THE DAVIES METHOD REVISITED FOR HEAT KERNEL UPPER BOUNDS OF REGULAR DIRICHLET FORMS ON METRIC MEASURE SPACES

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Abstract. We apply the Davies method to prove that for any regular Dirichlet form on a metric measure space, an off-diagonal stable-type upper bound of the heat kernel is equivalent to the conjunction of the on-diagonal upper bound, a cutoff inequality on any two concentric balls, and the jump kernel upper bound, for any walk dimension. If in addition the jump kernel vanishes, that is, if the Dirichlet form is strongly local, we obtain sub-Gaussian upper bound. This gives a unified approach to obtaining heat kernel upper bounds for both the non-local and the local Dirichlet forms.

Contents

1. Introduction

We are concerned with heat kernel upper bounds for both nonlocal and local Dirichlet forms on metric measure spaces.

Let (M, d) be a locally compact separable metric space and μ be a Radon measure on M with full support, and the triple (M, d, μ) is called a *metric measure space*. Let $(\mathcal{E}, \mathcal{F})$ be a regular *Dirichlet form* in $L^2(M, \mu)$, and $\mathcal{L}$ be its generator (non-positive definite self-adjoint). Let

$$\left\{P_t = e^{t\mathcal{L}}\right\}_{t\geq 0}$$

be the associated heat semigroup. Recall that the form $(\mathcal{E}, \mathcal{F})$ is *conservative* if $P_t 1 = 1$ holds for all $t > 0$.

Let Ω be a non-empty open set on M, let $\mathcal{F}(\Omega)$ be the closure of $\mathcal{F} \cap C_0(\Omega)$ in the norm of $\mathcal{F}$, where $C_0(\Omega)$ is the space of all continuous functions with compact supports in Ω. It is known that if $(\mathcal{E}, \mathcal{F})$ is regular, then $(\mathcal{E}, \mathcal{F}(\Omega))$ is a regular Dirichlet form in $L^2(\Omega, \mu)$ (cf. [14, Lemma 1.4.2 (ii) p.29]). We denote by $\mathcal{L}^\Omega$ the generator of $(\mathcal{E}, \mathcal{F}(\Omega))$ and by $\{P_t^\Omega\}$ the associated semigroup.

A family $\{p_t\}_{t>0}$ of non-negative $\mu \times \mu$-measurable functions on $M \times M$ is called the *heat kernel* of $(\mathcal{E}, \mathcal{F})$ if for any $f \in L^2(M, \mu)$ and $t > 0$,

$$P_t f(x) = \int_M p_t(x, y) f(y) d\mu(y)$$

for μ-almost all $x \in M$.

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Typically, there are two distinct types of heat kernel estimates on metric space, depending on whether the form $(\mathcal{E},\mathcal{F})$ is local or not. Indeed, assume that the heat kernel exists and satisfies the following estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\Phi\left(\frac{d(x,y)}{ct^{1/\beta}}\right) \tag{1.1}$$

with some function Φ and two positive parameters α,β, where the sign $\asymp$ means that both $\leq$ and $\geq$ are true but with different values of C, c. Then either $\Phi(s) = \exp\left(-s^{\frac{\beta}{\beta-1}}\right)$ (thus $(\mathcal{E},\mathcal{F})$ is local), or $\Phi(s) = (1+s)^{-(\alpha+\beta)}$ (thus $(\mathcal{E},\mathcal{F})$ is non-local), see [24]. For the local case, the heat kernel $p_t(x,y)$ admits the following *Gaussian* ($\beta = 2$)- or *Sub-Gaussian* ($\beta > 2$) estimate:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right), \tag{1.2}$$

where $\alpha > 0$ is the Hausdorff dimension and $\beta \geq 2$ is termed the walk dimension, see for example [2, 3, 4, 7, 26, 28]. Some equivalence conditions are stated in [6, 18, 23, 25]. On the other hand, for the non-local case, the heat kernel $p_t(x,y)$ admits the *stable-like* estimates:

$$p_t(x,y) \asymp \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{1.3}$$

where $\alpha > 0$ and $\beta > 0$, see for example, [5, 8, 10, 11] for $0 < \beta < 2$, and [12, 16, 17, 21] for any $\beta > 0$. Note that estimate (1.3) can also be obtained by using the subordination technique, see for example [15, 27, 29, 33]. It was shown in [24] that estimates (1.2) and (1.3) exhaust all possible two-sided estimates of heat kernels upon assuming (1.1).

