\partial_{\tilde{i}}, \overline{\nabla}_{\partial_{\tilde{k}}}\partial_{\tilde{j}}, \tilde{e}_\alpha\big) \\
&= (\nabla_{\partial_{\tilde{k}}}\nabla_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) - (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)\big(\partial_{\tilde{i}}, \partial_{\tilde{j}}, (\overline{\nabla}_{\partial_{\tilde{k}}}\tilde{e}_\alpha)^T\big) \\
&\quad - (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)\big((\overline{\nabla}_{\partial_{\tilde{k}}}\partial_{\tilde{i}})^\perp, \partial_{\tilde{j}}, \tilde{e}_\alpha\big) - (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)\big(\partial_{\tilde{i}}, (\overline{\nabla}_{\partial_{\tilde{k}}}\partial_{\tilde{j}})^\perp, \tilde{e}_\alpha\big) \\
&= (\nabla_{\partial_{\tilde{k}}}\nabla_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) - \tilde{h}_{\beta ij}\tilde{h}_{\beta kl}\tilde{h}_{\alpha kl} - \tilde{h}_{\beta kl}\tilde{h}_{\alpha lj}\tilde{h}_{\beta ki} - \tilde{h}_{\beta kj}\tilde{h}_{\beta kl}\tilde{h}_{\alpha il}\ .
\end{aligned}$$

Use (B.9) to rewrite the above computation as

$$(\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t - \nabla^*\nabla\mathrm{I\!I}^t)(\partial_{\tilde{i}}, \partial_{\tilde{j}}, \tilde{e}_\alpha) = \tilde{h}_{\beta ij}\tilde{h}_{\beta kl}\tilde{h}_{\alpha kl} + \tilde{h}_{\beta kl}\tilde{h}_{\alpha lj}\tilde{h}_{\beta ki} + \tilde{h}_{\beta kj}\tilde{h}_{\beta kl}\tilde{h}_{\alpha il} \tag{B.11}$$

The tensor $\nabla^*\nabla\mathrm{I\!I}^t$ does not have other components. However, $\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t$ does.

$$\begin{aligned}
(\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t)(\partial_{\tilde{i}},\partial_{\tilde{j}},\partial_{\tilde{l}}) &= -(\tilde{\nabla}_{\partial_{\tilde{k}}}\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}},\partial_{\tilde{j}},\partial_{\tilde{l}}) \\
&= -\partial_{\tilde{k}}\big((\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}},\partial_{\tilde{j}},\partial_{\tilde{l}})\big) + (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\overline{\nabla}_{\partial_{\tilde{k}}}\partial_{\tilde{i}},\partial_{\tilde{j}},\partial_{\tilde{l}}) \\
&\quad + (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}},\overline{\nabla}_{\partial_{\tilde{k}}}\partial_{\tilde{j}},\partial_{\tilde{l}}) + (\tilde{\nabla}_{\partial_{\tilde{k}}}\mathrm{I\!I}^t)(\partial_{\tilde{i}},\partial_{\tilde{j}},\overline{\nabla}_{\partial_{\tilde{k}}}\partial_{\tilde{l}}) \\
&= \partial_{\tilde{k}}(\tilde{h}_{\alpha ij}\tilde{h}_{\alpha kl}) + \tilde{h}_{\alpha ij,k}\tilde{h}_{\alpha kl} \\
&= 2\tilde{h}_{\alpha ij,k}\tilde{h}_{\alpha kl} + \tilde{h}_{\alpha ij}\tilde{h}_{\alpha kl,k} \\
&= 2\tilde{h}_{\alpha ij,k}\tilde{h}_{\alpha kl} + \tilde{h}_{\alpha ij}\tilde{h}_{\alpha,l} + \tilde{h}_{\alpha ij}R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{l}}\ . \qquad \text{(B.12)}
\end{aligned}$$

