PARABOLIC MEAN VALUE INEQUALITY AND ON-DIAGONAL UPPER BOUND OF THE HEAT KERNEL ON DOUBLING SPACES

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Abstract. We prove the diagonal upper bound of heat kernels for regular Dirichlet forms on metric measure spaces with volume doubling condition. As hypotheses, we use the Faber-Krahn inequality, the generalized capacity condition and an upper bound for the integrated tail of the jump kernel. The proof goes though a parabolic mean value inequality for subcaloric functions.

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1. Introduction

In this paper we are concerned with the existence and on-diagonal upper bounds of heat kernels in a rather general setting of metric measure spaces. The classical and best known heat kernel is the Gauss-Weierstrass function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right), \tag{1.1}$$

where $x, y \in \mathbb{R}^n$ and $t > 0$, which is the fundamental solution of the heat equation $\partial_t u = \Delta u$ in $\mathbb{R}^n$.

The heat equation can be considered on an arbitrary (connected) Riemannian manifold M where Δ denotes now the Laplace-Beltrami operator. Then the heat kernel $p_t(x,y)$ is defined as the minimal positive fundamental solution of the heat equation; it always exists and is a positive smooth function of $t \in \mathbb{R}_+$ and $x, y \in M$. There is a large literature devoted to heat kernel estimates on manifolds (see, for example, [16] and references therein).

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Here we deal only with on-diagonal upper estimates. In a simplest case such an estimate has the form

$$p_t(x,x) \leq Ct^{-\alpha/2}, \tag{1.2}$$

for all $t>0$, $x\in M$ and some constants $C,\alpha>0$. For example, the heat kernel (1.1) satisfies (1.2) with $\alpha=n$. A natural and important question is what geometric conditions ensure certain heat kernel bounds, for example, the estimate (1.2). One of the first results in this directions was proved by Varopoulos [29] and states the following: the estimate (1.2) with some $\alpha>2$ is equivalent to the following *Sobolev inequality*:

$$\int_M |\nabla f|^2 d\mu \geq c\left(\int_M |f|^{\frac{2\alpha}{\alpha-2}} d\mu\right)^{\frac{\alpha-2}{\alpha}} \quad \text{for all } f\in C_0^\infty(M), \tag{1.3}$$

where c is a positive constant, μ is the Riemannian measure, and ∇ is the Riemannian gradient. Carlen, Kusuoka and Stroock [8] gave an alternative equivalent condition: the estimate (1.2) with some $\alpha>0$ is equivalent to the following the *Nash inequality*:

$$\left(\int_M |\nabla f|^2 d\mu\right)\left(\int_M |f|\, d\mu\right)^{4/\alpha} \geq c\left(\int_M f^2 d\mu\right)^{1+2/\alpha} \quad \text{for all } f\in C_0^\infty(M). \tag{1.4}$$

It was proved in [15] by one of the authors that (1.2) is equivalent to the following *Faber-Krahn inequality*:

$$\lambda_1(U) \geq c\mu(U)^{-2/\alpha}, \tag{1.5}$$

for all precompact open sets $U\subset M$, where $\lambda_1(U)$ denotes the bottom eigenvalue of the Laplace operator in U with the Dirichlet boundary condition. Needless to say that all the conditions (1.3), (1.4) and (1.5) are satisfied in $\mathbb{R}^n$ with $\nu=n$ and with constants c depending on n.

However, there is a geometrically important class of manifolds, namely, complete manifolds of non-negative Ricci curvature, where none of the above conditions holds in general. In this case, in order to describe the on-diagonal behaviour of the heat kernel, one has to use the Riemannian distance function $d(x,y)$. Denote by $B(x,r)$ a geodesic ball of radius r centered at $x\in M$, that is,

$$B(x,r) = \{y\in M : d(x,y) < r\},$$

and set

$$V(x,r) = \mu(B(x,r)).$$

Li and Yau [28] proved that, on a complete manifold of non-negative Ricci curvature, the heat kernel satisfies the following on-diagonal estimate, for all $t>0$ and $x\in M$:

$$p_t(x,x) \simeq \frac{1}{V\left(x,\sqrt{t}\right)}, \tag{1.6}$$

where the sign $\simeq$ means that the ratio of the both sides is bounded above and below by positive constants, for the specified range of the variables. It was then proved in [15] that the upper bound in (1.6), that is, the estimate

$$p_t(x,y) \leq \frac{C}{V\left(x,\sqrt{t}\right)}, \tag{1.7}$$

in conjunction with the volume doubling condition

$$V(x,2r) \leq CV(x,r), \quad \text{for all } r>0 \text{ and } x\in M, \tag{VD}$$

is equivalent to the following *relative* form of the Faber-Krahn inequality: for any ball $B(x,r)$ and any open set $U\subset B(x,r)$,

$$\lambda_1(U) \geq \frac{c}{r^2}\left(\frac{V(x,r)}{\mu(U)}\right)^{\nu}, \tag{1.8}$$

for some constants $c, \nu > 0$. It is easy to see that in $\mathbb{R}^n$ (1.8) holds with $\nu = 2/n$, which is equivalent to (1.5) with $\alpha = n$. It was proved in [14] that (1.8) holds on any complete manifolds with non-negative Ricci curvature, thus reproving (1.6) (surprisingly enough, the lower bound in (1.6) follows from the upper bound (1.7) and (VD) – see [11]).

Since the discovery of strongly local Dirichlet forms on fractals spaces in 1980-90s (see, for example, [3, 5, 13, 24, 26, 27]), many efforts have been made for estimating heat kernel on metric measure spaces (see, for example, [2, 4, 23]).

Let (M, d, μ) be a metric measure space (as all fractals are). Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2 = L^2(M, \mu)$. It induces the heat semigroup $P_t = e^{-t\mathcal{L}}$, $t > 0$, where $\mathcal{L}$ is the (positive definite) generator of $(\mathcal{E}, \mathcal{F})$. If the operator P_t is an integral operator then its integral kernel is called *the heat kernel* of $(\mathcal{E}, \mathcal{F})$ and is denoted by $p_t(x, y)$. It was proved in the aforementioned papers that the heat kernel on many families of fractals exists and satisfies the following on-diagonal estimate:

$$p_t(x,x) \simeq \frac{1}{t^{\alpha/\beta}} \text{ for all } t > 0 \text{ and } x \in M, \tag{1.9}$$

where α is volume growth exponent, that is,

$$V(x,r) \simeq r^{\alpha} \text{ for all } r > 0 \text{ and } x \in M,$$

while β is a new parameter that is called the *walk dimension* and that satisfies $\beta \geq 2$. There is a very interesting issue of the off-diagonal bounds of the heat kernel but we do not touch such bounds here and refer the reader to the previously mentioned sources.

It is easy to extend the results of [8], [15] and [29] also to the present setting and to show that the upper bound of $p_t(x, x)$ in (1.9) is equivalent to appropriate versions of the Sobolev, Nash and Faber-Krahn inequalities, where $\int_M |\nabla f|^2 \, d\mu$ should be replaced by $\mathcal{E}(f, f)$. In particular, the Faber-Krahn inequality looks as follows:

$$\lambda_1(U) \geq c\mu(U)^{-\beta/\alpha},$$

for any precompact open set $U \subset M$, where $\lambda_1(U)$ is now the bottom of the spectrum of the generator $\mathcal{L}^U$ of the part Dirichlet form on U.

It would be natural to expect that a relative form of the Faber-Krahn inequality on metric measure spaces implies a certain analogue of the Li-Yau heat kernel estimate. However, there are great difficulties on this route. The major difference between the cases of manifolds and metric measure spaces is that the Riemannian distance function d is closely linked to the Dirichlet form $\mathcal{E}(f, f) = \int_M |\nabla f|^2 \, d\mu$ via the inequality $|\nabla d| \leq 1$, whereas in general there may be no relation between the metric d and the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Technically, the condition $|\nabla d| \leq 1$ is used in the proofs via a construction of a bump function φ of two concentric balls $B(x, R)$ and $B(x, R + r)$ that satisfies $|\nabla \varphi| \leq \frac{1}{r}$ and, hence,

$$\int_M u^2 |\nabla \varphi|^2 \, dx \leq \frac{1}{r^2} \int_M u^2 d\mu,$$

for any measurable function u. In the general setting one has to assume the existence of a cutoff function φ of such a pair of balls satisfying a similar condition, that we refer to as a *generalized capacity condition* and denote shortly by (Gcap) (see Definition 2.3 for the details).

For a regular strongly local Dirichlet form $(\mathcal{E}, \mathcal{F})$, the following result was proved by Andres and Barlow [1]. Assume that the metric measure space satisfies the volume doubling condition (VD), the following version of the relative Faber-Krahn inequality:

$$\lambda_1(U) \geq \frac{c}{r^{\beta}} \left(\frac{V(x,r)}{\mu(U)} \right)^{\nu} \tag{FK}$$

for some $\beta \geq 2$ and $\nu, c > 0$, as well as a matching condition (Gcap) (see also Definitions 2.3 and 2.4). Then the heat kernel of $(\mathcal{E}, \mathcal{F})$ exists and satisfies the upper estimate

$$p_t(x,x) \leq \frac{C}{V(x, t^{1/\beta})}. \tag{DUE}$$

The function r^β that appears in (FK) is called a *scaling function* because, as one can see from (DUE), the parameter β determines a space/time scaling for the heat kernel. A similar result with a more general scaling function $\Psi(r)$ was proved in [22].

Let now $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form of jump type that is given by a jump kernel $J(x, y)$. Assume again that (VD), (FK), (Gcap) are satisfied. In addition, assume that the jump kernel admits for all $x, y \in M$ the upper bound

$$J(x,y) \leq \frac{C}{V(x,r) r^\beta}, \tag{$J_\leq$}$$

where $r = d(x, y)$. Under these hypotheses, Chen, Kumagai and Wang proved in [10] that the heat kernel exists and satisfies (DUE) (this result is a combination of Proposition 4.13 and Theorem 4.25 in [10]).

In the present paper we deal with arbitrary (not necessarily local) regular Dirichlet forms without killing. We impose again the hypotheses (VD), (FK), (Gcap), while the jump kernel $J(x, y)$ has to satisfy instead of ($J_\leq$) some much weaker upper bound in the integral sense: for some $q \in [2, \infty]$ and for all balls $B(x, r)$,

$$\|J(x,\cdot)\|_{L^q(B(x,r)^c)} \leq \frac{C}{V(x,r)^{1/q'} r^\beta}, \tag{TJ_q}$$

where q' is the Hölder conjugate to q. We prove here that the heat kernel estimate (DUE) remains true under these weaker assumptions.

Hence, the major improvement that we achieve in this paper is replacement of the hypothesis ($J_\leq$) (that is equivalent to (TJ_q) for $q = \infty$) by the hypothesis (TJ_q) for any $q \geq 2$.

Besides, there are two other novelties in this paper as follows:

- (i) we work with an arbitrary scaling function $W(x, r)$ (in place of r^β) that may depend on $x \in M$ (see Definition 2.1);
- (ii) we allow the conditions (FK) and (Gcap) to be localized in the following sense: they should hold only for balls of restricted radii $< \overline{R}$ for some fixed $\overline{R}$.

Let us say a few words about the proof. The relative Faber-Krahn inequality (FK) allows to obtain rather straightforwardly a certain upper bound for Dirichlet heat kernels $p_t^{B(x,r)}$ in balls. In order to pass to the global heat kernel p_t, Kigami has devised in [25] an elaborated iteration argument based on a so called *survival estimate*. For the strongly local Dirichlet forms, this method was used in [20] by means of the following inequality: for all $R > r > 0$ and $x_0 \in M$

$$\operatorname*{esup}_{\frac{1}{4}B(x_0,r)} p_t^{B(x_0,R)} \leq \operatorname*{esup}_{B(x_0,r)} p_{t-s}^{B(x_0,r)} + \varepsilon \operatorname*{esup}_{B(x_0,r)} p_s^{B(x_0,R)}, \quad t > s > 0,$$

where the constant $\varepsilon \in (0, 1)$ comes from the survival estimate. Combining this inequality with upper bounds for the Dirichlet heat kernels and applying it recursively for a sequence of concentric balls with $R \to \infty$, one obtains a desired upper bound for p_t on the whole space M. Chen, Kumagai and Wang successfully applied in [10] the method of Kigami in the case of non-local Dirichlet forms (see also [7, 9] for more results on this topic).

However, the above iteration method requires the conditions (FK) and (Gcap) to be satisfied for *all* balls and does not work in our setting under the condition (ii). Hence, we use for obtaining (DUE) a different method based in a *parabolic mean value inequality* denoted by (PMV_q) where $q \in [1, \infty]$ (see Definition 2.9).

Our first main result – Theorem 2.10, says that, for any $q \geq 1$,

$$(\text{VD}) + (\text{FK}) + (\text{Gcap}) + (\text{TJ}_q) \Rightarrow (\text{PMV}_q)$$

Our second main result – Theorem 2.12, says that

$$(\text{VD}) + (\text{PMV}_2) \Rightarrow (\text{DUE})$$

and, hence, for any $q \geq 2$,

$$(\text{VD}) + (\text{FK}) + (\text{Gcap}) + (\text{TJ}_q) \Rightarrow (\text{DUE}),$$

as it was already mentioned above (cf. Corollary 2.13).

We do not touch here another very interesting question: off-diagonal upper estimates of the heat kernel. In the case of strongly local Dirichlet form, this question was solved in [1] and [22], and for jump type Dirichlet form satisfying ($\text{J}_\leq$) – in [10]. The jump type Dirichlet forms satisfying (TJ_q) will be dealt with in a companion paper.

The structure of the present paper is as follows. In Section 2 we give a detailed description of our results. In Section 3, we study some properties of the condition (TJ_q). In Section 4, we prove some auxiliary results. Theorem 2.10 (the parabolic mean value inequality) is proved in Section 5. Theorem 2.12 (on-diagonal upper estimate of the heat kernel) is proved in Section 6. Appendix contains some external results that are used in this paper.

Notation. Letters c, C, C', C_1, C_2, etc. are used to denote positive constants, whose values may change at any occurrence. For two open sets $U, V \subset M$ and a measurable function F on $M \times M$, in the double integral $\iint_{U \times V} F(x,y)dj(x,y)$, the variable x is taken in U and y in V. For a measurable function u on M, the notation $\text{supp}(u)$ means the support of u, that is, the minimal closed subset of M such that $u = 0$ a.e. outside it.

2. Main results

Let (M, d) be a locally compact separable metric space and let μ be a Radon measure on M with full support. The triple (M, d, μ) is called a *metric measure space*.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2 := L^2(M, \mu)$ and $\{P_t\}_{t>0}$ be the heat semigroup in L^2 associated with $(\mathcal{E}, \mathcal{F})$. The integral kernel $p_t(x, y)$ of $\{P_t\}$ (should it exist) is called the *heat kernel* of $(\mathcal{E}, \mathcal{F})$. The heat kernel coincides with the transition density of the Hunt process associated with $(\mathcal{E}, \mathcal{F})$.