Recently, Murugan and Saloff-Coste extend the Davies method developed in [9, 13] to obtain heat kernel upper bounds, for local Dirichlet forms on metric spaces in [32] and for non-local Dirichlet forms on infinite graphs in [31], where a cutoff inequality introduced in [1] plays an important role.

The purpose of this paper is twofold:

(1) to extend the result in [31] to the metric measure space;
(2) to unify the Davies method for both local and nonlocal Dirichlet forms.

More precisely, we give some equivalence characterizations of heat kernel upper bounds both in (1.3) for any $\beta > 0$ and in (1.2) for any $\beta > 1$, see Theorem 1.4 below, by applying the Davies method in a unified way. These characterization are stable under bounded perturbation of Dirichlet forms. We mention that one of our starting point here is from condition (CIB) below that is the weakest version among all the similar conditions in previous papers [1], [23], [32, 31], [12], [17].

Let us return to the general setup of a metric measure space (M,d,μ) equipped with a regular Dirichlet form $(\mathcal{E},\mathcal{F})$. Assume that $\mathcal{E}$ admits the following decomposition

$$\mathcal{E}(u,v) = \mathcal{E}^{(L)}(u,v) + \mathcal{E}^{(J)}(u,v), \tag{1.4}$$

where $\mathcal{E}^{(L)}$ denotes the *local part* and

$$\mathcal{E}^{(J)}(u,v) = \iint_{M\times M\setminus\mathrm{diag}} (u(x)-u(y))(v(x)-v(y))dj(x,y)$$

is a *jump part* with jump measure j defined on $M\times M\setminus\mathrm{diag}$. We assume that j has a density with respect to $\mu\times\mu$, denoted by $J(x,y)$, and so the jump part $\mathcal{E}^{(J)}$ can be written as

$$\mathcal{E}^{(J)}(u,v) = \iint_{M\times M}(u(x)-u(y))(v(x)-v(y))J(x,y)d\mu(y)d\mu(x). \tag{1.5}$$

For every $w \in \mathcal{F} \cap L^\infty$, there exists a unique positive finite Radon measure $\Gamma(w)$ on M, termed an *energy measure*, such that for any $\phi \in \mathcal{F} \cap L^\infty$[1],

$$\int \phi d\Gamma(w) = \mathcal{E}(w\phi, w) - \frac{1}{2}\mathcal{E}(\phi, w^2), \tag{1.6}$$

where and in the sequel the integration $\int$ means over M. The energy measure $\Gamma(w)$ can be uniquely extended to any $w \in \mathcal{F}$. For functions $v, w \in \mathcal{F}$, the signed measure $\Gamma(v, w)$ is defined by

$$\Gamma(v, w) = \frac{1}{2}\left(\Gamma(v + w) - \Gamma(v) - \Gamma(w)\right) \tag{1.7}$$

(see [30, formula 3,11]), and $\Gamma(v, v) \equiv \Gamma(v)$ and

$$\mathcal{E}(v, w) = \int_M d\Gamma(v, w).$$

For any $u, v, w \in \mathcal{F} \cap L^\infty$, we have by (1.6),

$$\int ud\Gamma(v, w) = \frac{1}{2}\left(\mathcal{E}(uv, w) + \mathcal{E}(v, uw) - \mathcal{E}(vw, u)\right), \tag{1.8}$$

and, from this,

$$\int d\Gamma(uv, w) = \mathcal{E}(uv, w) = \int ud\Gamma(v, w) + \int vd\Gamma(u, w). \tag{1.9}$$

Denote by $\Gamma_L(\cdot)$ the energy measure associated with *local part* $\mathcal{E}^{(L)}$. Then

$$d\Gamma(u)(x) = d\Gamma_L(u)(x) + \int_{M\setminus \mathrm{diag}} (u(x) - u(y))^2\, dj(x, y). \tag{1.10}$$

Denote by $B(x, r)$ the open metric ball of radius $r > 0$ centered at x. We always assume every ball $B(x, r)$ is precompact. In particular the volume function

$$V(x, r) := \mu\left(B(x, r)\right)$$

is finite and positive for any $x \in M$ and $r > 0$. Denote by λB a concentric ball of B with radius λr where r is the radius of B.