The second last equality uses the fact that $0 = \langle\nabla^{\perp}_{\partial_{\tilde{k}}}\tilde{e}_\alpha,\tilde{e}_\beta\rangle\tilde{h}_{\beta ij}\tilde{h}_{\alpha kl} - \langle\nabla^{\perp}_{\partial_{\tilde{k}}}\tilde{e}_\alpha,\tilde{e}_\beta\rangle\tilde{h}_{\alpha ij}\tilde{h}_{\beta kl}$. The last equality uses the Codazzi equation (2.2). Similarly,

$$\begin{aligned}
(\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t)(\tilde{e}_\beta,\partial_{\tilde{j}},\tilde{e}_\alpha) &= -2\tilde{h}_{\alpha lj,k}\tilde{h}_{\beta kl} - \tilde{h}_{\beta kl,k}\tilde{h}_{\alpha jl} \\
&= -2\tilde{h}_{\alpha lj,k}\tilde{h}_{\beta kl} - \tilde{h}_{\beta,l}\tilde{h}_{\alpha jl} - R_{\tilde{\beta}\tilde{k}\tilde{k}\tilde{l}}\tilde{h}_{\alpha jl} \\
(\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t)(\partial_{\tilde{i}},\tilde{e}_\beta,\tilde{e}_\alpha) &= -2\tilde{h}_{\alpha il,k}\tilde{h}_{\beta kl} - \tilde{h}_{\beta kl,k}\tilde{h}_{\alpha il} \\
&= -2\tilde{h}_{\alpha il,k}\tilde{h}_{\beta kl} - \tilde{h}_{\beta,l}\tilde{h}_{\alpha il} - R_{\tilde{\beta}\tilde{k}\tilde{k}\tilde{l}}\tilde{h}_{\alpha il} \\
(\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t)(\partial_{\tilde{i}},\tilde{e}_\alpha,\partial_{\tilde{j}}) &= -2\tilde{h}_{\alpha kl}\tilde{h}_{\beta il}\tilde{h}_{\beta kj} \\
(\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t)(\tilde{e}_\beta,\tilde{e}_\gamma,\tilde{e}_\alpha) &= -2\tilde{h}_{\beta kl}\tilde{h}_{\gamma kj}\tilde{h}_{\alpha lj} \\
(\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t)(\tilde{e}_\beta,\partial_{\tilde{j}},\partial_{\tilde{i}}) &= 2\tilde{h}_{\beta kl}\tilde{h}_{\alpha lj}\tilde{h}_{\alpha ki} \\
(\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t)(\tilde{e}_\beta,\tilde{e}_\alpha,\partial_{\tilde{j}}) &= 0
\end{aligned} \qquad \text{(B.13)}$$