Recall that any regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ in L^2 admits the following unique *Beurling-Deny decomposition* (cf. [12, Theorem 3.2.1 and Theorem 4.5.2]):

$$\mathcal{E}(u, v) = \mathcal{E}^{(L)}(u, v) + \mathcal{E}^{(J)}(u, v) + \mathcal{E}^{(K)}(u, v), \tag{2.1}$$

where $\mathcal{E}^{(L)}$ is the *local part* (or *diffusion part*), $\mathcal{E}^{(J)}$ is the jump part associated with a unique Radon measure j defined on $M \times M \setminus \text{diag}$:

$$\mathcal{E}^{(J)}(u, v) = \iint_{M \times M \setminus \text{diag}} (u(x) - u(y))(v(x) - v(y))dj(x, y), \tag{2.2}$$

and finally, $\mathcal{E}^{(K)}$ is the *killing part*. For simplicity, we set $j = 0$ on diag and will drop diag in expression $M \times M \setminus \text{diag}$ in (2.2) when no confusion arises. In this paper, we always assume that $\mathcal{E}^{(K)} \equiv 0$. Thus,

$$\mathcal{E}(u, v) = \mathcal{E}^{(L)}(u, v) + \mathcal{E}^{(J)}(u, v). \tag{2.3}$$

Let us give detailed definitions and statements of our main results that were mentioned in Introduction.

Define in M as above metric balls

$$B(x, r) := \{y \in M : d(y, x) < r\}$$

and their volumes $V(x,r) := \mu(B(x,r))$. For any ball $B = B(x,r)$ and a positive number λ, denote by

$$\lambda B := B(x, \lambda r).$$

We say that the measure μ satisfies the *volume doubling* condition (shortly denoted by (VD)), if there exists a constant $C \geq 1$ such that, for all $x \in M$ and all $r > 0$,

$$V(x, 2r) \leq CV(x,r). \tag{2.4}$$

Condition (VD) implies that $0 < V(x,r) < \infty$ for all $r > 0$. We set $V(x,0) = 0$ for all $x \in M$.

It is well known that condition (VD) is equivalent to the following: there exists a positive number α such that, for all $x, y \in M$ and all $0 < r \leq R < \infty$,

$$\frac{V(x,R)}{V(y,r)} \leq C\left(\frac{d(x,y)+R}{r}\right)^{\alpha}, \tag{2.5}$$

where constant C can be taken the same as in (VD). In particular, for all $x \in M$ and all $0 < r \leq R < \infty$,

$$\frac{V(x,R)}{V(x,r)} \leq C\left(\frac{R}{r}\right)^{\alpha}.$$

Definition 2.1 (Scaling function)**.** A function $W : M \times [0,\infty] \to [0,\infty]$ is called a *scaling function* if it satisfies the following conditions:

- for each $x \in M$, the function $W(x,\cdot)$ is strictly increasing, and $W(x,0) = 0$, $W(x,\infty) = \infty$;
- there exist three positive numbers C, β_1, β_2 ($\beta_1 \leq \beta_2$) such that, for all $0 < r \leq R < \infty$ and for all $x, y \in M$ with $d(x,y) \leq R$,

$$C^{-1}\left(\frac{R}{r}\right)^{\beta_1} \leq \frac{W(x,R)}{W(y,r)} \leq C\left(\frac{R}{r}\right)^{\beta_2}. \tag{2.6}$$

For any $x \in M$, $W^{-1}(x,\cdot)$ denotes the inverse function of $W(x,\cdot)$.

The function W will determine the space/time scaling of the Hunt process of the Dirichlet form $(\mathcal{E},\mathcal{F})$. A typical example of a scaling function is

$$W(x,r) = r^{\beta},$$

for some constant $\beta > 0$. In this case, β is called the *walk dimension* of the Dirichlet form. For example, if $M = \mathbb{R}^n$ and $(\mathcal{E},\mathcal{F})$ is the classical Dirichlet integral then $\beta = 2$; if M is the Sierpiński gasket in $\mathbb{R}^2$ and $(\mathcal{E},\mathcal{F})$ is the self-similar strongly local Dirichlet then $\beta = \frac{\log 5}{\log 2} > 2$.

For any metric ball $B := B(x,r)$ let us set

$$W(B) := W(x,r).$$

Note that $W(B)$ is still *not* a function of a ball as a subset of M, but is a function of a pair (x,r) as it may happen that $B(x_1,r_1) = B(x_2,r_2)$ whereas $W(x_1,r_1) \neq W(x_2,r_2)$.

Definition 2.2 (Cutoff function)**.** Let $U \subset M$ be an open set and A be a Borel subset of U. For any $\kappa \geq 1$, a *κ-cutoff function* of the pair (A,U) is any function ϕ in $\mathcal{F}$ such that

- $0 \leq \phi \leq \kappa$ μ-a.e. in M;
- $\phi \geq 1$ μ-a.e. in A;
- $\phi = 0$ μ-a.e. in U^c.

We denote by κ-cutoff(A,U) the collection of all κ-cutoff functions of the pair (A,U). Any 1-cutoff function for $\kappa = 1$ will be simply referred to as a *cutoff function*. Clearly, $\phi \in \mathcal{F}$ is a cutoff function of (A,U) if and only if $0 \leq \phi \leq 1$, $\phi|_A = 1$ and $\phi|_{U^c} = 0$. Denote by

$$\text{cutoff}(A,U) := 1\text{-}\,\text{cutoff}(A,U).$$

Note that for every $\kappa \geq 1$,

$$\mathrm{cutoff}(A,U) \subset \kappa\text{- cutoff}(A,U),$$

and that, if $\phi \in \kappa$-cutoff(A,U), then $1 \wedge \phi \in \mathrm{cutoff}(A,U)$. It is known that if $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in L^2, then cutoff(A,U) is not empty for any nonempty precompact $A \Subset U$. Here $A \Subset U$ means that A is precompact and $\overline{A} \subset U$.

Let $\mathcal{F}'$ be a *linear space* defined by

$$\mathcal{F}' := \{v + a : v \in \mathcal{F},\ a \in \mathbb{R}\},$$

which contains constant functions that are not in L^2 when $\mu(M) = \infty$.

Let diam$M \in (0,\infty]$ be the diameter of the metric space (M,d). We fix throughout this paper a parameter $\overline{R} \in (0, \mathrm{diam}M]$. Note that $\overline{R}$ can be infinite when M is unbounded.

Let us introduce the generalized capacity condition (Gcap).

Definition 2.3 (Generalized capacity condition)**.** We say that condition (Gcap) is satisfied if there exist two numbers $\kappa \geq 1, C > 0$ such that, for all $u \in \mathcal{F}' \cap L^\infty$ and for any pair of concentric balls $B_0 := B(x_0,R)$ and $B := B(x_0,R+r)$ with $x_0 \in M$ and $0 < R < R+r < \overline{R}$, there exists some $\phi \in \kappa$-cutoff(B_0,B) (see Fig. 1) such that

$$\mathcal{E}(u^2\phi,\phi) \leq \sup_{x\in B} \frac{C}{W(x,r)} \int_B u^2 d\mu. \tag{2.7}$$

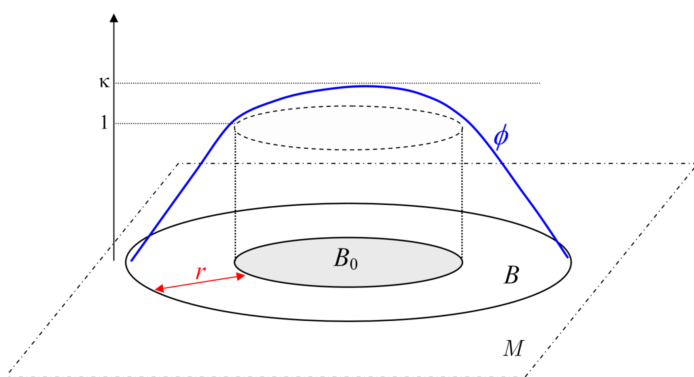

FIGURE 1. A κ-cutoff function ϕ of the pair B_0, B

We remark that the function ϕ in (Gcap) may depend on u, but the constants κ, C are independent of u, B_0, B. If the scaling function $W(x,r)$ is independent of space variable x, say, $W(x,r) = W(r)$, then the inequality (2.7) simplifies as follows:

$$\mathcal{E}(u^2\phi,\phi) \leq \frac{C}{W(r)} \int_B u^2 d\mu.$$

For a non-empty open subset U of M, denote by $C_0(U)$ the space of all continuous functions with compact supports in U. Define a linear space $\mathcal{F}(U)$ by

$$\mathcal{F}(U) = \text{ the closure of } \mathcal{F} \cap C_0(U) \text{ in the norm of } \sqrt{\mathcal{E}_1(\cdot,\cdot)},$$

where $\mathcal{E}_1(u,v) := \mathcal{E}(u,v) + (u,v)$ for all $u,v \in \mathcal{F}$. By the theory of Dirichlet forms, $(\mathcal{E},\mathcal{F}(U))$ is a regular Dirichlet form on $L^2(U,\mu)$ if $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(M,\mu)$ (see, for example, [12, Theorem 4.4.3]). Denote by $\mathcal{L}^U$ the generator of the Dirichlet form $(\mathcal{E},\mathcal{F}(U))$ and by $\lambda_1(U)$ – the *bottom* of the spectrum of $\mathcal{L}^U$ in $L^2(U,\mu)$. It is known that

$$\lambda_1(U) = \inf_{u\in\mathcal{F}(U)\setminus\{0\}} \frac{\mathcal{E}(u,u)}{\|u\|_{L^2}^2}. \tag{2.8}$$

Now we can introduce *Faber-Krahn inequality* (FK).

Definition 2.4 (Faber-Krahn inequality)**.** We say that condition (FK) is satisfied if there exist three numbers $\sigma \in (0,1]$ and $C, \nu > 0$ such that, for all balls B of radii $< \sigma \overline{R}$ and for all non-empty open subsets $U \subset B$,

$$\lambda_1(U) \geq \frac{C^{-1}}{W(B)} \left(\frac{\mu(B)}{\mu(U)} \right)^{\nu}. \tag{2.9}$$

Sometimes we use for (FK) the extended notation (FK_ν) in order to emphasize the role of constant ν.

Let $\mathcal{B}(M)$ be the sigma-algebra of all Borel sets of M. Recall that a *transition kernel* $J : M \times \mathcal{B}(M) \mapsto \mathbb{R}_+$ is a map satisfying the following two properties:

- for every fixed x in M, the map $E \mapsto J(x, E)$ is a measure on $\mathcal{B}(M)$;
- for every fixed E in $\mathcal{B}(M)$, the map $x \mapsto J(x, E)$ is a non-negative measurable function on M.

Let us define a *tail estimate* (TJ) of the jump measure j of $(\mathcal{E}, \mathcal{F})$.

Definition 2.5 (Tail estimate of jump measure)**.** We say that condition (TJ) is satisfied if there exists a transition kernel $J(\cdot, \cdot)$ on $M \times \mathcal{B}(M)$ such that

$$dj(x, y) = J(x, dy) d\mu(x) \ \text{ in } M \times M,$$

and, for all $x \in M$ and $R > 0$,

$$J(x, B(x, R)^c) = \int_{B(x,R)^c} J(x, dy) \leq \frac{C}{W(x, R)}, \tag{2.10}$$

where $C \in [0, \infty)$ is a constant independent of x, R.

For a given number $1 \leq q \leq \infty$, let q' be the *Hölder conjugate* of q, that is,

$$q' := \frac{q}{q-1}$$

so that $q' = 1$ if $q = \infty$, and $q' = \infty$ if $q = 1$.

For any $q \in [1, \infty]$, define a *tail estimate* (TJ_q) of the jump kernel.

Definition 2.6 (L^q-tail estimate of jump kernel)**.** For a given $q \in [1, \infty]$, we say that condition (TJ_q) is satisfied if there exists a non-negative measurable function $J(x, y)$ on $M \times M$ (called the *jump kernel*) such that

$$dj(x, y) = J(x, y) d\mu(y) d\mu(x) \ \text{ in } M \times M,$$

and, for all $x \in M$ and $R > 0$,

$$\| J(x, \cdot) \|_{L^q(B(x,R)^c)} \leq \frac{C}{V(x, R)^{1/q'} W(x, R)}, \tag{2.11}$$

where $C \in [0, \infty)$ is a constant independent of x, R.

Note that if $q < \infty$ then

$$\| J(x, \cdot) \|_{L^q(B(x,R)^c)} = \left(\int_{B(x,R)^c} J(x, y)^q d\mu(y) \right)^{1/q}.$$

If $B(x, R)^c$ is empty, then the inequalities (2.10) and (2.11) are automatically satisfied. We emphasize that the jump kernel $J(x, y)$ may not exist in condition (TJ), while it does in condition (TJ_q) for $1 \leq q \leq \infty$. It is also clear then (TJ_1) $\Rightarrow$ (TJ).

Example 2.7. Let $W(x, R) = R^\beta$ for all $x \in M$ and $R > 0$. The condition (TJ) becomes

$$J(x, B(x, R)^c) \leq \frac{C}{R^\beta}$$

for all $x \in M$ and $R > 0$. This condition was introduced and studied in [6] in the setting of ultra-metric spaces. Assume in addition that $V(x,R) \simeq R^\alpha$. Then the condition (TJ_q) becomes

$$\|J(x,\cdot)\|_{L^q(B(x,R)^c)} \le \frac{C}{R^{\alpha/q'+\beta}}.$$

In particular, for $q = \infty$ we have $q' = 1$ so that (TJ_∞) becomes

$$\|J(x,\cdot)\|_{L^\infty(B(x,R)^c)} \le \frac{C}{R^{\alpha+\beta}},$$

which is equivalent to the pointwise upper bound of the jump kernel

$$J(x,y) \le \frac{C}{d(x,y)^{\alpha+\beta}}.$$

The latter condition was used in many results in this area, see for example [10], [17] and the references therein.

Let us now recall the notions of the subcaloric and caloric functions. Let I be an interval in $\mathbb{R}$. A function $u : I \to L^2$ is said to be *weakly differentiable* at $t \in I$, if for any $\varphi \in L^2$, the function $(u(\cdot),\varphi)$ is differentiable at t, that is, the limit

$$\lim_{\varepsilon\to 0}\left(\frac{u(t+\varepsilon)-u(t)}{\varepsilon},\varphi\right)$$

exists. In this case, by the principle of uniform boundedness, there exists some $w \in L^2$ such that

$$\lim_{\varepsilon\to 0}\left(\frac{u(t+\varepsilon)-u(t)}{\varepsilon},\varphi\right) = (w,\varphi)$$

for any $\varphi \in L^2$. The function w is called the *weak derivative* of u at t, and we write $\partial_t u = w$ or $u'(t) = w$.

Note that weak differentiation defined in this way satisfies the *chain* and *product* rules, see Proposition 7.2 in Appendix. We remark that if $u : I \to L^2$ is weakly differentiable at some point $t \in I$ then the function $\|u\|_{L^2} : I \to \mathbb{R}$ is continuous at the point $t \in I$.

Definition 2.8. For an open subset $\Omega \subset M$, a function $u : I \to \mathcal{F}$ is called *subcaloric* in $I \times \Omega$ if u is weakly differentiable in L^2 at any $t \in I$ and if, for any $t \in I$ and any non-negative $\varphi \in \mathcal{F}(\Omega)$,

$$(\partial_t u,\varphi) + \mathcal{E}(u(t,\cdot),\varphi) \le 0.$$

A function u is said to be *caloric* in $I \times \Omega$ if the above inequality is replaced by equality, that is,

$$(\partial_t u,\varphi) + \mathcal{E}(u(t,\cdot),\varphi) = 0.$$

Note that for any $f \in L^2(\Omega)$, the function $u(t,\cdot) = P_t^\Omega f$ is caloric in $(0,\infty)\times\Omega$.