For a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ with a jump kernel J, we define for $\rho \geq 0$

$$\mathcal{E}_\rho(u, v) = \mathcal{E}^{(L)}(u, v) + \int_M \int_{B(x,\rho)} (u(x) - u(y))(v(x) - v(y))J(x, y)d\mu(y)d\mu(x). \tag{1.11}$$

It is known that $(\mathcal{E}_\rho, \mathcal{F})$ is a closable bilinear form and can be extended to a regular Dirichlet form $(\mathcal{E}_\rho, \mathcal{F}_\rho)$ with $\mathcal{F} \subset \mathcal{F}_\rho$ (see [21, Section 4]). Denote by $q_t(x, y)$, $\{Q_t\}_{t\geq 0}$, $\Gamma_\rho(\cdot)$ the heat kernel, heat semigroup and energy measure of $(\mathcal{E}_\rho, \mathcal{F}_\rho)$, respectively (we sometimes drop the superscript "ρ" from $q_t^{(\rho)}(x, y)$, $\{Q_t^{(\rho)}\}_{t\geq 0}$ for simplicity). Note that if $J \equiv 0$ or if $\rho = 0$, then $\left(\mathcal{E}_\rho, \mathcal{F}_\rho\right) = (\mathcal{E}, \mathcal{F}) = (\mathcal{E}^{(L)}, \mathcal{F})$, which is strongly local. Denote by

$$d\Gamma_\rho(u)(x) = d\Gamma_L(u)(x) + \left\{\int_{B(x,\rho)} (u(x) - u(y))^2 J(x, y)d\mu(y)\right\} d\mu(x). \tag{1.12}$$

Throughout this paper we fix some numbers $\alpha > 0, \beta > 0$ except otherwise is stated. In the sequel, the letters C, C', c, c' denote universal positive constants which may vary at each occurrence.

Introduce the following conditions.

Upper α-regularity. *For all $x \in M$ and all $r > 0$,*

$$V(x, r) \leq Cr^\alpha. \tag{$V_\leq$}$$

[1] Any function in $\mathcal{F}$ admits a quasi-continuous modification (cf. [14, Theorem 2.1.3,p.71]). Without loss of generality, every function in $\mathcal{F}$ will be replaced by its quasi-continuous modification in this paper.

On-diagonal upper estimate. *The heat kernel p_t exists and satisfies the on-diagonal upper estimate*

$$p_t(x, y) \le \frac{C}{t^{\alpha/\beta}} \tag{DUE}$$

for all $t > 0$ and μ-almost all x, $y \in M$.

Upper estimate of non-local type. *The heat kernel p_t exists and satisfies the off-diagonal upper estimate*

$$p_t(x, y) \le \frac{C}{t^{\alpha/\beta}}\left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \tag{UE}$$

for all $t > 0$ and μ-almost all $x, y \in M$.

Upper bound of jump density. *The jump density $J(x, y)$ exists and admits the estimate*

$$J(x, y) \le Cd(x, y)^{-(\alpha+\beta)} \tag{$\mathrm{J}_\le$}$$

for μ-almost all $x, y \in M$.

If $(\mathcal{E}, \mathcal{F})$ is local, we have $J \equiv 0$ so that ($\mathrm{J}_\le$) is trivially satisfied. In general, condition ($\mathrm{J}_\le$) restricts the long jumps and can be viewed as a measure of non-locality.

Upper estimate of local type. *The heat kernel p_t exists and satisfies the off-diagonal upper estimate*

$$p_t(x, y) \le \frac{C}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d(x, y)}{ct^{1/\beta}}\right)^{\beta/(\beta-1)}\right) \tag{$\mathrm{UE}_{\mathrm{loc}}$}$$

for all $t > 0$ and μ-almost all $x, y \in M$, where $\beta > 1$.

Let Ω be an open subset of M and $A \Subset \Omega$ be a Borel set (where $A \Subset \Omega$ means that A is precompact and its closure $\overline{A} \subset \Omega$). Recall that ϕ is a *cutoff function* of (A, Ω) if $\phi \in \mathcal{F}(\Omega)$, $0 \le \phi \le 1$ in M, and $\phi = 1$ in an open neighborhood of A. We denote the set of all cutoff functions of (A, Ω) by *cutoff*(A, Ω). It is known that if $(\mathcal{E}, \mathcal{F})$ is regular, then for any open set $\Omega \subset M$ and any nonempty $A \Subset \Omega$, cutoff(A, Ω) is non-empty (cf. [14, Lemma 1.4.2 (ii) p.29]).

Cutoff inequality on balls. *The* cutoff inequality on balls *holds on M if there exist constants $C_1 \ge 0, C_2 > 0$ such that for every $u \in \mathcal{F} \cap L^\infty$ and for every $x \in M$, $R, r > 0$ there exists a function $\phi \in$ cutoff$(B(x, R), B(x, R + r))$ satisfying that*

$$\int_M u^2 d\Gamma(\phi) \le C_1 \int_M d\Gamma(u) + \frac{C_2}{r^\beta}\int_M u^2 d\mu, \tag{CIB}$$

where $d\Gamma(u)$ is defined by (1.10).