The evolution equation for $\tilde{h}_{\alpha ij}$ was derived in [20, Proposition 7.1]. With (B.8) and (B.6), we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{I\!I}^t, \Psi\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tilde{g}^{ik}\tilde{g}^{jl}\tilde{h}_{\alpha kl}\tilde{\Psi}_{\alpha ij}\right)\\
&= 2\tilde{h}_{\beta ik}\tilde{h}_{\beta}\tilde{h}_{\alpha kj}\tilde{\Psi}_{\alpha ij} + 2\tilde{h}_{\beta jl}\tilde{h}_{\beta}\tilde{h}_{\alpha il}\tilde{\Psi}_{\alpha ij} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{h}_{\alpha ij}\right)\tilde{\Psi}_{\alpha ij} + \tilde{h}_{\alpha ij}\left(\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Psi}_{\alpha ij}\right)\\
&= 2\tilde{h}_{\beta ik}\tilde{h}_{\beta}\tilde{h}_{\alpha kj}\tilde{\Psi}_{\alpha ij} + 2\tilde{h}_{\beta jl}\tilde{h}_{\beta}\tilde{h}_{\alpha il}\tilde{\Psi}_{\alpha ij} - \langle \nabla^*\nabla \mathrm{I\!I}^t, \Psi\rangle\\
&\quad + (\overline{\nabla}_{\tilde{e}_k}R)_{\tilde{\alpha}\tilde{i}\tilde{j}\tilde{k}}\tilde{\Psi}_{\alpha ij} + (\overline{\nabla}_{\tilde{e}_j}R)_{\tilde{\alpha}\tilde{k}\tilde{i}\tilde{k}}\tilde{\Psi}_{\alpha ij} - 2R_{\widetilde{li}\tilde{j}\tilde{k}}\tilde{h}_{\alpha lk}\tilde{\Psi}_{\alpha ij} + 2R_{\tilde{\alpha}\tilde{\beta}\tilde{j}\tilde{k}}\tilde{h}_{\beta ik}\tilde{\Psi}_{\alpha ij}\\
&\quad + 2R_{\tilde{\alpha}\tilde{\beta}\tilde{i}\tilde{k}}\tilde{h}_{\beta jk}\tilde{\Psi}_{\alpha ij} - R_{\widetilde{lk}\tilde{i}\tilde{k}}\tilde{h}_{\alpha lj}\tilde{\Psi}_{\alpha ij} - R_{\widetilde{lk}\tilde{j}\tilde{k}}\tilde{h}_{\alpha li}\tilde{\Psi}_{\alpha ij} + R_{\tilde{\alpha}\tilde{k}\tilde{\beta}\tilde{k}}\tilde{h}_{\beta ij}\tilde{\Psi}_{\alpha ij}\\
&\quad - \tilde{h}_{\alpha il}\left(\tilde{h}_{\beta lj}\tilde{h}_{\beta} - \tilde{h}_{\beta lk}\tilde{h}_{\beta jk}\right)\tilde{\Psi}_{\alpha ij} - \tilde{h}_{\alpha lk}\left(\tilde{h}_{\beta lj}\tilde{h}_{\beta ik} - \tilde{h}_{\beta lk}\tilde{h}_{\beta ij}\right)\tilde{\Psi}_{\alpha ij}\\
&\quad - \tilde{h}_{\beta ik}\left(\tilde{h}_{\beta lj}\tilde{h}_{\alpha lk} - \tilde{h}_{\beta lk}\tilde{h}_{\alpha lj}\right)\tilde{\Psi}_{\alpha ij} - \tilde{h}_{\alpha jk}\tilde{h}_{\beta ik}\tilde{h}_{\beta}\tilde{\Psi}_{\alpha ij} + \tilde{h}_{\beta ij}\langle \tilde{e}_\beta, \overline{\nabla}_H\tilde{e}_\alpha\rangle\tilde{\Psi}_{\alpha ij}\\
&\quad - \langle \mathrm{I\!I}^t, \tilde{\nabla}^*\tilde{\nabla}\Psi\rangle + \langle \mathrm{I\!I}^t, \mathrm{tr}_{\Gamma_t}(\overline{\nabla}^*\overline{\nabla}\Psi)\rangle - \tilde{h}_{\alpha ij}\tilde{h}_{\alpha,k}\tilde{\Psi}_{kij} + \tilde{h}_{\alpha ij}\langle \overline{\nabla}_H\tilde{e}_\alpha, \tilde{e}_\beta\rangle\tilde{\Psi}_{\beta ij}\\
&\quad - \tilde{h}_{\alpha ij}\tilde{h}_{\gamma}\tilde{h}_{\gamma ik}\tilde{\Psi}_{\alpha kj} + \tilde{h}_{\alpha ij}\tilde{h}_{\gamma,i}\tilde{\Psi}_{\alpha\gamma j} - \tilde{h}_{\alpha ij}\tilde{h}_{\gamma}\tilde{h}_{\gamma jk}\tilde{\Psi}_{\alpha ik} + \tilde{h}_{\alpha ij}\tilde{h}_{\gamma,j}\tilde{\Psi}_{\alpha i\gamma}\\
&= -\langle \tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t, \Psi\rangle - \langle \mathrm{I\!I}^t, \tilde{\nabla}^*\tilde{\nabla}\Psi\rangle + \langle \mathrm{I\!I}^t, \mathrm{tr}_{\Gamma_t}(\overline{\nabla}^2\Psi)\rangle\\
&\quad + (\overline{\nabla}_{\tilde{e}_k}R)_{\tilde{\alpha}\tilde{i}\tilde{j}\tilde{k}}\tilde{\Psi}_{\alpha ij} + (\overline{\nabla}_{\tilde{e}_j}R)_{\tilde{\alpha}\tilde{k}\tilde{i}\tilde{k}}\tilde{\Psi}_{\alpha ij} - 2R_{\widetilde{li}\tilde{j}\tilde{k}}\tilde{h}_{\alpha lk}\tilde{\Psi}_{\alpha ij} + 2R_{\tilde{\alpha}\tilde{\beta}\tilde{j}\tilde{k}}\tilde{h}_{\beta ik}\tilde{\Psi}_{\alpha ij}\\
&\quad + 2R_{\tilde{\alpha}\tilde{\beta}\tilde{i}\tilde{k}}\tilde{h}_{\beta jk}\tilde{\Psi}_{\alpha ij} - R_{\widetilde{lk}\tilde{i}\tilde{k}}\tilde{h}_{\alpha lj}\tilde{\Psi}_{\alpha ij} - R_{\widetilde{lk}\tilde{j}\tilde{k}}\tilde{h}_{\alpha li}\tilde{\Psi}_{\alpha ij} + R_{\tilde{\alpha}\tilde{k}\tilde{\beta}\tilde{k}}\tilde{h}_{\beta ij}\tilde{\Psi}_{\alpha ij}\\
&\quad + R_{\tilde{\alpha}\tilde{k}\tilde{k}\tilde{l}}\tilde{h}_{\alpha ij}\tilde{\Psi}_{lij} - R_{\tilde{\beta}\tilde{k}\tilde{k}\tilde{l}}\tilde{h}_{\alpha jk}\tilde{\Psi}_{\alpha\beta j} - R_{\tilde{\beta}\tilde{k}\tilde{k}\tilde{l}}\tilde{h}_{\alpha il}\tilde{\Psi}_{\alpha i\beta}\\
&\quad - 2\tilde{h}_{\alpha il}\left(\tilde{h}_{\beta lj}\tilde{h}_{\beta} - \tilde{h}_{\beta lk}\tilde{h}_{\beta jk}\right)\tilde{\Psi}_{\alpha ij} - 2\tilde{h}_{\beta ik}\left(\tilde{h}_{\beta lj}\tilde{h}_{\alpha lk} - \tilde{h}_{\beta lk}\tilde{h}_{\alpha lj}\right)\tilde{\Psi}_{\alpha ij}\\
&\quad - 2\tilde{h}_{\alpha jk}\tilde{h}_{\beta ik}\tilde{h}_{\beta}\tilde{\Psi}_{\alpha ij} - 2\tilde{h}_{\alpha kl}\tilde{h}_{\beta il}\tilde{h}_{\beta kj}\tilde{\Psi}_{ji\alpha} - 2\tilde{h}_{\beta kl}\tilde{h}_{\gamma kj}\tilde{h}_{\alpha lj}\tilde{\Psi}_{\alpha\beta\gamma} + 2\tilde{h}_{\beta kl}\tilde{h}_{\alpha lj}\tilde{h}_{\alpha ki}\tilde{\Psi}_{i\beta j}\\
&\quad + 2\tilde{h}_{\alpha ij,k}\tilde{h}_{\alpha kl}\tilde{\Psi}_{lij} - 2\tilde{h}_{\alpha lj,k}\tilde{h}_{\beta kl}\tilde{\Psi}_{\alpha\beta j} - \tilde{h}_{\alpha il,k}\tilde{h}_{\beta kl}\tilde{\Psi}_{\alpha i\beta}\ .
\end{aligned}$$