Now we define the notion of *parabolic mean value* (PMV_q), involving a parameter $q \in [1,\infty]$. For any ball B in M and any $T > 0$, define two cylinders Q^-, Q by

$$Q^- := \tfrac{1}{2}B \times [T - \tfrac{1}{4}W(B), T] \ \text{ and } \ Q := B \times [T - \tfrac{1}{2}W(B), T]. \tag{2.12}$$

so that $Q^- \subset Q$. (see Figure 2).

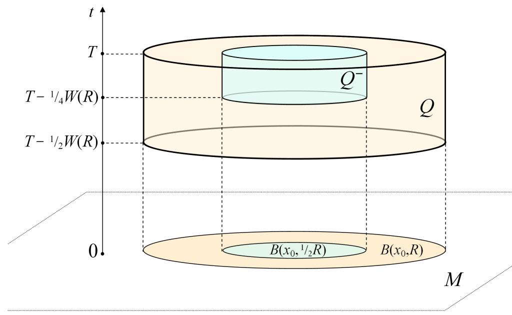

FIGURE 2. Cylinders Q^- and Q

Definition 2.9 (Parabolic mean value inequality)**.** Given $q \in [1,\infty]$, we say that condition (PMV_q) is satisfied, if there exist two constants $C > 0$ and $\sigma \in (0, 1]$ such that, for all balls B in M of radii $< \sigma\overline{R}$, for all $T \geq W(B)$ and for any function $u : (0, T] \to \mathcal{F} \cap L^\infty$ that is non-negative and subcaloric in $(0, T] \times B$, we have

$$\underset{Q^-}{\operatorname{esup}}\, u \leq C \left(\frac{1}{\mu(B)W(B)} \int_Q u^2(s, x)d\mu(x)ds \right)^{1/2} + \frac{K}{\mu(B)^{1/q'}} \sup_{s\in[T-\frac{1}{2}W(B),T]} \|u_+(s, \cdot)\|_{L^{q'}((\frac{1}{2}B)^c)}, \tag{2.13}$$

where K is defined by

$$K = \begin{cases} 1 & \text{if the measure } j \not\equiv 0, \\ 0 & \text{if the measure } j \equiv 0. \end{cases} \tag{2.14}$$

Here and in the sequel, we use following notion

$$\underset{I\times\Omega}{\operatorname{esup}}\, u := \sup_{t\in I} \underset{x\in\Omega}{\operatorname{esup}}\, u(t, x)$$

for an interval $I \subset \mathbb{R}$ and an open subset Ω of M.

In other words, the parabolic mean value inequality (PMV_q) says that the supremum of the function u (that is non-negative and subcaloric in $(0, T] \times B(x_0, R)$) over a smaller cylinder Q^- can be controlled by its L^2-norm in a larger cylinder Q plus a *tail term*, that is the $L^{q'}$-norm of the positive part u_+ outside the half ball $\frac{1}{2}B$ (see Figure 2).

In particular, if the Dirichlet form $(\mathcal{E}, \mathcal{F})$ has only the local part, then $K = 0$ and (2.13) becomes

$$\underset{Q^-}{\operatorname{esup}}\, u \leq C \left(\frac{1}{\mu(B)W(B)} \int_Q u^2(s, x)d\mu(x)ds \right)^{1/2} \simeq \left(\fint_Q u^2 \right)^{1/2},$$

which justifies the term "mean value inequality".

The next theorem is our first main result.

Theorem 2.10. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M, \mu)$ without killing part. Then*

$$(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}) \Rightarrow (\mathrm{PMV}_1),$$
$$(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{PMV}_q), \tag{2.15}$$

for any $q \in [1, \infty]$

The proof of Theorem 2.10 is given in the end of Section 5.

Let us recall the notion of a regular $\mathcal{E}$-*nest* (cf. [12, Section 2.1, p.66-69]). For an open set $U \subset M$, define the 1-capacity of U by

$$\mathrm{Cap}_1(U) := \inf \{\mathcal{E}_1(u, u) : u \in \mathcal{F} \text{ and } u \geq 1 \ \mu\text{-a.e. on } U\}$$

(note that $\mathrm{Cap}_1(U) = \infty$ if $\{u \in \mathcal{F},\ u \geq 1\ \mu\text{-a.e. on } U\} = \emptyset$).

An increasing sequence of closed subsets $\{F_k\}_{k=1}^{\infty}$ of M is called an $\mathcal{E}$-*nest* of M if

$$\lim_{k\to\infty} \mathrm{Cap}_1(M \setminus F_k) = 0.$$

An $\mathcal{E}$-nest $\{F_k\}$ is said to be *regular* with respect to μ if, for each k,

$$\mu(U(x) \cap F_k) > 0 \ \text{ for any } x \in F_k \text{ and any open neighborhood } U(x) \text{ of } x.$$

For an $\mathcal{E}$-nest $\{F_k\}_{k=1}^{\infty}$, define a function space

$$C(\{F_k\}) := \{u : M \to \mathbb{R} \cup \{\infty\} : u|_{F_k} \text{ is continuous for each } k\}. \tag{2.16}$$

A function $u : M \mapsto \mathbb{R} \cup \{\infty\}$ is said to be *quasi-continuous* if and only if $u \in C(\{F_k\})$ for some $\mathcal{E}$-nest $\{F_k\}_{k=1}^{\infty}$.

Recall that the heat kernel $p_t(x, y)$ of $(\mathcal{E}, \mathcal{F})$ is a non-negative function on $(0,\infty)\times M\times M$ such that, for any $t > 0$, the function $p_t(\cdot, \cdot)$ is measurable on $M \times M$ and, for any $f \in L^2$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y),$$

for μ-a.a. $x \in M$.

Now we can introduce condition (DUE): the *on-diagonal upper estimate* of the heat kernel, which, in particular, requires more regularity of the function $p_t(x, y)$.

Definition 2.11 (On-diagonal upper estimate)**.** We say that condition (DUE) is satisfied if the heat kernel $p_t(x, y)$ exists pointwise on $(0, \infty) \times M \times M$ and there exists a regular $\mathcal{E}$-nest $\{F_k\}$ such that

(1) for any $t > 0$ and any $x \in M$,
$$p_t(x, \cdot) \in C(\{F_k\});$$

(2) for any $C_0 \geq 1$, there exists a constant $C > 0$ such that for all $x \in M$ and $0 < t < C_0 W(x, \overline{R})$,
$$p_t(x, x) \leq \frac{C}{V(x, W^{-1}(x, t))}. \tag{2.17}$$

The next statement is our second main result.

Theorem 2.12. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(M, \mu)$ without killing part. Then*

$$(\mathrm{VD}) + (\mathrm{PMV}_2) \Rightarrow (\mathrm{DUE}). \tag{2.18}$$

A combination of Theorems 2.10 and 2.12 yields the following.

Corollary 2.13. *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form in $L^2(M, \mu)$ without killing part. Then, for any $q \in [2, \infty]$,*

$$(\mathrm{VD}) + (\mathrm{FK}) + (\mathrm{Gcap}) + (\mathrm{TJ}_q) \Rightarrow (\mathrm{DUE}). \tag{2.19}$$

Theorem 2.10 will be proved in Section 5, while Theorem 2.12 and Corollary 2.13 – in Section 6.

Our results have the following advantages in comparison with previous results.

(1) We use in (2.19) the condition (TJ_q) that is much weaker than the pointwise upper bounds of the jump kernel used in the previous works.
(2) The scaling function W may depend on the space variable x, which was not allowed in the previous results.
(3) The localization parameter $\overline{R}$ allows us to apply our results for both unbounded and compact metric spaces, for example, for compact fractals, which was problematic in other approaches. Moreover, in the case $\operatorname{diam} M = \infty$ and $\overline{R} < \infty$, when the hypotheses (FK) and (Gcap) are assumed for a restricted range of radii, we can still obtain the upper bound of the global heat kernel $p_t(x, y)$ although for a bounded range of t.

(4) Our method works for general regular Dirichlet forms without killing part. In particular, it works equally well for local and non-local Dirichlet forms. In contrast to that, the aforementioned Kigami's iteration method in the case of non-local Dirichlet forms requires dealing with a *truncated* jump kernel, which makes the argument much more complicated.

3. Monotonicity of condition (TJ_q)

In this section, we show that, under the standing assumption (VD), the condition (TJ_q) gets stronger when q increases. In particular, among the conditions (TJ), (TJ_1), (TJ_∞), condition (TJ) is the weakest, while condition (TJ_∞) is the strongest. In [6, Section 3] an example was given (in the setting of the ultra-metric spaces) where (TJ) holds but (TJ_∞) fails.

Proposition 3.1. *Let* (VD) *be satisfied. Then, for all* $1 \le q_1 \le q_2 \le \infty$,

$$(\mathrm{TJ}_{q_2}) \Rightarrow (\mathrm{TJ}_{q_1}) \Rightarrow (\mathrm{TJ}_1) \Rightarrow (\mathrm{TJ}). \tag{3.1}$$

Proof. It suffices to prove the implication (TJ_{q_2}) $\Rightarrow$ (TJ_{q_1}). Then it implies the implication (TJ_{q_1}) $\Rightarrow$ (TJ_1) for any $1 \le q_1 \le \infty$, while the implication (TJ_1) $\Rightarrow$ (TJ) is obvious.

Hence, assume that (TJ_{q_2}) holds. Fix a ball $B := B(x,R)$ in M and set

$$R_n = 2^n R \quad \text{and} \quad B_n = B(x, R_n),$$

for all non-negative integers n. By (TJ_{q_2}) and the Hölder inequality, we have

$$\begin{aligned}\int_{B^c} J(x,y)^{q_1} d\mu(y) &= \sum_{n=0}^{\infty} \int_{B_{n+1}\setminus B_n} J(x,y)^{q_1} d\mu(y) \\ &\le \sum_{n=0}^{\infty} \left(\int_{B_n^c} J(x,y)^{q_2} d\mu(y) \right)^{q_1/q_2} \mu(B_{n+1})^{1-q_1/q_2} \\ &\le \sum_{n=0}^{\infty} \left(\frac{C}{V(x,R_n)^{1-1/q_2} W(x,R_n)} V(x,R_{n+1})^{1/q_1-1/q_2} \right)^{q_1}.\end{aligned}$$

Note that if $\operatorname{diam} M < \infty$, then there exists an integer N such that $B_n = M$ for all $n \ge N$. In this case, the above summation terminates at $n = N$, and therefore, is finite.

On the other hand, using (VD) and $V(x,R_{n+1}) \ge V(x,R)$, we obtain

$$\frac{V(x,R_{n+1})^{1/q_1-1/q_2}}{V(x,R_n)^{1-1/q_2}} = \frac{1}{V(x,R_{n+1})^{1-1/q_1}} \left(\frac{V(x,R_{n+1})}{V(x,R_n)} \right)^{1-1/q_2} \le \frac{C}{V(x,R)^{1-1/q_1}}.$$

Moreover, using the left inequality in (2.6), we have

$$\frac{1}{W(x,R_n)} = \frac{1}{W(x,R)} \frac{W(x,R)}{W(x,2^n R)} \le \frac{2^{-n\beta_1} C}{W(x,R)}.$$

Combining the above three inequalities, we obtain

$$\begin{aligned}\int_{B^c} J(x,y)^{q_1} d\mu(y) &\le C \sum_{n=0}^{\infty} \left(\frac{2^{-n\beta_1}}{V(x,R)^{1-1/q_1} W(x,R)} \right)^{q_1} \\ &\simeq \left(\frac{1}{V(x,R)^{1-1/q_1} W(x,R)} \right)^{q_1},\end{aligned}$$

whence

$$\|J(x,\cdot)\|_{L^{q_1}(B^c)} \le \frac{C}{V(x,R)^{1-1/q_1} W(x,R)},$$

which is (TJ_{q_1}). □

4. Auxiliary lemmas

In this section we state some auxiliary results to be used in the next sections. Everywhere $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(M,\mu)$ without killing part.

Proposition 4.1 ([19, Proposition 9.1])**.** *Assume that a function $F\in C^2(\mathbb{R})$ satisfies*

$$\sup_{\mathbb{R}}|F'|<\infty,\ \ F''\geq 0,\ \ \sup_{\mathbb{R}}F''<\infty. \tag{4.1}$$

Then, for all $u,\varphi\in\mathcal{F}'\cap L^\infty$, both functions $F(u)$ and $F'(u)\varphi$ belong to the space $\mathcal{F}'\cap L^\infty$. Moreover, if in addition $\varphi\geq 0$ on M, then

$$\mathcal{E}(F(u),\varphi)\leq\mathcal{E}(u,F'(u)\varphi).$$

Lemma 4.2. *Let $I\subset\mathbb{R}$ be an interval and $\Omega\subset M$ be an open set. Assume that a function $u:I\to\mathcal{F}\cap L^\infty$ is subcaloric in $I\times\Omega$. Let $F\in C^2(\mathbb{R})$ be a function such that $F=0$ on $(-\infty,0]$ and*

$$F'\geq\ 0,\ \ \ \sup_{\mathbb{R}}F'<\infty,\ \ \ F''\geq 0,\ \sup_{\mathbb{R}}F''<\infty.$$

Then, for any $\varepsilon\geq 0$, the function

$$v:=F((u-\varepsilon)_+)$$

is also subcaloric in $I\times\Omega$.

Proof. Set

$$F_\varepsilon=F(\cdot-\varepsilon)$$

and observe that

$$v=F((u-\varepsilon)_+)=F\left(u-\varepsilon\right)=F_\varepsilon\left(u\right).$$

Note that the both functions F_ε and F_ε' are Lipschitz and vanish at 0. Hence, for any fixed $t\in I$, the functions $F_\varepsilon\left(u\right)$ and $F_\varepsilon'\left(u\right)$ belong to $\mathcal{F}$. By the chain rule of Proposition 7.2 (see Appendix), we have, for any fixed $t\in I$,

$$\partial_t v=\partial_t F_\varepsilon(u)=F_\varepsilon'(u)\partial_t u. \tag{4.2}$$

Since $F_\varepsilon'\left(u\right)\in\mathcal{F}\cap L^\infty$, we conclude by Proposition 7.1(*iii*) (see Appendix) that

$$F_\varepsilon'(u)\varphi\in\mathcal{F}\left(\Omega\right)\cap L^\infty$$

for any test function

$$\varphi\in\mathcal{F}\left(\Omega\right)\cap L^\infty.$$

Let in addition $\varphi\geq 0$. Since $u-\varepsilon\in\mathcal{F}'$, we obtain by Proposition 4.1 that

$$\mathcal{E}(v,\varphi)=\mathcal{E}(F(u-\varepsilon),\varphi)\leq\mathcal{E}\left(u-\varepsilon,F'(u-\varepsilon)\varphi\right)=\mathcal{E}\left(u,F_\varepsilon'(u)\varphi\right). \tag{4.3}$$

It follows from (4.2) and (4.3), that

$$(\partial_t v,\varphi)+\mathcal{E}(v,\varphi)\leq(\partial_t u,F_\varepsilon'(u)\varphi)+\mathcal{E}(u,F_\varepsilon'(u)\varphi). \tag{4.4}$$

Since u is subcaloric in Ω and the function $\psi=F_\varepsilon'\left(u\right)\varphi$ is non-negative and belongs to $\mathcal{F}\left(\Omega\right)\cap L^\infty$, we obtain that the right hand side of (4.4) is ≤ 0, which implies that v is subcaloric in $I\times\Omega$. $\square$

5. Proof of parabolic mean value inequality

In this section we prove the parabolic mean value inequality of Theorem 2.10. We present the proof in a sequence of lemmas so that at the end we only have to combine them to obtain Theorem 2.10. Assume everywhere that $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2(M,\mu)$ and that the jump measure j is given by

$$dj(x,y) = J(x,dy)d\mu(x).$$

Fix $x_0 \in M$, a number $T > 0$ and $0 < R' < \overline{R}$. Denote $\Omega = B(x_0, R')$ and let Q be the following cylinder

$$Q := (0,T] \times \Omega.$$

For $0 < t_1 < t_2 < T$, let χ be the Lipschitz function given by

$$\chi(t) = \begin{cases} 0, & 0 \le t \le t_1, \\ \frac{t-t_1}{t_2-t_1}, & t_1 \le t \le t_2, \\ 1, & t_2 \le t < T. \end{cases} \tag{5.1}$$

Clearly, we see that $\|\chi\|_{L^\infty} = 1$ and $\|\chi'\|_{L^\infty} \le \frac{1}{t_2-t_1}$.