Note that constants C_1, C_2 in (CIB) are universal (independent of u, ϕ, x, R, r) whilst the cutoff function ϕ may depend on the function u.

Remark 1.1. A similar condition to (CIB) is introduced in [17] for jump-type Dirichlet forms, which is termed *Condition* (AB), named after Andres and Barlow, who first introduced this kind of neat condition in [1] under the framework of local Dirichlet forms and who labelled it by (CSA) – a *cutoff Sobolev inequality in annulus* although it is actually unrelated to the classical *Sobolev* inequality. We emphasize that the condition (CIB) here is slightly weaker than Condition (AB) in [17] in that the second integral in (CIB) is over M against measure $d\Gamma(u)$, instead of over the larger ball $B(x, R + r)$ against measure $\phi^2 d\Gamma(u)$ in [17]. More variants than (CIB) were addressed in [12, Definition 2.2].

Remark 1.2. Condition (CIB) can be easily verified with $C_1 = 0$ for purely jump-type Dirichlet forms with $0 < \beta < 2$, provided that conditions $(\mathrm{J}_{\leq})$, $(\mathrm{V}_{\leq})$ are satisfied, using the standard bump function on balls (see [12, Remark 1.7] or [17, the proof of Corollary 2.12]).

The following is the main contribution of this paper.

Theorem 1.3. *Let (M, d, μ) be a metric measure space with precompact balls, and let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$ satisfying (1.4) with jump kernel J. If condition $(V_{\leq})$ is satisfied, then the following implication holds:*

$$(DUE) + (CIB) + (J_{\leq}) \Rightarrow (UE). \tag{1.13}$$

If in addition $\beta > 1$, then

$$(DUE) + (CIB) + (J \equiv 0) \Rightarrow (UE_{loc}). \tag{1.14}$$

We apply the Davies method to prove both (1.13) and (1.14). Particularly, in order to show the implication (1.14), we first derive a weaker upper bound of the heat kernel, see (3.75) below, and then obtain $(\mathrm{UE}_{\mathrm{loc}})$ by a self-improvement technique used in [19], see Lemma 3.5 and Remark 3.6 below.

As a consequence of Theorem 1.3, we have the following.

Theorem 1.4. *Let (M, d, μ) be a metric measure space with precompact balls and $(\mathcal{E}, \mathcal{F})$ be a regular conservative Dirichlet form in L^2 with a jump kernel J. If $(V_{\leq})$ holds, then*

$$(UE) \Leftrightarrow (DUE) + (CIB) + (J_{\leq}). \tag{1.15}$$

If in addition $\beta > 1$, then

$$(UE_{loc}) \Leftrightarrow (DUE) + (CIB) + (J \equiv 0). \tag{1.16}$$

The proof of Theorem 1.3 and Theorem 1.4 will be given in Section 3.

Remark 1.5. For the *nonlocal* case, a similar equivalence to (1.15) was obtained in [12] with (CIB) being replaced by condition $\mathrm{CSJ}(\phi)$ but for more general settings equipped with doubling measures and more general jump kernels involving ϕ, and also in [17] with condition (CIB) being replaced by condition $(Gcap)$ or condition (AB). For the *local* case, a similar equivalence to (1.16) was obtained in [1], [23] and [32] under different variants than condition (CIB).

2. Cutoff inequalities on balls

In this section, we first derive (CIB) from condition (S)-the survival estimate, see (S) below. We then state two inequalities, see (2.13), (2.14) below, which will be used in the Davies method. Inequality (2.13) can be viewed as a self-improvement of condition (CIB).

We need the following formula.

Proposition 2.1. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. Then, for any two functions $u, \varphi \in \mathcal{F} \cap L^\infty$ with $supp(\varphi) \subset \Omega$ for any open subset Ω of M,*

$$\int_\Omega u^2 d\Gamma_\Omega(\varphi) \leq 2\mathcal{E}(u^2\varphi, \varphi) + 4 \int_\Omega \varphi^2 d\Gamma_\Omega(u), \tag{2.1}$$

where $d\Gamma_\Omega(u)$ is defined by

$$d\Gamma_\Omega(u)(x) = d\Gamma_L(u)(x) + \int_{M \setminus diag} \mathbf{1}_\Omega(y) \, (u(x) - u(y))^2 \, dj(x, y). \tag{2.2}$$

