THE WEAK ELLIPTIC HARNACK INEQUALITY REVISITED

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ABSTRACT. In this paper we firstly derive the weak elliptic Harnack inequality from the generalized capacity condition, the tail estimate of jump measure and the Poincaré inequality, for any regular Dirichlet form without killing part on a measure metric space, by using the lemma of growth and the John-Nirenberg inequality. We secondly show several equivalent characterizations of the weak elliptic Harnack inequality for any (not necessarily regular) Dirichlet form. We thirdly present some consequences of the weak elliptic Harnack inequality.

Dedicated to the memory of Professor Ka-Sing Lau.

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1. INTRODUCTION AND MAIN RESULTS

In 1961, Moser showed in [36] that the following *elliptic Harnack inequality*, denoted by (H), is true: for any compact D' in a domain $D \subset \mathbb{R}^n$ and for any function u which is non-negative, harmonic (with respect to the symmetric, uniformly elliptic divergence-form operator) in D, we have

$$\sup_{D'} u \le C \inf_{D'} u,$$

where $C = C(D', D) \ge 1$ is a constant depending only on D', D. The importance of this inequality is that the constant C is independent of function u (but may depend on two domains D', D). If further D', D are two concentric balls, for example, if $D = B(x, R)$ and $D' = B(x, R/2)$, then

$$\sup_{B(x,R/2)} u \le C \inf_{B(x,R/2)} u, \tag{1.1}$$

where the constant $C \ge 1$ is independent not only of function u, but also of ball B. The inequality (1.1) says that a function, which is both non-negative and harmonic in a ball, is nearly constant around the center. The reader may consult a book [39, Theorem 2.1.1] for more details.

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A symmetric, uniformly elliptic operator gives arise to a strongly local, regular Dirichlet form in the Hilbert space $L^2(\mathbb{R}^n, dx)$ (see for example [19, Chapter 1] on the basic theory of Dirichlet forms on a Hilbert space). The elliptic Harnack inequality plays an important role in analysis, for example, in showing the uniformly local Hölder continuity of harmonic functions, or in obtaining the lower estimate of the heat kernel, for a given Dirichlet form on a metric space.

Since the Moser's celebrated paper [36], there has been an increasing interest in the study on the Harnack inequality for local Dirichlet forms. In 1972, Bombieri and Giusti [10] used the geometric analysis to prove a Harnack inequality for elliptic differential equations on minimal surfaces. In 1980, Safonov [37] obtained the elliptic Harnack inequalities for partial differential operators in non-divergence form. After that, the elliptic Harnack inequality was extended in various settings, see for example, by Benedetto and Trudinger [15, Theorem 3] in 1984 for De Giorgi classes on Euclidean spaces, by Biroli and Mosco [8] in 1995 for a certain class of local Dirichlet forms on discontinuous media, by Strum [41, Proposition 3.2] in 1996 for time-dependent local Dirichlet forms on compact metric spaces, and by Cabré [12] in 1997 for non-divergence elliptic operators on Riemannian manifolds with non-negative curvature. In 2005, Barlow [2, Theorem 2] showed that the elliptic Harnack inequality is equivalent to an annulus-type Harnack inequality for Green's functions in the context of random walks on graphs. In 2015, Grigor'yan, Hu and Lau [24] gave an equivalent characterization for the elliptic Harnack inequality and the mean exit time estimate combined, for any strongly local, regular Dirichlet form on a metric measure space, by using a more general Poincaré inequality and the generalized capacity inequality (see also an earlier work [22]). In 2018, Barlow and Murugan [4] showed that the elliptic Harnack inequality is stable under bounded perturbations for strongly local, regular Dirichlet forms on a length metric space, but assuming the existence of Green function. Recently, this result has been improved by Barlow, Chen and Murugan in [3], without assuming the existence of Green functions and a length but assuming the relative ball-connectedness.

The Harnack inequality above is investigated only for local Dirichlet forms. In recent years, the people have begun to study the elliptic Harnack inequality for non-local operators or non-local Dirichlet forms. It can be imagined that the classical Harnack inequality like the version (1.1) no longer holds for non-local operators (see, for example [5, Section 3] and [18, Theorem 2.2] for α-stable processes). Instead, a weak Harnack inequality different from (1.1) should take place. In this direction, the reader may refer to [16, Theorem 1.2], [17] and [18, Theorem 1.6] for non-local integro-differential operators, [34] for the fractional non-local linearized Monge-Ampère equation, and [13] for pure jump type Dirichlet forms.

In this paper, we are concerned with the weak elliptic Harnack inequality under a more general framework (We do not touch the parabolic Harnack inequality in this paper). Our underlying space is a metric measure space, which may be bounded or unbounded, and our Dirichlet form is mixed, which may be local or non-local, whose jump kernel may not exist. The main results of this paper are as follows:

- to establish the weak elliptic Harnack inequality for local or non-local regular Dirichlet forms (Theorem 1.8 below);
- to study the relationship among different versions of the weak elliptic Harnack inequality appearing in the literature (Theorem 1.9 below).

Let us state our framework of this paper. Let (M, d) be a locally compact separable metric space and μ be a Radon measure on M with full support. The triple (M, d, μ) is called a *metric measure space*. Denote by $B(x, r)$ an open metric ball of radius $r > 0$ centered at x, that is,

$$B_r(x) := B(x, r) := \{y \in M : d(y, x) < r\},$$

and its volume function is denoted by

$$V(x, r) := \mu(B(x, r)).$$

For a ball $B = B(x, r)$ and $\lambda > 0$, the letter $\lambda B := B(x, \lambda r)$ denotes the concentric ball of B. In this paper, we assume that every ball $B(x, r)$ is precompact.

Note that a ball in a metric space may not have a unique centre and radius, and even if the centre is fixed, the radius may not be unique. For this reason we always require a ball to have a fixed centre and radius in this paper. When we pick up a ball $B(y, s)$ contained in a bigger ball $B(x, r)$, we always assume that its radius s is less than $2r$. Let $\overline{R}$ be any number in $(0, \mathrm{diam}(M)]$. Since the metric space considered in this paper may be bounded or unbounded, the number $\overline{R}$ may be finite or infinite.

We say that the *volume doubling condition* (VD) holds if there exists a constant $C_\mu \geq 1$ such that for all $x \in M$ and $r > 0$,

$$V(x, 2r) \leq C_\mu V(x, r). \tag{1.2}$$

It is known that if condition (VD) holds, then there exists a positive number d_2 such that for all $x, y \in M$ and all $0 < r \leq R < \infty$,

$$\frac{V(x, R)}{V(y, r)} \leq C_\mu \left(\frac{d(x, y) + R}{r} \right)^{d_2} \tag{1.3}$$

with the same constant C_μ in (1.2), see for example [23, Proposition 5.1].

We say that the *reverse volume doubling condition* (RVD) holds if there exist two positive constants $C_d \leq 1$ and d_1 such that for all $x \in M$ and $0 < r \leq R < \overline{R}$

$$\frac{V(x, R)}{V(x, r)} \geq C_d \left(\frac{R}{r} \right)^{d_1}. \tag{1.4}$$

Let $w : M \times [0, \infty) \to [0, \infty)$ be a map such that $w(x, \cdot)$ is continuous, strictly increasing, $w(x, 0) = 0$, for any fixed x in M. Assume that there exist positive constants C_1, C_2 and $\beta_2 \geq \beta_1$ such that for all $0 < r \leq R < \infty$ and all $x, y \in M$ with $d(x, y) \leq R$,

$$C_1 \left(\frac{R}{r} \right)^{\beta_1} \leq \frac{w(x, R)}{w(y, r)} \leq C_2 \left(\frac{R}{r} \right)^{\beta_2}. \tag{1.5}$$

For convenience, we write for any metric ball $B = B(x, R)$

$$w(B) := w(x, R).$$

Note that the symbol $w(B)$ is *sensitive* to the center and radius of ball B.

Denote the norm in $L^p := L^p(M, \mu)$ $(1 \leq p < \infty)$ by

$$\|u\|_p := \left(\int_M |u(x)|^p \mu(dx) \right)^{1/p},$$

and $\|u\|_{L^\infty} := \mathrm{esup}_{x \in M} |u(x)|$, where esup is the essential supremum.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part, that is,

$$\mathcal{E}(u, v) = \mathcal{E}^{(L)}(u, v) + \mathcal{E}^{(J)}(u, v), \tag{1.6}$$

where $\mathcal{E}^{(L)}$ is the *local part* (or *diffusion part*) and $\mathcal{E}^{(J)}$ is the *jump part*. Let $\mathcal{F}_{\mathrm{loc}}$ be a space of all measurable functions u on M such that for every precompact open subset U of M, there exists some function $v \in \mathcal{F}$ such that $u = v$ for μ-almost everywhere in U. Then, there exists a unique Radon measure $d\Gamma^{(L)}\langle u \rangle := d\Gamma^{(L)}\langle u, u \rangle$ such that

$$\mathcal{E}^{(L)}(u, u) = \int_M d\Gamma^{(L)}\langle u \rangle$$

for $u \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty$, see for example [19, Lemma 3.2.3, and the first two paragraphs on p.130], wherein the symbols $\mathcal{E}^{(c)} = \mathcal{E}^{(L)}$ and $d\mu^{(c)}_{\langle u, u \rangle} = 2d\Gamma^{(L)}\langle u, u \rangle$ are used instead. For the jump part, there exists a unique Radon measure $J(dx, dy)$ defined on $M \times M \backslash \mathrm{diag}$ such that

$$\mathcal{E}^{(J)}(u,u) = \iint_{M\times M\backslash \mathrm{diag}} (u(x) - u(y))^2 J(dx, dy) \tag{1.7}$$

for all continuous functions $u \in \mathcal{F}$ with compact supports on M. For simplicity, we let the measure $J = 0$ on diag and will drop diag in expression $M \times M\backslash\mathrm{diag}$ in (1.7) when no confusion arises. In the sequel, set

$$\mathcal{E}(u) := \mathcal{E}(u,u)$$

for convenience.

For any non-empty open subset Ω of M, let $C_0(\Omega)$ be a space of all continuous functions with compact supports in Ω. Denote by $\mathcal{F}(\Omega)$ the closure of $\mathcal{F} \cap C_0(\Omega)$ in the norm of

$$\sqrt{\mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)}.$$

Recall that for any non-empty open subset Ω of M, the form $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega)$ if $(\mathcal{E}, \mathcal{F})$ is regular. Let $\left\{P_t^{\Omega}\right\}_{t\geq 0}$ be the heat semigroup associated with $(\mathcal{E}, \mathcal{F}(\Omega))$. Let

$$\mathcal{F}' := \{v + a : \ v \in \mathcal{F}, \ a \in \mathbb{R}\}$$

be a vector space that contains constant functions. We extend the domain of $\mathcal{E}$ to $\mathcal{F}'$ as follows: for all $u, v \in \mathcal{F}$ and $a, b \in \mathbb{R}$, set

$$\mathcal{E}(u + a, v + b) := \mathcal{E}(u, v).$$

We point that the extension is well defined by using (1.6).

Let $U \Subset V$ (that means U is precompact and the closure of U is contained in V) be two non-empty open subsets of M. We say that a measurable function ϕ is a *cutoff function* for $U \Subset V$, denoted by $\phi \in \mathrm{cutoff}(U, V)$, if $\phi \in \mathcal{F}$, and

$$\begin{aligned} \phi &= 1 \ \text{ on } \ U, \\ \phi &= 0 \ \text{ on } \ V^c, \\ 0 \leq \phi &\leq 1 \ \text{ on } \ M. \end{aligned}$$

It is known that if $(\mathcal{E}, \mathcal{F})$ is regular, the set $\mathrm{cutoff}(U, V)$ is non-empty for any two non-empty open subsets $U \Subset V$ of M.

We introduce conditions (Gcap) and ($\mathrm{Cap}_{\leq}$).

Definition 1.1 (condition (Gcap))**.** *We say that* condition (Gcap) *holds if for any $u \in \mathcal{F}' \cap L^\infty$ and any two concentric metric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ with $0 < R < R + r < \overline{R}$, there exists some $\phi \in \mathrm{cutoff}(B_0, B)$ such that*

$$\mathcal{E}(u^2\phi, \phi) \leq \frac{C}{w(x_0, r)} \int_B u^2 d\mu, \tag{1.8}$$

where $C > 0$ is a constant independent of u, B_0, B, but ϕ may depend on u.

Definition 1.2 (condition ($\mathrm{Cap}_{\leq}$))**.** *We say that* condition ($\mathrm{Cap}_{\leq}$) *holds if there exists a constant $C > 0$ such that for all balls B of radius R less than $\overline{R}$*

$$\mathrm{cap}((2/3)B, B) \leq C\frac{\mu(B)}{w(B)}, \tag{1.9}$$

where the capacity $\mathrm{cap}(A, \Omega)$ *for any two open subsets $A \Subset \Omega$ of M is defined by*

$$\mathrm{cap}(A, \Omega) := \inf\{\mathcal{E}(\varphi, \varphi) : \ \varphi \in \mathrm{cutoff}(A, \Omega)\}.$$

Clearly, condition (Gcap) implies condition ($\mathrm{Cap}_{\leq}$) by taking $u = 1$ in (1.8) and by using the second inequality in (1.5).

Definition 1.3 (condition (FK))**.** *We say that condition* (FK) *holds if there exist three positive constants C_F, ν and $\sigma \in (0,1)$ such that for any ball $B := B(x,r)$ with $0 < r < \sigma\overline{R}$ and any non-empty open subset $D \subset B$,*

$$\lambda_1(D) \geq \frac{C_F^{-1}}{w(B)}\left(\frac{\mu(B)}{\mu(D)}\right)^{\nu}, \tag{1.10}$$

where $\lambda_1(D)$ is defined by

$$\lambda_1(D) := \inf_{u\in\mathcal{F}(D)\setminus\{0\}} \frac{\mathcal{E}(u,u)}{\|u\|_2^2}.$$

Without loss of generality, we can assume that $0 < \nu < 1$ by noting that $\frac{\mu(B)}{\mu(D)} \geq 1$.

Definition 1.4 (condition (PI))**.** *We say that condition* (PI) *holds if there exist two constants $\kappa \geq 1$, $C > 0$ such that for any metric ball $B := B(x_0, r)$ with $0 < r < \overline{R}/\kappa$ and any $u \in \mathcal{F}' \cap L^\infty$,*

$$\int_B (u-u_B)^2 d\mu \leq Cw(B)\left\{\int_{\kappa B} d\Gamma^{(L)}\langle u\rangle + \int_{(\kappa B)\times(\kappa B)} (u(x)-u(y))^2 J(dx,dy)\right\}, \tag{1.11}$$

where u_B is the average of the function u over B, that is,

$$u_B = \frac{1}{\mu(B)}\int_B u d\mu =: \fint_B u d\mu$$

For a transition kernel $J(x,E)$ defined on $M\times\mathcal{B}(M)$ where $\mathcal{B}(M)$ is the collection of all Borel subsets of M, denote by

$$J(x,E) := \int_E J(x,dy). \tag{1.12}$$

We introduce condition (TJ).

Definition 1.5 (condition (TJ))**.** *We say that condition* (TJ) *holds if there exists a transition kernel $J(x,E)$ on $M\times\mathcal{B}(M)$ such that, for any point x in M and any $R > 0$,*

$$\begin{aligned} &J(dx,dy) = J(x,dy)\mu(dx) \ \text{ and} \\ &J(x, B(x,R)^c) \leq \frac{C}{w(x,R)} \end{aligned} \tag{1.13}$$

for a non-negative constant C independent of x, R.

For an open subset Ω of M and a function $f \in L^2(\Omega)$, we say that a function $u \in \mathcal{F}$ is *f-superharmonic* (resp. *f-subharmonic*) in Ω if for any non-negative $\varphi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u,\varphi) \geq (f,\varphi) \ \text{ (resp. } \mathcal{E}(u,\varphi) \leq (f,\varphi)\text{)}. \tag{1.14}$$

We say that a function $u \in \mathcal{F}$ is *f-harmonic* in Ω if u is both f-superharmonic and f-subharmonic in Ω. If $f \equiv 0$, an f-superharmonic is shortened *superharmonic*, and a similar notion applies to an f-subharmonic or an f-harmonic.
For any two open subsets $U \Subset \Omega$ of M and any measurable function v, denote by

$$T_{U,\Omega}(v) := \operatorname*{esup}_{x\in U} \int_{\Omega^c} |v(y)| J(x,dy). \tag{1.15}$$

We introduce *condition* (wEH), the *weak elliptic Harnack inequality.*

Definition 1.6 (condition (wEH))**.** *We say that condition* (wEH) *holds if there exist four universal constants p, δ, σ in $(0,1)$ and $C_H \geq 1$ such that, for any two concentric balls $B_r := B(x_0,r) \subset B_R := B(x_0,R)$ with $0 < r \leq \delta R$, $R < \sigma\overline{R}$, any function $f \in L^\infty(B_R)$, and for any $u \in \mathcal{F}' \cap L^\infty$ that is non-negative, f-superharmonic in B_R,*

$$\left(\fint_{B_r} u^p d\mu\right)^{1/p} \leq C_H\left(\operatorname*{einf}_{B_r} u + w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right), \tag{1.16}$$

where $u_- := 0 \vee (-u)$ *is the negative part of function* u*, and* $T_{\frac{3}{4}B_R,B_R}$ *is defined by (1.15), that is,*

$$T_{\frac{3}{4}B_R,B_R}(u_-) = \underset{x\in\frac{3}{4}B_R}{\mathrm{esup}} \int_{M\setminus B_R} u_-(y)J(x,dy).$$

We remark that the constants p, δ, σ, C_H *are all independent of* x_0, R, r, f *and* u*.*

Remark 1.7. *If* u *is superharmonic, non-negative in* B_R*, then (1.16) reads*

$$\left(\fint_{B_r} u^p d\mu\right)^{1/p} \leq C_H\left(\underset{B_r}{\mathrm{einf}}\, u + w(B_r)T_{\frac{3}{4}B_R,B_R}(u_-)\right). \tag{1.17}$$

If the form $(\mathcal{E},\mathcal{F})$ *is strongly local and* u *is harmonic, non-negative in* B_R*, then (1.16) becomes*

$$\left(\fint_{B_r} u^p d\mu\right)^{1/p} \leq C_H \underset{B_r}{\mathrm{einf}}\, u, \tag{1.18}$$

and in this situation, we in fact have that the weak Harnack inequality (1.18) is equivalent to the strong Harnack inequality (1.1), since the inequality (1.18) is equivalent to the following

$$\underset{B_r}{\mathrm{einf}}\, u \geq a \exp\left(-\frac{C}{\omega_{B_r}(\{u \geq a\})}\right) \text{ for any } a > 0 \tag{1.19}$$

by using the equivalence (wEH) $\Leftrightarrow$ (wEH2) *in Theorem 1.9 below where condition* (wEH2) *will be stated in Definition 5.3 and by using the fact that (1.19)* $\Rightarrow$ (H) *in* [24, from Corollary 7.3 to Theorem 7.8 on pages 1525-1535].

The weak elliptic Harnack inequality says that for any function u, which is non-negative and superharmonic in a ball B_R, its mean value over a smaller concentric ball B_r in the L^p-quantity (not a norm) for a small $p \in (0,1)$, can be controlled by its essential infimum over the smaller ball B_r, plus a tail estimate outside the ball B_R.

The main results of this paper are stated in Theorems 1.8 and 1.9 below.

Theorem 1.8. *Let* $(\mathcal{E},\mathcal{F})$ *be a regular Dirichlet form in* $L^2(M,\mu)$ *without killing part. Then*

$$\text{(VD)} + \text{(RVD)} + \text{(Gcap)} + \text{(TJ)} + \text{(PI)} \Rightarrow \text{(wEH)}. \tag{1.20}$$

We will prove Theorem 1.8 at the end of Section 4. For this, we need to show the following implications:

$$\text{(VD)} + \text{(RVD)} + \text{(PI)} \quad \Rightarrow \quad \text{(FK) (see Section 2)}, \tag{1.21}$$

$$\text{(VD)} + \text{(FK)} + \text{(Gcap)} + \text{(TJ)} \quad \Rightarrow \quad \text{(LG) (see Section 3)}, \tag{1.22}$$

$$\text{(VD)} + \text{(LG)} + (\text{Cap}_\leq) + \text{(PI)} \quad \Rightarrow \quad \text{(wEH) (see Section 4)}, \tag{1.23}$$

where condition (LG) is a refinement of the *lemma of growth* to be stated in Lemma 3.5 below.

We remark that if the metric space (M,d) is unbounded and the scaling function $w(x,r)$ is independent of point x, a similar implication to (1.21) was obtained for strongly local Dirichlet forms (cf. [24, Theorem 5.1]), and for purely jump Dirichlet forms (cf. [14, Propositions 7.3 and 7.4]). Here we generalize this result to the case when the scaling function $w(x,\cdot)$ may depend on point x and the metric space may be bounded or unbounded.

Our Theorem 1.8 is an extension of a similar result in [13, Theorem 3.1] in the sense that, instead of assuming condition (TJ) in this paper, the following stronger hypothesis than condition (TJ) was assumed in [13]: the jump kernel $J(x,y)$ exists and satisfies the following *pointwise* upper estimate

$$J(x,y) \leq \frac{C}{V(x,d(x,y))w(x,d(x,y))}$$

for $\mu\times\mu$-almost all (x,y) in $M\times M \setminus \text{diag}$. Also the metric space (M,d) considered in [13] is assumed to be unbounded. We emphasize that we do not assume the jump kernel $J(x,y)$ exists, neither the boundedness of the metric space.

Liu and Murugan [31, Theorem 1.2] show that the parabolic Harnack inequality implies the existence of the jump kernel $J(x, y)$ for a pure jump regular Dirichlet form. A natural question arises whether the weak elliptic Harnack inequality also implies the existence of the jump kernel. The answer is negative. In fact, the paper [6, Section 15] has given an example on the ultra-metric space where the jump measure satisfies both conditions (PI) and (TJ) (noting that condition (Gcap) automatically holds since it follows directly from condition (TJ) and the ultra-metric property), but the jump kernel does not exist. By Theorem 1.8 above, the weak elliptic Harnack inequality is true, however, the jump kernel does not exist in this case. We will give the details in Section 7.

Let us explain the idea of proving the weak elliptic Harnack inequality in Theorem 1.8. The proof essentially consists of the following two steps (under the case when $f \equiv 0$).

(1) To obtain the so-called *measure-to-point lemma* as follows: for some $\varepsilon \in (0, 1)$ and for any non-negative superharmonic function u in a ball B, there exists a constant $\eta > 0$, depending only on ε but independent of the ball B and the function u, such that

$$\frac{\mu(B \cap \{u > 1\})}{\mu(B)} \geq \varepsilon \quad \Rightarrow \quad \inf_{\frac{1}{2}B} u \geq \eta. \tag{1.24}$$

(2) To obtain the so-called *crossover lemma* as follows: there exist three universal numbers p, δ in $(0, 1)$ and $C > 0$ such that, for any non-negative superharmonic function u in a ball B_R, any concentric ball B_r of B_R with $0 < r \leq \delta R$ and for any positive number $\lambda \geq w(B_R) T_{\frac{3}{4}B_R, B_R}(u_-)$,

$$\left(\fint_{B_r} (u + \lambda)^p d\mu \right)^{1/p} \left(\fint_{B_r} (u + \lambda)^{-p} d\mu \right)^{1/p} \leq C. \tag{1.25}$$

The implication (1.24) says that, if the occupation *measure* of a superlevel set

$$\{u \geq a\} \ \text{ for } a > 0$$

in a ball B for a function u, which is non-negative, superharmonic in B, is bounded from below by a constant ε, then the function u should be also bounded from below by a positive number ηa at almost all *points* near the center.

The measure-to-point lemma is essentially the same as the *Lemma of growth* introduced by Landis in [29], [30] in studying solutions of elliptic second order PDEs (local Dirichlet form) in $\mathbb{R}^n$. This Lemma of growth has been reformulated and extended to the case for pure jump type (non-local) Dirichlet forms on the metric measure space in [20, Lemma 4.1], see also a forthcoming paper [21] for mixed (either local or non-local) Dirichlet forms defined by (1.6) without killing part (cf. Lemma 3.5 below). An alternative version of Lemma of growth for pure jump type (non-local) Dirichlet forms on metric space was stated in [13, Proposition 3.6].