Recall that a function $v : (0,T] \to \mathcal{F}$ is called subcaloric in Q if, for any $0 \le \varphi \in \mathcal{F}(\Omega)$ and any $t \in (0,T]$,

$$(\partial_t v, \varphi) + \mathcal{E}(v,\varphi) \le 0. \tag{5.2}$$

We need the following condition (EP′) that plays a crucial role in the proof.

Definition 5.1 (Condition (EP′))**.** We say that the condition (EP′) is satisfied if there exists a constant $C > 0$ such that, for any three concentric balls $B_0 := B(x_0,R)$, $B := B(x_0, R+r)$ and $\Omega := B(x_0,R')$ with $0 < R < R+r < R' < \overline{R}$, there exists $\phi \in \operatorname{cutoff}(B_0,B)$ such that, for any $u \in \mathcal{F} \cap L^\infty$,

$$\mathcal{E}(u\phi) \le \frac{3}{2}\mathcal{E}(u,u\phi^2) + \sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega u^2 d\mu + 3\iint_{\Omega\times\Omega^c} u(x)u(y)\phi^2(x)dj.$$

Remark 5.2. The condition (EP′) is stronger than a similar condition (EP) in [19], where the cutoff function ϕ was allowed to depend on the weight function u, while the cutoff function ϕ in (EP′) is universal. It is important to observe that, by [19, Corollary 14.2], we have the following implication.

$$(\mathrm{VD}) + (\mathrm{Gcap}) + (\mathrm{FK}) + (\mathrm{TJ}) \Rightarrow (\mathrm{EP}'). \tag{5.3}$$

Note that, in spite of the fact that the cutoff function ϕ in condition (Gcap) may depend on u, the function ϕ in (EP′) does not depend on u. Because of that, the implication (5.3) is highly non-trivial, and its proof in [19] uses an *elliptic* mean value inequality.

Lemma 5.3. *If condition (EP′) holds, then, for any pair of concentric balls $B_0 = B(x_0,R)$ and $B := B(x_0,R+r)$ with $0 < R < R+r < R'$, there exists some $\phi \in \operatorname{cutoff}(B_0,B)$, such that the following assertion is true. Let $u : (0,T] \to \mathcal{F}\cap L^\infty$ be non-negative and subcaloric in Q, and $F : \mathbb{R} \to \mathbb{R}_+$ be a function such that*

$$F \in C^2(\mathbb{R}), \quad F = 0 \quad \text{on } (-\infty, 0],$$
$$F' \ge 0, \quad F'' \ge 0, \quad \sup_{\mathbb{R}} F' < \infty, \quad \text{and } \sup_{\mathbb{R}} F'' < \infty. \tag{5.4}$$

Choose any $\rho > 0$ and set

$$v = F(u-\rho).$$

Then, for all $s \in (0,T]$,

$$\int_M \phi^2 \partial_s(\chi^2(s)v^2(s,\cdot))d\mu + \frac{4}{3}\chi^2(s)\mathcal{E}(v(s,\cdot)\phi)$$
$$\le 4(\sup_{\mathbb{R}} F')^2 A_0 \int_\Omega (u(s,\cdot)-\rho)_+ d\mu$$

$$+\frac{C(\sup_{\mathbb{R}} F')^2}{\inf_{x\in\Omega} W(x,r)\wedge(t_2-t_1)}\int_\Omega (u(s,\cdot)-\rho)_+^2 d\mu, \tag{5.5}$$

where function χ *is given by* (5.1),

$$A_0 := \sup_{t_1\le s'\le T}\ \underset{x\in B}{\mathrm{esup}}\int_{\Omega^c} u_+(s',y)J(x,dy), \tag{5.6}$$

and the constant $C>0$ *depends only the constants in* (EP′).

Proof. By Lemma 4.2, the function $v=F(u-\rho)\ge 0$ is subcaloric in Q and, hence, it satisfies (5.2). Denote

$$\widetilde{B} := B(x_0, R+r/2).$$

Applying condition (EP′) to the triple $(B_0,\widetilde{B},\Omega)$, we see that there exists some $\phi\in$ cutoff$(B_0,\widetilde{B})$ such that

$$\mathcal{E}(v\phi)\le \frac{3}{2}\mathcal{E}(v,v\phi^2)+\sup_{x\in\Omega}\frac{C}{W(x,r/2)}\int_\Omega v^2 d\mu + 3\iint_{\Omega\times\Omega^c} v(x)v(y)\phi^2(x)dj. \tag{5.7}$$

Since v is bounded in $\Omega=B(x_0,R')$ for each $s\in(0,T]$, we have $v\phi^2\in\mathcal{F}(\Omega)\cap L^\infty$ (cf. Proposition 7.1(iii) in Appendix). Substituting $\varphi=v\phi^2$ into (5.2) and using the chain rule (7.1) (see Appendix), we obtain

$$\mathcal{E}(v,v\phi^2)\le -\int_M (v\phi^2)\partial_s v d\mu = -\frac{1}{2}\int_M \phi^2\partial_s(v^2)d\mu.$$

Plugging this into (5.7) and then using the right inequality in (2.6), we obtain

$$\mathcal{E}(v\phi)\le -\frac{3}{4}\int_M \phi^2\partial_s(v^2)d\mu+\sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega v^2 d\mu+3\iint_{\Omega\times\Omega^c} v(x)v(y)\phi^2(x)dj,$$

from which, it follows that

$$\int_M \phi^2\partial_s(v^2)d\mu\le \sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega v^2 d\mu-\frac{4}{3}\mathcal{E}(v\phi)+4\iint_{\Omega\times\Omega^c} v(x)v(y)\phi^2(x)dj$$

for each $s\in(0,T]$, where we omit the variable s in $v(s,x)$ for simplicity. Therefore,

$$\begin{aligned}\int_M \phi^2\partial_s(\chi^2v^2)d\mu &= \chi^2\int_M\phi^2\partial_s(v^2)d\mu+2\chi\chi'\int_M\phi^2v^2d\mu\\ &\le \chi^2\left(\sup_{x\in\Omega}\frac{C}{W(x,r)}\int_\Omega v^2d\mu-\frac{4}{3}\mathcal{E}(v\phi)+4\iint_{\Omega\times\Omega^c}v(x)v(y)\phi^2(x)dj\right)\\ &\quad+2\chi\chi'\int_M\phi^2v^2d\mu.\end{aligned}$$

From this and using the facts that $\|\chi\|_{L^\infty}=1$, $\|\chi'\|_{L^\infty}\le\frac{1}{t_2-t_1}$ and $0\le\phi\le 1$ in M, supp$(\phi)\subset\widetilde{B}$, we obtain, for each $s\in(0,T]$,

$$\begin{aligned}&\int\phi^2\partial_s(\chi^2v^2)d\mu+\frac{4}{3}\chi^2\mathcal{E}(v\phi)\\ &\le \sup_{x\in\Omega}\frac{C\chi^2}{W(x,r)}\int_\Omega v^2d\mu+2\chi\chi'\int\phi^2v^2d\mu+4\chi^2\iint_{\Omega\times\Omega^c}v(x)v(y)\phi^2(x)dj\\ &\le C\left(\sup_{x\in\Omega}\frac{1}{W(x,r)}+\frac{1}{t_2-t_1}\right)\int_\Omega v^2d\mu+4\iint_{\widetilde{B}\times\Omega^c}v(x)v(y)dj.\end{aligned} \tag{5.8}$$

Note that by (5.4), for any $s\in(0,T]$ and $y\in M$,

$$v(s,y)=F((u(s,y)-\rho)_+)\le(\sup_{\mathbb{R}}F')(u(s,y)-\rho)_+\le(\sup_{\mathbb{R}}F')u_+(s,y). \tag{5.9}$$

It follows that

$$\sup_{x\in\widetilde{B}}\int_{\Omega^c} v(s,y)J(x,dy) \le (\sup_{\mathbb{R}} F')\sup_{x\in\widetilde{B}}\int_{\Omega^c} u_+(s,y)J(x,dy) \le (\sup_{\mathbb{R}} F')A_0$$

where we have used the definition (5.6) of A_0. Moreover, by (5.9), we have, for any $s\in(0,T]$,

$$\begin{aligned}\iint_{\widetilde{B}\times\Omega^c} v(s,x)v(s,y)dj &= \int_{\widetilde{B}} v(s,x)\left(\int_{\Omega^c} v(s,y)J(x,dy)\right)d\mu(x)\\ &\le \left(\sup_{x\in\widetilde{B}}\int_{\Omega^c} v(s,y)J(x,dy)\right)\int_{\widetilde{B}} v(s,x)d\mu(x)\\ &\le (\sup_{\mathbb{R}} F')^2 A_0\int_{\Omega}(u(s,x)-\rho)_+d\mu(x), \end{aligned}\tag{5.10}$$

Therefore, substituting (5.9) and (5.10) into (5.8), we obtain (5.5). □

Fix a point $x_0\in M$, some numbers

$$0<b_1<b_2<\infty,\quad 0<r_2<r_1<\overline{R},\quad 0<t_1<t_2<T$$

and consider two balls $B_2:=B(x_0,r_2)$, $B_1:=B(x_0,r_1)$ as well as two cylinders

$$Q_1:=[t_1,T]\times B_1 \text{ and } Q_2:=[t_2,T]\times B_2.$$

For a function $u:(0,T]\times B(x_0,r_1)\to\mathbb{R}$, we set

$$a_1=\int_{Q_1}(u(s,x)-b_1)_+^2d\mu(x)ds,\qquad a_2=\int_{Q_2}(u(s,x)-b_2)_+^2d\mu(x)ds. \tag{5.11}$$

Clearly, we have $a_2\le a_1$. In the next lemma we show that a_2 can be controlled by $a_1^{1+\nu}$ (see Fig. 3).

Lemma 5.4. *Assume that conditions* (FK_ν) *and* (EP') *hold. Let* $u:(0,T]\to\mathcal{F}\cap L^\infty$ *be non-negative, subcaloric in* $(0,T]\times B(x_0,r_1)$ *with* $r_1<\sigma\overline{R}$ *where* σ *comes from* (FK_ν), *and let* a_1,a_2 *be defined by (5.11) for* $0<r_2<r_1$. *Then*

$$a_2\le\frac{CW(B_1)}{(b_2-b_1)^{2\nu}\mu(B_1)^\nu}\left(\frac{1}{\inf_{x\in B_1}W(x,r_1-r_2)\wedge(t_2-t_1)}+\frac{A}{b_2-b_1}\right)^{1+\nu}a_1^{1+\nu}, \tag{5.12}$$

where $C>0$ *depends only on the constants in assumptions, and* A *is given by*

$$A:=\sup_{t_1\le s\le T}\ \operatorname*{esup}_{x\in\widetilde{B}}\int_{B_1^c}u_+(s,y)J(x,dy) \tag{5.13}$$

with the intermediate ball $\widetilde{B}:=B(x_0,r_2+\frac{3}{4}(r_1-r_2))$ *so that* $B_2\subset\widetilde{B}\subset B_1$.

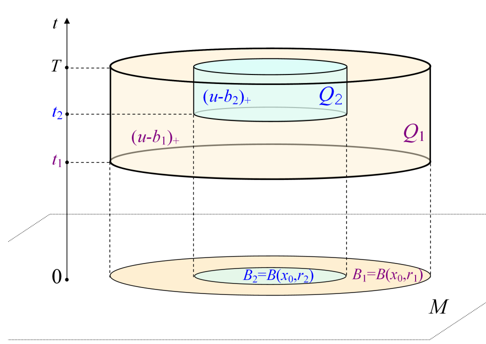

FIGURE 3. Functions $(u-b_i)_+$ in cylinders Q_i

The inequality (5.12) plays an important role in the proof of the parabolic mean value inequality by means of De Giorgi's iterations.

Proof. For simplicity set $r := r_1 - r_2$ and

$$T_0 := \inf_{x\in B_1} W(x,r) \wedge (t_2 - t_1). \tag{5.14}$$

Let U be another concentric ball given by

$$U := B(x_0, r_2 + \frac{1}{2}r)$$

so that $B_2 \subset U \subset \widetilde{B} \subset B_1$. Consider also the number

$$\xi := b_1 + \frac{1}{2}(b_2 - b_1) \tag{5.15}$$

so that $0 < b_1 < \xi < b_2$. Choose a sequence $\{F_n\}_{n=1}^{\infty}$ of C^2-functions satisfying (5.4) such that and

$$c_0 := \sup_k \sup_{\mathbb{R}} F_k' < \infty,$$

and

$$F_n(r) \overset{\mathbb{R}}{\rightrightarrows} r \vee 0 \text{ as } n \to \infty. \tag{5.16}$$

Step 1. Applying Lemma 5.3 with three sets B_0, B, Ω being respectively replaced by B_2, U, B_1 and with $\rho = b_2$ and $F = F_n$ $(n \geq 1)$, we obtain from (5.5), (5.14) and the right inequality in (2.6) that there exists some $\phi_0 \in \mathrm{cutoff}(B_2, U)$ such that, for any $s \in (0, T]$,

$$\begin{aligned}
&\int_U \phi_0^2 \partial_s(\chi^2(s) F_n(u(s,\cdot) - b_2)^2) d\mu + \frac{4}{3}\chi^2(s)\mathcal{E}(F_n(u(s,\cdot) - b_2)\phi_0) \\
&\leq 4(\sup_{\mathbb{R}} F_n')^2 \left(A_1 \int_{B_1} (u - b_2)_+ d\mu + \frac{C(\sup_{\mathbb{R}} F_n')^2}{\inf_{x\in B_1} W(x, r/2) \wedge (t_2 - t_1)} \int_{B_1} (u - b_2)_+^2 d\mu \right) \\
&\leq 4c_0^2 \left(A_1 \int_{B_1} (u - b_2)_+ d\mu + \frac{Cc_0^2}{\inf_{x\in B_1} W(x, r) \wedge (t_2 - t_1)} \int_{B_1} (u - b_2)_+^2 d\mu \right) \\
&\leq 4c_0^2 \left(A_1 \int_{B_1} (u - b_2)_+ d\mu + \frac{Cc_0^2}{T_0} \int_{B_1} (u - b_2)_+^2 d\mu \right),
\end{aligned} \tag{5.17}$$

where χ is define by (5.1), and where by (5.6)

$$A_1 = \sup_{t_1 \leq s' \leq T} \operatorname*{esup}_{x\in U} \int_{B_1^c} u_+(s', y) J(x, dy) \leq \sup_{t_1 \leq s' \leq T} \operatorname*{esup}_{x\in \widetilde{B}} \int_{B_1^c} u_+(s', y) J(x, dy) = A.$$