Proof. We first show that

$$\int_\Omega u^2 d\Gamma_L(\varphi) \leq 2\mathcal{E}^{(L)}(u^2\varphi, \varphi) + 4 \int_\Omega \varphi^2 d\Gamma_L(u). \tag{2.3}$$

Indeed, using the Leibniz and chain rules of $d\Gamma_L(\cdot)$ (cf. [14, Lemma 3.2.5, Theorem 3.2.2]) and using Cauchy-Schwarz, we have

$$\begin{aligned}\int_M u^2 d\Gamma_L(\varphi) &= \int_M d\Gamma_L(u^2\varphi,\varphi) - 2\int_M u\varphi d\Gamma_L(u,\varphi)\\ &\le \mathcal{E}^{(L)}(u^2\varphi,\varphi) + \frac{1}{2}\int_M u^2 d\Gamma_L(\varphi) + 2\int_M \varphi^2 d\Gamma_L(u),\end{aligned}$$

which gives that

$$\int_M u^2 d\Gamma_L(\varphi) \le 2\mathcal{E}^{(L)}(u^2\varphi,\varphi) + 4\int_M \varphi^2 d\Gamma_L(u).$$

Since φ is supported in Ω, we see that $d\Gamma_L(\varphi) = 0$ outside Ω (cf. [14, formula (3.2.26) p.128]), thus proving (2.3).

Next we show that

$$\int_\Omega u^2 d\Gamma_\Omega^{(J)}(\varphi) \le 2\mathcal{E}^{(J)}(u^2\varphi,\varphi) + 4\int_\Omega \varphi^2 d\Gamma_\Omega^{(J)}(u), \tag{2.4}$$

where the measure $d\Gamma_\Omega^{(J)}$ is defined by

$$d\Gamma_\Omega^{(J)}(f,g)(x) = \int_{\Omega\setminus\mathrm{diag}} (f(x)-f(y))\,(g(x)-g(y))\,dj(x,y).$$

Indeed, noting that

$$u^2(x)(\varphi(x)-\varphi(y))^2 = \left\{\left[(u^2\varphi)(x)-(u^2\varphi)(y)\right] - \left[u^2(x)-u^2(y)\right]\varphi(y)\right\}(\varphi(x)-\varphi(y)),$$

we have

$$\begin{aligned}\int_{\Omega\times\Omega\setminus\mathrm{diag}} u^2(x)\,[\varphi(x)-\varphi(y)]^2\,dj(x,y) = &\int_{\Omega\times\Omega\setminus\mathrm{diag}} \left[(u^2\varphi)(x)-(u^2\varphi)(y)\right](\varphi(x)-\varphi(y))dj(x,y)\\ &-\int_{\Omega\times\Omega\setminus\mathrm{diag}} \left[u^2(x)-u^2(y)\right]\varphi(y)(\varphi(x)-\varphi(y))dj(x,y),\end{aligned}$$

which gives that

$$\int_\Omega u^2 d\Gamma_\Omega^{(J)}(\varphi) = \int_\Omega d\Gamma_\Omega^{(J)}(u^2\varphi,\varphi) - \int_\Omega \varphi d\Gamma_\Omega^{(J)}(u^2,\varphi). \tag{2.5}$$

To estimate the last term, note that

$$\begin{aligned}-\int_\Omega \varphi d\Gamma_\Omega^{(J)}(u^2,\varphi) &= -\int_{\Omega\times\Omega\setminus\mathrm{diag}} (u(x)+u(y))\varphi(y)(u(x)-u(y))(\varphi(x)-\varphi(y))dj(x,y)\\ &= -\int_{\Omega\times\Omega\setminus\mathrm{diag}} u(x)\varphi(y)(u(x)-u(y))(\varphi(x)-\varphi(y))dj(x,y)\\ &\quad -\int_{\Omega\times\Omega\setminus\mathrm{diag}} u(y)\varphi(y)(u(x)-u(y))(\varphi(x)-\varphi(y))dj(x,y).\end{aligned}$$