The last equality uses (B.11), (B.12) and (B.13) to replace $\nabla^*\nabla\mathrm{I\!I}^t$ by $\tilde{\nabla}^*\tilde{\nabla}\mathrm{I\!I}^t$.

By the Cauchy–Schwarz inequality,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{I\!I}^t, \Psi\rangle - \Delta^{\Gamma_t}\langle \mathrm{I\!I}^t, \Psi\rangle + 2\langle \tilde{\nabla}\mathrm{I\!I}^t, \tilde{\nabla}\Psi\rangle\right| \le \frac{1}{2}|\nabla\mathrm{I\!I}^t|^2 + c(|\mathrm{I\!I}^t|^4 + |\mathrm{I\!I}^t|^2 + 1)\ .$$

This together with (6.7) and (B.7) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{I\!I}^t - \Psi|^2 \le \Delta^{\Gamma_t}|\mathrm{I\!I}^t - \Psi|^2 - 2|\tilde{\nabla}(\mathrm{I\!I}^t - \Psi)|^2 + |\nabla\mathrm{I\!I}^t|^2 + c'(|\mathrm{I\!I}^t|^4 + |\mathrm{I\!I}^t|^2 + 1)\ .$$

According to (B.10),

$$|\tilde{\nabla}(\mathrm{I\!I}^t - \Psi)|^2 \ge |\tilde{\nabla}\mathrm{I\!I}^2| - |\tilde{\nabla}\Psi|^2 \ge |\nabla\mathrm{I\!I}^t|^2 + c''|\mathrm{I\!I}^t|^4 - |\overline{\nabla}\Psi|^2\ .$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{I\!I}^t - \Psi|^2 \le \Delta^{\Gamma_t}|\mathrm{I\!I}^t - \Psi|^2 - |\tilde{\nabla}(\mathrm{I\!I}^t - \Psi)|^2 + c'''(|\mathrm{I\!I}^t|^4 + |\mathrm{I\!I}^t|^2 + 1)$$

By the triangle inequality $|\mathrm{I\!I}^t|^2 \leq |\mathrm{I\!I}^t - \Psi|^2 + |\Psi|^2$, it finishes the proof of the proposition. $\square$

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