We estimate the middle integral in (5.17). A simple calculation shows that for any $u \in \mathbb{R}$, $0 < b_1 \leq b_2$,

$$(u - b_2)_+ \leq \frac{(u - b_1)_+^2}{b_2 - b_1},$$

which implies that

$$\int_{B_1} (u(s,\cdot) - b_2)_+ d\mu \leq \frac{1}{b_2 - b_1} \int_{B_1} (u(s,\cdot) - b_1)_+^2 d\mu.$$

Substituting this to (5.17) and using in the last integral in (5.17) $(u - b_2)_+^2 \leq (u - b_1)_+^2$, we obtain that for any $s \in (0, T]$ and $n \geq 1$,

$$\begin{aligned}
\frac{4}{3}\chi^2(s)\mathcal{E}(F_n(u(s,\cdot) - b_2)\phi_0) \leq & -\int_U \phi^2 \partial_s(\chi^2(s) F_n(u(s,\cdot) - b_2)^2) d\mu \\
& + C\left(\frac{A}{b_2 - b_1} + \frac{1}{T_0}\right) \int_{B_1} (u(s,\cdot) - b_1)_+^2 d\mu.
\end{aligned} \tag{5.18}$$

Step 2. Applying Lemma 5.3 again but this time with three sets B_0, B, Ω being respectively replaced by $U, \widetilde{B}, B_1$ and with $\rho = \xi$ and $F = F_n$ $(n \geq 1)$, we obtain from (5.5), (5.14) and the right inequality in (2.6) that there exists some $\phi_1 \in \mathrm{cutoff}(U, \widetilde{B})$ such that, for any $s \in (0, T]$,

$$\begin{aligned}
&\int_M \phi^2 \partial_s(\chi^2(s) F_n(u(s,\cdot)-\xi)^2) d\mu \\
&\le 4(\sup_{\mathbb{R}} F_n')^2 A \int_{B_1} (u(s,\cdot)-\xi)_+ d\mu + \frac{C(\sup_{\mathbb{R}} F_n')^2}{\inf_{x\in B_1} W(x, r/4)\wedge(t_2-t_1)} \int_{B_1} (u(s,\cdot)-\xi)_+^2 d\mu \\
&\le C\left(A\int_{B_1}(u(s,\cdot)-\xi)_+ d\mu + \frac{1}{\inf_{x\in B_1} W(x,r)\wedge(t_2-t_1)}\int_{B_1}(u(s,\cdot)-\xi)_+^2 d\mu\right) \\
&= C\left(A\int_{B_1}(u(s,\cdot)-\xi)_+ d\mu + \frac{1}{T_0}\int_{B_1}(u(s,\cdot)-\xi)_+^2 d\mu\right), \qquad (5.19)
\end{aligned}$$

where we have dropped the second term in (5.5) since it is non-negative. Now we need to estimate the middle integral in (5.19). Indeed, a simple computation shows that, for any $u \in \mathbb{R}$, $0 < b_1 \le b_2$,

$$(u-\xi)_+ = \left(u - \frac{b_1+b_2}{2}\right)_+ \le \frac{2}{b_2-b_1}(u-b_1)_+^2,$$

which implies

$$\int_{B_1}(u-\xi)_+ d\mu ds \le \frac{2}{b_2-b_1}\int_{B_1}(u-b_1)_+^2 d\mu. \qquad (5.20)$$

Plugging (5.20) into (5.19) and integrating (5.19) over $s \in (t_1, t]$ with respect to ds for any $t_2 \le t < T$, we obtain for any $n \ge 1$,

$$\begin{aligned}
&\int_M \phi^2 \chi^2(t) F_n(u(t,\cdot)-\xi)^2 d\mu \\
&\le C\left(A\int_{t_1}^t\int_{B_1}(u(s,\cdot)-\xi)_+ d\mu ds + \frac{1}{T_0}\int_{t_1}^t\int_{B_1}(u(s,\cdot)-\xi)_+^2 d\mu ds\right) \\
&\le C\left(\frac{A}{b_2-b_1}\int_{t_1}^T\int_{B_1}(u(s,\cdot)-b_1)_+^2 d\mu ds + \frac{1}{T_0}\int_{t_1}^T\int_{B_1}(u(s,\cdot)-b_1)_+^2 d\mu ds\right) \\
&= C\left(\frac{A}{b_2-b_1} + \frac{1}{T_0}\right) a_1, \qquad (5.21)
\end{aligned}$$

Hence, noticing that $\phi_1 = 1$ in U and $\chi = 1$ on $[t_2, T]$, and using (5.16), we obtain from (5.21) that, for any $t \in [t_2, T]$,

$$\begin{aligned}
\int_U (u(t,\cdot)-\xi)_+^2 d\mu &= \lim_{n\to\infty}\int_U F_n(u(t,\cdot)-\xi)^2 d\mu \\
&\le \liminf_{n\to\infty}\int_M \phi_1^2 F_n(u(t,\cdot)-\xi)^2 d\mu \\
&= \liminf_{n\to\infty}\int_M \phi_1^2\chi^2(t) F_n(u(t,\cdot)-\xi)^2 d\mu \\
&\le C\left(\frac{A}{b_2-b_1}+\frac{1}{T_0}\right) a_1. \qquad (5.22)
\end{aligned}$$

Step 3. For any $s \in (0, T]$, consider the set

$$E_s := U \cap \{u(\cdot, s) \ge b_2\},$$

where $\phi_0 \in \mathrm{cutoff}(B_2, U)$ is the same as in (5.18), so that $E_s \subset U$. By the outer regularity of μ, for any $\varepsilon > 0$, there exists a non-empty open set U_s such that $E_s \subset U_s \subset U \subset B_1$, and

$$\mu(U_s) \le \mu(E_s) + \varepsilon. \qquad (5.23)$$

On the other hand, as

$$F_n(u(s,\cdot)-b_2)\phi_0 = 0 (n\geq 1) \quad \text{q.e. in} \quad (E_s)^c = M\setminus E_s,$$

we see that

$$F_n(u(s,\cdot)-b_2)\phi_0 = 0 \ \ \text{q.e. in} \ \ (U_s)^c \subset (E_s)^c,$$

and hence (cf. [12, Corollary 2.3.1 on p.98]),

$$(u(s,\cdot)-\xi_2)_+\phi_0 \in \mathcal{F}(U_s).$$

Therefore, using that $\phi_0 = 1$ in $B_2 \subset B_1$ and $\chi(s) = 1$ for $s \in [t_2, T]$, we obtain from (5.18) that for any $s \in [t_2, T]$

$$\begin{aligned}
\int_{B_2} F_n(u(s,\cdot)-b_2)^2 d\mu &\leq \int_{B_1} \phi_0^2 (u(s,\cdot)-b_2)_+^2 d\mu \\
&= \int_{E_s} \phi_0^2 F_n(u(s,\cdot)-b_2)^2 d\mu \leq \int_{U_s} \phi_0^2 F_n(u(s,\cdot)-b_2)^2 d\mu \\
&\leq \frac{\mathcal{E}(F_n(u(s,\cdot)-b_2)\phi_0)}{\lambda_{\min}(U_s)} \quad \text{(by (2.8))} \\
&\leq \sup_{t_2\leq s\leq T} \lambda_{\min}(U_s)^{-1} \cdot \chi^2(s)\mathcal{E}(F_n(u(s,\cdot)-b_2)\phi_0) \\
&\leq \sup_{t_2\leq s\leq T} \lambda_{\min}(U_s)^{-1} \left(-\frac{3}{4}\int_U \phi_0^2 \partial_s(\chi^2(s)F_n(u(s,\cdot)-b_2)^2) d\mu \right. \\
&\qquad \left. + C\left(\frac{A}{b_2-b_1}+\frac{1}{T_0}\right)\int_{B_1}(u(s,\cdot)-b_1)_+^2 d\mu \right).
\end{aligned}$$

Integrating it over $[t_2, T]$, and using the fact that $\chi(t_2) = 0$, we obtain that for any $n \geq 1$,

$$\begin{aligned}
\int_{t_2}^T\int_{B_2} F_n(u(s,\cdot)-b_2)^2 d\mu ds &\leq \sup_{t_2\leq s\leq T} \lambda_{\min}(U_s)^{-1} \left(-\frac{3}{4}\int_U \phi_0^2 \chi^2(T) F_n(u(T,\cdot)-b_2)^2) d\mu \right. \\
&\qquad \left. + C\left(\frac{A}{b_2-b_1}+\frac{1}{T_0}\right)\int_{t_2}^T\int_{B_1}(u(s,\cdot)-b_1)_+^2 d\mu \right) \\
&\leq \sup_{t_2\leq s\leq T} \lambda_{\min}(U_s)^{-1} \cdot C\left(\frac{A}{b_2-b_1}+\frac{1}{T_0}\right) \\
&\qquad \times \int_{t_1}^T\int_{B_1}(u(s,\cdot)-b_1)_+^2 d\mu \\
&= \sup_{t_2\leq s\leq T} \lambda_{\min}(U_s)^{-1} \cdot C\left(\frac{A}{b_2-b_1}+\frac{1}{T_0}\right) a_1.
\end{aligned}$$

This inequality together with (5.16) yields that

$$\begin{aligned}
a_2 &= \lim_{n\to\infty}\int_{t_2}^T\int_{B_2} F_n(u(s,\cdot)-b_2)^2 d\mu ds \\
&\leq \sup_{t_2\leq s\leq T} \lambda_{\min}(U_s)^{-1} \cdot C\left(\frac{A}{b_2-b_1}+\frac{1}{T_0}\right) a_1.
\end{aligned} \tag{5.24}$$

Step 4. Since $U_s \subset B_1$ and the ball B_1 has a radius $0 < r_1 < \sigma\overline{R}$, we apply ($\text{FK}_\nu$) and obtain

$$\lambda_{\min}(U_s)^{-1} \leq CW(B_1)\left(\frac{\mu(U_s)}{\mu(B_1)}\right)^\nu \leq CW(B_1)\left(\frac{\mu(E_s)+\varepsilon}{\mu(B_1)}\right)^\nu,$$

where we have used also (5.23). Substituting this into (5.24) and then letting $\varepsilon \to 0$, we conclude

$$a_2 \leq \frac{CW(B_1)}{\mu(B_1)^\nu}\sup_{t_2\leq s\leq T}\mu(E_s)^\nu \cdot \left(\frac{1}{T_0}+\frac{A}{b_2-b_1}\right) a_1. \tag{5.25}$$

In order to bound $\mu(E_s)$ for every $s \in [t_2, T]$, observe that, by $E_s \subset U_s \subset U$ and (5.15),

$$\int_U (u(s,\cdot)-\xi)_+^2 d\mu \geq \int_{E_s} (u(s,\cdot)-\xi)_+^2 d\mu \geq \int_{E_s} (b_2-\xi)^2 d\mu = \frac{(b_2-b_1)^2}{4}\mu(E_s).$$

From this and (5.22), we obtain, for any $s \in [t_2, T]$,

$$\mu(E_s) \leq \frac{4}{(b_2-b_1)^2}\int_U (u(s,\cdot)-\xi)_+^2 d\mu \leq \frac{C}{(b_2-b_1)^2}\left(\frac{1}{T_0}+\frac{A}{b_2-b_1}\right) a_1. \tag{5.26}$$

Finally, substituting (5.26) into (5.25), we conclude that

$$a_2 \leq \frac{CW(B_1)}{\mu(B_1)^\nu (b_2-b_1)^{2\nu}}\left(\frac{1}{T_0}+\frac{A}{b_2-b_1}\right)^{1+\nu} a_1^{1+\nu},$$

which is (5.12). □

The next step in the proof of Theorem 2.10 is iterating of (5.12) as we do below in Lemma 5.5. For that, let us fix some

$$0 < R < \sigma \overline{R},$$

where σ comes from (FK_ν), as well as some constant $\rho > 0$ to be determined later. Let $\{R_k\}_{k=0}^\infty$, $\{\rho_k\}_{k=0}^\infty$ be two sequences of positive numbers given by

$$R_k = \left(2^{-1}+2^{-k-1}\right) R \ \text{ and } \ \rho_k = \left(1-2^{-k}\right)\rho \ \text{ for any } k \geq 0. \tag{5.27}$$

Then $\{R_k\}$ is decreasing with $R_0 = R$, $R_k \downarrow \frac{1}{2}R$, and

$$R_{k-1} - R_k = 2^{-k-1}R < R_k, \tag{5.28}$$

while $\{\rho_k\}$ is increasing with $\rho_0 = 0$, $\rho_k \uparrow \rho$, and

$$\rho_k - \rho_{k-1} = 2^{-k}\rho. \tag{5.29}$$

Let $\{T_k\}_{k=0}^\infty$ be an increasing sequence of positive numbers such that $0 < T_0 < T$ and

$$T_{k+1} = T_k + \delta W(x_0, R_k - R_{k+1}), \tag{5.30}$$

where $\delta \in (0,1]$ will be chosen later on. By (5.28), the left inequality in (2.6), we have

$$\begin{aligned} W(x_0, R_k - R_{k+1}) &= W(x_0, 2^{-k-2}R) = W(x_0,R)\cdot \frac{W(x_0, 2^{-k-2}R)}{W(x_0,R)} \\ &\leq CW(x_0,R)\left(\frac{2^{-k-2}R}{R}\right)^{\beta_1} = C2^{-(k+2)\beta_1}W(x_0,R). \end{aligned}$$

It follows from (5.30) that

$$\begin{aligned} T_\infty &:= \lim_{k\to\infty} T_k = T_0 + \sum_{k=0}^\infty (T_{k+1}-T_k) = T_0 + \delta\sum_{k=0}^\infty W(x_0, R_k - R_{k+1}) \\ &\leq T_0 + \delta \sum_{k=0}^\infty C2^{-(k+2)\beta_1} W(x_0,R) = T_0 + \frac{1}{4}W(x_0,R), \end{aligned}$$

where we have chosen δ to satisfy

$$\delta \sum_{k=0}^\infty C2^{-(k+2)\beta_1} = \frac{1}{4}. \tag{5.31}$$

In order to guarantee $T_\infty < T$, we take

$$T \geq W(x_0,R) \quad \text{ and } \quad T_0 = T - \frac{1}{2}W(x_0,R) \tag{5.32}$$

so that

$$T_\infty \leq T_0 + \frac{1}{4}W(x_0,R) = T - \frac{1}{4}W(x_0,R).$$

Finally, set as in (2.12)

$$Q := B \times [T - \tfrac{1}{2}W(B), T], \quad Q^- := \tfrac{1}{2}B \times [T - \tfrac{1}{4}W(B), T]$$

and consider for all integers $k \geq 0$, balls $B_k := B(x_0, R_k)$ and cylinders $Q_k := B_k \times [T_k, T]$ (see Fig. 4).

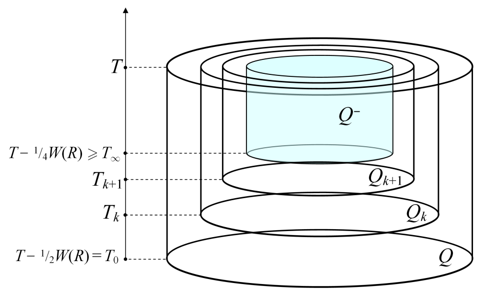

FIGURE 4. Cylinders Q, Q^-, Q_k

Now we are ready to prove a version of the parabolic mean value inequality with a parameter $\varepsilon > 0$.