From this and using the Cauchy-Schwarz inequality, we derive

$$\begin{aligned}&-\int_\Omega \varphi d\Gamma_\Omega^{(J)}(u^2,\varphi)\\ &\le\frac{1}{4}\int_{\Omega\times\Omega\setminus\mathrm{diag}} u^2(x)(\varphi(x)-\varphi(y))^2 dj(x,y) + \int_{\Omega\times\Omega\setminus\mathrm{diag}} \varphi^2(y)(u(x)-u(y))^2 dj(x,y)\\ &\quad+\frac{1}{4}\int_{\Omega\times\Omega\setminus\mathrm{diag}} u^2(y)(\varphi(x)-\varphi(y))^2 dj(x,y) + \int_{\Omega\times\Omega\setminus\mathrm{diag}} \varphi^2(y)(u(x)-u(y))^2 dj(x,y)\\ &=\frac{1}{2}\int_\Omega u^2 d\Gamma_\Omega^{(J)}(\varphi) + 2\int_\Omega \varphi^2 d\Gamma_\Omega^{(J)}(u).\end{aligned}$$

Plugging this into (2.5), we have

$$\int_\Omega u^2 d\Gamma_\Omega^{(J)}(\varphi) \le 2\int_\Omega d\Gamma_\Omega^{(J)}(u^2\varphi,\varphi) + 4\int_\Omega \varphi^2 d\Gamma_\Omega^{(J)}(u). \tag{2.6}$$

As φ vanishes outside Ω, we see that

$$\begin{aligned}
\mathcal{E}^{(J)}(u^2\varphi,\varphi) &= \int_{M\times M\setminus \mathrm{diag}} \left[(u^2\varphi)(x)-(u^2\varphi)(y)\right](\varphi(x)-\varphi(y))dj(x,y)\\
&= \int_{\Omega\times\Omega\setminus\mathrm{diag}} +2\int_{\Omega\times\Omega^c} + \int_{\Omega^c\times\Omega^c\setminus\mathrm{diag}} \cdots\\
&= \int_\Omega d\Gamma_\Omega^{(J)}(u^2\varphi,\varphi) + 2\int_{\Omega\times\Omega^c}(u^2\varphi^2)(x)dj(x,y)\\
&\ge \int_\Omega d\Gamma_\Omega^{(J)}(u^2\varphi,\varphi),
\end{aligned}$$

which together with (2.6) implies (2.4).

Finally, summing up (2.3), (2.4) we conclude from (2.2) that (2.1) is true. □

We introduce condition (S).

Survival estimate. *There exist constants ε, $\delta \in (0,1)$ such that, for all balls B of radius $r>0$ and for all $t^{1/\beta} \le \delta r$,*

$$1 - P_t^B 1_B(x) \le \varepsilon \tag{S}$$

for μ-almost all $x \in \frac{1}{4}B$.

Lemma 2.2. *Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form in $L^2(M,\mu)$. Then*

$$(S) \Rightarrow (CIB).$$

Proof. Fix $x_0 \in M$ and set $B_0 = B(x_0,R)$, $B' = B(x_0,R+r)$ for $R>0$, $r>0$ and let $B'\subset\Omega$ for any open subset Ω of M. It suffices to show that there exists some $\phi \in$cutoff(B_0,B') such that

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \le C_1\int_\Omega \phi^2 d\Gamma_\Omega(u) + \frac{C_2}{r^\beta}\int_\Omega \phi u^2 d\mu \tag{2.7}$$

for any $u \in \mathcal{F}\cap L^\infty$, where the measure $d\Gamma_\Omega$ is defined by (2.2), since this inequality, on taking $\Omega = M$ and using the fact that $\phi \le 1$ in M, will imply (CIB).

To do this, let

$$w := \int_0^{+\infty} e^{-\lambda t} P_t^{B'} 1_{B'} dt,$$

where $\lambda = r^{-\beta}$. It is known that

$$\mathcal{E}(w,\varphi) + \lambda\int_{B'} w\varphi d\mu = \int_{B'}\varphi d\mu, \tag{2.8}$$

for any $\varphi \in \mathcal{F}(B')$. By [23, (3.6) p.1503], we have that

$$te^{-\lambda t}P_t^{B'}1_{B'} \le w \le r^\beta \text{ in } M.$$

Let $z \in B_0$ be any point and set $B_z = B(z,r)\subset B'$. An application of (S) with $t = (\delta r)^\beta$ yields that for almost all $x \in \frac{1}{4}B_z$,

$$w(x) \ge te^{-\lambda t}P_t^{B'}1_{B'}(x) \ge te^{-\lambda t}P_t^{B_z}1_{B_z}(x) \ge (\delta r)^\beta e^{-\delta^\beta}(1-\varepsilon) = C_0^{-1}r^\beta,$$

for $C_0 = \left[\delta^\beta e^{-\delta^\beta}(1-\varepsilon)\right]^{-1} > 1$. Hence,

$$\begin{aligned}
w &\le r^\beta \text{ in } M,\\
w &\ge C_0^{-1}r^\beta \text{ in } B_0.
\end{aligned}$$

Set $v := C_0 \frac{w}{r^\beta}$. Then $v \le C_0$ in M, and $v \ge 1$ in B_0. Define

$$\phi = v \wedge 1 \text{ in } M.$$

We see that $\phi \in \text{cutoff}\,(B_0, B')$. It suffices to show such a function ϕ satisfies (2.7).