We remark that the measure-to-point lemma is originated from the work by Moser [35, Theorem 2], and developed by Krylov and Safonov [27], [28], [37]. The reader may consult the reference [33, Section 3] for the classical case.

Once the measure-to-point lemma has been established, one needs further to show the crossover lemma (1.25), where the Poincaré inequality comes into a stage. To achieve (1.25), one needs to show that, for any $u \in \mathcal{F}' \cap L^\infty$ that is superharmonic and non-negative in a ball B and for any positive number λ bounded from below by a tail (in the case of local Dirichlet forms, any number $\lambda > 0$ will be fine), the logarithm function

$$\ln(u + \lambda)$$

belongs to the space $\mathrm{BMO}(\delta B)$, for some number $\delta \in (0, 1)$ that is independent of u, λ and ball B. After that, the rest of the proof is standard: one makes use of Lemma 8.3 in Appendix for an exponential function

$$\exp\left(\frac{c}{b} g\right) \ \text{ for any } b \geq \|g\|_{\mathrm{BMO}}$$

for $g := \ln(u+\lambda)$, which is valid from the John-Nirenberg inequality (see Lemma 8.2 in Appendix), and we are eventually led to the desired crossover lemma (1.25) (see Lemma 4.4 below).

Besides the version of the weak elliptic Harnack inequality stated in Definition 1.6, there are several other versions in the literature, see for example [13, Proposition 3.6], [24, Lemma 7.2], [20, Lemma 4.5]. We list all of them in Section 5 and term as conditions (wEH1), (wEH2), (wEH3), (wEH4). We shall show that the first three conditions (wEH1), (wEH2), (wEH3) are equivalent one another, each of which implies condition (wEH4).

Theorem 1.9. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form in $L^2(M,\mu)$. If condition* (VD) *holds, then*

$$\begin{aligned}(\text{wEH}) \quad &\Leftrightarrow \quad (\text{wEH1}) \Leftrightarrow (\text{wEH2}) \Leftrightarrow (\text{wEH3}) &(1.26)\\ &\Rightarrow \quad (\text{wEH4}). &(1.27)\end{aligned}$$

We will prove Theorem 1.9 at the end of Section 5.

2. Faber-Krahn inequality and Dirichlet heat kernel

In this section, we show that for a regular Dirichlet form without killing part on a metric space, if the measure satisfies conditions (VD) and (RVD), then the Poincaré inequality implies the Faber-Krahn inequality. Although this conclusion is known to the expert, there is no a direct proof in the literature, and we will give a self-contained proof for convenience. Here we do not assume the existence of the jump kernel, neither the independence of point for the scaling function w. Our result can be viewed as an extension of the previous work [24, Theorem 5.1] for a local Dirichlet form for the doubling measure, and [6, Lemmas 5.2, 5.3] for a non-local Dirichlet form for the Ahlfors-regular measure. See also [11, Proposition 3.4.1]. As a by-product, we derive that the Dirichlet heat kernel $p_t^B(x,y)$ exists and satisfies an upper bound, for any ball B of radius less than $\sigma\overline{R}$.

We introduce condition (Nash_B), which is the *Nash inequality on a ball B.*

Definition 2.1 (condition (Nash_B))**.** *We say that condition* (Nash_B) *holds if there exist three positive constants $\sigma\in(0,1)$ and ν, C such that for any metric ball B of radius $r\in(0,\sigma\overline{R})$ and any $u\in\mathcal{F}(B)$,*

$$\|u\|_2^{2+2\nu} \le \frac{C}{\mu(B)^\nu}\|u\|_1^{2\nu}\left(\|u\|_2^2 + w(B)\mathcal{E}(u,u)\right). \tag{2.1}$$

We remark that constants C and ν,σ are all independent of ball B and function u.

We show that the Poincaré inequality implies the Nash inequality on a ball.

Lemma 2.2. *Assume that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2(M,\mu)$ without killing part. If conditions* (VD) *and* (PI) *are satisfied, then condition* (Nash_B) *holds, that is,*

$$(\text{VD}) + (\text{PI}) \Rightarrow (\text{Nash}_B).$$

Proof. Since the proof is quite long, we divided into two steps.

Step 1. We show that there exists a constant $C>0$ such that for all $s>0$ and all $u\in\mathcal{F}\cap L^1$ with $\|u\|_1>0$

$$\|u_s\|_2^2 \le \frac{C\|u\|_1^2}{\inf_{z\in \text{supp}\,(u)} V(z,s)}, \tag{2.2}$$

where $u_s(x)$ is the average of function u over a ball $B(x,s)$, that is,

$$u_s(x) = \frac{1}{V(x,s)}\int_{B(x,s)} u(z)\mu(dz) \ \text{ for } x\in M, s>0.$$

The proof is motivated by [38, Theorem 2.4]. At this step, we do not need condition (PI). To this end, let $\|u\|_1>0$, and denote by

$$A_s := \{x\in M : d(x,\text{supp}\,(u)) < s\},$$

the s-neighborhood of the support of u. Clearly, we see that $u_s(x) \equiv 0$ when x lies outside the set A_s, since $u(z) = 0$ for $z \in B(x, s) \subset M \setminus \operatorname{supp}(u)$. It follows that

$$\|u_s\|_\infty \le \frac{\|u\|_1}{\inf_{x\in A_s} V(x,s)} \le \frac{2^{d_2} C_\mu \|u\|_1}{\inf_{x\in \operatorname{supp}(u)} V(x,s)}, \tag{2.3}$$

where we have used the fact that for any $x \in A_s$,

$$\frac{1}{\inf_{x\in A_s} V(x,s)} \le \frac{2^{d_2} C_\mu}{\inf_{x\in \operatorname{supp}(u)} V(x,s)},$$

since there exists a point $z \in \operatorname{supp}(u)$ such that $d(z, x) < s$, and thus by (1.3)

$$\frac{V(z,s)}{V(x,s)} \le C_\mu \left(\frac{d(z,x)+s}{s}\right)^{d_2} \le C_\mu \left(\frac{s+s}{s}\right)^{d_2} = 2^{d_2} C_\mu, \tag{2.4}$$

from which,

$$\frac{1}{\inf_{x\in A_s} V(x,s)} = \sup_{x\in A_s} \frac{1}{V(x,s)} \le 2^{d_2} C_\mu \sup_{z\in \operatorname{supp}(u)} \frac{1}{V(z,s)} = \frac{2^{d_2} C_\mu}{\inf_{x\in \operatorname{supp}(u)} V(x,s)}.$$

On the other hand,

$$\begin{aligned}
\|u_s\|_1 &\le \int_{A_s} \frac{1}{V(x,s)} \left(\int_{B(x,s)} |u(z)|\mu(dz)\right)\mu(dx) \\
&= \int_{A_s} \frac{1}{V(x,s)} \left(\int_{\operatorname{supp}(u)} |u(z)| 1_{B(x,s)}(z)\mu(dz)\right)\mu(dx) \\
&= \int_{\operatorname{supp}(u)} |u(z)| \left(\int_{A_s} \frac{1_{B(x,s)}(z)}{V(x,s)}\mu(dx)\right)\mu(dz) \\
&= \int_{\operatorname{supp}(u)} |u(z)| \left(\int_{A_s\cap B(z,s)} \frac{1}{V(x,s)}\mu(dx)\right)\mu(dz) \\
&\le \int_{\operatorname{supp}(u)} |u(z)| \frac{V(z,s)}{\inf_{x\in B(z,s)} V(x,s)}\mu(dz) \\
&= \int_{\operatorname{supp}(u)} |u(z)| \sup_{x\in B(z,s)} \frac{V(z,s)}{V(x,s)}\mu(dz) \le 2^{d_2} C_\mu \|u\|_1,
\end{aligned} \tag{2.5}$$

since for any $z \in \operatorname{supp}(u)$ and any $x \in B(z, s)$,

$$\frac{V(z,s)}{V(x,s)} \le 2^{d_2} C_\mu$$

by virtue of (2.4). Therefore, it follows from (2.3), (2.5) that

$$\|u_s\|_2^2 \le \|u_s\|_\infty \|u_s\|_1 \le \frac{(2^{d_2} C_\mu)^2 \|u\|_1^2}{\inf_{z\in \operatorname{supp}(u)} V(z,s)},$$

thus showing (2.2) with $C := (2^{d_2} C_\mu)^2$.

Step 2. We show that condition (Nash_B) holds. We assume that condition (PI) holds.

Fix a ball $B := B(x_0, r)$ with $r \in (0, \frac{\overline{R}}{\kappa})$, where constant κ is the same as in condition (PI). Let $s \in (0, \frac{\overline{R}}{2\kappa})$ be a number to be determined later on, and fix a function $u \in \mathcal{F}(B) \cap L^1(M, \mu)$. Since M is separable, there is a countable family of points $\{y_i\}_{i=1}^\infty$ such that $M \subset \bigcup_{i=1}^\infty B(y_i, s)$. By the doubling property, we can find a subsequence $\{x_i\}_{i=1}^\infty \subset \{y_i\}_{i=1}^\infty$ such that $M = \bigcup_{i=1}^\infty B_i$ with $B_i := B(x_i, s)$, and $\{\frac{1}{5} B_i\}_{i=1}^\infty$ are pairwise disjoint (see [25, Theorem 1.16]). The over-lapping number

$\sum_{i=1}^\infty 1_{2\kappa B_i}$ is bounded by some integer N_0 depending only on κ and C_μ, that is, $\sum_{i=1}^\infty 1_{2\kappa B_i} \le N_0$. From this, we have for any measurable function $g \ge 0$,

$$\begin{aligned}\sum_{i=1}^\infty \iint_{(2\kappa B_i)\times M} g(x,y)J(dx,dy) &= \iint_{M\times M} g(x,y)\sum_{i=1}^\infty 1_{2\kappa B_i}(x)J(dx,dy)\\ &\le N_0 \iint_{M\times M} g(x,y)J(dx,dy). \end{aligned} \tag{2.6}$$

We estimate the term $\|u-u_s\|_2^2$ by

$$\begin{aligned}\|u-u_s\|_2^2 &\le \sum_{i=1}^\infty \int_{B_i} |u(x)-u_s(x)|^2\mu(dx)\\ &\le 2\sum_{i=1}^\infty \left(\int_{B_i} (|u(x)-u_{2B_i}|^2 + |u_{2B_i}-u_s(x)|^2)\mu(dx)\right) =: 2(I_1+I_2). \end{aligned} \tag{2.7}$$

For I_1, we have by condition (PI),

$$\begin{aligned} I_1 &= \sum_{i=1}^\infty \int_{B_i} |u(x)-u_{2B_i}|^2\mu(dx) \le \sum_{i=1}^\infty \int_{2B_i} |u(x)-u_{2B_i}|^2\mu(dx)\\ &\le C\sum_{i=1}^\infty w(x_i,2s)\left\{\int_{2\kappa B_i} d\Gamma^{(L)}\langle u\rangle + \iint_{(2\kappa B_i)\times(2\kappa B_i)} (u(x)-u(y))^2J(dx,dy)\right\}. \end{aligned} \tag{2.8}$$

For I_2, note that for any $x \in B_i = B(x_i,s)$, the function $1_{B(x,s)}(z) = 0$ when $z \in (2B_i)^c \subset B(x,s)^c$. Using the Cauchy-Schwarz inequality and condition (PI), we have for any $x \in B_i$

$$\begin{aligned} |u_s(x)-u_{2B_i}|^2 &= \left|\int_M \frac{1_{B(x,s)}(z)}{V(x,s)}(u(z)-u_{2B_i})\mu(dz)\right|^2 \le \int_M \frac{1_{B(x,s)}(z)}{V(x,s)}|u(z)-u_{2B_i}|^2\mu(dz)\\ &\le \int_{2B_i} \frac{1}{V(x,s)}|u(z)-u_{2B_i}|^2\mu(dz) \le \frac{2^{d_2}C_\mu}{V(x_i,s)}\int_{2B_i} |u(z)-u_{2B_i}|^2\mu(dz)\\ &\le \frac{Cw(x_i,2s)}{V(x_i,s)}\left\{\int_{2\kappa B_i} d\Gamma^{(L)}\langle u\rangle + \iint_{(2\kappa B_i)\times(2\kappa B_i)} (u(x)-u(y))^2J(dx,dy)\right\}, \end{aligned} \tag{2.9}$$

where we have used the fact that for any $x \in B_i$,

$$\frac{V(x_i,s)}{V(x,s)} \le C_\mu\left(\frac{d(x_i,x)+s}{s}\right)^{d_2} \le 2^{d_2}C_\mu$$

by virtue of (1.3). Therefore, it follows that

$$\begin{aligned} I_2 &= \sum_{i=1}^\infty \int_{B_i} (u_{2B_i}-u_s(x))^2\mu(dx)\\ &\le \sum_{i=1}^\infty \int_{B_i} \frac{Cw(x_i,2s)}{V(x_i,s)}\left\{\int_{2\kappa B_i} d\Gamma^{(L)}\langle u\rangle + \iint_{(2\kappa B_i)\times(2\kappa B_i)} (u(x)-u(y))^2J(dx,dy)\right\}\mu(dx)\\ &= C\sum_{i=1}^\infty w(x_i,2s)\left\{\int_{2\kappa B_i} d\Gamma^{(L)}\langle u\rangle + \iint_{(2\kappa B_i)\times(2\kappa B_i)} (u(x)-u(y))^2J(dx,dy)\right\}. \end{aligned} \tag{2.10}$$

Combining (2.8) and (2.10), we conclude from (2.7) that

$$\begin{aligned} \|u-u_s\|_2^2 &\le 2(I_1+I_2)\\ &\le C\sum_{i=1}^\infty w(x_i,2s)\left\{\int_{2\kappa B_i} d\Gamma^{(L)}\langle u\rangle + \iint_{(2\kappa B_i)\times(2\kappa B_i)} (u(x)-u(y))^2J(dx,dy)\right\} \end{aligned} \tag{2.11}$$

for a positive constant C depending only on the constants from condition (VD) (independent of u, s, B_i).

Since $u \in \mathcal{F}(B)$, if $2\kappa B_i \subset B^c$, we see that $u(x) = u(y) = 0$ when $x, y \in 2\kappa B_i$, thus $1_{2\kappa B_i} d\Gamma^{(L)}\langle u \rangle = 0$, and so the integral in the above summation vanishes. In other words, the summation in (2.11) is taken only over the indices i such that $(2\kappa B_i) \cap B \neq \emptyset$. Set

$$Q := \sup_{i:(2\kappa B_i)\cap B\neq\emptyset} w(x_i, s). \tag{2.12}$$

Since $w(x_i, 2s) \le C_2 2^{\beta_2} w(x_i, s)$ by using (1.5), we obtain that

$$\begin{aligned}
&\sum_{i=1}^{\infty} w(x_i, s) \iint_{(2\kappa B_i)\times(2\kappa B_i)} (u(x) - u(y))^2 J(dx, dy)\\
&\le Q \sum_{i=1}^{\infty} \iint_{(2\kappa B_i)\times M} (u(x) - u(y))^2 J(dx, dy)\\
&\le Q \cdot N_0 \iint_{M\times M} (u(x) - u(y))^2 J(dx, dy) \ \text{ (using (2.6))}\\
&= N_0 Q \mathcal{E}^{(J)}(u, u).
\end{aligned} \tag{2.13}$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{\infty} w(x_i, s) \int_{2\kappa B_i} d\Gamma^{(L)}\langle u \rangle &\le Q \sum_{i=1}^{\infty} \int_{2\kappa B_i} d\Gamma^{(L)}\langle u \rangle = Q \int_M \sum_{i=1}^{\infty} 1_{2\kappa B_i} d\Gamma^{(L)}\langle u \rangle\\
&\le N_0 Q \int_M d\Gamma^{(L)}\langle u \rangle = N_0 Q \mathcal{E}^{(L)}(u, u).
\end{aligned} \tag{2.14}$$

Therefore, combining (2.13) and (2.14), we conclude from (2.11) that for all $s \in (0, \frac{\overline{R}}{2\kappa})$

$$\|u - u_s\|_2^2 \le C \left\{ N_0 Q \mathcal{E}^{(L)}(u, u) + N_0 Q \mathcal{E}^{(J)}(u, u) \right\} = C N_0 Q \mathcal{E}(u). \tag{2.15}$$

It is left to estimate Q for any $s \in (0, \frac{\overline{R}}{2\kappa})$. We distinguish two cases when $s \le r$ or not.

Indeed, let $z_0 \in (2\kappa B_i) \cap B$. By (1.5), we have

$$\frac{w(x_i, s)}{w(z_0, s)} = \frac{w(x_i, s)}{w(x_i, 2\kappa s)} \cdot \frac{w(x_i, 2\kappa s)}{w(z_0, s)} \le C_1^{-1} \left(\frac{s}{2\kappa s}\right)^{\beta_1} \cdot C_2 \left(\frac{2\kappa s}{s}\right)^{\beta_2} = c'(\kappa),$$

whilst for $s \le r$

$$\frac{w(z_0, s)}{w(x_0, r)} \le C_1^{-1} \left(\frac{s}{r}\right)^{\beta_1}.$$

Thus,

$$\frac{w(x_i, s)}{w(x_0, r)} = \frac{w(x_i, s)}{w(z_0, s)} \cdot \frac{w(z_0, s)}{w(x_0, r)} \le c \left(\frac{s}{r}\right)^{\beta_1}$$

if $(2\kappa B_i) \cap B \neq \emptyset$ and $s \le r$. From this, we obtain

$$Q = \sup_{i:(2\kappa B_i)\cap B\neq\emptyset} w(x_i, s) \le c' \left(\frac{s}{r}\right)^{\beta_1} w(x_0, r) \ \text{ if } s \le r. \tag{2.16}$$

Plugging (2.16) into (2.15), we have

$$\|u - u_s\|_2^2 \le C' \left(\frac{s}{r}\right)^{\beta_1} w(x_0, r) \mathcal{E}(u) \tag{2.17}$$

if $s \le r$. Note that if $s \le r$, then for any $x \in \text{supp}\,(u) \subset B(x_0, r)$

$$\frac{V(x_0, r)}{V(x, s)} \le C_\mu \left(\frac{d(x_0, x) + r}{s}\right)^{d_2} \le 2^{d_2} C_\mu \left(\frac{r}{s}\right)^{d_2},$$

which gives by (2.2) that

$$\|u_s\|_2^2 \le \frac{C\|u\|_1^2}{\inf\limits_{x\in \operatorname{supp}(u)} V(x,s)} \le C\left(\frac{r}{s}\right)^{d_2} \frac{\|u\|_1^2}{V(x_0,r)}. \tag{2.18}$$

Therefore, we conclude from (2.17), (2.18) that for all $0 < s < r \wedge \frac{\overline{R}}{2\kappa}$,

$$\begin{aligned}\|u\|_2^2 &\le 2\left(\|u-u_s\|_2^2 + \|u_s\|_2^2\right)\\ &\le C\left(\left(\frac{s}{r}\right)^{\beta_1} w(x_0,r)\mathcal{E}(u) + \left(\frac{r}{s}\right)^{d_2} \frac{\|u\|_1^2}{V(x_0,r)}\right).\end{aligned} \tag{2.19}$$

On the other hand, if $r \le s < \frac{\overline{R}}{2\kappa}$, it is clear that

$$\|u\|_2^2 \le \left(\frac{s}{r}\right)^{\beta_1} \|u\|_2^2. \tag{2.20}$$

Summing up (2.19) and (2.20), we obtain for all $0 < s < \frac{\overline{R}}{2\kappa}$,

$$\begin{aligned}\|u\|_2^2 &\le C\left(\left(\frac{s}{r}\right)^{\beta_1}\left(w(x_0,r)\mathcal{E}(u) + \|u\|_2^2\right) + \left(\frac{r}{s}\right)^{d_2} \frac{\|u\|_1^2}{V(x_0,r)}\right)\\ &\le C2^{d_2}\left(\left(\frac{2s}{r}\right)^{\beta_1}\left(w(x_0,r)\mathcal{E}(u) + \|u\|_2^2\right) + \left(\frac{r}{2s}\right)^{d_2} \frac{\|u\|_1^2}{V(x_0,r)}\right).\end{aligned} \tag{2.21}$$

We minimize the right-hand side of (2.21) in $s \in (0, \frac{\overline{R}}{2\kappa})$, for example, by choosing s such that

$$\left(\frac{2s}{r}\right)^{\beta_1}\left(w(x_0,r)\mathcal{E}(u) + \|u\|_2^2\right) = \left(\frac{r}{2s}\right)^{d_2} \frac{\|u\|_1^2}{V(x_0,r)},$$

that is,

$$s = \frac{r}{2}\left(\frac{\|u\|_1^2}{V(x_0,r)\left(w(x_0,r)\mathcal{E}(u) + \|u\|_2^2\right)}\right)^{\frac{1}{\beta_1+d_2}}. \tag{2.22}$$

We postpone verifying that $s \in (0, \frac{\overline{R}}{2\kappa})$. Therefore, it follows that

$$\|u\|_2^2 \le C'\left(\frac{\|u\|_1^2}{V(x_0,r)}\right)^{\frac{\beta_1}{\beta_1+d_2}}\left(\|u\|_2^2 + w(x_0,r)\mathcal{E}(u)\right)^{\frac{d_2}{\beta_1+d_2}},$$

thus showing that

$$\|u\|_2^{2(1+\frac{\beta_1}{d_2})} \le C\left(\|u\|_2^2 + w(x_0,r)\mathcal{E}(u)\right)\left(\frac{\|u\|_1^2}{V(x_0,r)}\right)^{\frac{\beta_1}{d_2}},$$

for all $u \in \mathcal{F}(B) \cap L^1$. Hence, condition (Nash$_B$) holds with $\sigma = \frac{1}{\kappa}$ and $\nu = \frac{\beta_1}{d_2}$.

It remains to verify that the number s given by (2.22) satisfies condition $s \in (0, \frac{\overline{R}}{2\kappa})$. Indeed, by the Cauchy-Schwarz inequality, we have for any $u \in \mathcal{F}(B)$,

$$\|u\|_1^2 \le V(x_0,r)\|u\|_2^2,$$

from which, we see that, using the fact that $r \in (0, \overline{R}/\kappa)$,

$$s = \frac{r}{2}\left(\frac{\|u\|_1^2}{V(x_0,r)\left(w(x_0,r)\mathcal{E}(u) + \|u\|_2^2\right)}\right)^{\frac{1}{\beta_1+d_2}} \le \frac{r}{2}\left(\frac{\|u\|_1^2}{V(x_0,r)\|u\|_2^2}\right)^{\frac{1}{\beta_1+d_2}} \le \frac{r}{2} < \frac{\overline{R}}{2\kappa}.$$

The proof is complete. □

We derive the on-diagonal upper bound of the Dirichlet heat kernel on any ball by using condition (Nash_B). In particular, we derive the Faber-Krahn inequality.