Lemma 5.5. *Let $q \in [1, \infty]$. Assume that conditions (VD), (FK$_\nu$), (EP′), are satisfied. Assume in addition that (TJ) is satisfied when $q = 1$, and (TJ$_q$) is satisfied when $q \in (1, \infty]$. Let $u : (0, T] \to \mathcal{F} \cap L^\infty$ be non-negative, subcaloric in $(0, T] \times B$, $B := B(x_0, R)$ with $0 < R < \sigma\overline{R}$ and $T \geq W(x_0, R)$, where σ comes from condition (FK$_\nu$). Then, for any $\varepsilon > 0$,*

$$\operatorname*{esup}_{Q^-} u \leq C(1 + \varepsilon^{-\frac{1+\nu}{2\nu}}) \left(\frac{1}{\mu(B)W(B)} \int_Q u^2(s, x) d\mu(x) ds \right)^{1/2} + \frac{\varepsilon K}{\mu(B)^{1/q'}} \sup_{s \in [T - \frac{1}{2}W(B), T]} \|u_+(s, \cdot)\|_{L^{q'}((\frac{1}{2}B)^c)}, \tag{5.33}$$

where the number K is given by (2.14), and $q' = \frac{q}{q-1}$ as before, and where C depends only on the constants in the hypothesis but is independent of B, T, u, ε.

Proof. Fix a ball $B = B(x_0, R)$ with $R \in (0, \sigma\overline{R})$. Using he above definitions of Q_k and ρ_k, set

$$a_k := \int_{Q_k} (u(s, x) - \rho_k)_+^2 d\mu(x) ds,$$

for any $k \geq 0$.

We apply (5.12) with $r_2 = R_k, r_1 = R_{k-1}, t_1 = T_{k-1}, t_2 = T_k$, and b_1, b_2 being replaced by ρ_{k-1}, ρ_k respectively. Note that by (5.30)

$$\inf_{x \in B_{k-1}} W(x, R_{k-1} - R_k) \wedge (T_k - T_{k-1}) \geq \delta \inf_{x \in B_{k-1}} W(x, R_{k-1} - R_k),$$

where δ is given by (5.31). Therefore, for any $k \geq 1$,

$$a_k \leq \frac{CW(B_{k-1})}{(\rho_k - \rho_{k-1})^{2\nu} \mu(B_{k-1})^\nu} \left(\frac{\delta^{-1}}{\inf_{x \in B_{k-1}} W(x, R_{k-1} - R_k)} + \frac{A_k}{\rho_k - \rho_{k-1}} \right)^{1+\nu} a_{k-1}^{1+\nu}, \tag{5.34}$$

where, by (5.13) and $T_0 \leq T_k \leq T$,

$$A_k := \sup_{T_{k-1} \leq s \leq T} \operatorname*{esup}_{x \in \widetilde{B}_k} \int_{B_{k-1}^c} u_+(s, y) J(x, dy) \leq \sup_{T_0 \leq s \leq T} \operatorname*{esup}_{x \in \widetilde{B}_k} \int_{B_{k-1}^c} u_+(s, y) J(x, dy) \tag{5.35}$$

and $\widetilde{B}_k := B(x_0, \widetilde{R}_k)$ with

$$\widetilde{R}_k := R_k + \frac{3}{4}(R_{k-1} - R_k).$$

Let us estimate the term on the right-hand side of (5.35). Observe that, for any $k \geq 1$,

$$B_{k-1} \supseteq \frac{1}{2}B \ \text{ so that } \ B_{k-1}^c \subseteq \left(\frac{1}{2}B\right)^c = \left(\frac{1}{2}B_0\right)^c, \tag{5.36}$$

while by (5.28), for any $x \in \widetilde{B}_k$,

$$B_{k-1}^c \subseteq B(x, (R_{k-1} - R_k)/4)^c = B(x, 2^{-(k+3)}R)^c. \tag{5.37}$$

If $q = 1$, we see by (5.36) and the above inclusion that, for any $x \in [T_0, T]$ and $x \in \widetilde{B}_k$,

$$\begin{aligned}\int_{B_{k-1}^c} u_+(s,y)J(x,dy) \leq\ & \|u_+(s,\cdot)\|_{L^\infty(B_{k-1}^c)} \int_{B_{k-1}^c} J(x,dy) \\ \leq\ & \|u_+(s,\cdot)\|_{L^\infty((\frac{1}{2}B)^c)} \int_{B(x,2^{-(k+3)}R)^c} J(x,dy) \\ \leq\ & \|u_+(s,\cdot)\|_{L^\infty((\frac{1}{2}B_0)^c)} \frac{CK}{W(x,2^{-(k+3)}R)} \quad (\text{by condition(TJ)}) \\ \leq\ & \frac{C'2^{k\beta_2}K}{W(B_0)} \|u_+(s,\cdot)\|_{L^\infty((\frac{1}{2}B_0)^c)} \quad (\text{by the right inequality in (2.6)}),\end{aligned}$$

where the constant K is defined in (2.14), and in the last inequality, we have used that, by the right inequality in (2.6),

$$\frac{1}{W(x,2^{-(k+3)}R)} = \frac{1}{W(x,R)} \cdot \frac{W(x,R)}{W(x,2^{-(k+3)}R)} \leq \frac{2^{k\beta_2}C}{W(x,R)}. \tag{5.38}$$

If $1 < q \leq \infty$, using Hölder's inequality and condition (TJ_q), we obtain by (5.37) and (5.36) that, for any $x \in [T_0, T]$ and $x \in \widetilde{B}_k$,

$$\begin{aligned}\int_{B_{k-1}^c} u_+(s,y)J(x,dy) =\ & \int_{B_{k-1}^c} u_+(s,y)J(x,y)d\mu(y) \\ \leq\ & \|J(x,\cdot)\|_{L^q(B_{k-1}^c)} \|u_+(s,\cdot)\|_{L^{q'}(B_{k-1}^c)} \\ \leq\ & \|J(x,\cdot)\|_{L^q(B(x,2^{-(k+3)}R)^c)} \|u_+(s,\cdot)\|_{L^{q'}(B_{k-1}^c)} \\ \leq\ & \frac{CK}{V(x,2^{-(k+3)}R)^{1/q'}W(x,2^{-(k+3)}R)} \|u_+(s,\cdot)\|_{L^{q'}(\frac{1}{2}B_0^c)}.\end{aligned} \tag{5.39}$$

By (VD), we have

$$\frac{1}{V(x,2^{-(k+3)}R)^{1/q'}} = \frac{1}{V(x,R)^{1/q'}} \left(\frac{V(x,R)}{V(x,2^{-(k+3)}R)}\right)^{1/q'} \leq \frac{2^{k\alpha/q'}C}{V(x,R)^{1/q'}}.$$

Combining the above two inequalities and (5.38), we have in the case when $1 < q \leq \infty$ that for any $x \in [T_0, T]$ and $x \in \widetilde{B}_k$,

$$\int_{B_{k-1}^c} u_+(s,y)J(x,dy) \leq \frac{C2^{k(\alpha/q'+\beta_2)}K}{\mu(B_0)^{1/q'}W(B_0)} \|u_+(s,\cdot)\|_{L^{q'}((\frac{1}{2}B_0)^c)}.$$

Therefore, in the both cases when either $q = 1$ and (TJ) holds or $1 < q \leq \infty$ and (TJ_q) holds, we always obtain from above and (5.35) that

$$A_k \leq \frac{C2^{k(\alpha/q'+\beta_2)}K}{\mu(B_0)^{1/q'}W(B_0)} \sup_{T_0 \leq s \leq T} \|u_+(s,\cdot)\|_{L^{q'}(\frac{1}{2}B_0)^c} = C2^{k(\alpha/q'+\beta_2)} \frac{\Lambda}{W(B_0)},$$

where

$$\Lambda := \frac{K}{\mu(B_0)^{1/q'}} \sup_{T_0 \leq s \leq T} \|u_+(s,\cdot)\|_{L^{q'}(\frac{1}{2}B_0)^c}. \tag{5.40}$$

Let us now estimate the rest terms on the right hand side of (5.34). Observe that the right inequality in (2.6) and (5.28) imply

$$\frac{W(B_0)}{\inf_{x\in B_{k-1}} W(x,R_{k-1}-R_k)} = \sup_{x\in B_{k-1}} \frac{W(x_0,R)}{W(x,R_{k-1}-R_k)} \le C\left(\frac{R}{R_{k-1}-R_k}\right)^{\beta_2} = C2^{k\beta_2}.$$

Therefore, using

$$W(B_{k-1}) \le W(B_0) \text{ and } \mu(B_{k-1}) \ge \mu(\frac{1}{2}B_0) \ge C^{-1}\mu(B_0) \text{ for any } k\ge 1$$

(5.29) and (5.27), we obtain from (5.34) that

$$\begin{aligned} a_k \le & \frac{CW(B_0)}{(2^{-k}\rho)^{2\nu}\left(C^{-1}\mu(B_0)\right)^{\nu}}\left(\frac{C\delta^{-1}2^{k\beta_2}}{W(B_0)}+\frac{C2^{k(\alpha/q'+\beta_2)}\Lambda}{2^{-k}\rho W(B_0)}\right)^{1+\nu} a_{k-1}^{1+\nu} \\ \le & \frac{C'}{\rho^{2\nu}\mu(B_0)^{\nu}W(B_0)^{\nu}}2^{ks}\left(1+\frac{\Lambda}{\rho}\right)^{1+\nu} a_{k-1}^{1+\nu}, \end{aligned} \tag{5.41}$$

where $s = 2\nu + (\alpha/q' + \beta_2 + 1)(1+\nu)$ is an exponent that is unimportant.

For any $\varepsilon > 0$, we choose ρ as follows:

$$\rho = \varepsilon\Lambda + C(\varepsilon)\left(\frac{a_0}{\mu(B_0)W(B_0)}\right)^{1/2},$$

where $C(\varepsilon)$ is a constant yet to be determined (see (5.45) below). With this choice of ρ, it is obvious that

$$1+\frac{\Lambda}{\rho} \le 1+\varepsilon^{-1},$$

which implies by (5.41) that

$$a_k \le \frac{(1+\varepsilon^{-1})^{1+\nu}C'}{\rho^{2\nu}\mu(B_0)^{\nu}W(B_0)^{\nu}}2^{ks}a_{k-1}^{1+\nu} = D2^{ks}a_{k-1}^{1+\nu},$$

where

$$D := \frac{(1+\varepsilon^{-1})^{1+\nu}C'}{\rho^{2\nu}\mu(B_0)^{\nu}W(B_0)^{\nu}}. \tag{5.42}$$

Therefore, applying Proposition 7.3 (from Appendix) with $\lambda = 2^s > 1$, we obtain

$$a_k \le D^{-\frac{1}{\nu}}\left(D^{\frac{1}{\nu}}\lambda^{\frac{1+\nu}{\nu^2}}a_0\right)^{(1+\nu)^k} \le D^{-\frac{1}{\nu}}\left(\frac{1}{2}\right)^{(1+\nu)^k}. \tag{5.43}$$

provided that

$$D^{\frac{1}{\nu}}\lambda^{\frac{1+\nu}{\nu^2}}a_0 \le \frac{1}{2}. \tag{5.44}$$

By definition (5.42) of D, condition (5.44) can be guaranteed if

$$\begin{aligned} \rho \ge & \left(2\left((1+\varepsilon^{-1})^{1+\nu}C'\right)^{1/\nu}(2^s)^{(1+\nu)/\nu^2}\frac{a_0}{\mu(B_0)W(B_0)}\right)^{1/2} \\ = & C(\varepsilon)\left(\frac{a_0}{\mu(B_0)W(B_0)}\right)^{1/2}, \end{aligned}$$

where the number $C(\varepsilon)$ is given by

$$C(\varepsilon) = \sqrt{2\left\{(1+\varepsilon^{-1})^{1+\nu}C'\right\}^{1/\nu}(2^s)^{(1+\nu)/\nu^2}} = C(1+\varepsilon^{-1})^{\frac{1+\nu}{2\nu}}. \tag{5.45}$$

Finally, it follows from (5.43) that

$$\int_{T_\infty}^{T}\int_{B(x_0,R/2)}(u(s,\cdot)-\rho)_+^2 d\mu ds \le a_k \to 0 \text{ as } k\to\infty.$$

Note that the function $s \mapsto \int_{B(x_0,R/2)}(u(s,\cdot)-\rho)_+^2 d\mu$ is continuous on $[T_\infty, T]$. Hence, by the above formula and using definitions (5.40), (5.45), (5.32), we conclude that for each $s \in [T_\infty, T] \supseteq [T - \frac{1}{4}W(B_0), T]$ we have

$$\begin{aligned}\operatorname*{esup}_{B(x_0,R/2)} u(s,\cdot) \le \rho = \varepsilon\Lambda + C(\varepsilon)\left(\frac{a_0}{\mu(B_0)W(B_0)}\right)^{1/2} \\ = \varepsilon \frac{K}{\mu(B_0)^{1/q'}} \sup_{T_0 \le s \le T} \|u_+(s,\cdot)\|_{L^{q'}(\frac{1}{2}B_0)^c} \\ + C(1+\varepsilon^{-1})^{\frac{1+\nu}{2\nu}}\left(\frac{a_0}{\mu(B_0)W(B_0)}\right)^{1/2},\end{aligned}$$

thus proving (5.33). □

Proof of Theorem 2.10. The parabolic mean value inequlaity (PMV_q) of Theorem 2.10 concides with the inequality (5.33) of Lemma 5.5 with $\varepsilon = 1$. Hence, we only need to verify that the hypotheses Theorem 2.10 imply those of Lemma 5.5.

In the case $q = 1$ the hypotheses of Theorem 2.10 are

$$(\text{VD}) + (\text{Gcap}) + (\text{FK}) + (\text{TJ}),$$

and they imply (EP') by (5.3). Consequenlty, we obtain

$$(\text{VD}) + (\text{Gcap}) + (\text{FK}) + (\text{TJ}) \Rightarrow (\text{VD}) + (\text{FK}) + (\text{EP}') + (\text{TJ}),$$

where the right hand side constitutes the hypotheses of Lemma 5.5 for $q = 1$.

In the case $q > 1$, conditions (VD) and (TJ_q) imply (TJ) by (3.1); consequently (EP') is also satisfied. Hence, we obtain

$$(\text{VD}) + (\text{Gcap}) + (\text{FK}) + (\text{TJ}_q) \Rightarrow (\text{VD}) + (\text{FK}) + (\text{EP}') + (\text{TJ}_q),$$

so that we have the hypotheses of Lemma 5.5 for $q > 1$. □

6. Proof of on-diagonal upper bound

In this section we will prove the on-diagonal upper estimate of the heat kernel of Theorem 2.12 by applying the parabolic mean value inequality of Theorem 2.10.

Let us first introduce the notion of a *pointwise heat kernel* on a general metric measure space. As before, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(M, \mu)$ without killing term. $\{P_t\}_{t\ge 0}$ is the associated heat semigroup, that is, $P_t = e^{-t\mathcal{L}}$ where $\mathcal{L}$ is the generator of $(\mathcal{E}, \mathcal{F})$.