Indeed, using (2.1) with $\varphi = v$,

$$\int_\Omega u^2 d\Gamma_\Omega(v) \le 2\mathcal{E}(u^2 v, v) + 4\int_\Omega v^2 d\Gamma_\Omega(u). \tag{2.9}$$

Observe that in M

$$C_0\phi = C_0\,(v \wedge 1) = (C_0 v) \wedge C_0 \ge v \wedge C_0 = v, \tag{2.10}$$

which gives that

$$\int_\Omega v^2 d\Gamma_\Omega(u) \le C_0^2 \int_\Omega \phi^2 d\Gamma_\Omega(u). \tag{2.11}$$

On the other hand, using (2.8) with $\varphi = u^2 v$ and using (2.10)

$$\begin{aligned}
\mathcal{E}(u^2 v, v) &= \frac{C_0}{r^\beta}\mathcal{E}(u^2 v, w) \\
&= \frac{C_0}{r^\beta}\left\{\int_{B'} u^2 v d\mu - \lambda \int_{B'} \left(u^2 v\right) w d\mu\right\} \\
&\le \frac{C_0}{r^\beta}\int u^2 v d\mu \le \frac{C_0^2}{r^\beta}\int u^2 \phi d\mu. \qquad (2.12)
\end{aligned}$$

Thus, plugging (2.12), (2.11) into (2.9), we conclude that

$$\int_\Omega u^2 d\Gamma_\Omega(v) \le \frac{2C_0^2}{r^\beta}\int_\Omega u^2 \phi d\mu + 4C_0^2 \int_\Omega \phi^2 d\Gamma_\Omega(u).$$

Finally, using the facts that $|\phi(x) - \phi(y)| \le |v(x) - v(y)|$ and that $\phi(x) \le v(x)$ for any $x, y \in M$ and then using [14, formula (3.2.12), p.122] and (2.2),

$$\int_\Omega u^2 d\Gamma_\Omega(\phi) \le \int_\Omega u^2 d\Gamma_\Omega(v).$$

Therefore, we obtain (2.7) with $C_1 = 4C_0^2$, $C_2 = 2C_0^2$. □

The same result in Lemma 2.2 was proved in [1, 23] for the local case.

We show the following two inequalities (2.13) and (2.14) by using condition (CIB).

Proposition 2.3. *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in $L^2(M, \mu)$. Let $B_0 = B(x_0, R)$, $B' = B(x_0, R + r)$ be two balls. If conditions (CIB), $(V_\le)$, $(J_\le)$ hold, then for every positive integer n and for every $u \in \mathcal{F} \cap L^\infty$, there exists some function $\phi = \phi_n \in$ cutoff(B_0, B') satisfying that*

$$\int u^2 d\Gamma(\phi) \le \frac{C_3}{n}\int d\Gamma(u) + \frac{C_4 n^\beta}{r^\beta}\int u^2 d\mu, \tag{2.13}$$

and that

$$\|\phi - \Phi\|_\infty \le 1/n \tag{2.14}$$

with

$$\Phi(y) := \left(\frac{R + r - d(x_0, y)}{r}\right)_+ \wedge 1, \tag{2.15}$$

where $C_3 \ge 1, C_4 \ge 1$ are universal constants (independent of B_0, B', n, u).

Proof. Fix positive integer n and for integers $0 \le k \le n$ set $r_k = kr/n$, $B_k := B(x_0, R + r_k)$. Define

$$U_k := B_k \setminus B_{k-1} (1 \le k \le n).$$

For a function $u \in \mathcal{F} \cap L^\infty$, we apply (CIB) to each pair (B_{k-1}, B_k) $(k \ge 1)$ and obtain

$$\int u^2 d\Gamma(\phi_k) \le C_1 \int d\Gamma(u) + \frac{C_2}{(r/n)^\beta} \int u^2 d\mu \tag{2.16}$$

for some $\phi_k \in$cutoff(B_{k-1}, B_k).