Lemma 2.3. *Let* $(\mathcal{E},\mathcal{F})$ *be a regular Dirichlet form in* L^2. *If conditions* (VD), (RVD) *and* (Nash_B) *hold, then the Dirichlet heat kernel* $p_t^B(x,y)$ *exists and satisfies*

$$\operatorname*{esup}_{x,y\in B} p_t^B(x,y) \le \frac{C}{\mu(B)}\left(\frac{w(B)}{t}\right)^{1/\nu} \quad \textit{for all } t>0 \tag{2.23}$$

for any ball B *of radius* $r<\frac{\sigma\overline{R}}{A}$, *where* C, A *are two universal constants independent of* B, t *and constant* σ *comes from condition* (Nash_B). *Consequently, we have*

$$(\text{VD}) + (\text{RVD}) + (\text{Nash}_B) \Rightarrow (\text{FK}).$$

Proof. Assume that $A>1$ is a number to be chosen, see (2.27) below. Let $B:=B(x_0,r)$ with

$$0<r<\frac{\sigma\overline{R}}{A}. \tag{2.24}$$

Since Ar is less than $\sigma\overline{R}$, we can apply condition (Nash_B) on a ball $B(x_0,Ar)$ and obtain for any $u\in\mathcal{F}(B(x_0,Ar))$

$$\|u\|_2^{2+2\nu} \le \frac{C\|u\|_1^{2\nu}}{V(x_0,Ar)^\nu}\left(\|u\|_2^2 + w(x_0,Ar)\mathcal{E}(u)\right). \tag{2.25}$$

Note that for all $u\in\mathcal{F}(B(x_0,r))$,

$$\|u\|_1 = \int_{B(x_0,r)} |u|d\mu \le V(x_0,r)^{1/2}\|u\|_2. \tag{2.26}$$

Since $\mathcal{F}(B(x_0,r)) \subset \mathcal{F}(B(x_0,Ar))$ for any $A>1$, it follows (2.26), (2.25) that for any $u\in\mathcal{F}(B(x_0,r))$

$$\begin{aligned} \|u\|_2^{2+2\nu} &\le \frac{C\left(V(x_0,r)^{1/2}\|u\|_2\right)^{2\nu}}{V(x_0,Ar)^\nu}\|u\|_2^2 + \frac{C\|u\|_1^{2\nu}}{V(x_0,Ar)^\nu}w(x_0,Ar)\mathcal{E}(u) \\ &= C\left(\frac{V(x_0,r)}{V(x_0,Ar)}\right)^\nu \|u\|_2^{2(1+\nu)} + \frac{Cw(x_0,Ar)}{V(x_0,Ar)^\nu}\|u\|_1^{2\nu}\mathcal{E}(u). \end{aligned}$$

By condition (RVD), we have

$$\frac{V(x_0,r)}{V(x_0,Ar)} \le \frac{1}{C_d A^{d_1}} = \left(\frac{1}{2C}\right)^{1/\nu},$$

provided that

$$A = C_d^{-1/d_1}(2C)^{\frac{1}{\nu d_1}} > 1. \tag{2.27}$$

Therefore, for all $u\in\mathcal{F}(B(x_0,r))$,

$$\|u\|_2^{2+2\nu} \le 2C\frac{w(x_0,Ar)}{V(x_0,r)^\nu}\|u\|_1^{2\nu}\mathcal{E}(u), \tag{2.28}$$

which gives that

$$\mathcal{E}(u) \ge \frac{1}{2C}\frac{V(x_0,r)^\nu}{w(x_0,Ar)}\|u\|_2^{2+2\nu}\|u\|_1^{-2\nu} \ge \frac{C'\mu(B)^\nu}{w(B)}\|u\|_2^{2+2\nu}\|u\|_1^{-2\nu}.$$

Applying [23, Lemma 5.5] with $U=B(x_0,r)$, $a=C'\frac{\mu(B)^\nu}{w(B)}$, we conclude that the Dirichlet heat kernel $p_t^B(x,y)$ exists and satisfies (2.23).

We will show that condition (FK) follows from (2.28).

Indeed, let $D\subset B$ be an open subset, and let $u\in\mathcal{F}(D)$. Noting that

$$\|u\|_1^2 \le \mu(D)\|u\|_2^2$$

by the Cauchy-Schwarz inequality, we see from (2.28) and (1.5) that

$$\|u\|_2^{2+2\nu} \le 2Cw(x_0, Ar)\left(\frac{\mu(D)}{\mu(B)}\right)^{\nu} \|u\|_2^{2\nu}\mathcal{E}(u) \le C''w(B)\left(\frac{\mu(D)}{\mu(B)}\right)^{\nu} \|u\|_2^{2\nu}\mathcal{E}(u),$$

thus showing that

$$\lambda_1(D) = \inf_{u\in\mathcal{F}(D)\setminus\{0\}} \frac{\mathcal{E}(u)}{\|u\|_2^2} \ge \frac{c'}{w(B)}\left(\frac{\mu(B)}{\mu(D)}\right)^{\nu}.$$

Therefore, the Faber-Krahn inequality holds for any ball B of radius r satisfying (2.24). □

We remark that if the metric space (M, d) is connected and unbounded, then condition (VD) implies condition (RVD), see for example [23, Corollary 5.3]. In this case, we have that conditions (VD), (PI) will imply condition (FK), since condition (RVD) is automatically true.

3. A refinement of lemma of growth

In this section we shall derive the lemma of growth for any two concentric balls $B, \delta B$ with $0 < \delta < 1$, which is a refinement of the version stated in a forthcoming paper [21], see also [20, Lemma 4.1]. The lemma of growth will follow from conditions (VD), (Gcap), (FK), (TJ). The basic tool in the proof is to use the celebrated De-Giorgi iteration technique for occupation measures (instead of for L^2-norms). Although the idea is essentially the same as in [21], [20, Proof of Lemma 4.1], we sketch the proof for the reader's convenience.

Before we address the lemma of growth, we give the following preliminary. For each $n \ge 1$, let F_n be a function on $[0, \infty)$ given by

$$F_n(r) = \frac{1}{2}\left(r + \sqrt{r^2 + \frac{1}{n^2}}\right) - \frac{1}{2n} \quad \text{for } r \in (-\infty, \infty). \tag{3.1}$$

Clearly, $F_n(0) = 0$, and for any $r \in (-\infty, \infty)$,

$$\begin{aligned} 0 \le F_n'(r) &= \frac{1}{2}\left(1 + \frac{r}{\sqrt{r^2 + n^{-2}}}\right) \le 1, \\ 0 \le F_n''(r) &= \frac{1}{2n^2(r^2 + n^{-2})^{3/2}} \le \frac{n}{2}, \\ F_n(r) &\rightrightarrows r_+ \ \text{ uniformly in } (-\infty, \infty) \text{ as } n \to \infty. \end{aligned} \tag{3.2}$$

Proposition 3.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ without killing part and let F_n be given by (3.1). Then for any $u \in \mathcal{F}' \cap L^\infty$ and any $0 \le \varphi \in \mathcal{F} \cap L^\infty$,*

$$\mathcal{E}(u_+, \varphi) \le \limsup_{k\to\infty} \mathcal{E}(u, F_{n_k}'(u)\varphi) \tag{3.3}$$

for a subsequence $\{n_k\}_{k\ge1}$ of $\{n\}_{n\ge1}$.

Proof. Note that the functions $F_n(u)$, $F_n'(u)\varphi$ belong to $\mathcal{F} \cap L^\infty$ for each $n \ge 1$ by using Proposition 8.4 in Appendix. Since $\varphi \ge 0$ in M, we have

$$\mathcal{E}(F_n(u), \varphi) \le \mathcal{E}(u, F_n'(u)\varphi) \ \ (n \ge 1) \tag{3.4}$$

by using (8.2) in Appendix.

Write $u = v + a$ for some $v \in \mathcal{F}$ and $a \in \mathbb{R}$. Since $F_n(v + a) - F_n(a)$ is a normal contraction of $v \in \mathcal{F}$, we have

$$f_n := F_n(u) - F_n(a) = F_n(v + a) - F_n(a) \in \mathcal{F} \ \text{ and } \ \mathcal{E}(f_n, f_n) \le \mathcal{E}(v, v).$$

Since $(v + a)_+ - a_+$ is also a normal contraction of $v \in \mathcal{F}$, we also have

$$f := u_+ - a_+ = (v + a)_+ - a_+ \in \mathcal{F}.$$

On the other hand, by the dominated convergence theorem,

$$f_n \xrightarrow{L^2} f \quad \text{as } n \to \infty.$$

Since $f_n \in \mathcal{F}$ and

$$\sup_n \mathcal{E}(f_n, f_n) \leq \mathcal{E}(v, v) < \infty,$$

there exists a subsequence $\{f_{n_k}\}_{k\geq 1}$ converging to f weakly in terms of the energy norm $\mathcal{E}$ by using Lemma 8.5 in Appendix. Therefore,

$$\begin{aligned}\mathcal{E}(u_+, \varphi) &= \mathcal{E}(f + a_+, \varphi) = \mathcal{E}(f, \varphi) = \lim_{k\to\infty} \mathcal{E}(f_{n_k}, \varphi) \\ &= \lim_{k\to\infty} \mathcal{E}(F_{n_k}(u) - F_{n_k}(a), \varphi) = \limsup_{k\to\infty} \mathcal{E}(F_{n_k}(u), \varphi) \leq \limsup_{k\to\infty} \mathcal{E}(u, F'_{n_k}(u)\varphi)\end{aligned}$$

by virtue of (3.4), thus showing (3.3). The proof is complete. □

We recall condition (LG), termed the *lemma of growth*, which was introduced in [20, Lemma 4.1] for the case when $w(x, r) = r^\beta$ and $f \equiv 0$. Note that the following notion of lemma of growth involves a given function f.

Definition 3.2. *For any two fixed numbers ε, δ in $(0, 1)$, we say that* condition $\mathrm{LG}(\varepsilon, \delta)$ *holds if there exist four constants $\sigma \in (0, 1), \varepsilon_0 \in (0, \frac{1}{2})$ and $\theta, C_L > 0$ such that, for any ball $B := B(x_0, R)$ with radius $R \in (0, \sigma\overline{R})$, any function $f \in L^\infty(B)$, and for any $u \in \mathcal{F}' \cap L^\infty$ which is f-superharmonic and non-negative in B, if for some $a > 0$*

$$\frac{\mu(B \cap \{u < a\})}{\mu(B)} \leq \varepsilon_0(1 - \varepsilon)^{2\theta}(1 - \delta)^{C_L\theta} \left(1 + \frac{w(B)\left(T_{\frac{3+\delta}{4}B,B}(u_-) + \|f\|_{L^\infty(B)}\right)}{\varepsilon a}\right)^{-\theta}, \tag{3.5}$$

then

$$\operatorname*{einf}_{\delta B} u \geq \varepsilon a, \tag{3.6}$$

where the tail $T_{\frac{3+\delta}{4}B,B}(u_-)$ is defined by (1.15), that is

$$T_{\frac{3+\delta}{4}B,B}(u_-) = \operatorname*{esup}_{x\in\frac{3+\delta}{4}B} \int_{M\setminus B} u_-(y)J(x, dy).$$

For simplicity, we write condition $\mathrm{LG}(\varepsilon, \delta)$ *by condition* (LG) *without mentioning ε, δ.*

We remark that the constants $\sigma, \varepsilon_0, \theta, C_L$ are all independent of ε, δ. Recall that condition (EP), termed the *energy product* of a function u with some cutoff function ϕ, was introduced in [21].

Definition 3.3 (Condition (EP))**.** *We say that the* condition (EP) *is satisfied if there exist two universal constants $C > 0, C_0 \geq 0$ such that, for any three concentric balls $B_0 := B(x_0, R)$, $B := B(x_0, R + r)$ and $\Omega := B(x_0, R')$ with $0 < R < R + r < R' < \overline{R}$, and for any $u \in \mathcal{F}' \cap L^\infty$, there exists some $\phi \in$* cutoff(B_0, B) *such that*

$$\mathcal{E}(u\phi) \leq \frac{3}{2}\mathcal{E}(u, u\phi^2) + \frac{C}{w(x_0, r)}\left(\frac{R'}{r}\right)^{C_0} \int_\Omega u^2 d\mu + 3\int_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)J(dx, dy). \tag{3.7}$$

Condition (EP) plays an important role in deriving condition (LG). The following has been proved in [21].

Lemma 3.4 ([21])**.** *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in L^2 without killing part. Then*

$$(\mathrm{Gcap}) + (\mathrm{TJ}) \Rightarrow (\mathrm{EP}). \tag{3.8}$$

We shall prove the lemma of growth, where condition (EP) is our starting point, instead of from condition (Gcap). The idea is essentially adopted from [20, 21].

Lemma 3.5. *Let* $(\mathcal{E},\mathcal{F})$ *be a regular Dirichlet form in* L^2 *without killing part. If conditions* (VD), (FK), (TJ) *and* (EP) *are satisfied, then condition* (LG) *holds with* $\theta = 1/\nu$ *and* $C_L = C_0 + \beta_2 + d_2$, *where the constants* σ, ν *are taken same as in condition* (FK) *and* C_0 *same as in condition* (EP). *Namely, we have*

$$\text{(VD)} + \text{(FK)} + \text{(TJ)} + \text{(EP)} \Rightarrow \text{(LG)}. \tag{3.9}$$

Consequently,

$$\text{(VD)} + \text{(Gcap)} + \text{(FK)} + \text{(TJ)} \Rightarrow \text{(LG)}. \tag{3.10}$$

Proof. Note that any function $u \in \mathcal{F}$ admits a *quasi-continuous* version $\widetilde{u}$ [19, Theorem 2.1.3, p.71]. We will use the same letter u to denote some quasi-continuous modification of u. For any $u \in \mathcal{F}$ and any open subset Ω of M, a function u belongs to the space $\mathcal{F}(\Omega)$ if and only if $\widetilde{u} = 0$ q.e. in Ω^c, where q.e. means *quasi-everywhere* (see [19, Corollary 2.3.1, p.98]).

We shall show the implication (3.9).

Fix a ball $B := B(x_0, R)$ with radius $0 < R < \sigma\overline{R}$ and a function $f \in L^\infty(B)$. Let $u \in \mathcal{F}' \cap L^\infty$ be a function that is f-superharmonic and non-negative in B. We will show that (3.6) is true if condition (3.5) is satisfied for some $a > 0$.

To do this, denote

$$B_r := B(x_0, r) \text{ for any } r > 0,$$

so that $B_R = B = B(x_0, R)$. Fix four numbers a, b and r_1, r_2 such that

$$0 < a < b < \infty \ \text{ and } \ \frac{r_2}{2} \le r_1 < r_2 < R, \tag{3.11}$$

and set

$$m_1 = \frac{\mu(B_{r_1} \cap \{u < a\})}{\mu(B_{r_1})} \ \text{ and } \ m_2 = \frac{\mu(B_{r_2} \cap \{u < b\})}{\mu(B_{r_2})}.$$

Set also $v := (b - u)_+$ and

$$\widetilde{m}_1 := \mu(B_{r_1} \cap \{u < a\}), \ \ \widetilde{m}_2 := \mu(B_{r_2} \cap \{u < b\}).$$

Let $\widetilde{B}$ be any intermediate concentric ball between B_{r_1} and B_{r_2}, so that

$$B_{r_1} \subset \widetilde{B} := B_{r_1+\rho} \subset B_{r_2} \ (0 < \rho < r_2 - r_1).$$

Applying condition (EP) to the triple B_{r_1}, $\widetilde{B}$, B_{r_2} and the function v, we see that there exists some function $\phi \in \text{cutoff}(B_{r_1}, \widetilde{B})$ such that

$$\begin{aligned}\mathcal{E}(v\phi) \ &\le \ \frac{3}{2}\mathcal{E}(v, v\phi^2) + \frac{C}{w(x_0,\rho)}\left(\frac{r_2}{\rho}\right)^{C_0} \int_{B_{r_2}} v^2 d\mu \\ &\quad + 3\int_{B_{r_2}\times B_{r_2}^c} v(x)v(y)\phi^2(x) J(dx,dy).\end{aligned} \tag{3.12}$$

Without loss of generality, we can assume that ϕ is quasi-continuous. Then we have

$$\widetilde{m}_1 = \int_{B_{r_1}\cap\{u<a\}} \phi^2 d\mu \le \int_{B_{r_1}} \phi^2 \underbrace{\left(\frac{(b-u)_+}{b-a}\right)^2}_{\ge 1 \text{ on } \{u<a\}} d\mu = \frac{1}{(b-a)^2} \int_{B_{r_1}} (\phi v)^2 d\mu. \tag{3.13}$$

Consider the set

$$E := \widetilde{B} \cap \{u < b\}.$$

By the outer regularity of μ, for any $\epsilon > 0$, there is an open set Ω such that $E \subset \Omega \subset B_{r_2}$ and

$$\mu(\Omega) \le \mu(E) + \epsilon \le \widetilde{m}_2 + \epsilon. \tag{3.14}$$

On the other hand, since $\phi = 0$ q.e. outside $\widetilde{B}$ and $v = 0$ outside $\{u < b\}$, we see that $\phi v = 0$ q.e. in E^c. Since $\phi v \in \mathcal{F}$ and $\phi v = 0$ q.e. in $\Omega^c \subset E^c$, we conclude that

$$\phi v \in \mathcal{F}(\Omega). \tag{3.15}$$

By the definition of $\lambda_1(\Omega)$, we have

$$\int_\Omega (\phi v)^2 d\mu \le \frac{\mathcal{E}(\phi v)}{\lambda_1(\Omega)}.$$

Using again the fact that ϕv vanishes outside Ω and combining this inequality with (3.13), we obtain that

$$\widetilde{m}_1 \le \frac{1}{(b-a)^2}\int_{B_{r_1}} (\phi v)^2 d\mu \le \frac{1}{(b-a)^2}\int_\Omega (\phi v)^2 d\mu \le \frac{\mathcal{E}(\phi v)}{(b-a)^2\lambda_1(\Omega)}. \tag{3.16}$$

By condition (FK) and (3.14),

$$\lambda_1(\Omega) \ge \frac{C_F^{-1}}{w(B_{r_2})}\left(\frac{\mu(B_{r_2})}{\mu(\Omega)}\right)^\nu \ge \frac{C_F^{-1}}{w(B_{r_2})}\left(\frac{\mu(B_{r_2})}{\widetilde{m}_2+\epsilon}\right)^\nu, \tag{3.17}$$

from which, it follows by (3.16) that

$$\widetilde{m}_1 \le \frac{\mathcal{E}(\phi v)}{(b-a)^2}\cdot\frac{w(B_{r_2})}{C_F^{-1}}\left(\frac{\mu(B_{r_2})}{\widetilde{m}_2+\epsilon}\right)^{-\nu}.$$

Letting $\epsilon \to 0$, we obtain that, using the fact that $m_2 = \frac{\widetilde{m}_2}{\mu(B_{r_2})}$,

$$\widetilde{m}_1 \le \frac{C_F}{(b-a)^2}\left(\frac{\widetilde{m}_2}{\mu(B_{r_2})}\right)^\nu\cdot w(B_{r_2})\mathcal{E}(\phi v) = \frac{C_F\,(m_2)^\nu}{(b-a)^2}\cdot w(B_{r_2})\mathcal{E}(\phi v), \tag{3.18}$$

where the constants ν and C_F are the same as in condition (FK).

We estimate the term $\mathcal{E}(\phi v)$ on the right-hand side of (3.18) by applying the inequality (3.12). For this, we need to estimate the term $\mathcal{E}(v, v\phi^2)$. This can be done by using the f-superharmonicity of u and using condition (TJ).

Indeed, since $v\phi \in \mathcal{F}(\Omega)\cap L^\infty$ and $\phi \in \mathcal{F}\cap L^\infty$, the function $v\phi^2 = v\phi\cdot\phi \in \mathcal{F}(\Omega) \subset \mathcal{F}(B)$, which is non-negative. Let F_n be given by (3.2) for $n \ge 1$. Since u is f-superharmonic in B and $\|F_n'\|_\infty \le 1$ and since the function $F_n'(b-u)v\phi^2$ is non-negative and belongs to the space $\mathcal{F}(\Omega)$ so that it can be used as a test function, we have

$$\begin{aligned}
\mathcal{E}(b-u, F_n'(b-u)v\phi^2) &= -\mathcal{E}(u, F_n'(b-u)v\phi^2) \le -(f, F_n'(b-u)v\phi^2)\\
&\le \int_M |f|v\phi^2 d\mu \le \|f\|_{L^\infty(B_{r_2})}\int_{B_{r_2}} v d\mu\\
&\le \|f\|_{L^\infty(B_{r_2})} b\mu(B_{r_2}\cap\{u<b\}) \quad (\text{using } v \le b1_{\{u<b\}})\\
&= b\|f\|_{L^\infty(B_{r_2})}\widetilde{m}_2.
\end{aligned} \tag{3.19}$$

Applying (3.3) with u replaced by $b-u$ and with $\varphi = v\phi^2$, we obtain by (3.19)

$$\mathcal{E}(v, v\phi^2) = \mathcal{E}((b-u)_+, v\phi^2) \le \limsup_{k\to\infty}\mathcal{E}(b-u, F_{n_k}'(b-u)v\phi^2) \le b\|f\|_{L^\infty(B_{r_2})}\widetilde{m}_2. \tag{3.20}$$

Therefore, plugging (3.20) into (3.12) and then using the facts that

$$\operatorname{supp}(\phi) \subset \widetilde{B} \ \text{ and } \ J(dx, dy) = J(x, dy)\mu(dx),$$

we see that

$$\begin{aligned}
\mathcal{E}(v\phi) \le{}& \frac{3}{2}b\|f\|_{L^\infty(B_{r_2})}\widetilde{m}_2 + \frac{C}{w(x_0,\rho)}\left(\frac{r_2}{\rho}\right)^{C_0}\int_{B_{r_2}} v^2 d\mu\\
&+ 3\int_{\widetilde{B}} v(x)\,\mu(dx)\cdot\operatorname*{esup}_{x\in\widetilde{B}}\int_{B_{r_2}^c} v(y)J(x, dy).
\end{aligned} \tag{3.21}$$

Since $v=(b-u)_+\le b$ in $B_{r_2}\subset B_R=B$, we have

$$\int_{B_{r_2}} v^2 d\mu \le b^2\mu(B_{r_2}\cap\{u<b\})=b^2\widetilde{m}_2.$$

Since $v=(b-u)_+\le b+u_-$ in M, we also have

$$\begin{aligned}\int_{\widetilde{B}} v(x)\mu(dx)\operatorname*{esup}_{x\in\widetilde{B}}\int_{B_{r_2}^c} v(y)J(x,dy) &\le b\widetilde{m}_2\operatorname*{esup}_{x\in\widetilde{B}}\int_{B_{r_2}^c} v(y)J(x,dy)\\ &\le b\widetilde{m}_2\operatorname*{esup}_{x\in\widetilde{B}}\int_{B_{r_2}^c}(b+u_-(y))J(x,dy)\\ &= b\widetilde{m}_2\left(b\operatorname*{esup}_{x\in\widetilde{B}}\int_{B_{r_2}^c}J(x,dy)+T_{\widetilde{B},B_{r_2}}(u_-)\right).\end{aligned}$$

Thus, using the fact that for any point x_0 in M and any $0<\rho<r_2-r_1$,

$$\frac{w(B_{r_2})}{w(x_0,\rho)}=\frac{w(x_0,r_2)}{w(x_0,\rho)}\le C_2\left(\frac{r_2}{\rho}\right)^{\beta_2}$$

by virtue of (1.5), it follows from (3.21) that

$$\begin{aligned}\mathcal{E}(v\phi) &\le \frac{3}{2}b\|f\|_{L^\infty(B_{r_2})}\widetilde{m}_2+\frac{CC_2}{w(B_{r_2})}\left(\frac{r_2}{\rho}\right)^{C_0+\beta_2}\cdot b^2\widetilde{m}_2\\ &\quad +3b\widetilde{m}_2\left(b\operatorname*{esup}_{x\in\widetilde{B}}\int_{B_{r_2}^c}J(x,dy)+T_{\widetilde{B},B_{r_2}}(u_-)\right).\end{aligned}\tag{3.22}$$

We look at the third term on the right-hand side of (3.22).