Definition 6.1. A function $p_t(x,y)$ of three variables $(t,x,y) \in (0,\infty) \times M \times M$ is said to be a *pointwise heat kernel* of $(\mathcal{E}, \mathcal{F})$ if there exists a regular $\mathcal{E}$-nest $\{F_k\}_{k=1}^\infty$ (independent of t) such that the following statements are true for all $t, s > 0$ and all $x, y \in M$.

(1) If one of points x, y lies outside $\cup_{k=1}^\infty F_k$, then

$$p_t(x,y) = 0.$$

(2) The continuity in one variable:

$$p_t(x,\cdot) \in C(\{F_k\}),$$

where $C(\{F_k\})$ is defined in (2.16).

(3) The measurability: $p_t(\cdot,\cdot)$ is jointly measurable on $M \times M$.

(4) The Markov property: $p_t(x,y) \ge 0$ and

$$\int_M p_t(x,y)d\mu(y) \le 1.$$

(5) The symmetry: $p_t(x,y) = p_t(y,x)$.

(6) The semigroup property:

$$p_{s+t}(x,y) = \int_M p_s(x,z)p_t(z,y)d\mu(z).$$

(7) The continuity in integral form: for any $f \in L^2$ and any $t > 0$,

$$\int_M p_t(\cdot,y)f(y)d\mu(y) \in C(\{F_k\}).$$

Moreover, we have

$$P_tf(x) = \int_M p_t(x,y)f(y)d\mu(y) \tag{6.1}$$

for a.a. $x \in M$.

It follows from (6.1) that, for any $f \in L^2$,

$$\int_M p_t(\cdot,y)f(y)d\mu(y) \overset{L^2}{\to} f \quad \text{as } t \to 0+.$$

Any pointwise heat kernel $p_t(x,y)$ on a metric measure space will give arise to a unique *pointwise heat semigroup.* Indeed, let us redefine P_t by using the identity (6.1) as definition, so that $P_tf(x)$ is now defined for any $t > 0$ and any $x \in M$ whenever the integral in (6.1) converges. For example, this is the case for $f \in L^\infty$ and $f \in L^2$, which implies that $P_tf(x)$ is defined pointwise for any $f \in L^q$ with $q \in [2,\infty]$.

We will derive (DUE) from (PMV_2). Before that, we prove the following estimate of the heat semigroup.

Lemma 6.2. *If conditions (VD) and (PMV_2) are satisfied then, for all $x \in M$ and $0 < t < W(x,\overline{R})$,*

$$\|P_tf\|_{L^\infty(B(x,\frac{1}{2}W^{-1}(x,t)))} \le \frac{C}{\sqrt{V(x,W^{-1}(x,t))}}\|f\|_{L^2} \quad f \in L^2, \tag{6.2}$$

where $C > 0$ is a constant independent of t, x, f.

Recall that $W^{-1}(x,\cdot)$ denotes the inverse function of $W(x,\cdot)$, for any fixed $x \in M$.

Proof. It suffices to prove (6.2) assuming that $0 \le f \in L^2 \cap L^\infty$. Then the function

$$u(t,x) := P_tf(x)$$

is non-negative, bounded, caloric in $(0,\infty) \times M$.

Fix now $x \in M$ and assume first that

$$0 < t < W(x,\sigma\overline{R}),$$

where constant σ comes from condition (PMV_2). Set $B := B(x,R)$, where

$$R := W^{-1}(x,t)$$

so that $t = W(B)$ and $R < \sigma\overline{R}$. Applying (PMV_2) to the function u in the cylinder $(0,t] \times B$, we obtain by (2.13), with $q' = 2$ and $T = t = W(B)$, that

$$\begin{aligned} \operatorname*{esup}_{\frac{1}{2}B} P_tf = \operatorname*{esup}_{\frac{1}{2}B} u(t,\cdot) &\le C\left(\frac{1}{\mu(B)W(B)}\int_{\frac{t}{2}}^t\int_B u(s,x)^2d\mu(x)ds\right)^{1/2} \\ &\quad + \frac{1}{\mu(B)^{1/2}}\sup_{\frac{t}{2}\le s\le t}\left(\int_{(\frac{1}{2}B)^c} u(s,y)^2d\mu(y)\right)^{1/2}. \end{aligned} \tag{6.3}$$

Since for any $s > 0$

$$\|u(s,\cdot)\|_{L^2} = \|P_sf\|_{L^2} \le \|f\|_{L^2},$$

we obtain

$$\int_{\frac{t}{2}}^{t}\int_B u(s,x)^2 d\mu(x)ds \leq \int_{\frac{t}{2}}^{t} \|u(s,\cdot)\|_{L^2}^2 ds \leq \frac{t}{2}\|f\|_{L^2}^2 = \frac{1}{2}W(B)\|f\|_{L^2}^2,$$

and, for the same reason,

$$\sup_{\frac{t}{2}\leq s\leq t}\left(\int_{(\frac{1}{2}B)^c} u(s,y)^2 d\mu(y)\right)^{1/2} \leq \sup_{\frac{t}{2}\leq s\leq t}\|P_s f\|_{L^2} \leq \|f\|_{L^2}.$$

Substituting into (6.3), we obtain

$$\operatorname*{esup}_{\frac{1}{2}B} P_t f \leq \frac{C}{\mu(B)^{1/2}}\|f\|_{L^2},$$

which proves (6.2) for $0 < t < W(x,\sigma\overline{R})$. If $\overline{R} = \infty$ then this finishes the proof.

Let now assume that $\overline{R} < \infty$. Then we still need to prove (6.2) in the case when

$$W(x,\sigma\overline{R}) \leq t < W(x,\overline{R}). \tag{6.4}$$

Set

$$t_0 := \frac{1}{2}W(x,\sigma\overline{R}) < \infty.$$

Since we have already proved (6.2) for $t = t_0$, we have that, for any t as in (6.4) and $\xi \in B(x,\frac{1}{2}W^{-1}(x,t))$,

$$\begin{aligned}\|P_t f\|_{L^\infty(B(\xi,\frac{1}{2}W^{-1}(\xi,t_0)))} =& \ \|P_{t_0}(P_{t-t_0}f)\|_{L^\infty(B(\xi,\frac{1}{2}W^{-1}(\xi,t_0)))}\\ &\leq \frac{C}{\sqrt{V(\xi,W^{-1}(\xi,t_0))}}\|P_{t-t_0}f\|_{L^2}\\ &\leq \frac{C}{\sqrt{V(\xi,W^{-1}(\xi,t_0))}}\|f\|_{L^2}.\end{aligned}$$

Let us estimate the term $V(\xi,W^{-1}(\xi,t_0))$. Indeed, if $W^{-1}(\xi,t_0)) \geq W^{-1}(x,t)$, then by (2.5)

$$\frac{V(x,W^{-1}(x,t))}{V(\xi,W^{-1}(\xi,t_0))} \leq \frac{V(x,W^{-1}(x,t))}{V(\xi,W^{-1}(x,t))} \leq C\left(\frac{2W^{-1}(x,t)}{W^{-1}(x,t)}\right)^\alpha = 2^\alpha C.$$

In the opposite case $W^{-1}(\xi,t_0) < W^{-1}(x,t)$ we have by $t < W(x,\overline{R})$

$$\begin{aligned}\frac{V(x,W^{-1}(x,t))}{V(\xi,W^{-1}(\xi,t_0))} &\leq C\left(\frac{W^{-1}(x,t)}{W^{-1}(\xi,t_0)}\right)^\alpha \quad \text{(using (2.5))}\\ &\leq C'\left(\frac{W(x,W^{-1}(x,t))}{W(\xi,W^{-1}(\xi,t_0))}\right)^{\alpha/\beta_1} \quad \text{(using the left inequality in (2.6))}\\ &= C'\left(\frac{t}{t_0}\right)^{\alpha/\beta_1} \leq C'\left(\frac{W(x,\overline{R})}{W(x,\sigma\overline{R})/2}\right)^{\alpha/\beta_1} \leq C.\end{aligned} \tag{6.5}$$

Hence, combining the above three inequality, we obtain

$$\|P_t f\|_{L^\infty(B(\xi,\frac{1}{2}W^{-1}(\xi,t_0)))} \leq \frac{C'}{\sqrt{V(x,W^{-1}(x,t))}}\|f\|_{L^2}. \tag{6.6}$$

On the other hand, the inequality (6.5) show that

$$\inf_{\xi\in B(x,\frac{1}{2}W^{-1}(x,t))} W^{-1}(\xi,t_0) \geq C^{-1}W^{-1}(x,t).$$

Therefore, since the set $B(x,\frac{1}{2}W^{-1}(x,t))$ can be covered by a countable family of balls like $B(\xi,\frac{1}{2}W^{-1}(\xi,t_0))$, we conclude from (6.6) that (6.2) also holds for any t from (6.4).
□

Now prove an upper bound of P_tf inside an arbitrary ball.

Lemma 6.3. *Assume that condition (VD) holds. Then inequality (6.2) is equivalent to the following: for any ball $B := B(x,R)$ of radius $R < \overline{R}$ and any $t > 0$,*

$$\|P_tf\|_{L^\infty(B)} \le \varphi(B,t)\|f\|_{L^2}, \quad f \in L^2, \tag{6.7}$$

where

$$\varphi(B,t) = \frac{C}{\sqrt{\mu(B)}}\left(\frac{W(B)}{t}+1\right)^{\frac{\alpha}{2\beta_1}},$$

and $C > 0$ is a constant that depends only on the constants in the hypotheses.

Proof. Fix $x \in M$, $R < \overline{R}$ and $t > 0$. We distinguish two cases when $t < \eta W(x,\overline{R})$ or not, where $\eta \in (0,1]$ is some constant to be determined below.

Case $0 < t < \eta W(x,\overline{R})$. Assume first that

$$R \le \frac{1}{2}W^{-1}(x,t)$$

so that $B = B(x,R) \subseteq B(x,\frac{1}{2}W^{-1}(x,t))$. Then (6.7) follows directly from (6.2).

Assume now that

$$R > \frac{1}{2}W^{-1}(x,t).$$

Then we can choose a number $\eta \in (0,1]$ so small that

$$t < W(z,\overline{R}) \ \text{ for any } z \in B(x,R). \tag{6.8}$$

Indeed, by the right inequality in (2.6) we have $\frac{W(x,\overline{R})}{W(z,\overline{R})} \le C$ whence

$$t < \eta W(x,\overline{R}) = \eta\frac{W(x,\overline{R})}{W(z,\overline{R})}W(z,\overline{R}) \le \eta C W(z,\overline{R}) < W(z,\overline{R})$$

provided $\eta < \frac{1}{C}$. Using (6.8) and (6.2), we obtain that

$$\|P_tf\|_{L^\infty(B(z,\frac{1}{2}W^{-1}(z,t)))} \le \frac{C}{\sqrt{V(z,W^{-1}(z,t))}}\|f\|_{L^2}. \tag{6.9}$$

Now, we claim that, for any $z \in B(x,R)$,

$$\frac{V(x,R)}{V(z,W^{-1}(z,t))} \le C\left(\frac{W(B)}{t}+1\right)^{\frac{\alpha}{\beta_1}} \tag{6.10}$$

for some universal constant $C > 0$ independent of t, B, z. Indeed, if $R \le W^{-1}(z,t)$, then by condition (VD)

$$\frac{V(x,R)}{V(z,W^{-1}(z,t))} \le \frac{V(x,R)}{V(z,R)} \le C.$$

In the opposite case $R > W^{-1}(z,t)$ we have by (VD) and the left inequality in (2.6) that

$$\frac{V(x,R)}{V(z,W^{-1}(z,t))} \le C\left(\frac{R}{W^{-1}(z,t)}\right)^{\alpha} \le C'\left(\frac{W(x,R)}{W(z,W^{-1}(z,t))}\right)^{\frac{\alpha}{\beta_1}} = C'\left(\frac{W(B)}{t}\right)^{\frac{\alpha}{\beta_1}}. \tag{6.11}$$

Hence, (6.10) is proved in both cases.

Therefore, plugging (6.10) into (6.9), we obtain, for any $z \in B(x,R)$

$$\|P_tf\|_{L^\infty(B(z,\frac{1}{2}W^{-1}(z,t)))} \le \frac{C}{\mu(B)^{1/2}}\left(\frac{W(B)}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|f\|_{L^2} = \varphi(B,t)\|f\|_{L^2}. \tag{6.12}$$

On the other hand, the inequality (6.11) also shows that $W^{-1}(z,t)$ has a uniform lower bound for all $z \in B(x,R)$. Hence, we can cover B by a countable family of balls like

$B(z, \frac{1}{2}W^{-1}(z,t))$ with z varying in B, and obtain from (6.12) that, for any $0 < t < \eta W(x, \overline{R})$,

$$\|P_t f\|_{L^\infty(B)} \le \varphi(B,t)\|f\|_{L^2}.$$

Case $t \ge \eta W(x, \overline{R})$ and $\overline{R} < \infty$. Using the semigroup property and the latter inequality for $t_0 := \frac{\eta}{2}W(x, \overline{R})$, we obtain

$$\begin{aligned}\|P_t f\|_{L^\infty(B)} = \; & \|P_{t_0}P_{t-t_0}f\|_{L^\infty(B)} \le \varphi(B,t_0)\,\|P_{t-t_0}f\|_{L^2} \\ \le \; & \frac{C}{\mu(B)^{1/2}}\left(\frac{W(B)}{t_0}+1\right)^{\frac{\alpha}{2\beta_1}}\|f\|_{L^2} \\ \le \; & \frac{C}{\mu(B)^{1/2}}\left(\frac{2W(x,\overline{R})}{\eta W(x,\overline{R})}+1\right)^{\frac{\alpha}{2\beta_1}}\|f\|_{L^2} \\ \le \; & \frac{C'}{\mu(B)^{1/2}}\|f\|_{L^2} \le \frac{C'}{\mu(B)^{1/2}}\left(\frac{W(B)}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|f\|_{L^2},\end{aligned}$$

which finishes the proof of (6.7). □

Corollary 6.4. *Assume that condition* (VD) *holds. Then inequality (6.7) is equivalent to the following: for any ball* $B := B(x,R)$ *of radius* $R > 0$ *and any* $t > 0$,

$$\|P_t f\|_{L^\infty(B)} \le \frac{C}{\sqrt{\mu(B)}}\left(\frac{R}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}}\left(\frac{W(B)}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|f\|_{L^2} \quad f \in L^2. \tag{6.13}$$

where $C > 0$ *is a constant independent of* t, B, f.

Proof. Indeed, it suffices to consider the case when $\overline{R} < \infty$, otherwise, this corollary follows directly from Lemma 6.3. Fix $x \in M$ and $t > 0$. Let $B := B(x,R)$ with $R \ge \overline{R}$ and $B_0 := B(z, \overline{R}/2)$ for $z \in B$. By (6.7), condition (VD) and (2.6), we have for any $f \in L^2$

$$\begin{aligned}\|P_t f\|_{L^\infty(B_0)} &\le \frac{C}{\sqrt{\mu(B_0)}}\left(\frac{W(B_0)}{t}+1\right)^{\frac{\alpha}{2\beta_1}}\|f\|_{L^2} \\ &\le \frac{C'}{\sqrt{\mu(B)}}\left(\frac{2R}{\overline{R}}\right)^{\frac{\alpha}{2}}\left(\frac{W(B)}{t}\left(\frac{\overline{R}}{2R}\right)^{\beta_1}+1\right)^{\frac{\alpha}{2\beta_1}}\|f\|_{L^2}.\end{aligned}$$

Covering B by an at most countable family of balls like $B(z, \overline{R}/2)$, we obtain (6.13) from the above inequality. The proof is complete. □

The following theorem is [18, Theorem 2.2 (with $T_0 = \infty$)], which will be used in Lemma 6.6.