Observing by (1.5) that for any $x\in\widetilde{B}\subset B_{r_2}$,

$$\frac{w(B_{r_2})}{w(x,r_2-r_1-\rho)}=\frac{w(x_0,r_2)}{w(x,r_2-r_1-\rho)}\le C_2\left(\frac{r_2}{r_2-r_1-\rho}\right)^{\beta_2},\tag{3.23}$$

we have by condition (TJ) that

$$\begin{aligned}\operatorname*{esup}_{x\in\widetilde{B}}\int_{B_{r_2}^c}J(x,dy) &\le \operatorname*{esup}_{x\in\widetilde{B}}\int_{B(x,r_2-r_1-\rho)^c}J(x,dy)\le\operatorname*{esup}_{x\in\widetilde{B}}\frac{C}{w(x,r_2-r_1-\rho)}\\ &\le \frac{CC_2}{w(B_{r_2})}\left(\frac{r_2}{r_2-r_1-\rho}\right)^{\beta_2}.\end{aligned}\tag{3.24}$$

Plugging (3.24) into (3.22), we obtain

$$\begin{aligned}\mathcal{E}(v\phi) &\le \frac{3}{2}b\|f\|_{L^\infty(B_{r_2})}\widetilde{m}_2+\frac{CC_2}{w(B_{r_2})}\left(\frac{r_2}{\rho}\right)^{C_0+\beta_2}\cdot b^2\widetilde{m}_2\\ &\quad +3b\widetilde{m}_2\left(b\frac{CC_2}{w(B_{r_2})}\left(\frac{r_2}{r_2-r_1-\rho}\right)^{\beta_2}+T_{\widetilde{B},B_{r_2}}(u_-)\right)\\ &\le \frac{C'b^2\widetilde{m}_2}{w(B_{r_2})}\left(\frac{r_2}{\rho}\right)^{C_0+\beta_2}\left(1+\frac{w(B_{r_2})\left(T_{\widetilde{B},B_{r_2}}(u_-)+\|f\|_{L^\infty(B_{r_2})}\right)}{b}\right),\end{aligned}$$

provided that $0<\rho\le(r_2-r_1)/2$, since in this case

$$\left(\frac{r_2}{r_2-r_1-\rho}\right)^{\beta_2}\le\left(\frac{r_2}{\rho}\right)^{\beta_2}\le\left(\frac{r_2}{\rho}\right)^{C_0+\beta_2}.$$

From this, we obtain by (3.18) that

$$\widetilde{m}_1 \le C' C_F\, m_2^{\nu}\, \widetilde{m}_2 \left(\frac{b}{b-a}\right)^2 \left(\frac{r_2}{\rho}\right)^{C_0+\beta_2} \left(1 + \frac{w(B_{r_2})\left(T_{\widetilde{B},B_{r_2}}(u_-) + \|f\|_{L^\infty(B_{r_2})}\right)}{b}\right).$$

Dividing this inequality by $\mu\,(B_{r_1})$ and then using the facts that

$$m_1 = \frac{\widetilde{m}_1}{\mu(B_{r_1})} \ \text{ and } \ m_2 = \frac{\widetilde{m}_2}{\mu(B_{r_2})}$$

and that, for any $\frac{r_2}{2} \le r_1 < r_2$,

$$\frac{\mu(B_{r_2})}{\mu(B_{r_1})} \le C_\mu \left(\frac{r_2}{r_1}\right)^{d_2} \le C_\mu \left(\frac{r_2}{r_2-r_1}\right)^{d_2} \le C_\mu \left(\frac{r_2}{\rho}\right)^{d_2} \ \text{(by using (1.3))}$$

we conclude that, for all $0 < \rho \le (r_2 - r_1)/2$ with $\frac{r_2}{2} \le r_1 < r_2$,

$$\begin{aligned} m_1 &\le C' C_F m_2^{1+\nu} \left(\frac{b}{b-a}\right)^2 \frac{\mu(B_{r_2})}{\mu(B_{r_1})} \left(\frac{r_2}{\rho}\right)^{C_0+\beta_2} \left(1 + \frac{w(B_{r_2})\left(T_{\widetilde{B},B_{r_2}}(u_-) + \|f\|_{L^\infty(B_{r_2})}\right)}{b}\right) \\ &\le C \left(\frac{b}{b-a}\right)^2 \left(\frac{r_2}{\rho}\right)^{C_0+\beta_2+d_2} \left(1 + \frac{w(B_{r_2})\left(T_{B_{r_1+\rho},B_{r_2}}(u_-) + \|f\|_{L^\infty(B_{r_2})}\right)}{b}\right) \cdot m_2^{1+\nu}, \end{aligned} \tag{3.25}$$

where $C := C' C_F C_\mu > 0$ depends only on the constants from the hypotheses (but is independent of the numbers ρ, a, b, r_1, r_2 and the functions f, u). We will apply (3.25) to show (3.5).

In fact, let δ, ε be any two fixed numbers in $(0, 1)$. Consider the following sequences

$$R_k := \left(\delta + 2^{-k}(1-\delta)\right) R \quad \text{ and } \quad a_k := \left(\varepsilon + 2^{-k}(1-\varepsilon)\right) a \ \text{ for } k \ge 0.$$

Clearly, $R_0 = R$, $a_0 = a$, $R_k \searrow \delta R$, and $a_k \searrow \varepsilon a$ as $k \to \infty$, and

$$\frac{R_{k-1}}{2} < R_k < R_{k-1} \ \text{ for any } k \ge 1.$$

Set

$$m_k := \frac{\mu(B_{R_k} \cap \{u < a_k\})}{\mu(B_{R_k})}.$$

Applying (3.25) with

$$\begin{aligned} &a = a_k,\ b = a_{k-1},\ r_1 = R_k,\ r_2 = R_{k-1} \ \text{ and} \\ &\rho = \rho_k := (R_{k-1} - R_k)/2 = 2^{-k-1}(1-\delta)R, \end{aligned}$$

we obtain for all $k \ge 1$

$$m_k \le C A_k \left(\frac{a_{k-1}}{a_{k-1} - a_k}\right)^2 \left(\frac{R_{k-1}}{R_{k-1} - R_k}\right)^{C_0+\beta_2+d_2} m_{k-1}^{1+\nu}, \tag{3.26}$$

where A_k is given by

$$A_k := 1 + \frac{w(B_{R_{k-1}})\left(T_{B_{R_k+\rho_k},B_{R_{k-1}}}(u_-) + \|f\|_{L^\infty(B_{R_{k-1}})}\right)}{a_{k-1}}.$$

Since $B_{R_k+\rho_k} \subset B_{(3+\delta)R/4}$ for any $k \ge 1$ by noting that

$$R_k + \rho_k = \left(\delta + 2^{-k}(1-\delta)\right) R + 2^{-k-1}(1-\delta)R \le \frac{(3+\delta)R}{4}$$

and since $u_- = 0$ in $B = B_R \supseteq B_{R_{k-1}}$ by using the fact that u is non-negative, we have

$$T_{B_{R_k+\rho_k},B_{R_{k-1}}}(u_-) = T_{B_{R_k+\rho_k},B_R}(u_-) \le T_{B_{(3+\delta)R/4},B_R}(u_-).$$

Since $a_{k-1} \geq \varepsilon a$ and $w(B_{R_{k-1}}) \leq w(B)$, it follows that

$$A_k \leq 1 + \frac{w(B)\left(T_{B_{(3+\delta)R/4},B}(u_-) + \|f\|_{L^\infty(B)}\right)}{\varepsilon a} =: A \text{ for any } k \geq 1. \tag{3.27}$$

Since for any $k \geq 1$

$$\frac{a_{k-1}}{a_{k-1} - a_k} = \frac{\varepsilon + 2^{-(k-1)}(1-\varepsilon)}{(2^{-(k-1)} - 2^{-k})(1-\varepsilon)} \leq \frac{2^k}{1-\varepsilon} \text{ and } \frac{R_{k-1}}{R_{k-1} - R_k} \leq \frac{2^k}{1-\delta},$$

we obtain from (3.26), (3.27) that

$$m_k \leq CA\left(\frac{2^k}{1-\varepsilon}\right)^2 \left(\frac{2^k}{1-\delta}\right)^{C_0+\beta_2+d_2} m_{k-1}^{1+\nu} =: DA \cdot 2^{\lambda k} \cdot m_{k-1}^q, \tag{3.28}$$

where the constants D, λ, q are respectively given by

$$D := C(1-\varepsilon)^{-2}(1-\delta)^{-(C_0+\beta_2+d_2)}, \quad \lambda := 2 + C_0 + \beta_2 + d_2 \quad \text{and} \quad q := 1 + \nu. \tag{3.29}$$

Iterating the inequality (3.28), we have for all $k \geq 1$

$$\begin{aligned} m_k &\leq (DA) \cdot 2^{\lambda k} \cdot m_{k-1}^q \leq (DA) \cdot 2^{\lambda k} \cdot \left(DA \cdot 2^{\lambda(k-1)} \cdot m_{k-2}^q\right)^q \\ &= (DA)^{1+q} \cdot 2^{\lambda k + \lambda q(k-1)} \cdot m_{k-2}^{q^2} \leq \cdots \\ &\leq (DA)^{1+q+\cdots+q^{k-1}} \cdot 2^{\lambda(k+q(k-1)+\cdots+q^{k-1})} \cdot m_0^{q^k}. \end{aligned}$$

Since

$$\begin{aligned} k + q(k-1) + \cdots + q^{k-1} &= \frac{q^{k+1} - (k+1)q + k}{(q-1)^2} \leq \frac{q}{(q-1)^2} q^k, \\ 1 + q + \cdots + q^{k-1} &= \frac{q^k - 1}{q-1} \leq \frac{q^k}{q-1}, \end{aligned}$$

it follows that

$$m_k \leq \left((DA)^{\frac{1}{q-1}} \cdot 2^{\frac{\lambda q}{(q-1)^2}} \cdot m_0\right)^{q^k}, \tag{3.30}$$

from which, we conclude that if

$$2^{\frac{\lambda q}{(q-1)^2}} \cdot (DA)^{\frac{1}{q-1}} \cdot m_0 \leq \frac{1}{2}, \tag{3.31}$$

then

$$\lim_{k\to\infty} m_k = 0 \tag{3.32}$$

by using the fact that $q > 1$. Note that (3.31) is equivalent to

$$m_0 \leq 2^{-\frac{\lambda q}{(q-1)^2} - 1} \cdot (DA)^{-\frac{1}{q-1}},$$

that is,

$$\begin{aligned} \frac{\mu(B \cap \{u < a\})}{\mu(B)} &= m_0 \leq 2^{-\frac{\lambda q}{(q-1)^2} - 1} D^{-\frac{1}{q-1}} A^{-1/\nu} \\ &=: \varepsilon_0 (1-\varepsilon)^{2\theta} (1-\delta)^{C_L \theta} \left(1 + \frac{w(B)\left(T_{\frac{3+\delta}{4}B,B}(u_-) + \|f\|_{L^\infty(B)}\right)}{\varepsilon a}\right)^{-\theta}, \end{aligned} \tag{3.33}$$

where $\varepsilon_0, \theta, C_L$ are universal constants given by

$$\varepsilon_0 := 2^{-\lambda q/(q-1)^2 - 1} C^{-1/(q-1)} < 1/2, \ \theta := 1/\nu, \text{ and } C_L := C_0 + \beta_2 + d_2, \tag{3.34}$$

since by (3.29)

$$\begin{aligned} 2^{-\frac{\lambda q}{(q-1)^2}-1} D^{-\frac{1}{q-1}} &= 2^{-\frac{\lambda q}{(q-1)^2}-1}\left(C(1-\varepsilon)^{-2}(1-\delta)^{-(C_0+\beta_2+d_2)}\right)^{-\frac{1}{q-1}} \\ &= \varepsilon_0\left((1-\varepsilon)^2(1-\delta)^{C_0+\beta_2+d_2}\right)^{1/\nu}. \end{aligned}$$

Note that the constants $\varepsilon_0, \theta, C_L$ are all universal, all of which are independent of the numbers ε, δ, the ball B and the functions f, u.

The inequality (3.33) is just the hypothesis (3.5). With a choice of $\varepsilon_0, \theta, C_L$ in (3.34), the assumption (3.31) is satisfied, and hence, we have (3.32). Therefore,

$$\frac{\mu(\delta B \cap \{u < \varepsilon a\})}{\mu(\delta B)} = 0,$$

thus showing that (3.6) is true.

Finally, the implication (3.10) follows directly from (3.8) and (3.9). The proof is complete. □

4. Proof of weak elliptic Harnack inequality

In this section, we prove Theorem 1.8.

Proposition 4.1. *If $v \in \mathcal{F}'$ and $v \geq 0$ in B_R with $0 < R < \overline{R}$, then*

$$T_{\frac{3}{4}B,B}(v_-) \leq T_{\frac{3}{4}B_R,B_R}(v_-) \tag{4.1}$$

for any $B \subset \frac{3}{4}B_R$, where $T_{U,\Omega}(v)$ is defined by (1.15).

Proof. Since $v \geq 0$ in B_R, we see that $v_- = 0$ in B_R, and hence,

$$\begin{aligned} T_{\frac{3}{4}B,B}(v_-) &= \operatorname*{esup}_{x\in\frac{3}{4}B}\int_{B^c} v_-(y)J(x,dy) = \operatorname*{esup}_{x\in\frac{3}{4}B}\int_{B_R^c} v_-(y)J(x,dy) \\ &\leq \operatorname*{esup}_{x\in B}\int_{B_R^c} v_-(y)J(x,dy) \leq \operatorname*{esup}_{x\in\frac{3}{4}B_R}\int_{B_R^c} v_-(y)J(x,dy) \\ &= T_{\frac{3}{4}B_R,B_R}(v_-), \end{aligned}$$

thus showing (4.1). □

We remark that an alternative version of the tail for a function v outside a ball $B(x_0, R)$ is defined in [13] by

$$\mathrm{Tail}_w(v; x_0, R) := \int_{B(x_0,R)^c} \frac{|v(z)|}{V(x_0, d(x_0,z))w(x_0, d(x_0,z))}\mu(dz). \tag{4.2}$$

If *condition* ($J_\leq$) holds, that is, if $J(dx, dy) = J(x,y)\mu(dx)\mu(dy)$ for a non-negative function $J(x,y)$ with

$$J(x,y) \leq \frac{C}{V(x, d(x,y))w(x, d(x,y))} \tag{4.3}$$

for any (x,y) in $M \times M \setminus \mathrm{diag}$, for some constant $C \geq 0$, then for any function v and any ball $B_R := B(x_0, R)$ with $0 < R < \overline{R}$,

$$T_{\frac{3}{4}B_R,B_R}(v) \leq C' \,\mathrm{Tail}_w(v; x_0, R) \tag{4.4}$$

for a constant $C' > 0$ independent of B_R, v.

Indeed, for any two points $x \in \frac{3}{4}B_R$ and $y \in B_R^c$, since $d(x,y) \geq \frac{R}{4}$ and $d(x_0, x) \leq \frac{3}{4}R$, it follows by using (1.3) and the triangle inequality that

$$\frac{V(x_0, d(x_0,y))}{V(x, d(x,y))} \leq \frac{V(x_0, d(x_0,x) + d(x,y))}{V(x, d(x,y))} \leq C_\mu\left(\frac{d(x_0,x) + d(x_0,x) + d(x,y)}{d(x,y)}\right)^{d_2}$$

$$= C_\mu \left(1 + \frac{2d(x_0, x)}{d(x, y)}\right)^{d_2} \le C_\mu \left(1 + \frac{2 \cdot \frac{3}{4}R}{\frac{1}{4}R}\right)^{d_2} = C_\mu 7^{d_2}, \tag{4.5}$$

whilst by using (1.5),

$$\frac{w(x_0, d(x_0, y))}{w(x, d(x, y))} \le \frac{w(x_0, d(x_0, x) + d(x, y))}{w(x, d(x, y))} \le C_2 \left(\frac{d(x_0, x) + d(x, y)}{d(x, y)}\right)^{\beta_2}$$
$$= C_2 \left(1 + \frac{d(x_0, x)}{d(x, y)}\right)^{\beta_2} \le C_2 \left(1 + \frac{\frac{3}{4}R}{\frac{1}{4}R}\right)^{\beta_2} = C_2 4^{\beta_2}. \tag{4.6}$$

Therefore, by (4.3), (4.5), (4.6)

$$T_{\frac{3}{4}B_R, B_R}(v) = \operatorname*{esup}_{x \in \frac{3}{4}B_R} \int_{B_R^c} |v(y)| J(x, dy) \le \operatorname*{esup}_{x \in \frac{3}{4}B_R} \int_{B_R^c} \frac{C|v(y)|}{V(x, d(x, y)) w(x, d(x, y))} \mu(dy)$$
$$\le \int_{B_R^c} \frac{C(C_\mu 7^{d_2})(C_2 4^{\beta_2})|v(y)|}{V(x_0, d(x_0, y)) w(x_0, d(x_0, y))} \mu(dy) = C' \mathrm{Tail}_w(v; x_0, R),$$

thus showing (4.4).

The inequality (4.4) says that the tail of a function v defined in this paper is slightly weaker than that defined in [13], and therefore, so is the weak elliptic Harnack inequality introduced in Definition 1.6.

Proposition 4.2. *If $u \in \mathcal{F}' \cap L^\infty$ and $\lambda > 0$, then $\ln(u_+ + \lambda) \in \mathcal{F}' \cap L^\infty$.*

Proof. For $s \in \mathbb{R}$, we define

$$F(s) = \ln(s_+ + \lambda).$$

Since $u \in L^\infty$, we see that $F(u) \in L^\infty$. For any $s_1, s_2 \in \mathbb{R}$, we assume $(s_1)_+ \ge (s_2)_+$ without loss of generality. Then

$$|F(s_1) - F(s_2)| = \ln\left(1 + \frac{(s_1)_+ - (s_2)_+}{(s_2)_+ + \lambda}\right) \le \frac{(s_1)_+ - (s_2)_+}{(s_2)_+ + \lambda} \le \frac{|s_1 - s_2|}{\lambda}$$

by using the elementary inequality

$$\ln(1 + x) \le x \quad \text{for any } x \ge 0.$$

Thus, F is Lipschitz on $\mathbb{R}$. Therefore, by [21] (see also [20, Proposition A.2 in Appendix] for a purely jump Dirichlet form), we conclude that

$$F(u) \in \mathcal{F}',$$

thus showing that

$$F(u) = \ln(u_+ + \lambda) \in \mathcal{F}' \cap L^\infty.$$

The proof is complete. □

The following will be used shortly.

Proposition 4.3. *(see [20, Lemma 3.7]) Let a function $u \in \mathcal{F}' \cap L^\infty$ be non-negative in an open set $B \subset M$ and $\phi \in \mathcal{F}' \cap L^\infty$ be such that $\phi = 0$ in B^c. Let $\lambda > 0$ and set $u_\lambda := u + \lambda$. Then we have $\phi^2 u_\lambda^{-1} \in \mathcal{F}(B)$ and*

$$\mathcal{E}^{(J)}\left(u, \phi^2 u_\lambda^{-1}\right) \le -\frac{1}{2} \iint_{B \times B} (\phi^2(x) \wedge \phi^2(y)) \left| \ln \frac{u_\lambda(y)}{u_\lambda(x)} \right|^2 J(dx, dy)$$
$$+3\mathcal{E}^{(J)}(\phi, \phi) - 2 \iint_{B \times B^c} u_\lambda(y) \frac{\phi^2(x)}{u_\lambda(x)} J(dx, dy).$$

We show the following *crossover lemma*.

Lemma 4.4 (the crossover lemma)**.** *Assume that conditions* (VD), ($\mathrm{Cap}_{\leq}$) *and* (PI) *are satisfied. Let $u \in \mathcal{F}' \cap L^{\infty}$ be f-superharmonic and non-negative in a ball $B_R := B(x_0, R)$ with radius less than $\overline{R}$. If*

$$\lambda \geq w(B_R)\Big(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\Big), \tag{4.7}$$

then we have

$$\left(\fint_{B_r} u_\lambda^p d\mu\right)^{1/p}\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{1/p} \leq C \ \text{ where } u_\lambda = u + \lambda \tag{4.8}$$

for any $B_r := B(x_0, r)$ with $0 < r \leq \frac{R}{16(4\kappa+1)}$, where $C > 0$, $p \in (0,1)$ are two constants independent of B_R, r, u, f, and the constant $\kappa \geq 1$ comes from condition (PI).

Proof. The proof is motivated by [36, Section 4] and [7, Proposition 5.7] for diffusions. The key is to show that the logarithm function $\ln u_\lambda$ is a BMO function (cf. Definition 8.1 in Appendix). Our result covers both a diffusion and a jump process.

Let $B := B(z, r)$ be an arbitrary ball contained in $\frac{3}{4(4\kappa+1)}B_R$. Without loss of generality, we may assume that

$$r \leq 2 \cdot \frac{3}{4(4\kappa+1)}R = \frac{3}{2(4\kappa+1)}R < R, \tag{4.9}$$

see for example [9, Remark 3.16]. Then

$$2\kappa B \subset \frac{3}{4}B_R = B(x_0, \frac{3}{4}R), \tag{4.10}$$

since by the triangle inequality, for any point $x \in 2\kappa B = B(z, 2\kappa r)$,

$$\begin{aligned} d(x, x_0) &\leq d(x,z) + d(z, x_0) < 2\kappa r + \frac{3}{4(4\kappa+1)}R \\ &\leq 2\kappa \cdot \frac{3}{2(4\kappa+1)}R + \frac{3}{4(4\kappa+1)}R = \frac{3}{4}R. \end{aligned}$$

Let $u \in \mathcal{F}' \cap L^\infty$ be f-superharmonic and non-negative in B_R. Applying Proposition 4.1 with B replaced by $2\kappa B$, we have

$$T_{\frac{3}{2}\kappa B, 2\kappa B}(u_-) \leq T_{\frac{3}{4}B_R, B_R}(u_-). \tag{4.11}$$

Let λ be a number satisfying (4.7). Without loss of generality, we assume that

$$w(B_R)\Big(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\Big) > 0 \ \ (\text{thus } \lambda > 0).$$

Otherwise, we consider $\lambda + \varepsilon$ for some $\varepsilon > 0$ and then let $\varepsilon \to 0$. We shall show that

$$\ln u_\lambda \subset \mathrm{BMO}\left(\frac{3}{4(4\kappa+1)}B_R\right). \tag{4.12}$$

Indeed, note that $\ln(u_+ + \lambda) \in \mathcal{F}' \cap L^\infty$ by using Proposition 4.2. Applying condition (PI) to the function $\ln(u_+ + \lambda)$, we have by (1.11) that

$$\begin{aligned} &\int_B \big(\ln(u_+ + \lambda) - (\ln(u_+ + \lambda))_B\big)^2 d\mu \\ &\leq C w(B)\left\{\int_{\kappa B} d\Gamma^{(L)}\langle \ln(u_+ + \lambda)\rangle + \iint_{(\kappa B)\times(\kappa B)} \big(\ln(u_+(x) + \lambda) - \ln(u_+(y) + \lambda)\big)^2 J(dx, dy)\right\} \\ &= C w(B)\left\{\int_{\kappa B} d\Gamma^{(L)}\langle \ln u_\lambda\rangle + \iint_{(\kappa B)\times(\kappa B)} \left(\ln\frac{u_\lambda(x)}{u_\lambda(y)}\right)^2 J(dx, dy)\right\}, \end{aligned} \tag{4.13}$$

where we have used the fact that $u \geq 0$ (thus $u_+ = u$) in $B_R \supset \kappa B$.

We estimate the right-hand side of (4.13). Indeed, using condition ($\mathrm{Cap}_{\leq}$) to the two concentric balls $(\kappa B, \frac{3}{2}\kappa B)$, we have by (1.3), (1.5)

$$\mathcal{E}(\phi,\phi) \leq C\frac{\mu\left(\frac{3}{2}\kappa B\right)}{w\left(\frac{3}{2}\kappa B\right)} \leq C'\frac{\mu(B)}{w(B)} \tag{4.14}$$

for some $\phi \in \text{cutoff}\left(\kappa B, \frac{3}{2}\kappa B\right)$.

On the other hand, using the Leibniz and chain rules of $d\Gamma^{(L)}\langle\cdot\rangle$, we see that

$$\begin{aligned}\int \phi^2 d\Gamma^{(L)}\langle \ln u_\lambda\rangle &= -\int \phi^2 d\Gamma^{(L)}\langle u_\lambda, u_\lambda^{-1}\rangle \\ &= -\int d\Gamma^{(L)}\langle u_\lambda, \phi^2 u_\lambda^{-1}\rangle + 2\int \phi u_\lambda^{-1} d\Gamma^{(L)}\langle u_\lambda, \phi\rangle \\ &= -\mathcal{E}^{(L)}(u_\lambda, \phi^2 u_\lambda^{-1}) + 2\int \phi u_\lambda^{-1} d\Gamma^{(L)}\langle u_\lambda, \phi\rangle. \end{aligned}\tag{4.15}$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}2\int \phi u_\lambda^{-1} d\Gamma^{(L)}\langle u_\lambda, \phi\rangle = 2\int \phi d\Gamma^{(L)}\langle \ln u_\lambda, \phi\rangle &\leq \frac{1}{2}\int \phi^2 d\Gamma^{(L)}\langle \ln u_\lambda\rangle + 2\int d\Gamma^{(L)}\langle\phi\rangle \\ &= \frac{1}{2}\int \phi^2 d\Gamma^{(L)}\langle \ln u_\lambda\rangle + 2\mathcal{E}^{(L)}(\phi,\phi), \end{aligned}\tag{4.16}$$

from which, it follows by (4.15) that

$$\int \phi^2 d\Gamma^{(L)}\langle \ln u_\lambda\rangle \leq -2\mathcal{E}^{(L)}(u_\lambda, \phi^2 u_\lambda^{-1}) + 4\mathcal{E}^{(L)}(\phi,\phi). \tag{4.17}$$

We estimate the first term on the right-hand side.

Indeed, since $\phi = 0$ in $\left(\frac{3}{2}\kappa B\right)^c \supset (2\kappa B)^c$, using Proposition 4.3 with B being replaced by $2\kappa B$, we obtain that $0 \leq \phi^2 u_\lambda^{-1} \in \mathcal{F}(2\kappa B)$, and

$$\begin{aligned}\mathcal{E}^{(J)}(u, \phi^2 u_\lambda^{-1}) \quad \leq \quad & -\frac{1}{2}\iint_{(2\kappa B)\times(2\kappa B)} (\phi^2(x)\wedge\phi^2(y))\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 J(dx,dy) \\ & +3\mathcal{E}^{(J)}(\phi,\phi) - 2\iint_{(2\kappa B)\times(2\kappa B)^c} u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)} J(dx,dy). \end{aligned}\tag{4.18}$$

Noting that $\mathcal{E}(u_\lambda, \phi^2 u_\lambda^{-1}) \geq (f, \phi^2 u_\lambda^{-1})$ since u is f-superharmonic in $B_R \supset 2\kappa B$, we see by (1.6), (4.18) that

$$\begin{aligned}-\mathcal{E}^{(L)}(u_\lambda, \phi^2 u_\lambda^{-1}) &= -\mathcal{E}(u_\lambda, \phi^2 u_\lambda^{-1}) + \mathcal{E}^{(J)}(u_\lambda, \phi^2 u_\lambda^{-1}) \\ &\leq -(f, \phi^2 u_\lambda^{-1}) + \mathcal{E}^{(J)}(u_\lambda, \phi^2 u_\lambda^{-1}) \\ &\leq -(f, \phi^2 u_\lambda^{-1}) - \frac{1}{2}\iint_{(2\kappa B)\times(2\kappa B)} (\phi^2(x)\wedge\phi^2(y))\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 J(dx,dy) \\ &\quad + 3\mathcal{E}^{(J)}(\phi,\phi) - 2\iint_{(2\kappa B)\times(2\kappa B)^c} u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)} J(dx,dy). \end{aligned}\tag{4.19}$$

Since $\phi = 1$ in κB, we have

$$-\iint_{(2\kappa B)\times(2\kappa B)} (\phi^2(x)\wedge\phi^2(y))\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 J(dx,dy) \leq -\iint_{(\kappa B)\times(\kappa B)} \left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2 J(dx,dy), \tag{4.20}$$

whilst, since $\phi = 0$ in $(\frac{3}{2}\kappa B)^c$ and $0 \leq \phi \leq 1$ in M,

$$-\iint_{(2\kappa B)\times(2\kappa B)^c} u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)} J(dx,dy) = -\iint_{(\frac{3}{2}\kappa B)\times(2\kappa B)^c} u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)} J(dx,dy)$$

$$
\begin{aligned}
&\le \iint_{(\frac{3}{2}\kappa B)\times(2\kappa B)^c} u_-(y)\frac{1}{u_\lambda(x)}J(dx,dy)\\
&\le \frac{1}{\lambda}\int_{(\frac{3}{2}\kappa B)}\left\{\operatorname*{esup}_{x\in(\frac{3}{2}\kappa B)}\int_{(2\kappa B)^c} u_-(y)J(x,dy)\right\}\mu(dx)\\
&= \frac{1}{\lambda}\mu\left(\frac{3}{2}\kappa B\right)T_{\frac{3}{2}\kappa B,2\kappa B}(u_-)\\
&\le \frac{1}{\lambda}C_\mu\left(\frac{3}{2}\kappa\right)^{d_2}\mu(B)T_{\frac{3}{4}B_R,B_R}(u_-),
\end{aligned}
\tag{4.21}
$$

where in the last inequality we have used condition (VD) and inequality (4.11). From this, condition (4.7), and using the fact that $\frac{w(B_R)}{w(B)} \ge C_1$ by (1.5), (4.9), we obtain

$$
\begin{aligned}
-\iint_{(2\kappa B)\times(2\kappa B)^c} u_\lambda(y)\frac{\phi^2(x)}{u_\lambda(x)}J(dx,dy) &\le \frac{1}{\lambda}C_\mu\left(\frac{3}{2}\kappa\right)^{d_2}\mu(B)T_{\frac{3}{4}B_R,B_R}(u_-)\\
&\le \frac{1}{w(B_R)T_{\frac{3}{4}B_R,B_R}(u_-)}C_\mu\left(\frac{3}{2}\kappa\right)^{d_2}\mu(B)T_{\frac{3}{4}B_R,B_R}(u_-)\\
&= C_\mu\left(\frac{3}{2}\kappa\right)^{d_2}\frac{\mu(B)}{w(B_R)} \le C\frac{\mu(B)}{w(B)}.
\end{aligned}
\tag{4.22}
$$

Therefore, plugging (4.20) and (4.22) into (4.19), we obtain

$$
-\mathcal{E}^{(L)}(u_\lambda,\phi^2u_\lambda^{-1}) \le -(f,\phi^2u_\lambda^{-1})-\frac{1}{2}\iint_{(\kappa B)\times(\kappa B)}\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2J(dx,dy)+3\mathcal{E}^{(J)}(\phi,\phi)+2C\frac{\mu(B)}{w(B)}. \tag{4.23}
$$

Plugging (4.23), (4.14) into (4.17), it follows that

$$
\begin{aligned}
\int \phi^2 d\Gamma^{(L)}\langle \ln u_\lambda\rangle &\le -2\mathcal{E}^{(L)}(u_\lambda,\phi^2u_\lambda^{-1}) + 4\mathcal{E}^{(L)}(\phi,\phi)\\
&\le -2(f,\phi^2u_\lambda^{-1}) - \iint_{(\kappa B)\times(\kappa B)}\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2J(dx,dy) + 6\mathcal{E}^{(J)}(\phi,\phi)\\
&\quad +4C\frac{\mu(B)}{w(B)} + 4\mathcal{E}^{(L)}(\phi,\phi)\\
&\le -2(f,\phi^2u_\lambda^{-1}) - \iint_{(\kappa B)\times(\kappa B)}\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2J(dx,dy) + C'\frac{\mu(B)}{w(B)},
\end{aligned}
$$

from which, using the fact that $\phi = 1$ in κB, we have

$$
\int_{\kappa B} d\Gamma^{(L)}\langle \ln u_\lambda\rangle + \iint_{(\kappa B)\times(\kappa B)}\left|\ln\frac{u_\lambda(y)}{u_\lambda(x)}\right|^2J(dx,dy) \le -2(f,\phi^2u_\lambda^{-1}) + C'\frac{\mu(B)}{w(B)}. \tag{4.24}
$$

Since

$$
\begin{aligned}
-2(f,\phi^2u_\lambda^{-1}) &= -2\int_{\frac{3}{2}\kappa B} f\phi^2u_\lambda^{-1}d\mu \le 2\int_{\frac{3}{2}\kappa B}|f|u_\lambda^{-1}d\mu\\
&\le 2\int_{\frac{3}{2}\kappa B}\frac{\|f\|_{L^\infty(B_R)}}{\lambda}d\mu \le \frac{2\mu(\frac{3}{2}\kappa B)}{w(B_R)} \quad \text{(by using (4.7))}\\
&\le C\frac{\mu(B)}{w(B)} \qquad \text{(by condition (VD) and (1.5))},
\end{aligned}
$$

we have by plugging (4.24) into (4.13),

$$
\int_B (\ln u_\lambda - (\ln u_\lambda)_B)^2 d\mu \le Cw(B)\cdot\Big(-2(f,\phi^2u_\lambda^{-1}) + C'\frac{\mu(B)}{w(B)}\Big) \le C\mu(B),
$$

which yields that, using the Cauchy-Schwarz inequality,

$$\left(\int_B |\ln u_\lambda - (\ln u_\lambda)_B|\, d\mu\right)^2 \le \mu(B)\left(\int_B \left(\ln u_\lambda - (\ln u_\lambda)_B\right)^2 d\mu\right) \le C''\mu(B)^2,$$

that is,

$$\fint_B |\ln u_\lambda - (\ln u_\lambda)_B|\, d\mu \le C_3 \tag{4.25}$$

for all balls B in $\frac{3}{4(4\kappa+1)}B_R$ and all λ satisfying (4.7), where C_3 is a universal constant independent of B_R, B, λ and the functions u, f, thus proving (4.12).

Applying Corollary 8.3 in Appendix with function $\ln u_\lambda$ and $B_0 = \frac{3}{4(4\kappa+1)}B_R$, we have

$$\left\{\fint_B \exp\left(\frac{c_2}{2b}\ln u_\lambda\right) d\mu\right\}\left\{\fint_B \exp\left(-\frac{c_2}{2b}\ln u_\lambda\right) d\mu\right\} \le (1+c_1)^2$$

for any ball B satisfies $12B \subseteq \frac{3}{4(4\kappa+1)}B_R$ and any

$$b \ge \|\ln u_\lambda\|_{\mathrm{BMO}\left(\frac{3}{4(4\kappa+1)}B_R\right)}. \tag{4.26}$$

In particular, for any $B_r := B(x_0, r)$ with $0 < r \le \frac{R}{16(4\kappa+1)}$ so that $12B_r \subseteq \frac{3}{4(4\kappa+1)}B_R$ and for any number b satisfying (4.26),

$$\left\{\fint_{B_r} \exp\left(\frac{c_2}{2b}\ln u_\lambda\right) d\mu\right\}\left\{\fint_{B_r} \exp\left(-\frac{c_2}{2b}\ln u_\lambda\right) d\mu\right\} \le (1+c_1)^2. \tag{4.27}$$

Finally, choosing $b = \frac{c_2}{2} + C_3$ so that (4.26) is satisfied and letting $p := \frac{c_2}{2b} \in (0,1)$, we conclude from (4.27) that

$$\left\{\fint_{B_r} \exp\left(p\ln u_\lambda\right) d\mu\right\}\left\{\fint_{B_r} \exp\left(-p\ln u_\lambda\right) d\mu\right\} \le (1+c_1)^2,$$

thus showing (4.8). The proof is complete. □

We are now in a position to prove Theorem 1.8.

Proof of Theorem 1.8. We need to show the implication (1.20). Indeed, by Lemma 3.5, condition (LG) is true. Let $B_R := B(x_0, R)$ be a metric ball in M with $0 < R < \sigma\overline{R}$, where constant σ comes from condition (LG). Let $B_r := B(x_0, r)$ with

$$0 < r \le \delta R \ \text{ where } \delta := \frac{1}{32(4\kappa+1)} \tag{4.28}$$

and κ is the same constant as in condition (PI). Let $u \in \mathcal{F}' \cap L^\infty$ be a function that is non-negative, f-superharmonic in B_R. We need to show that

$$\left(\frac{1}{\mu(B_r)}\int_{B_r} u^p d\mu\right)^{1/p} \le C\left(\operatorname*{einf}_{B_r} u + w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right) \tag{4.29}$$

for some universal numbers $p \in (0,1)$ and $C \ge 1$, both of which are independent of B_R, r, u, f.

To do this, let λ be a number determined by

$$\lambda = w(B_R)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right). \tag{4.30}$$

We claim that for any $r \in (0, \delta R]$

$$\left(\fint_{B_r} u_\lambda^p d\mu\right)^{1/p} \le C \operatorname*{einf}_{B_r} u_\lambda \text{ with } u_\lambda = u + \lambda \tag{4.31}$$

for some constant C independent of B_R, r, u, f.

Indeed, by Lemma 4.4, there exist two positive constants $p \in (0,1)$ and c', independent of B_R, r, u, f, such that

$$\left(\fint_{B_r} u_\lambda^p d\mu\right)^{1/p}\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{1/p} \le c' \tag{4.32}$$

for any $0 < r \le 2\delta R$. Let $s = p/\theta$, where $\theta = \frac{1}{\nu}$ with constant ν coming from condition (FK). Without loss of generality, assume $\theta \ge 1$. Thus $s \in (0,1)$. Let

$$b := w(B_r)\left(T_{\frac{3}{2}B_r, 2B_r}((u_\lambda)_-) + \|f\|_{L^\infty(B_r)}\right).$$

Define the function g by

$$g(a) = a^s(1 + \frac{2b}{a}) \text{ for any } a \in (0, +\infty). \tag{4.33}$$

Using the facts that $(u_\lambda)_- \le u_-$ in M and $2B_r \subset \frac{3}{4}B_R$, we have, by Proposition 4.1 with B being replaced by $2B_r$, that

$$\begin{aligned} b &= w(B_r)\left(T_{\frac{3}{2}B_r, 2B_r}((u_\lambda)_-) + \|f\|_{L^\infty(B_r)}\right) \le w(B_r)\left(T_{\frac{3}{4}(2B_r), 2B_r}(u_-) + \|f\|_{L^\infty(B_r)}\right) \\ &\le w(B_R)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) = \lambda. \end{aligned} \tag{4.34}$$

Clearly, for any $a > \lambda$,

$$\frac{\mu(B_r \cap \{u_\lambda < a\})}{\mu(B_r)} = \frac{\mu(B_r \cap \{u_\lambda^{-p} > a^{-p}\})}{\mu(B_r)} \le a^p \fint_{B_r} u_\lambda^{-p} d\mu. \tag{4.35}$$

Note that $u_\lambda \in \mathcal{F}' \cap L^\infty$ is f-superharmonic, non-negative in $2B_r \subset B_R$. To look at whether the hypotheses in condition (LG) are satisfied or not, we consider two cases.

Case 1. Assume that there exists a number a ($> \lambda$) such that

$$\begin{aligned} \varepsilon_0 2^{-(2+C_L)\theta}\left(1 + \frac{2b}{a}\right)^{-\theta} &= \varepsilon_0 2^{-(2+C_L)\theta}\left(1 + \frac{2w(B_r)\left(T_{\frac{3}{2}B_r, 2B_r}((u_\lambda)_-) + \|f\|_{L^\infty(B_r)}\right)}{a}\right)^{-\theta} \\ &= a^p \fint_{B_r} u_\lambda^{-p} d\mu, \end{aligned} \tag{4.36}$$

that is,

$$(g(a))^{1/s} = a\left(1 + \frac{2b}{a}\right)^{1/s} = \varepsilon_1^{1/p}\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{-1/p}, \tag{4.37}$$

where the constant C_L comes from condition (LG) and $\varepsilon_1 := \varepsilon_0 2^{-(2+C_L)\theta}$. In this case, by using (4.35) and (4.36), we have

$$\begin{aligned} \frac{\mu(B_r \cap \{u_\lambda < a\})}{\mu(B_r)} &\le a^p \fint_{B_r} u_\lambda^{-p} d\mu = \varepsilon_0 2^{-(2+C_L)\theta}\left(1 + \frac{2w(B_r)\left(T_{\frac{3}{2}B_r, 2B_r}((u_\lambda)_-) + \|f\|_{L^\infty(B_r)}\right)}{a}\right)^{-\theta} \\ &\le \varepsilon_0 2^{-(2+C_L)\theta}\left(1 + \frac{2w(B_r)\left(T_{\frac{7}{8}B_r, B_r}((u_\lambda)_-) + \|f\|_{L^\infty(B_r)}\right)}{a}\right)^{-\theta} \\ &= \varepsilon_0(1 - 1/2)^{2\theta}(1 - 1/2)^{C_L\theta}\left(1 + \frac{w(B_r)\left(T_{\frac{3+1/2}{4}B_r, B_r}((u_\lambda)_-) + \|f\|_{L^\infty(B_r)}\right)}{\frac{1}{2}a}\right)^{-\theta}, \end{aligned}$$

since $T_{\frac{7}{8}B_r,B_r}((u_\lambda)_-) \leq T_{\frac{3}{2}B_r,2B_r}((u_\lambda)_-)$ by noting that u_λ is non-negative in $2B_r$. Therefore, we see that the assumption (3.5), with B being replaced by B_r and u replaced by u_λ, is true. Thus, all the hypothesis in condition LG(ε, δ) are satisfied with $\varepsilon = \delta = 1/2$. Therefore, it follows that

$$\begin{aligned}
\operatorname*{einf}_{\frac{1}{2}B_r} u_\lambda &\geq \frac{1}{2}a = \frac{1}{2}\left(1+\frac{2b}{a}\right)^{-1/s} \varepsilon_1^{1/p}\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{-1/p} \quad \text{(using (4.37))} \\
&\geq \frac{1}{2}\left(1+\frac{2b}{a}\right)^{-1/s} \varepsilon_1^{1/p} c'^{-1}\left(\fint_{B_r} u_\lambda^{p} d\mu\right)^{1/p} \quad \text{(using (4.32))} \\
&\geq \frac{1}{2}3^{-1/s}\varepsilon_1^{1/p} c'^{-1}\left(\fint_{B_r} u_\lambda^{p} d\mu\right)^{1/p} \quad \text{(using (4.34) and } a > \lambda\text{)},
\end{aligned}$$

which gives that

$$\left(\fint_{B_r} u_\lambda^{p} d\mu\right)^{1/p} \leq 2c'\varepsilon_1^{-1/p} 3^{1/s} \operatorname*{einf}_{\frac{1}{2}B_r} u_\lambda.$$

Thus, the inequality (4.31) is true in this case.

Case 2. Assume that (4.37) is not satisfied for any $a \in (\lambda, +\infty)$. In this case, noting that

$$\lim_{a\to+\infty} g(a) = +\infty$$

and g is continuous on $(0, +\infty)$, we have that

$$(g(a))^{1/s} > \varepsilon_1^{1/p}\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{-1/p}, \tag{4.38}$$

for any $a \in (\lambda, +\infty)$.

If $\lambda = 0$, then $b = 0$ by (4.34). By definition (4.33), we have $g(a) = a^s$ for any $a > 0$. From this and using (4.38), it follows that

$$\varepsilon_1^{1/p}\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{-1/p} < g(a)^{1/s} = a \text{ for any } a \in (0, +\infty).$$

Letting $a \to 0$, we have $\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{-1/p} = 0$, which gives that

$$\left(\fint_{B_r} u_\lambda^{p} d\mu\right)^{1/p} = 0$$

by using (4.32), thus showing (4.31).

In the sequel, we assume that $\lambda > 0$. Since g is continuous on $(0, +\infty)$, we have from (4.38) by letting $a \searrow \lambda$ that

$$(g(\lambda))^{1/s} \geq \varepsilon_1^{1/p}\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{-1/p},$$

from which, we see by using (4.34)

$$3^{1/s}\lambda \geq \lambda\left(1+\frac{2b}{\lambda}\right)^{1/s} = (g(\lambda))^{1/s} \geq \varepsilon_1^{1/p}\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{-1/p}. \tag{4.39}$$

Thus, we have

$$\begin{aligned}
\left(\fint_{B_r} u_\lambda^{p} d\mu\right)^{1/p} &\leq c'\left(\fint_{B_r} u_\lambda^{-p} d\mu\right)^{-1/p} \quad \text{(using (4.32))} \\
&\leq c'\varepsilon_1^{-1/p} 3^{1/s}\lambda \quad \text{(using (4.39))}.
\end{aligned} \tag{4.40}$$

Therefore, combining Case 1 and Case 2, we always have

$$\left(\fint_{B_r} u_\lambda^p d\mu\right)^{1/p} \le C(\lambda + \operatorname*{einf}_{\frac{1}{2}B_r} u_\lambda) \le 2C \operatorname*{einf}_{\frac{1}{2}B_r} u_\lambda \tag{4.41}$$

for any $0 < r \le 2\delta R$.

On the other hand, by condition (VD),

$$\fint_{B_r} u_\lambda^p d\mu \ge \frac{1}{\mu(B_r)} \int_{\frac{1}{2}B_r} u_\lambda^p d\mu \ge \frac{1}{C_\mu \mu\left(\frac{1}{2}B_r\right)} \int_{\frac{1}{2}B_r} u_\lambda^p d\mu, \tag{4.42}$$

from which, it follows by (4.41) that

$$\left(\fint_{\frac{1}{2}B_r} u_\lambda^p d\mu\right)^{1/p} \le C' \operatorname*{einf}_{\frac{1}{2}B_r} u_\lambda$$

for $0 < r \le 2\delta R$, thus proving our claim (4.31) by renaming $r/2$ by r, as desired.

Therefore, we obtain by (4.31)

$$\begin{aligned} \left(\frac{1}{\mu(B_r)} \int_{B_r} u^p d\mu\right)^{1/p} &\le \left(\frac{1}{\mu(B_r)} \int_{B_r} u_\lambda^p d\mu\right)^{1/p} \le C' \operatorname*{einf}_{B_r} u_\lambda \\ &= C' \left(\operatorname*{einf}_{B_r} u + w(B_R)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right) \end{aligned} \tag{4.43}$$

for $0 < r \le \delta R$.

Finally, we show that the term $w(B_R)$ on the right-hand side of (4.43) can be replaced by a smaller one $w(B_r)$ for any $0 < r \le \delta R$, by adjusting the value of constant C'.

Indeed, fix a number r in $(0, \delta R)$. Let $i \ge 1$ be an integer such that, setting $r_k = \delta^k R$ for any $k \ge 0$,

$$r_{i+1} = \delta^{i+1} R \le r < \delta^i R = r_i. \tag{4.44}$$

By Proposition 4.1, we see that

$$T_{\frac{3}{4}B_{r_{i-1}}, B_{r_{i-1}}}(u_-) \le T_{\frac{3}{4}B_R, B_R}(u_-). \tag{4.45}$$

By (1.5) and (4.44),

$$w(B_{r_{i-1}}) = \frac{w(x_0, \delta^{i-1}R)}{w(x_0, r)} w(x_0, r) \le C_2 \left(\frac{\delta^{i-1}R}{r}\right)^{\beta_2} w(B_r) \le C_2 \delta^{-2\beta_2} w(B_r). \tag{4.46}$$

Since u is f-superharmonic in $B_{r_{i-1}}$, applying (4.43) with R being replaced by r_{i-1} and then using (4.45), (4.46), we conclude that

$$\begin{aligned} \left(\frac{1}{\mu(B_r)} \int_{B_r} u^p d\mu\right)^{1/p} &\le C' \left(\operatorname*{einf}_{B_r} u + w(B_{r_{i-1}})\left(T_{\frac{3}{4}B_{r_{i-1}}, B_{r_{i-1}}}(u_-) + \|f\|_{L^\infty(B_{r_{i-1}})}\right)\right) \\ &= C' \left(\operatorname*{einf}_{B_r} u + C_2 \delta^{-2\beta_2} w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right) \\ &\le C \left(\operatorname*{einf}_{B_r} u + w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right), \end{aligned}$$

thus showing that condition (wEH) holds. The proof is complete. □

5. Other equivalent characterizations

In this section, we prove Theorem 1.9. Denote by

$$\omega_B(A) := \frac{\mu(A \cap B)}{\mu(B)}, \tag{5.1}$$

the *occupation measure* of the set A in B.