Theorem 6.5. *Let* $q \in [1,2]$. *Assume that there exist a countable family* $\mathcal{S}$ *of open sets with* $M = \cup_{U\in\mathcal{S}}U$ *and a function* $\varphi : \mathcal{S} \times (0,\infty) \mapsto \mathbb{R}_+$ *such that, for each* $t \in (0,\infty)$, $U \in \mathcal{S}$ *and each* $f \in L^q \cap L^2$

$$\|P_t f\|_{L^\infty(U)} \le \varphi(U,t)\|f\|_{L^q}.$$

Then $\{P_t\}_{t>0}$ *possesses a pointwise heat kernel* $p_t(x,y)$ *(in the sense of Definition 6.1 except that Property (7) holds for* $f \in L^q$ *instead of* $f \in L^2$*) defined in* $(0,\infty) \times M \times M$ *that satisfies for each* $t \in (0,\infty)$ *and* $x \in U$

$$\|p_t(x,\cdot)\|_{L^{q'}} \le \varphi(U,t)$$

where where $q' = \frac{q}{q-1}$ *is the Hölder conjugate of* q.

In the next lemma, we obtain (DUE).

Lemma 6.6. *If condition* (VD) *and* (6.13) *hold, then there exists the heat kernel* $p_t(x,y)$ *of* $(\mathcal{E},\mathcal{F})$ *that satisfies all the conditions of Definition* 6.1 *for a regular* $\mathcal{E}$*-nest* $\{F_k\}$*. Moreover, for any* $C_0 \geq 1$*, there exists a constant* $C > 0$ *such that, for any* $x \in M$ *and any* $0 < t < C_0 W(x, \overline{R})$*,*

$$p_t(x,x) \leq \frac{C}{V(x, W^{-1}(x,t))}. \tag{6.14}$$

In particular, we have the following implication:

$$\text{(VD)} + (\text{PMV}_2) \Rightarrow \text{(DUE)}.$$

Proof. Since (6.13) holds, we see the hypothesis of Theorem 6.5 is satisfied with $q = 2$ and

$$\mathcal{S} = \{B(y,R) : R \in \mathbb{Q}_+\},$$

for any fixed $y \in M$.

Hence, by Theorem 6.5, the heat kernel $p_t(x,y)$ of the form $(\mathcal{E},\mathcal{F})$ exists, which satisfies all the properties in Definition 6.1, and moreover, for every $z \in B := B(x,R)$ ($x \in M$ and $R \in \mathbb{Q}_+$) and every $t > 0$

$$\|p_t(z,\cdot)\|_{L^2(M)} \leq \varphi(B,t) := \frac{C}{\sqrt{\mu(B)}} \left(\frac{R}{\overline{R}} \vee 1\right)^{\frac{\alpha}{2}} \left(\frac{W(B)}{t} + 1\right)^{\frac{\alpha}{2\beta_1}}. \tag{6.15}$$

Let us show (6.14). Indeed, fix any $C_0 \geq 1$ and $t < C_0 W(x,\overline{R})$ so that

$$W^{-1}(x, C_0^{-1}t) < \overline{R}.$$

Then, by (6.15) and (VD), we have for any $R \in \mathbb{Q}_+ \cap (0, W^{-1}(x, C_0^{-1}t))$

$$\begin{aligned}
\sqrt{p_t(x,x)} &= \|p_{t/2}(x,\cdot)\|_{L^2(M)} \leq \varphi(B(x,R), t/2) \\
&\leq \frac{C}{\sqrt{V(x,R)}} \left(\frac{W(x, W^{-1}(x, C_0^{-1}t))}{t/2} + 1\right)^{\frac{\alpha}{2\beta_1}} \\
&= \frac{C\left(2C_0^{-1}+1\right)^{\frac{\alpha}{2\beta_1}}}{\sqrt{V(x,R)}} = \frac{C'}{\sqrt{V(x,W^{-1}(x,t))}} \sqrt{\frac{V(x,W^{-1}(x,t))}{V(x,R)}} \\
&\leq \frac{C}{\sqrt{V(x,W^{-1}(x,t))}} \left(\frac{W^{-1}(x,t)}{R}\right)^{\frac{\alpha}{2}}.
\end{aligned}$$

Since $R \in \mathbb{Q}_+ \cap (0, W^{-1}(x, C_0^{-1}t))$ is arbitrary, by passing to the limit in the above inequality as $R \to W^{-1}(x, C_0^{-1}t) (< \overline{R})$ and using the left inequality in (2.6), we obtain

$$\begin{aligned}
\sqrt{p_t(x,x)} &\leq \frac{C}{\sqrt{V(x,W^{-1}(x,t))}} \left(\frac{W^{-1}(x,t)}{W^{-1}(x, C_0^{-1}t)}\right)^{\frac{\alpha}{2}} \\
&\leq \frac{C}{\sqrt{V(x,W^{-1}(x,t))}} \left(\frac{W(x, W^{-1}(x,t))}{W(x, W^{-1}(x, C_0^{-1}t))}\right)^{\frac{\alpha}{2\beta_1}} \\
&= \frac{C}{\sqrt{V(x,W^{-1}(x,t))}} \left(\frac{t}{C_0^{-1}t}\right)^{\frac{\alpha}{2\beta_1}} = \frac{C C_0^{\frac{\alpha}{2\beta_1}}}{\sqrt{V(x,W^{-1}(x,t))}},
\end{aligned}$$

thus showing (6.14). □

Proof of Theorem 2.12. Combining Lemmas 6.2, 6.3, Corollary 6.4 and Lemma 6.6, we obtain the following implication:

$$\text{(VD)} + (\text{PMV}_2) \Rightarrow \text{(VD)} + (6.2) \Rightarrow \text{(VD)} + (6.7) \Rightarrow \text{(VD)} + (6.13) \Rightarrow \text{(DUE)},$$

thus proving the implication (2.18). □

Lemma 6.7. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2 without killing part. Assume that the heat kernel $p_t(x,y)$ of the form $(\mathcal{E},\mathcal{F})$ exists, and assume that, there exist constants $\delta>0$ and $C>0$ such that for μ-almost all $x\in M$ and any $0<t<\delta W(x,\overline{R})$,*

$$p_t(x,x)\le \frac{C}{V(x,W^{-1}(x,t))}. \tag{6.16}$$

then, the inequality (6.2) holds for all $x\in M$, $0<t<W(x,\overline{R})$, and any $f\in L^2$.

Proof. Fix $x\in M$, $0<t<\frac{1}{2}\delta W(x,\overline{R})$ and $f\in L^2$.

By (6.16) and Hölder inequality, we have for μ-a.a. $y\in M$,

$$|P_tf(y)|\le\sqrt{p_{2t}(y,y)}\|f\|_{L^2}\le\frac{C}{\sqrt{V(y,W^{-1}(y,2t))}}\|f\|_{L^2}\le\frac{C}{\sqrt{V(y,W^{-1}(y,t))}}\|f\|_{L^2}.$$

Let $y\in B(x,\frac{1}{2}W^{-1}(x,t))$. If $W^{-1}(y,t)\ge W^{-1}(x,t)$, then by (VD), we have

$$\frac{V(x,W^{-1}(x,t))}{V(y,W^{-1}(y,t))}\le\frac{V(x,W^{-1}(x,t))}{V(y,W^{-1}(x,t))}\le C\left(\frac{W^{-1}(x,t)}{W^{-1}(x,t)}\right)^{\alpha}=C.$$

If $W^{-1}(y,t)<W^{-1}(x,t)$, then by (VD) and the left inequality in (2.6), we have

$$\frac{V(x,W^{-1}(x,t))}{V(y,W^{-1}(y,t))}\le C\left(\frac{W^{-1}(x,t)}{W^{-1}(y,t)}\right)^{\alpha}\le C'\left(\frac{W(x,W^{-1}(x,t))}{W(y,W^{-1}(y,t))}\right)^{\alpha/\beta_1}=C'\left(\frac{t}{t}\right)=C'.$$

Combining the above three inequalities, we have

$$\|P_tf\|_{L^\infty(B(x,\frac{1}{2}W^{-1}(x,t)))}\le\frac{C}{\sqrt{V(x,W^{-1}(x,t))}}\|f\|_{L^2}.$$

Finally, following the arguments in the second part of the proof of Lemma 6.2, we can extend the range of t in the above inequality from $(0,\frac{1}{2}\delta W(x,\overline{R}))$ to $(0,W(x,\overline{R}))$. The proof is complete. □

Remark 6.8. Under condition (VD), by Lemma 6.3, Corollary 6.4 and Lemmas 6.6, 6.7, the following equivalences are true:

$$(6.2)\Leftrightarrow(6.7)\Leftrightarrow(6.13)\Leftrightarrow(\text{DUE}).$$

In particular, we see that under condition (VD), condition (DUE) holds true for any $C_0\ge1$ if and only if it holds true for $C_0=1$.

Corollary 6.9. *Let condition (VD) be satisfied. Let $q\in[1,2]$. Assume that for any $x\in M$ and $t<W(x,\overline{R})$,*

$$\|P_tf\|_{L^\infty(B(x,\frac{1}{2}W^{-1}(x,t)))}\le\frac{C}{V(x,W^{-1}(x,t))^{1/q}}\|f\|_{L^q}\qquad\forall\ f\in L^q\cap L^2(M). \tag{6.17}$$

Then, the heat kernel $p_t(x,y)$ exists, and for any open set $U\subset M$, the heat kernel $p_t^U(x,y)$ of Dirichlet form $(\mathcal{E},\mathcal{F}(U))$ exists. Moreover, the following is true

$$p_t^U(x,y)\le p_t(x,y)\quad\forall t>0,\ x,y\in M. \tag{6.18}$$

In particular, condition (DUE) implies (6.18).

Proof. Denote the heat semigroup of $(\mathcal{E},\mathcal{F}(U))$ by $\{P_t^U\}$. Following the proof of Lemma 6.3, we obtain by condition (VD) that for any ball $B:=B(x,R)$ of radius $R<\overline{R}$ and $t<W(x,\overline{R})$,

$$\left\|P_t^Uf\right\|_{L^\infty(B)}\le\|P_tf\|_{L^\infty(B)}\le\varphi(B,t)\|f\|_{L^q}\qquad\forall\ f\in L^q\cap L^2(M),$$

where

$$\varphi(B,t)=\frac{C}{V(x,R)^{1/q}}\left(\frac{R}{\overline{\overline{R}}}\vee1\right)^{\frac{\alpha}{q}}\left(\frac{W(x,R)}{t}\vee1\right)^{\frac{\alpha}{q\beta_1}}.$$

Then, applying Theorem 6.5 to the heat semigroup $\{P_t\}$ and $\{P_t^U\}$, we obtain that both the heat kernel $p_t(x,y)$ of $\{P_t\}$ and the Dirichlet heat kernel $p_t^U(x,y)$ of $\{P_t^U\}$ (or Dirichlet form $(\mathcal{E},\mathcal{F}(U))$) exist. Moreover, there is a regular $\mathcal{E}$-nest $\{F'_k\}_{k=1}^\infty$ such that for any $x \in M$ and $t > 0$,

$$p_t(x,\cdot), p_t^U(x,\cdot) \in C(\{F'_k\}).$$

Then, by setting $p_t^U(x,y) = 0 = p_t(x,y)$ whenever $x \notin \cup_{k=1}^\infty F'_k$ or $y \notin \cup_{k=1}^\infty F'_k$ and by using [12, Theorem 2.1.2(ii), p. 69], we obtain (6.18).

It remains to prove that (DUE) implies (6.18). Indeed, by Remark 6.8, (DUE) implies (6.2) which is exactly (6.17) with $q = 2$. Then, by the result in the first part, there is a regular $\mathcal{E}$-nest $\{F'_k\}_{k=1}^\infty$ such that $F'_k \subset F_k$ for $k \geq 1$ and for any $x \in M$ and $t > 0$, $p_t(x,\cdot), p_t^U(x,\cdot) \in C(\{F'_k\})$, where $\{F_k\}_{k=1}^\infty$ is the $\mathcal{E}$-nest as in condition (DUE). Again, by using [12, Theorem 2.1.2(ii), p. 69], we obtain (6.18). □

Remark 6.10. Corollary 6.9 shows that whenever the global heat kernel $p_t(x,y)$ exists and has the on-diagonal upper estimate, for any open set $U \subset M$, the Dirichlet heat kernel $p_t^U(x,y)$ also exists. Moreover, usually $p_t^U(x,y) \leq p_t(x,y)$ for $\mu \times \mu$-a.a. $(x,y) \in M \times M$. While, Corollary 6.9 also shows that one can choose a quasi-continuous version of $p_t^U(x,y)$ such that $p_t^U(x,y) \leq p_t(x,y)$ for all $(x,y) \in M \times M$.

Proof of Corollary 2.13. This corollary follows directly from Proposition 3.1, the implication (2.15) with $q = 2$ and Theorem 2.12. □

7. Appendix

In this appendix, we collect some facts that have been used in this paper.

Proposition 7.1 ([19, Proposition 15.1 in Appendix])**.** *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in L^2. Then the following statements are true.*

(*i*) *If $u \in \mathcal{F}'$ and $F : \mathbb{R} \mapsto \mathbb{R}$ is a Lipschitz function, then $F(u) \in \mathcal{F}'$.*
(*ii*) *If $u \in \mathcal{F}' \cap L^\infty$ and $v \in \mathcal{F} \cap L^\infty$ then $uv \in \mathcal{F} \cap L^\infty$*
(*iii*) *Let Ω be an open subset of M. If $u \in \mathcal{F}' \cap L^\infty$ and $v \in \mathcal{F}(\Omega) \cap L^\infty$, then $uv \in \mathcal{F}(\Omega)$.*

The following properties on weak differentiation of a function u were proved in [21, Lemma 5.1].

Proposition 7.2. *Assume that both functions $u : I \to L^2$ and $v : I \to L^2$ are weakly differentiable at t. Then we have the following.*

(*i*) (Product rule) *The inner product (u,v) is also differentiable at t, and*

$$(u,v)' = (u',v) + (u,v').$$

(*ii*) (Chain rule) *Let Φ be a smooth real-valued function on $\mathbb{R}$ such that*

$$\Phi(0) = 0, \quad \sup_{\mathbb{R}} |\Phi'| < \infty, \quad \sup_{\mathbb{R}} |\Phi''| < \infty.$$

Then $\Phi(u)$ is also weakly differentiable at t, and

$$\Phi(u)' = \Phi'(u)u'. \tag{7.1}$$

Proposition 7.3 ([19, Proposition 15.4 in Appendix])**.** *Let $\{a_k\}_{k=0}^\infty$ be a sequence of non-negative numbers such that*

$$a_k \leq D\lambda^k a_{k-1}^{1+\nu} \quad \textit{for } k = 1, 2, \cdots$$

for some constants $D, \nu > 0$ and $\lambda \geq 1$. Then for any $k \geq 0$,

$$a_k \leq D^{-\frac{1}{\nu}} \left(D^{\frac{1}{\nu}} \lambda^{\frac{1+\nu}{\nu^2}} a_0 \right)^{(1+\nu)^k}.$$

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