The following version of the weak elliptic Harnack inequality was introduced in [13, Proposition 3.6] when $f \equiv 0$, and we label it by condition (wEH1).

Definition 5.1 (condition (wEH1))**.** *We say that condition* (wEH1) *holds if there exist two universal constants $\sigma \in (0,1)$ and $\delta_1 \in (0,1/4)$ such that, for any two concentric balls $B_R := B(x_0,R) \supset B(x_0,r) =: B_r$ with $R \in (0,\sigma\overline{R})$, $r \in (0,\delta_1 R)$, any function $f \in L^\infty(B_R)$, any number $\eta_1 \in (0,1]$ and for any function $u \in \mathcal{F}' \cap L^\infty$ which is non-negative, f-superharmonic in B_R, if for some $a > 0$,*

$$\omega_{B_r}(\{u \geq a\}) = \frac{\mu(B_r \cap \{u \geq a\})}{\mu(B_r)} \geq \eta_1,$$

then

$$\operatorname*{einf}_{B_{4r}} u > \varepsilon_1 a - w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right), \tag{5.2}$$

where $\varepsilon_1 = \varepsilon_1(\eta_1) \in (0,1)$ depends only on η_1 (independent of x_0, r, R, f, u, a).

We show that condition (wEH) defined in Definition 1.6 is equivalent to condition (wEH1).

Proposition 5.2. *Assume that $(\mathcal{E},\mathcal{F})$ is a Dirichlet form in $L^2(M,\mu)$. If condition* (VD) *holds, then*

$$\text{(wEH)} \Leftrightarrow \text{(wEH1)}.$$

Proof. The proof was essentially given in [13, Proof of Theorem 3.1 and Remark 3.9] wherein the jump kernel is assumed to exist and $f \equiv 0$. For the reader's convenience, we sketch the proof. We mention that the jump kernel here may not exist.

We first show (wEH) $\Rightarrow$ (wEH1).

Assume that condition (wEH) holds. Let $u \in \mathcal{F}' \cap L^\infty$ be non-negative, f-superharmonic in a ball $B_R(x_0)$ with $R \in (0,\sigma\overline{R})$. Let η_1 be any number in $(0,1]$ and r any number in $(0,\delta R/4)$, where constant δ is the same as in condition (wEH). Assume that

$$\omega_{B_r}(\{u \geq a\}) \geq \eta_1 \tag{5.3}$$

for some $a > 0$. We will show that condition (wEH1) holds with $\delta_1 = \frac{\delta}{4}$ and

$$\varepsilon_1(\eta_1) = \left(C_2 4^{\beta_2} C_H\right)^{-1}\left(\frac{\eta_1}{C_\mu^2}\right)^{1/p}, \tag{5.4}$$

where constants C_2, β_2 are the same as in (1.5) and C_H, p the same as in condition (wEH), while the number C_μ comes from (1.2). It suffices to show (5.2).

Indeed, we have by (1.16), with r replaced by $4r$, that

$$\left(\fint_{B_{4r}} u^p d\mu\right)^{1/p} \leq C_H\left(\operatorname*{einf}_{B_{4r}} u + w(x_0,4r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right). \tag{5.5}$$

Since $\mu(B_{4r}) \leq C_\mu^2 \mu(B_r)$ by condition (VD), we have by (5.3)

$$\begin{aligned}\left(\fint_{B_{4r}} u^p d\mu\right)^{1/p} &\geq \left(\frac{1}{C_\mu^2\mu(B_r)}\int_{B_r} u^p d\mu\right)^{1/p} \geq \left(\frac{1}{C_\mu^2\mu(B_r)}\int_{B_r\cap\{u\geq a\}} a^p d\mu\right)^{1/p} \\ &= a\left(\frac{\omega_{B_r}(\{u \geq a\})}{C_\mu^2}\right)^{1/p} \geq \left(\frac{\eta_1}{C_\mu^2}\right)^{1/p} a. \end{aligned} \tag{5.6}$$

By the second inequality in (1.5),

$$w(B_r) = \frac{w(x_0, r)}{w(x_0, 4r)} w(x_0, 4r) \geq \frac{1}{C_2 4^{\beta_2}} w(x_0, 4r). \tag{5.7}$$

Therefore, plugging (5.6) and (5.7) into (5.5), we obtain

$$\begin{aligned}
\left(\frac{\eta_1}{C_\mu^2}\right)^{1/p} a \quad &\leq \left(\fint_{B_{4r}} u^p d\mu\right)^{1/p} \leq C_H \left(\operatorname*{einf}_{B_{4r}} u + w(x_0, 4r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right) \\
&\leq C_H \left(\operatorname*{einf}_{B_{4r}} u + C_2 4^{\beta_2} w(x_0, r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right) \\
&\leq C_2 4^{\beta_2} C_H \left(\operatorname*{einf}_{B_{4r}} u + w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right),
\end{aligned}$$

which gives that

$$\begin{aligned}
\operatorname*{einf}_{B_{4r}} u &\geq \left(C_2 4^{\beta_2} C_H\right)^{-1} \left(\frac{\eta_1}{C_\mu^2}\right)^{1/p} a - w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) \\
&= \varepsilon_1 a - w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right),
\end{aligned}$$

thus showing (5.2) with ε_1 given by (5.4). Hence, condition (wEH1) holds.

We show the opposite implication (wEH1) $\Rightarrow$ (wEH).

We will use the Krylov-Safonov covering lemma on the doubling space as follows, see for example [13, Lemma 3.8] or [26, Lemma 7.2]. Suppose that condition (VD) holds. Let r be a number in $(0, \overline{R}/5)$ and $E \subset B_r(x_0)$ a measurable set. For any number $\eta \in (0, 1)$, we define

$$[E]_\eta = \bigcup_{0<\rho<r} \left\{B_{5\rho}(x) \cap B_r(x_0) : x \in B_r(x_0) \text{ and } \frac{\mu(E \cap B_{5\rho}(x))}{\mu(B_\rho(x))} > \eta\right\}.$$

Then either

$$[E]_\eta = B_r(x_0)$$

or

$$\mu([E]_\eta) \geq \frac{1}{\eta}\mu(E).$$

Assume that condition (wEH1) holds. We show (wEH).

To do this, let η be any fixed number in $(0, 1)$. Let $\sigma \in (0, 1)$ and $\delta_1 \in (0, 1/4)$ be the constants coming from condition (wEH1). Let $B_R := B(x_0, R)$ be any metric ball with $0 < R < \sigma\overline{R}$ and r any number in $(0, \frac{\delta_1}{10}R]$. Let $u \in \mathcal{F}' \cap L^\infty$ be any function that is non-negative, f-superharmonic in B_R. We define

$$A_t^i := \left\{x \in B_r(x_0) : \; u(x) > t\varepsilon^i - \frac{T}{1-\varepsilon}\right\}$$

for any $t > 0$ and $i \geq 0$, where constant $\varepsilon \in (0, 1)$ will be determined later and T is given by

$$T = C_2 w(x_0, 5r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) \tag{5.8}$$

with constant C_2 as in (1.5).

Obviously, we have $A_t^{i-1} \subset A_t^i$ for any $i \geq 1$. Let x be any point in $B_r(x_0)$ and ρ be any number in $(0, r)$. If

$$B_{5\rho}(x) \cap B_r(x_0) \subset [A_t^{i-1}]_\eta, \tag{5.9}$$

which is equivalent to the fact that $\mu(A_t^{i-1} \cap B_{5\rho}(x)) > \eta\mu(B_\rho(x))$ by the definition of $[A_t^{i-1}]_\eta$, then

$$\mu(A_t^{i-1} \cap B_{5\rho}(x)) > \eta\mu(B_\rho(x)) \geq C_\mu^{-3}\eta\mu(B_{5\rho}(x)),$$

since $\mu(B_\rho(x)) \geq C_\mu^{-3}\mu(B_{5\rho}(x))$ by using condition (VD). Let $\varepsilon := \varepsilon_1(C_\mu^{-3}\eta)$. Since $B(x, \frac{R}{2}) \subset B(x_0, R)$, the function u is non-negative, f-superharmonic in $B(x, \frac{R}{2})$. Noting that $5\rho < 5r \leq 5\frac{\delta_1}{10}R = \delta_1\frac{R}{2}$ and

$$\frac{\mu\left(B_{5\rho}(x) \cap \{u \geq t\varepsilon^{i-1} - \frac{T}{1-\varepsilon}\}\right)}{\mu(B_{5\rho}(x))} \geq \frac{\mu\left(B_{5\rho}(x) \cap A_t^{i-1}\right)}{\mu(B_{5\rho}(x))} \geq C_\mu^{-3}\eta,$$

we apply condition (wEH1) on two concentric balls $B(x, \frac{R}{2}), B(x, 5\rho)$ for $\eta_1 = C_\mu^{-3}\eta$ and for those $t > 0$ such that

$$a := t\varepsilon^{i-1} - \frac{T}{1-\varepsilon} > 0.$$

It follows that, using the fact that $w(x, 5\rho) < w(x, 5r) \leq C_2 w(x_0, 5r)$ by (1.5),

$$\begin{aligned}
\operatorname*{einf}_{B_{20\rho}(x)} u &> \varepsilon\left(t\varepsilon^{i-1} - \frac{T}{1-\varepsilon}\right) - w(x, 5\rho)\left(T_{\frac{3}{4}B(x,\frac{R}{2}),B(x,\frac{R}{2})}(u_-) + \|f\|_{L^\infty(B(x,\frac{R}{2}))}\right) \\
&\geq \varepsilon\left(t\varepsilon^{i-1} - \frac{T}{1-\varepsilon}\right) - C_2 w(x_0, 5r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) \\
&= \varepsilon\left(t\varepsilon^{i-1} - \frac{T}{1-\varepsilon}\right) - T = t\varepsilon^{i} - \frac{T}{1-\varepsilon},
\end{aligned} \tag{5.10}$$

where we have used the fact that

$$T_{\frac{3}{4}B(x,\frac{R}{2}),B(x,\frac{R}{2})}(u_-) \leq T_{\frac{3}{4}B_R,B_R}(u_-)$$

by Proposition 4.1 since $B(x, \frac{R}{2}) \subset \frac{3}{4}B_R = B(x_0, \frac{3}{4}R)$ for any x in $B_r(x_0)$. Clearly, the inequality (5.10) also holds for those t when $t\varepsilon^{i-1} - \frac{T}{1-\varepsilon} \leq 0$, and hence, it is true for any $t > 0$, provided that (5.9) is satisfied.

Therefore, for any ball $B_{5\rho}(x)$ satisfying (5.9), we have $B_{5\rho}(x) \cap B_r(x_0) \subset A_t^i$, which implies that

$$[A_t^{i-1}]_\eta \subset A_t^i \ \text{ for any } t > 0 \text{ and any } i \geq 1.$$

By the Krylov-Safonov covering lemma with $E = A_t^{i-1}$, we must have that for any $t > 0$ and any $i \geq 1$, either $A_t^{i-1} = B_r(x_0)$ (thus $A_t^i = B_r(x_0)$) or

$$\frac{1}{\eta}\mu(A_t^{i-1}) \leq \mu([A_t^{i-1}]_\eta) \leq \mu(A_t^i). \tag{5.11}$$

We choose an integer $j \geq 1$ such that

$$\eta^j < \frac{\mu(A_t^0)}{\mu(B_r(x_0))} \leq \eta^{j-1}.$$

Suppose that $A_t^{j-1} \neq B_r(x_0)$. Using the fact that $A_t^{i-1} \subset A_t^i$, we have $A_t^k \neq B_r(x_0)$ for all $0 \leq k \leq j-1$. Hence, we obtain from (5.11) that

$$\mu(A_t^{j-1}) \geq \frac{1}{\eta}\mu(A_t^{j-2}) \geq \cdots \geq \frac{1}{\eta^{j-1}}\mu(A_t^0) \geq \eta\mu(B_r(x_0)).$$

Since $\eta \in (0, 1)$, this inequality holds trivially when $A_t^{j-1} = B_r(x_0)$. Therefore, using condition (wEH1) again, we have

$$\begin{aligned}
\operatorname*{einf}_{B_{4r}(x_0)} u &> \varepsilon_1(\eta)\left(t\varepsilon^{j-1} - \frac{T}{1-\varepsilon}\right) - w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) \\
&\geq \varepsilon_1(\eta)\left(t\varepsilon^{j-1} - \frac{T}{1-\varepsilon}\right) - T \geq \varepsilon_1(\eta)t\varepsilon^{j-1} - \frac{\varepsilon_1(\eta)+1}{1-\varepsilon}T \\
&\geq \varepsilon_1(\eta)t\left(\frac{\mu(A_t^0)}{\mu(B_r(x_0))}\right)^{\frac{1}{\gamma}} - \frac{\varepsilon_1(\eta)+1}{1-\varepsilon}T,
\end{aligned}$$

where $\gamma = \log_\varepsilon \eta$. From this, it follows that, for any $t > 0$ and any $r \in (0, \frac{\delta_1}{10}R]$,

$$\frac{\mu(A_t^0)}{\mu(B_r(x_0))} \leq \frac{c_3}{t^\gamma}\left(\operatorname*{einf}_{B_{4r}(x_0)} u + \frac{T}{1-\varepsilon}\right)^\gamma$$

for some positive constant c_3 depending only on η (for example, $c_3 = \frac{\varepsilon_1(\eta)+1}{\varepsilon_1(\eta)}$).

Therefore, for any $0 < p < \gamma$ and any $a > 0$,

$$\begin{aligned}
\fint_{B_r(x_0)} u^p d\mu &= p\int_0^\infty t^{p-1}\frac{\mu(B_r(x_0)\cap\{u>t\})}{\mu(B_r(x_0))}dt \leq p\int_0^\infty t^{p-1}\frac{\mu(A_t^0)}{\mu(B_r(x_0))}dt \\
&\leq p\left[\int_0^a t^{p-1}dt + c_3\left(\operatorname*{einf}_{B_{4r}(x_0)} u + \frac{T}{1-\varepsilon}\right)^\gamma \int_a^\infty t^{p-1-\gamma}dt\right] \\
&\leq c_4(p,\eta,\varepsilon)\left[a^p + \left(\operatorname*{einf}_{B_{4r}(x_0)} u + \frac{T}{1-\varepsilon}\right)^\gamma a^{p-\gamma}\right].
\end{aligned}$$

By choosing a such that

$$a = \operatorname*{einf}_{B_{4r}(x_0)} u + \frac{T}{1-\varepsilon},$$

we conclude by using (5.8), (1.5) that for any $0 < r \leq \frac{\delta_1}{10}R$

$$\begin{aligned}
\fint_{B_r(x_0)} u^p d\mu &\leq 2c_4(p,\eta,\varepsilon)\left(\operatorname*{einf}_{B_{4r}(x_0)} u + \frac{T}{1-\varepsilon}\right)^p \\
&\leq 2c_4(p,\eta,\varepsilon)\left(\operatorname*{einf}_{B_{4r}(x_0)} u + \frac{C_2^2 5^{\beta_2} w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)}{1-\varepsilon}\right)^p \\
&\leq c_5(p,\eta,\varepsilon)\left(\operatorname*{einf}_{B_r(x_0)} u + w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right)^p,
\end{aligned}$$

thus showing that condition (wEH) holds with $\delta := \frac{\delta_1}{10}$. The proof is complete. □

We introduce condition (wEH2) (cf. [24, Lemma 7.2] for the local Dirichlet form).

Definition 5.3 (condition (wEH2))**.** *We say that condition* (wEH2) *holds if there exist three universal constants σ, δ_2 in $(0,1)$ and $C > 0$ such that, for any two concentric balls $B_R := B(x_0,R) \supset B(x_0,r) =: B_r$ with $R \in (0,\sigma\overline{R})$, $r \in (0,\delta_2 R)$, any number $a > 0$, any function $f \in L^\infty(B_R)$, and for any $u \in \mathcal{F}' \cap L^\infty$ which is non-negative, f-superharmonic in B_R, we have*

$$\operatorname*{einf}_{B_r} u \geq a\exp\left(-\frac{C}{\omega_{B_r}(\{u \geq a\})}\right) - w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right). \tag{5.12}$$

We remark that the constants σ, δ_2, C are all independent of B_R, B_r, a, f and u.

Remark 5.4. *Let $a > 0$. If*

$$\omega_{B_r}(\{u \geq a\}) = 0,$$

then (5.12) is trivially satisfied since $u \geq 0$ in B_r. On the other hand, if

$$\omega_{B_r}(\{u \geq a\}) = 1,$$

then (5.12) is also trivially satisfied since $u \geq a$ in B_r. Thus, in order to show (5.12), it suffices to consider the case $0 < \omega_{B_r}(\{u \geq a\}) < 1$ only.

We have the following.

Proposition 5.5. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form in L^2. Then*

$$\text{(wEH)} \Rightarrow \text{(wEH2)}.$$

Proof. Assume that condition (wEH) holds with four constants p, δ, σ in $(0,1)$ and $C_H \geq 1$. Fix a ball $B_R := B(x_0, R)$ with $R \in (0, \sigma\overline{R})$ and fix another concentric ball $B_r := B(x_0, r)$ with $0 < r \leq \delta R$. Let $u \in \mathcal{F}' \cap L^\infty$ be non-negative, f-superharmonic in B_R. Then

$$\left(\frac{1}{\mu(B_r)}\int_{B_r} u^p d\mu\right)^{1/p} \leq C_H \left(\operatorname*{einf}_{B_r} u + w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\right). \tag{5.13}$$

In order to show condition (wEH2), we shall prove that (5.12) holds with

$$\delta_2 = \delta \text{ and } C := \ln C_H + 1/p. \tag{5.14}$$

To see this, let a be any positive number. By Remark 5.4, we may assume

$$0 < \omega_{B_r}(\{u \geq a\}) < 1.$$

Note that, using the elementary inequality $\ln x \geq 1 - \frac{1}{x}$ for any $0 < x \leq 1$,

$$\begin{aligned}
\left(\frac{1}{\mu(B_r)}\int_{B_r} u^p d\mu\right)^{1/p} &\geq \left(\frac{1}{\mu(B_r)}\int_{B_r\cap\{u\geq a\}} a^p d\mu\right)^{1/p} = \left(a^p \omega_{B_r}(\{u \geq a\})\right)^{1/p} \\
&= a \exp\left(\frac{1}{p}\ln \omega_{B_r}(\{u \geq a\})\right) \geq a\exp\left(\frac{1}{p}\left(1 - \frac{1}{\omega_{B_r}(\{u \geq a\})}\right)\right) \\
&\geq a\exp\left(-\frac{1/p}{\omega_{B_r}(\{u \geq a\})}\right) = a\exp\left(-\frac{C - \ln C_H}{\omega_{B_r}(\{u \geq a\})}\right) \\
&= a\exp\left(\frac{\ln C_H}{\omega_{B_r}(\{u \geq a\})}\right)\cdot \exp\left(-\frac{C}{\omega_{B_r}(\{u \geq a\})}\right) \\
&\geq aC_H \exp\left(-\frac{C}{\omega_{B_r}(\{u \geq a\})}\right) \ \ (\text{since } \omega_{B_r}(\{u \geq a\}) \leq 1).
\end{aligned} \tag{5.15}$$

Plugging (5.15) into (5.13) and then dividing by C_H on the both sides, we conclude that

$$a\exp\left(-\frac{C}{\omega_{B_r}(\{u \geq a\})}\right) \leq \operatorname*{einf}_{B_r} u + w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right),$$

thus showing that (5.12) holds with constants δ_2, C chosen as in (5.14). The proof is complete. □

We introduce condition (wEH3).

Definition 5.6 (condition (wEH3))**.** *We say that condition* (wEH3) *holds if there exist two universal constants σ, δ_3 in* $(0,1)$ *such that, for any two concentric balls $B_R := B(x_0, R) \supset B(x_0, r) =: B_r$ with $R \in (0, \sigma\overline{R})$, $r \in (0, \delta_3 R]$, any number $\eta_3 \in (0,1]$, any function $f \in L^\infty(B_R)$, and for any $u \in \mathcal{F}' \cap L^\infty$ which is non-negative, f-superharmonic in B_R, if for some $a > 0$,*

$$\omega_{B_r}(\{u \geq a\}) = \frac{\mu(B_r \cap \{u \geq a\})}{\mu(B_r)} \geq \eta_3$$

and

$$w(B_r)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) \leq F(\eta_3)a$$

for a map $F : (0,1] \longmapsto (0,1]$, then we have

$$\operatorname*{einf}_{B_r} u \geq F(\eta_3)a. \tag{5.16}$$

We show condition (wEH2) alone implies condition (wEH3) for any Dirichlet form in L^2.

Proposition 5.7. *Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in L^2, then*

$$(\text{wEH2}) \Rightarrow (\text{wEH3}).$$

Proof. Assume that condition (wEH2) holds with constants σ, δ_2, C. We shall show that condition (wEH3) holds with the same σ and with δ_3, F being given by

$$\delta_3 = \delta_2 \text{ and } F(\eta_3) = \frac{1}{2}\exp\left(-\frac{C}{\eta_3}\right). \tag{5.17}$$

To see this, fix a ball $B_R := B(x_0, R)$ with $R \in (0, \sigma\overline{R})$ and fix another concentric ball $B_r := B(x_0, r)$ with $0 < r \le \delta_2 R$. Let $\eta_3 \in (0, 1]$ and $r \in (0, \delta_2 R]$ be any two numbers. Let $u \in \mathcal{F}' \cap L^\infty$ be any function that is non-negative, f-superharmonic in B_R. If for some $a > 0$,

$$\omega_{B_r}(\{u \ge a\}) \ge \eta_3$$

and if

$$w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) \le F(\eta_3)a = \frac{1}{2}\exp\left(-\frac{C}{\eta_3}\right)a,$$

then by condition (wEH2),

$$\begin{aligned}\operatorname*{einf}_{B_r} u &\ge a\exp\left(-\frac{C}{\omega_{B_r}(\{u \ge a\})}\right) - w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right)\\ &\ge a\exp\left(-\frac{C}{\eta_3}\right) - \frac{1}{2}\exp\left(-\frac{C}{\eta_3}\right)a = \frac{1}{2}\exp\left(-\frac{C}{\eta_3}\right)a = F(\eta_3)a.\end{aligned}$$

This proves that (5.16) is true, and so condition (wEH3) holds. The proof is complete. □

The following shows that condition (wEH3) implies condition (wEH1).

Proposition 5.8. *Let* $(\mathcal{E}, \mathcal{F})$ *be a Dirichlet form in* L^2. *If condition* (VD) *holds, then*

$$\text{(wEH3)} \Rightarrow \text{(wEH1)}.$$

Proof. Assume that condition (wEH3) holds with constants σ, δ_3 in $(0, 1)$ and a map $F : (0, 1] \to (0, 1]$. We show that condition (wEH1) holds with the same σ and with constants

$$\delta_1 = \frac{\delta_3}{8} \text{ and } \varepsilon_1 := \varepsilon_1(\eta_1) = \frac{F(\eta_1/C_\mu^3)}{C_2 8^{\beta_2}}, \tag{5.18}$$

so that $\delta_1 \in (0, \frac{1}{4})$ and $\varepsilon_1 = \varepsilon_1(\eta_1) \in (0, 1)$, where constants C_μ is the same as in (1.2) and C_2, β_2 same as in (1.5).

To see this, fix two concentric balls $B_R := B(x_0, R) \supset B(x_0, r) =: B_r$ with $R \in (0, \sigma\overline{R})$, $r \in (0, \delta_1 R]$. Let $\eta_1 \in (0, 1]$ be any fixed number. Let $u \in \mathcal{F}' \cap L^\infty$ be any function that is non-negative, f-superharmonic in B_R. Suppose that for some $a > 0$,

$$\omega_{B_r}(\{u \ge a\}) \ge \eta_1. \tag{5.19}$$

We need to show that (5.2) is satisfied.

Indeed, since $r \le \delta_1 R = \delta_3 R/8$ so that

$$B_{\frac{8r}{\delta_3}} \subseteq B_R, \tag{5.20}$$

the function u is non-negative and f-superharmonic in $B_{\frac{8r}{\delta_3}}$. By (5.19) and condition (VD)

$$\begin{aligned}\omega_{\delta_3 B_{\frac{8r}{\delta_3}}}(u \ge a) = \omega_{B_{8r}}(\{u \ge a\}) &= \frac{\mu(B_{8r} \cap \{u \ge a\})}{\mu(B_{8r})} \ge \frac{\mu(B_r \cap \{u \ge a\})}{\mu(B_{8r})}\\ &= \frac{\omega_{B_r}(\{u \ge a\})\mu(B_r)}{\mu(B_{8r})} \ge \frac{\eta_1 \mu(B_r)}{\mu(B_{8r})} \ge \frac{\eta_1}{C_\mu^3} := \eta_3.\end{aligned} \tag{5.21}$$

We distinguish two cases.

Case 1 when

$$w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) \le \varepsilon_1 a = \frac{F(\eta_1/C_\mu^3)}{C_2 8^{\beta_2}}a. \tag{5.22}$$

In this case, we have

$$\begin{aligned} w(\delta_3 B_{\frac{8r}{\delta_3}})\left(T_{\frac{3}{4}B_{\frac{8r}{\delta_3}},B_{\frac{8r}{\delta_3}}}(u_-)+\|f\|_{L^\infty(B_{\frac{8r}{\delta_3}})}\right) &= w(B_{8r})\left(T_{B_{\frac{6r}{\delta_3}},B_{\frac{8r}{\delta_3}}}(u_-)+\|f\|_{L^\infty(B_{\frac{8r}{\delta_3}})}\right) \\ &\le w(B_{8r})\left(T_{\frac{3}{4}B_R,B_R}(u_-)+\|f\|_{L^\infty(B_R)}\right) \\ &\le C_2 8^{\beta_2} w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-)+\|f\|_{L^\infty(B_R)}\right) \\ &\le F(\eta_1/C_\mu^3)a = F(\eta_3)a, \end{aligned}$$

where in the first inequality we have used the fact

$$T_{B_{\frac{6r}{\delta_3}},B_{\frac{8r}{\delta_3}}}(u_-) \le T_{\frac{3}{4}B_R,B_R}(u_-)$$

since $B_{\frac{6r}{\delta_3}} \subseteq \frac{3}{4}B_R$ and u is non-negative in B_R, whilst in the second inequality we have used the fact that

$$\frac{w(B_{8r})}{w(B_r)} = \frac{w(x_0,8r)}{w(x_0,r)} \le C_2\left(\frac{8r}{r}\right)^{\beta_2} = C_2 8^{\beta_2}$$

by virtue of (1.5). Therefore, applying (wEH3) with B_R being replaced by $B_{\frac{8r}{\delta_3}}$, we obtain

$$\operatorname*{einf}_{B_{4r}} u \ge F(\eta_3)a. \tag{5.23}$$

Noting that

$$\varepsilon_1 = \frac{F(\eta_1/C_\mu^3)}{C_2 8^{\beta_2}} < F(\eta_1/C_\mu^3) = F(\eta_3), \tag{5.24}$$

we see that

$$\begin{aligned} F(\eta_3)a &\ge F(\eta_3)a - w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-)+\|f\|_{L^\infty(B_R)}\right) \\ &> \varepsilon_1 a - w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-)+\|f\|_{L^\infty(B_R)}\right). \end{aligned} \tag{5.25}$$

Plugging (5.25) into (5.23), it follows that

$$\operatorname*{einf}_{B_{4r}} u \ge F(\eta_3)a > \varepsilon_1 a - w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-)+\|f\|_{L^\infty(B_R)}\right),$$

thus showing that (5.2) is true in this case.

Case 2 when

$$w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-)+\|f\|_{L^\infty(B_R)}\right) > \varepsilon_1 a.$$

In this case, we immediately see that

$$\operatorname*{einf}_{B_{4r}} u \ge 0 > \varepsilon_1 a - w(B_r)\left(T_{\frac{3}{4}B_R,B_R}(u_-)+\|f\|_{L^\infty(B_R)}\right),$$

thus showing that (5.2) is true again.

Therefore, we always have that (5.2) holds, no matter the *Case* 1 happens or not. This proves condition (wEH1). The proof is complete. ◻

The following another version of the weak elliptic Harnack inequality was essentially introduced in [20, Lemma 4.5] when $f \equiv 0$, and the jump kernel J exists and satisfies the upper and lower bounds.

Definition 5.9 (condition (wEH4))**.** *We say that condition* (wEH4) *holds if there exist three universal constants* $\sigma, \varepsilon_4, \delta_4$ *in* $(0,1)$ *such that, for any ball* $B_R := B(x_0, R)$ *with* $R \in (0, \sigma\overline{R})$*, any function* $f \in L^\infty(B_R)$*, and for any* $u \in \mathcal{F}' \cap L^\infty$ *which is non-negative and* f*-superharmonic in* B_R*, if for some* $a > 0$*,*

$$\omega_{\delta_4 B_R}(\{u \geq a\}) = \frac{\mu(\delta_4 B_R \cap \{u \geq a\})}{\mu(\delta_4 B_R)} \geq \frac{1}{2}$$

and

$$w(\delta_4 B_R)\left(T_{\frac{3}{4}B_R, B_R}(u_-) + \|f\|_{L^\infty(B_R)}\right) \leq \varepsilon_4 a,$$

then

$$\operatorname*{einf}_{\frac{1}{2}(\delta_4 B_R)} u \geq \varepsilon_4 a. \tag{5.26}$$

Proposition 5.10. *Let* $(\mathcal{E}, \mathcal{F})$ *be a Dirichlet form in* L^2*, then*

$$\text{(wEH3)} \Rightarrow \text{(wEH4)}.$$

Proof. In fact, condition (wEH4) is a special case of condition (wEH3) with $\eta_3 = \frac{1}{2}$, $\varepsilon_4 = F(1/2)$ and $\delta_4 = \delta_3$. The proof is complete. □

We are now in a position to prove Theorem 1.9.

Proof of Theorem 1.9. We have the following conclusions:

$$\begin{aligned} \text{(wEH)} &\Leftrightarrow \text{(wEH1)} \ \text{(Proposition 5.2)},\\ \text{(wEH)} &\Rightarrow \text{(wEH2)} \ \text{(Proposition 5.5)},\\ &\Rightarrow \text{(wEH3)} \ \text{(Proposition 5.7)},\\ &\Rightarrow \text{(wEH1)} \ \text{(Proposition 5.8)}, \end{aligned}$$

thus showing that the equivalences in (1.26) are all true.

Finally, the implication (wEH3) ⇒ (wEH4) in (1.27) follows immediately by Proposition 5.10. The proof of Theorem 1.9 is complete. □

6. Consequences of weak Harnack inequality

In this section, we look at two consequences of the weak Harnack inequality. One is that we obtain the Hölder continuity of any harmonic function if conditions (wEH) and (TJ) hold for any regular Dirichlet form without killing part, see Lemma 6.2 below. The Hölder continuity of harmonic functions was investigated in various settings, see for example [40, Theorem 5.3] for a certain class of integro-differential equations in $\mathbb{R}^n$ (see also [18, Theorem 1.7] in $\mathbb{R}^n$ under a weaker assumption), and [13, Theorem 2.1] for a pure-jump Dirichlet form. Here we have extended this conclusion to a more general situation where the jump kernel does not necessarily exist. Although the proof is standard, we sketch the proof for completeness of this paper.

The other consequence of the weak Harnack inequality is that we can obtain a Lemma of growth for any globally non-negative, superharmonic function (Lemma 6.4 below), which leads to a lower bound of the mean exit time on a ball (Lemma 6.3 below). The lower bound of the mean exit time plays an important role in obtaining the heat kernel estimate.

Recall that for an open subset Ω of M, a function $u \in \mathcal{F}$ is *harmonic* in Ω if for any non-negative $\varphi \in \mathcal{F}(\Omega)$,

$$\mathcal{E}(u, \varphi) = 0.$$

For any ball $B \subseteq M$ and any function $u \in L^\infty(B, \mu)$, we define

$$\operatorname*{eosc}_B u := \operatorname*{esup}_B u - \operatorname*{einf}_B u.$$

Lemma 6.1. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2(M,\mu)$ without killing part. If conditions* (wEH) *and* (TJ) *hold, then there exist two constants $\beta\in(0,1)$ and $C>0$ such that, for any $x_0\in M$, $0<r<\sigma\overline{R}$ and any harmonic function u in $B(x_0,r)$,*

$$\operatorname*{eosc}_{B(x_0,\rho)} u \le C\|u\|_{L^\infty}\left(\frac{\rho}{r}\right)^{\beta},\quad 0<\rho\le r. \tag{6.1}$$

We remark that constants C,β are independent of $\overline{R},u,x_0,r,\rho$.

Proof. Fix a ball $B(x_0,r)$ for $0<r<\sigma\overline{R}$. Set

$$B_\rho := B(x_0,\rho)\quad\text{for any } \rho>0.$$

Let u be a harmonic function in B_r. Without loss of generality, we assume that $\|u\|_{L^\infty(M)}<\infty$. Let

$$M_0 := \|u\|_{L^\infty},\quad m_0 := \operatorname*{einf}_M u,\ \ K := M_0-m_0$$

so that $0\le K\le 2\|u\|_{L^\infty}$.

We will construct two sequences $\{m_n\}_{n\ge0}$, $\{M_n\}_{n\ge0}$ of positive numbers such that for each n,

$$\begin{aligned} m_{n-1} &\le m_n\le M_n\le M_{n-1}\quad\text{and}\quad M_n-m_n=K\theta^{-n\beta},\\ m_n &\le u(x)\le M_n\qquad\text{for any } x\in B_{r\theta^{-n}}, \end{aligned}\tag{6.2}$$

where θ,β are two constants to be determined so that

$$\theta\ge\delta^{-1},\quad \beta\in(0,1),\quad\text{and}\quad \frac{2-\lambda}{2}\theta^{\beta}\le 1, \tag{6.3}$$

where $\lambda:=(2^{1+1/p}C_H)^{-1}\in(0,1)$ and $p,\delta\in(0,1)$ and $C_H\ge1$ come from condition (wEH). Once this is true, then we are done by noting that (6.1) follows, since for any $0<\rho<r$, there is some integer $j\ge0$ such that

$$\theta^{-j-1}\le\frac{\rho}{r}<\theta^{-j},$$

from which, we see by (6.2) that

$$\operatorname*{eosc}_{B_\rho} u\le \operatorname*{eosc}_{B_{r\theta^{-j}}} u\le M_j-m_j=K\theta^{-j\beta}\le 2\theta^{\beta}\|u\|_{L^\infty}\left(\frac{\rho}{r}\right)^{\beta}.$$

We will show (6.2) inductively. Indeed, assume that there exists an integer $k\ge1$ such that (6.2) holds for any $n\le k-1$. We need to construct m_k,M_k such that (6.2) still holds for $n=k$ and for θ,β satisfying (6.3).

To do this, set for any $x\in M$

$$v(x)=\left(u(x)-\frac{M_{k-1}+m_{k-1}}{2}\right)\frac{2\theta^{(k-1)\beta}}{K}. \tag{6.4}$$

Clearly, we have by (6.2) for $n=k-1$ that

$$|v(x)|\le\frac{M_{k-1}-m_{k-1}}{2}\frac{2\theta^{(k-1)\beta}}{K}=\frac{K\theta^{-(k-1)\beta}}{2}\frac{2\theta^{(k-1)\beta}}{K}=1 \tag{6.5}$$

for almost all $x\in B_{r\theta^{-(k-1)}}$.

Note that for any point $y\in B(x_0,r\theta^{-(k-1)})^c$, there is some integer $j\ge1$ such that

$$r\theta^{-k+j}\le d(y,x_0)<r\theta^{-k+j+1}.$$

For simplicity, set $M_{-n}=M_0$ and $m_{-n}=m_0$ for any $n\ge1$. By (6.2), for any $y\in B(x_0,r\theta^{-(k-j-1)})\setminus B(x_0,r\theta^{-(k-j)})$ $(j\ge1)$,

$$\begin{aligned}\frac{K}{2\theta^{(k-1)\beta}}v(y)=u(y)-\frac{M_{k-1}+m_{k-1}}{2}&\le M_{k-j-1}-\frac{M_{k-1}+m_{k-1}}{2}\\ &=M_{k-j-1}-m_{k-j-1}+m_{k-j-1}-\frac{M_{k-1}+m_{k-1}}{2}\end{aligned}$$

$$\le M_{k-j-1} - m_{k-j-1} - \frac{M_{k-1} - m_{k-1}}{2}$$
$$\le K\theta^{-(k-j-1)\beta} - \frac{K}{2}\theta^{-(k-1)\beta},$$

from which, it follows that

$$v(y) \le 2\theta^{j\beta} - 1 \le 2\left(\frac{d(y, x_0)}{r\theta^{-k}}\right)^{\beta} - 1 \text{ for any } y \in B(x_0, r\theta^{-(k-1)})^c. \tag{6.6}$$

On the other hand, we similarly have that, for any $y \in B(x_0, r\theta^{-(k-j-1)}) \setminus B(x_0, r\theta^{-(k-j)})$ $(j \ge 1)$,

$$\frac{K}{2\theta^{(k-1)\beta}} v(y) = u(y) - \frac{M_{k-1} + m_{k-1}}{2} \ge m_{k-j-1} - \frac{M_{k-1} + m_{k-1}}{2}$$
$$= m_{k-j-1} - M_{k-j-1} + M_{k-j-1} - \frac{M_{k-1} + m_{k-1}}{2}$$
$$\ge -\left(M_{k-j-1} - m_{k-j-1}\right) + \frac{M_{k-1} - m_{k-1}}{2}$$
$$\ge -K\theta^{-(k-j-1)\beta} + \frac{K}{2}\theta^{-(k-1)\beta},$$

which gives that

$$v(y) \ge 1 - 2\theta^{j\beta} \ge 1 - 2\left(\frac{d(y, x_0)}{r\theta^{-k}}\right)^{\beta} \text{ for any } y \in B(x_0, r\theta^{-(k-1)})^c. \tag{6.7}$$

We distinguish two cases: either

$$\mu(B_{r\theta^{-k}} \cap \{v \le 0\}) \ge \mu(B_{r\theta^{-k}})/2, \tag{6.8}$$

or

$$\mu(B_{r\theta^{-k}} \cap \{v > 0\}) \ge \mu(B_{r\theta^{-k}})/2. \tag{6.9}$$

If (6.8) holds, we will show that for almost every $z \in B_{r\theta^{-k}}$

$$v(z) \le 1 - \lambda. \tag{6.10}$$

Temporally assume that (6.10) holds true. Then by (6.4), we see that for any point $z \in B_{r\theta^{-k}}$,

$$u(z) = \frac{K}{2\theta^{(k-1)\beta}} v(z) + \frac{M_{k-1} + m_{k-1}}{2} \le \frac{K(1-\lambda)}{2\theta^{(k-1)\beta}} + \frac{M_{k-1} + m_{k-1}}{2}$$
$$= \frac{K(1-\lambda)}{2}\theta^{-(k-1)\beta} + \frac{M_{k-1} - m_{k-1}}{2} + m_{k-1}$$
$$= \frac{K(1-\lambda)}{2}\theta^{-(k-1)\beta} + \frac{K}{2}\theta^{-(k-1)\beta} + m_{k-1} \text{ (using (6.2))}$$
$$= \frac{(2-\lambda)\theta^{\beta}}{2} K\theta^{-k\beta} + m_{k-1} \le K\theta^{-k\beta} + m_{k-1},$$

where in the last inequality we have used the fact that $\frac{2-\lambda}{2}\theta^{\beta} \le 1$ in (6.3). Therefore, setting

$$m_k = m_{k-1} \quad \text{and} \quad M_k = m_k + K\theta^{-k\beta} \le M_{k-1},$$

we obtain that $m_k \le u(z) \le M_k$ for a.e $z \in B_{r\theta^{-k}}$, thus showing that (6.2) holds when $n = k$, which finishes the induction step from $n \le k-1$ to $n = k$ in the case when (6.8) holds.

We turn to show (6.10). Indeed, consider $h := 1 - v$. Clearly, the function h is harmonic in $B_{r\theta^{-(k-1)}}$ and also is non-negative in $B_{r\theta^{-(k-1)}}$ by using (6.5). Applying (wEH) to the function h in $B_{r\theta^{-(k-1)}}$ and $f = 0$, we find that

$$\left(\fint_{B_{r\theta^{-k}}} h^p d\mu\right)^{1/p} \le C_H \left(\operatorname*{einf}_{B_{r\theta^{-k}}} h + w\left(x_0, r\theta^{-k}\right) T_{\frac{3}{4}B_{r\theta^{-(k-1)}}, B_{r\theta^{-(k-1)}}}(h_-)\right), \tag{6.11}$$

where we have used the fact that $\theta^{-1} \le \delta$ so that $r\theta^{-k} \le \delta \cdot r\theta^{-(k-1)}$. Note that by (6.8),

$$\begin{aligned}\left(\fint_{B_{r\theta^{-k}}} h^p du\right)^{1/p} &\ge \left(\frac{1}{\mu\left(B_{r\theta^{-k}}\right)}\int_{B_{r\theta^{-k}}\cap\{v\le 0\}}(1-v)^p du\right)^{1/p} \\ &\ge \left(\frac{\mu(B_{r\theta^{-k}}\cap\{v\le 0\})}{\mu\left(B_{r\theta^{-k}}\right)}\right)^{1/p} \ge 2^{-1/p}. \end{aligned} \tag{6.12}$$

Also note that by (6.6),

$$h_-(y) = (1-v(y))_- = (v(y)-1)_+ \le 2\left[\left(\frac{d\left(y,x_0\right)}{r\theta^{-k}}\right)^{\beta} - 1\right]$$

for any $y \in B(x_0, r\theta^{-(k-1)})^c = B^c_{r\theta^{-(k-1)}}$. From this, we have by condition (TJ)

$$\begin{aligned} T_{\frac{3}{4}B_{r\theta^{-(k-1)}},B_{r\theta^{-(k-1)}}}(h_-) &= \sup_{x\in\frac{3}{4}B_{r\theta^{-(k-1)}}}\int_{B^c_{r\theta^{-(k-1)}}} h_-(y)J(x,dy) \\ &\le 2\sup_{x\in\frac{3}{4}B_{r\theta^{-(k-1)}}}\int_{B^c_{r\theta^{-(k-1)}}}\left[\left(\frac{d\left(y,x_0\right)}{r\theta^{-k}}\right)^{\beta}-1\right]J(x,dy) \\ &= 2\sup_{x\in\frac{3}{4}B_{r\theta^{-(k-1)}}}\sum_{j=1}^{\infty}\int_{B_{r\theta^{-k+j+1}}\setminus B_{r\theta^{-k+j}}}\left[\left(\frac{d\left(y,x_0\right)}{r\theta^{-k}}\right)^{\beta}-1\right]J(x,dy) \\ &\le 2\sum_{j=1}^{\infty}\sup_{x\in\frac{3}{4}B_{r\theta^{-(k-1)}}}\int_{B_{r\theta^{-k+j+1}}\setminus B_{r\theta^{-k+j}}}\left(\theta^{(j+1)\beta}-1\right)J(x,dy) \\ &\le 2\sum_{j=1}^{\infty}\left(\theta^{(j+1)\beta}-1\right)\sup_{x\in\frac{3}{4}B_{r\theta^{-(k-1)}}}\int_{B^c_{r\theta^{-k+j}}}J(x,dy) \\ &\le 2\sum_{j=1}^{\infty}\left(\theta^{(j+1)\beta}-1\right)\sup_{x\in\frac{3}{4}B_{r\theta^{-(k-1)}}}\int_{B(x,r\theta^{-k+j}/4)^c}J(x,dy) \\ &\le 2C\sum_{j=1}^{\infty}\left(\theta^{(j+1)\beta}-1\right)\sup_{x\in\frac{3}{4}B_{r\theta^{-(k-1)}}}\frac{1}{w(x,r\theta^{-k+j}/4)}, \end{aligned} \tag{6.13}$$

where $C > 0$ is the same constant as in (1.13). Since

$$\frac{w(x,r\theta^{-k+j}/4)}{w(x_0,r\theta^{-k})} \ge \frac{w(x,r\theta^{-k+j})}{C_2 4^{\beta_2} w(x_0,r\theta^{-k})} \ge \frac{C_1\theta^{j\beta_1}}{C_2 4^{\beta_2}}$$

for any $x \in \frac{3}{4}B_{r\theta^{-(k-1)}}$ by using (1.5), it follows from (6.13) that

$$w\left(x_0,r\theta^{-k}\right)T_{\frac{3}{4}B_{r\theta^{-(k-1)}},B_{r\theta^{-(k-1)}}}(h_-) \le \frac{2CC_2 4^{\beta_2}}{C_1}\sum_{j=1}^{\infty}\frac{\theta^{(j+1)\beta}-1}{\theta^{j\beta_1}}. \tag{6.14}$$

Therefore, substituting (6.12), (6.14) into (6.11), we obtain

$$\begin{aligned} \operatorname*{einf}_{B_{r\theta^{-k}}} h &\ge \left(C_H 2^{1/p}\right)^{-1} - w\left(x_0,r\theta^{-k}\right)T_{\frac{3}{4}B_{r\theta^{-(k-1)}},B_{r\theta^{-(k-1)}}}(h_-) \\ &\ge \left(C_H 2^{1/p}\right)^{-1} - \frac{2CC_2 4^{\beta_2}}{C_1}\sum_{j=1}^{\infty}\frac{\theta^{(j+1)\beta}-1}{\theta^{j\beta_1}}. \end{aligned} \tag{6.15}$$

Since $\theta^{-1} \le \delta$, we see that for any $\beta \in (0, \beta_1/2)$

$$\sum_{j=l+1}^{\infty} \theta^{-j\beta_1}\left(\theta^{(j+1)\beta}-1\right) \le \sum_{j=l+1}^{\infty} \theta^{-j\beta_1}\theta^{(j+1)\beta_1/2} = \frac{\theta^{-l\beta_1/2}}{1-\theta^{-\beta_1/2}} \le \frac{\delta^{l\beta_1/2}}{1-\delta^{\beta_1/2}} \le \left(\frac{8CC_2 4^{\beta_2}}{C_1} C_H 2^{1/p}\right)^{-1},$$

provided that the number l is sufficiently large, which depends only on δ but is independent of β, θ. For such a number l, we now choose $\beta \in (0, \beta_1/2)$ to be so small that

$$\begin{aligned}\sum_{j=1}^{l} \theta^{-j\beta_1}\left(\theta^{(j+1)\beta}-1\right) &\le \theta^{-\beta_1}\sum_{j=1}^{l}\left(\theta^{(j+1)\beta}-1\right) \le l\theta^{-\beta_1}\left(\theta^{(l+1)\beta}-1\right) \\ &\le l\delta^{\beta_1}\left(\theta^{(l+1)\beta}-1\right) \le \left(\frac{8CC_2 4^{\beta_2}}{C_1} C_H 2^{1/p}\right)^{-1}.\end{aligned}$$

It follows that

$$\sum_{j=1}^{\infty} \frac{\theta^{(j+1)\beta}-1}{\theta^{j\beta_1}} = \sum_{j=1}^{l} \theta^{-j\beta_1}\left(\theta^{(j+1)\beta}-1\right) + \sum_{j=l+1}^{\infty} \theta^{-j\beta_1}\left(\theta^{(j+1)\beta}-1\right) \le 2\left(\frac{8CC_2 4^{\beta_2}}{C_1} C_H 2^{1/p}\right)^{-1},$$

from which, we see by (6.15) that

$$\operatorname*{einf}_{B_{r\theta^{-k}}} h \ge \left(C_H 2^{1/p}\right)^{-1} - \frac{2CC_2 4^{\beta_2}}{C_1}\cdot 2\left(\frac{8CC_2 4^{\beta_2}}{C_1} C_H 2^{1/p}\right)^{-1} = \left(2C_H 2^{1/p}\right)^{-1} = \lambda.$$

Therefore, $v \le 1-\lambda$ in $B_{r\theta^{-k}}$, thus showing (6.10) when (6.8) is satisfied, as desired.

It remains to consider the case when (6.9) is satisfied. We need to show

$$v \ge -1+\lambda \ \text{ in } B_{r\theta^{-k}}. \tag{6.16}$$

Indeed, consider the function $h = 1+v$. Similar to the argument above, setting $M_k = M_{k-1}$ and $m_k = M_k - K\theta^{-k\beta}$, one can obtain (6.16). The proof is complete. □

From the above, we immediately get the Hölder continuity of harmonic functions.

Lemma 6.2. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. If conditions* (wEH) *and* (TJ) *hold, then there exist three constants $C > 0, \theta \in (0, 1]$ and $\varepsilon \in (0, 1)$ such that, for any ball $B(x_0, r)$ with $r < \sigma\overline{R}$ and for any globally bounded function u, which is harmonic in $B(x_0, r)$,*

$$|u(x)-u(y)| \le C\left(\frac{d(x,y)}{r}\right)^{\theta}\|u\|_{L^\infty}. \tag{6.17}$$

for almost every points $x, y \in B(x_0, \varepsilon r)$.

Proof. Let the function $u \in L^\infty$ be harmonic in $B(x_0, r)$ with $r < \sigma\overline{R}$. By Lemma 6.1,

$$\operatorname*{eosc}_{B(x_0,\rho)} u \le C\|u\|_{L^\infty}\left(\frac{\rho}{r}\right)^{\beta}, \quad 0 < \rho \le r. \tag{6.18}$$

We show that (6.17) holds for $\theta = \beta, \varepsilon = 1/4$.

Indeed, let x be any point x in $B(x_0, r/4)$, the function u is harmonic in $B(x, \frac{3}{4}r) \subseteq B(x_0, r)$. Let y be a point in $B(x_0, r/4)$. Applying (6.18) with x_0 replaced by x, r by $\frac{3}{4}r$ and with $\rho = \frac{3}{2}d(x,y)$, we obtain

$$|u(x)-u(y)| \le \operatorname*{eosc}_{B(x,\frac{3}{2}d(x,y))} u \le C\|u\|_{L^\infty}\left(\frac{3d(x,y)/2}{3r/4}\right)^{\beta} = C2^{\beta}\left(\frac{d(x,y)}{r}\right)^{\beta}\|u\|_{L^\infty},$$

thus showing (6.17). The proof is complete. □

Another consequence of the weak elliptic Harnack inequality is that it implies a lower bound of the mean exit time on a ball, as we will see below.

Recall that the operator $\mathcal{L}^{\Omega}$ is the generator of the Dirichlet form $(\mathcal{E}, \mathcal{F}(\Omega))$ for any non-empty open subset Ω of M. For a ball $B \subset M$, let the function E^B be a weak solution of the Poisson-type equation $-\mathcal{L}^B u = 1$ in B, that is

$$\mathcal{E}(E^B, \varphi) = (1, \varphi) \ \text{ for any } 0 \leq \varphi \in \mathcal{F}(B). \tag{6.19}$$

We say that *condition* $(\mathrm{E}_{\geq})$ holds if there exist three constants $C > 0$ and σ, δ in $(0,1)$ such that, for all balls $B \subset M$ with radius less than $\sigma\overline{R}$,

$$\operatorname*{einf}_{x \in \delta B} E^B(x) \geq C w(B). \tag{6.20}$$

We say that *condition* $(\mathrm{E}_{\leq})$ holds if there exist two constants $C > 0$ and σ in $(0,1)$ such that, for all balls $B \subset M$ with radius less than $\sigma\overline{R}$,

$$\|E^B\|_{L^\infty} \leq C w(B). \tag{6.21}$$

Lemma 6.3. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in L^2. Then*

$$(\mathrm{VD}) + (\mathrm{Cap}_{\leq}) + (\mathrm{FK}) + (\mathrm{wEH}) \Rightarrow (\mathrm{E}_{\geq}) + (\mathrm{E}_{\leq}). \tag{6.22}$$

Proof. Note that by [24, Theorem 9.4 p.1542]

$$(\mathrm{FK}) \Rightarrow (\mathrm{E}_{\leq}), \tag{6.23}$$

(observing that we only use condition (FK) at this stage).

It remains to show the implication

$$(\mathrm{VD}) + (\mathrm{Cap}_{\leq}) + (\mathrm{FK}) + (\mathrm{wEH}) \Rightarrow (\mathrm{E}_{\geq}). \tag{6.24}$$

Let δ be the same constant as in condition (wEH). Without loss of generality, assume that $\delta < \frac{2}{3}$. Let $B := B(x_0, R)$ be a ball in M with $R < \sigma\overline{R}$. Let u be the unique weak solution such that

$$\mathcal{E}(u, \varphi) = (1_{\delta B}, \varphi) \ \text{ for any } 0 \leq \varphi \in \mathcal{F}(B). \tag{6.25}$$

It is known that $u \in \mathcal{F}(B)$, $u \geq 0$ in M, and u is superharmonic in B, see for example [22, Lemma 5.1]. Applying (1.16) in condition (wEH) on the function u and the ball B, and with $f = 0$, $r = \delta R$, and noting that $u_- = 0$ in M, we obtain

$$\left(\fint_{\delta B} u^p d\mu \right)^{1/p} \leq C_H \operatorname*{einf}_{\delta B} u. \tag{6.26}$$

On the other hand, we have by condition $(\mathrm{Cap}_{\leq})$

$$\mathcal{E}(\phi, \phi) \leq C \frac{\mu(B)}{w(B)} \tag{6.27}$$

for some $\phi \in \text{cutoff}\,((2/3)B, B)$.

Taking $\varphi = \phi$ in (6.25) and using condition (VD), we see that

$$\mathcal{E}(u, \phi) = (1_{\delta B}, \phi) = \int_{\delta B} \phi d\mu = \mu(\delta B) \geq C_\mu^{-1} \delta^{d_2} \mu(B). \tag{6.28}$$

Taking $\varphi = u$ in (6.25) and using the Cauchy-Schwarz inequality and (6.27), it follows that

$$\mathcal{E}(u, \phi) \leq \sqrt{\mathcal{E}(u,u)\mathcal{E}(\phi,\phi)} = \sqrt{(1_{\delta B}, u)}\sqrt{\mathcal{E}(\phi,\phi)} \leq C \sqrt{\int_{\delta B} u d\mu} \sqrt{\frac{\mu(B)}{w(B)}}. \tag{6.29}$$

Therefore, combining (6.28) and (6.29), we obtain

$$\int_{\delta B} u d\mu \geq C \mu(B) w(B). \tag{6.30}$$

Since by (6.23)

$$\|u\|_{L^\infty} \le \|E^B\|_{L^\infty} \le Cw(B),$$

we conclude by (6.26) that

$$\int_{\delta B} u d\mu = \int_{\delta B} u^p \cdot u^{1-p} d\mu \le (Cw(B))^{1-p} \int_{\delta B} u^p d\mu = (Cw(B))^{1-p} \mu(\delta B) \fint_{\delta B} u^p d\mu$$
$$\le C' w(B)^{1-p} \mu(B) (\operatorname{einf}_{\delta B} u)^p \le C'' w(B)^{1-p} \mu(B) (\operatorname{einf}_{\delta B} E^B)^p,$$

thus showing (6.20) by (6.30). The proof is complete. □

Finally, we show that the weak elliptic Harnack inequality also implies a Lemma of growth, termed *condition* (LG$_0$), for any *global non-negative superharmonic* function.

We say that *condition* (LG$_0$) holds if there exist four constants $\sigma, \epsilon_0, \eta, \delta \in (0, 1)$ such that, for any ball $B := B(x_0, R)$ with radius $R \in (0, \sigma \overline{R})$ and for any $u \in \mathcal{F}' \cap L^\infty$ that is superharmonic in B and non-negative globally in M, if

$$\frac{\mu(\delta B \cap \{u < a\})}{\mu(\delta B)} \le \epsilon_0 \tag{6.31}$$

for some $a > 0$, then

$$\operatorname{einf}_{\delta B} u \ge \eta a. \tag{6.32}$$

We remark that the superharmonic function u in condition (LG$_0$) is required to be non-negative *globally* in M, instead of being non-negative *locally* in condition (LG) given in Definition 3.2.

Lemma 6.4. *Let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirichlet form in* L^2. *Then*

$$\text{(wEH)} \Rightarrow (\text{LG}_0). \tag{6.33}$$

Proof. Let $u \in \mathcal{F}' \cap L^\infty$ be superharmonic in B and non-negative globally in M. Assume that (6.31) holds, namely,

$$\frac{\mu(\delta B \cap \{u \ge a\})}{\mu(\delta B)} = 1 - \frac{\mu(\delta B \cap \{u < a\})}{\mu(\delta B)} \ge 1 - \epsilon_0.$$

Since $u_- = 0$ in M, we see that

$$T_{\frac{3}{4}B_R, B_R}(u_-) \equiv 0.$$

Applying (1.16) with $r = \delta R$ and $f \equiv 0$, it follows that

$$\operatorname{einf}_{\delta B} u \ge C_H^{-1} \left(\fint_{\delta B} u^p d\mu \right)^{1/p} \ge C_H^{-1} a \left(\frac{\mu(\delta B \cap \{u \ge a\})}{\mu(\delta B)} \right)^{1/p} \ge C_H^{-1} (1 - \epsilon_0)^{1/p} a,$$

thus showing that (6.32) is true with $\eta = C_H^{-1}(1 - \epsilon_0)^{1/p}$. The proof is complete. □

Lemma 6.3 above gives a direct, simpler proof of obtaining a lower bound of the mean exit time from the weak elliptic Harnack inequality. We remark that this conclusion can also be obtained in a more indirect way, without recourse to condition (FK). Indeed, the implication

$$\text{(VD)} + (\text{Cap}_\le) + (\text{LG}_0) \Rightarrow (\text{E}_\ge)$$

has been proved in a forthcoming paper [21] for any regular Dirichlet form in L^2. Combining this with (6.33), we have

$$\text{(VD)} + (\text{Cap}_\le) + \text{(wEH)} \Rightarrow \text{(VD)} + (\text{Cap}_\le) + (\text{LG}_0) \Rightarrow (\text{E}_\ge),$$

from which, we also obtain condition (E$_\ge$) from the weak elliptic Harnack inequality but without using (FK). We do need condition (FK) in Lemma 6.3, not only in deriving condition (E$_\le$) but also in deriving condition (E$_\ge$).

7. An example

In this section we give an example to illustrate Theorem 1.8. We show that the assumptions (VD), (RVD), (Gcap), (TJ), (PI) are all satisfied so that the weak Harnack inequality holds, but the jump kernel does not exist. This example is essentially taken from [6, Section 15], see also [21].

Example 7.1 (Ultra-metric space)**.** *Let $\beta, \alpha_1, \alpha_2$ be three positive numbers. Let (M_i, d_i, μ_i) for $i = 1, 2$ be two ultrametric spaces, where d_i is an ultra-metric:*

$$d_i(x, y) \le \max\{d_i(x, z), d_i(z, y)\} \text{ for all } x, y, z \in M_i,$$

and the measure μ_i is Ahlfors-regular:

$$C^{-1} r^{\alpha_i} \le \mu_i (B(x_i, r)) \le C r^{\alpha_i} \text{ for all } x_i \in M_i \text{ and all } r > 0 \tag{7.1}$$

for some constant $C \ge 1$. Let J_i be a function on $M_i \times M_i$ for $i = 1, 2$ such that for μ_i-almost all $x_i, y_i \in M_i$,

$$J_i (x_i, y_i) = d_i (x_i, y_i)^{-(\alpha_i+\beta)}. \tag{7.2}$$

Consider the product space $M := M_1 \times M_2$ equipped with product measure $\mu := \mu_1 \times \mu_2$ and the metric

$$d(x, y) := \max \{d_1 (x_1, y_1), d_2 (x_2, y_2)\} \text{ for } x = (x_1, x_2), y = (y_1, y_2) \text{ in } M.$$

Clearly, (M, d, μ) is an ultrametric space and for any point $x = (x_1, x_2)$ in M, the metric ball $B(x, r)$ in M can be written as

$$B(x, r) = B(x_1, r) \times B(x_2, r) \text{ for any } r > 0. \tag{7.3}$$

From this, we see that for any point $x = (x_1, x_2)$ in M and any $r > 0$,

$$V(x, r) = \mu(B(x, r)) = \mu_1 (B (x_1, r)) \mu_2 (B (x_2, r)) \asymp r^{\alpha_1+\alpha_2} = r^{\alpha}, \tag{7.4}$$

where $\alpha := \alpha_1 + \alpha_2$. For simplicity, let the scaling function $w(x, r)$ be defined by

$$w(x, r) = a(x) r^{\beta} \text{ for any point } x \in M \text{ and any } r > 0,$$

where $a(x)$ is a measurable function on M with $C^{-1} \le a(x) \le C$ for all $x \in M$ ($C \ge 1$). Clearly, such a function w satisfies (1.5) and

$$C^{-1} r^{\beta} \le w(x, r) \le C r^{\beta}. \tag{7.5}$$

Define the measure J on $\mathcal{B}(M \times M)$ by $J(dx, dy) = J(x, dy)\mu(dx)$, where $J(x, dy)$ is a transition function on $M \times \mathcal{B}(M)$ given by

$$J(x, dy) = J_1(x_1, y_1)\mu_1(dy_1)\delta_{x_2}(dy_2) + J_2(x_2, y_2)\mu_2(dy_2)\delta_{x_1}(dy_1) \tag{7.6}$$

for any points $x = (x_1, x_2), y = (y_1, y_2)$ in M, where $\delta_b(dx)$ is the Dirac measure concentrated at point b. By (7.4) and (7.6), we have for any $r > 0$ and any point $x = (x_1, x_2) \in M$,

$$\begin{aligned} \int_{B(x,r)^c} J(x, dy) &= \int_{B(x,r)^c} (J_1(x_1, y_1)\mu_1(dy_1)\delta_{x_2}(dy_2) + J_2(x_2, y_2)\mu_2(dy_2)\delta_{x_1}(dy_1)) \\ &= \int_{B(x_1,r)^c} J_1(x_1, y_1)\mu_1(dy_1) + \int_{B(x_2,r)^c} J_2(x_2, y_2)\mu_2(dy_2) \\ &\le \frac{C}{r^{\beta}} + \frac{C}{r^{\beta}} = \frac{2C}{r^{\beta}} \le \frac{C'}{w(x, r)} \quad \text{(using (7.1), (7.2) and (7.5))}, \end{aligned} \tag{7.7}$$

which is exactly condition (TJ).

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in $L^2 (M, \mu)$ defined by

$$\mathcal{E}(u, v) = \iint_{M\times M} (u (x) - u (y)) (v (x) - v (y)) J (x, dy) \mu(dx), \quad u, v \in \mathcal{F},$$

where the space $\mathcal{F}$ is the closure of the set

$$\left\{\sum_{i=0}^{n} c_i \mathbf{1}_{B_i} : n \in \mathbb{N}, c_i \in \mathbb{R}, B_i \text{ is a compact ball}\right\}$$

under the inner product

$$\sqrt{\mathcal{E}(\cdot,\cdot) + (\cdot,\cdot)_{L^2(M,\mu)}}.$$

By [6, Theorem 2.2], *the form* $(\mathcal{E},\mathcal{F})$ *is regular and non-local. By (7.4), the measure* μ *satisfies conditions* (VD) *and* (RVD), *whilst condition* (Gcap) *automatically holds since it follows directly from condition* (TJ) *and the ultrametric property. Hence, conditions* (VD), (RVD), (Gcap), (TJ) *in Theorem 1.8 are satisfied.*

It remains to verify condition (PI). *Indeed, let* $B := B(x', r)$ *be a metric ball in* M. *Writing up* $x' = (x_0, y_0)$ *with* $x_0 \in M_1, y_0 \in M_2$, *we see* $B = B(x_0, r) \times B(y_0, r)$ *by using (7.3). By (7.6), (7.2)*

$$\begin{aligned}\int_B \int_B (u(x)-u(y))^2 J(x,dy)\mu(dx) &= \int_B \left\{\int_{B(x_0,r)} \frac{(u(x_1,x_2)-u(y_1,x_2))^2}{d_1(x_1,y_1)^{\alpha_1+\beta}}\mu_1(dy_1)\right.\\ &\quad \left. + \int_{B(y_0,r)} \frac{(u(x_1,x_2)-u(x_1,y_2))^2}{d_2(x_2,y_2)^{\alpha_2+\beta}}\mu_2(dy_2)\right\}\mu(dx).\end{aligned}$$

The first integral on the right-hand side is estimated as follows: for any $(x_1, x_2) \in B(x_0, r)\times B(y_0, r)$,

$$\begin{aligned}&\int_{B(x_0,r)} \frac{(u(x_1,x_2)-u(y_1,x_2))^2}{d_1(x_1,y_1)^{\alpha_1+\beta}}\mu_1(dy_1)\\ \geq\ & \int_{B(x_0,r)} \frac{(u(x_1,x_2)-u(y_1,x_2))^2}{r^{\alpha_1+\beta}}\mu_1(dy_1)\\ \geq\ & C^{-1}\int_{B(y_0,r)}\int_{B(x_0,r)} \frac{(u(x_1,x_2)-u(y_1,x_2))^2}{r^{\alpha_1+\alpha_2+\beta}}\mu_1(dy_1)\mu_2(dy_2) \ \ (\text{using } (7.1))\\ =\ & C^{-1}\int_B \frac{(u(x_1,x_2)-u(y_1,x_2))^2}{r^{\alpha+\beta}}\mu(dy)\end{aligned}$$

by using the fact that $\alpha_1 + \alpha_2 = \alpha$, *from which, we have*

$$\int_B\int_{B(x_0,r)} \frac{(u(x_1,x_2)-u(y_1,x_2))^2}{d_1(x_1,y_1)^{\alpha_1+\beta}}\mu_1(dy_1)\mu(dx) \geq C^{-1}\int_B\int_B \frac{(u(x_1,x_2)-u(y_1,x_2))^2}{r^{\alpha+\beta}}\mu(dy)\mu(dx).$$

Similarly, the second integral is estimated by

$$\begin{aligned}&\int_B\int_{B(y_0,r)} \frac{(u(x_1,x_2)-u(x_1,y_2))^2}{d_2(x_2,y_2)^{\alpha_2+\beta}}\mu_2(dy_2)\mu(dx)\\ \geq\ & C^{-1}\int_B\int_B \frac{(u(x_1,x_2)-u(x_1,y_2))^2}{r^{\alpha+\beta}}\mu(dy)\mu(dx)\\ =\ & C^{-1}\int_B\int_B \frac{(u(y_1,y_2)-u(y_1,x_2))^2}{r^{\alpha+\beta}}\mu(dx)\mu(dy) \ \ (\text{swapping } (x_1,x_2) \text{ with } (y_1,y_2)).\end{aligned}$$

Therefore, we conclude from above that, using the elementary inequality $a^2 + b^2 \geq (a + b)^2/2$,

$$\begin{aligned}\int_B\int_B (u(x)-u(y))^2 J(x,dy)\mu(dx) &\geq C^{-1}\left\{\int_B\int_B \frac{(u(x_1,x_2)-u(y_1,x_2))^2}{r^{\alpha+\beta}}\mu(dy)\mu(dx)\right.\\ &\quad \left. + \int_B\int_B \frac{(u(y_1,y_2)-u(y_1,x_2))^2}{r^{\alpha+\beta}}\mu(dx)\mu(dy)\right\}\\ &\geq C^{-1}\int_B\int_B \frac{(u(x_1,x_2)-u(y_1,y_2))^2}{2r^{\alpha+\beta}}\mu(dx)\mu(dy)\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{C'r^{-\beta}}{\mu(B)}\int_B\int_B (u(x)-u(y))^2\mu(dx)\mu(dy) \ \textit{(using (7.4))}\\
&= 2C'r^{-\beta}\int_B (u-u_B)^2 d\mu\\
&\geq \frac{C}{w(B)}\int_B (u-u_B)^2 d\mu \ \textit{(using (7.5))},
\end{aligned}
$$

thus showing that condition (PI) *with* $\kappa = 1$ *is satisfied.*

Therefore, all the hypotheses in Theorem 1.8 are satisfied, and the weak elliptic Harnack inequality follows. We mention that the jump kernel does not exist by (7.6) in this case.

8. Appendix

In this appendix, we collect some known results that have been cited in this paper. Recall the John-Nirenberg inequality for BMO functions on a doubling space.

Definition 8.1 (BMO function)**.** *For a locally integrable function* u *on an open set* Ω*, the seminorm* $\|u\|_{\mathrm{BMO}(\Omega)}$ *is defined by*

$$
\|u\|_{\mathrm{BMO}(\Omega)} := \sup_{B\subset\Omega} \fint_B |u-u_B| d\mu,
$$

where the supremum is taken over all the balls contained in Ω*. The space* $\mathrm{BMO}(\Omega)$ *consists of all locally integrable functions* u *on* Ω *such that* $\|u\|_{\mathrm{BMO}(\Omega)} < \infty$.

The following was addressed in [1, Theorem 5.2].

Lemma 8.2 (John-Nirenberg inequality)**.** *Let* (M, d, μ) *be a metric measure space satisfying condition* (VD)*. If* $u \in \mathrm{BMO}(\Omega)$ *for a non-empty open subset* Ω *of* M*, then*

$$
\mu(\{x\in B : |u-u_B| > \lambda\}) \geq c_1\mu(B)\exp\left(-\frac{c_2\lambda}{\|u\|_{\mathrm{BMO}(\Omega)}}\right)
$$

for any ball with $12B \subseteq \Omega$ *and any* $\lambda > 0$*, where constants* c_1, c_2 *are independent of* u, λ, Ω *and ball* B*.*

The following is a folklore, see for example [7, Corollary 5.6].

Lemma 8.3. *Let* (M, d, μ) *be a metric measure space satisfying condition* (VD)*. Let* $B_0 := B(x_0, R)$ *be a ball in* M*. Then for any* $u \in \mathrm{BMO}(B_0)$

$$
\left\{\fint_B \exp\left(\frac{c_2}{2b}u\right)d\mu\right\}\left\{\fint_B \exp\left(-\frac{c_2}{2b}u\right)d\mu\right\} \leq (1+c_1)^2 \tag{8.1}
$$

for any ball B *with* $12B \subseteq B_0$ *and any* $b \geq \|u\|_{\mathrm{BMO}(B_0)}$*, where the constants* c_1, c_2 *are the same as in Lemma 8.2.*

The following has been proved in a forthcoming paper [21].

Proposition 8.4. *Let* $(\mathcal{E}, \mathcal{F})$ *be a regular Dirichlet form in* L^2 *without killing part. Assume that a function* $F \in C^2(\mathbb{R})$ *satisfies*

$$
F'' \geq 0, \ \sup_{\mathbb{R}} |F'| < \infty, \ \sup_{\mathbb{R}} F'' < \infty.
$$

Then for any $u, \varphi \in \mathcal{F}' \cap L^\infty$*, both functions* $F(u), F'(u)\varphi$ *belong to the space* $\mathcal{F}' \cap L^\infty$*. Moreover, if further* $\varphi \geq 0$ *in* M*, then*

$$
\mathcal{E}(F(u), \varphi) \leq \mathcal{E}(u, F'(u)\varphi). \tag{8.2}
$$

The following is taken from in [32, Lemma 2.12].

Lemma 8.5. *Let $(\mathcal{E},\mathcal{F})$ be a Dirichlet form in L^2. If each $f_n \in \mathcal{F}$ and*

$$f_n \xrightarrow{L^2} f, \quad \sup_n \mathcal{E}(f_n) < \infty,$$

then $f \in \mathcal{F}$, and there exists a subsequence, still denoted by $\{f_n\}$, such that $f_n \overset{\mathcal{E}}{\rightharpoonup} f$ weakly, that is,

$$\mathcal{E}(f_n,\varphi) \to \mathcal{E}(f,\varphi)$$

as $n \to \infty$ for any $\varphi \in \mathcal{F}$. Moreover, we have

$$\mathcal{E}(f) \leq \liminf_{n\to\infty} \mathcal{E}(f_n).$$

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