We define

$$\phi = \phi_n := \frac{1}{n} \sum_{k=1}^{n} \phi_k.$$

Clearly, $\phi \in$cutoff(B_0, B') and for each $1 \le k \le n$,

$$\frac{n-k}{n} \le \phi = \frac{\phi_k + (\phi_{k+1} + \cdots + \phi_n)}{n} \le \frac{n-k+1}{n} \text{ in } U_k.$$

On the other hand, for any $y \in U_k$ we have $R + r_{k-1} \le d(x_0, y) < R + r_k$, and by definition (2.15) of Φ,

$$\frac{n-k}{n} = 1 - \frac{r_k}{r} \le \Phi(y) \le 1 - \frac{r_{k-1}}{r} = \frac{n-k+1}{n}.$$

Hence, we see that (2.14) holds on each U_k. Both functions ϕ and Φ take values 1 in B_0, and 0 outside B', and (2.14) is also true in the set $B_0 \cup (M \setminus B')$.

It remains to prove (2.13) with such choice of ϕ.

To do this, note that, using the fact that $1_\Omega d\Gamma_L(u_1, u_2) = 0$ for $u_1, u_2 \in \mathcal{F}$ if u_1 is constant on Ω (cf. [14, formula (3.2.26) p. 128]),

$$\int u^2 d\Gamma_L(\phi) = \frac{1}{n^2} \sum_{k=1}^{n} \int u^2 d\Gamma_L(\phi_k). \tag{2.17}$$

On the other hand, for any $x, y \in M$,

$$\begin{aligned}(\phi(x) - \phi(y))^2 &= \frac{1}{n^2} \left(\sum_{k=1}^{n} (\phi_k(x) - \phi_k(y)) \right)^2 \\ &= \frac{1}{n^2} \left\{ \sum_{k=1}^{n} (\phi_k(x) - \phi_k(y))^2 + 2 \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (\phi_k(x) - \phi_k(y))(\phi_j(x) - \phi_j(y)) \right\}. \end{aligned} \tag{2.18}$$

The last double summation contains the following terms ($j = k + 1$ and $1 \le k \le n - 1$):

$$\begin{aligned} 2\sum_{k=1}^{n-1} (\phi_k(x) - \phi_k(y))(\phi_{k+1}(x) - \phi_{k+1}(y)) &\le \sum_{k=1}^{n-1} (\phi_k(x) - \phi_k(y))^2 + \sum_{k=1}^{n-1} (\phi_{k+1}(x) - \phi_{k+1}(y))^2 \\ &\le 2 \sum_{k=1}^{n} (\phi_k(x) - \phi_k(y))^2, \end{aligned}$$

where we have used the Cauchy-Schwarz. From this, we obtain from (2.18) that

$$(\phi(x) - \phi(y))^2 \le \frac{3}{n^2} \sum_{k=1}^{n} (\phi_k(x) - \phi_k(y))^2 + \frac{2}{n^2} \sum_{k=1}^{n-2} \sum_{j=k+2}^{n} (\phi_k(x) - \phi_k(y))(\phi_j(x) - \phi_j(y)).$$

Multiplying by $u^2(x)J(x, y)$ then integrating over $M \times M$ on both sides, we obtain that

$$\int_{M \times M} u^2(x)(\phi(x) - \phi(y))^2 J(x, y) d\mu(y) d\mu(x)$$

$$\leq \frac{3}{n^2}\sum_{k=1}^{n}\int_{M\times M} u^2(x)\left(\phi_k(x)-\phi_k(y)\right)^2 J(x,y)d\mu(y)d\mu(x)$$
$$+\frac{2}{n^2}\sum_{k=1}^{n-2}\sum_{j=k+2}^{n}\int_{M\times M} u^2(x)(\phi_k(x)-\phi_k(y))(\phi_j(x)-\phi_j(y))J(x,y)d\mu(y)d\mu(x). \tag{2.19}$$

Noting that for any $1\leq k\leq n-2$ and any $k+2\leq j\leq n$

$$\phi_j\phi_k=\phi_k \text{ in } M,$$

we have that for any $x,y\in M$

$$\begin{aligned}(\phi_k(x)-\phi_k(y))(\phi_j(x)-\phi_j(y)) &= \phi_k(x)-\phi_k(x)\phi_j(y)-\phi_k(y)\phi_j(x)+\phi_k(y)\\ &= \phi_k(x)(1-\phi_j(y))+\phi_k(y)(1-\phi_j(x)).\end{aligned}$$

Plugging this into (2.19) and then summing up with (2.17), we obtain that

$$\begin{aligned}\int u^2 d\Gamma(\phi) &= \int u^2 d\Gamma_L(\phi)+\int_{M\times M} u^2(x)(\phi(x)-\phi(y))^2 J(x,y)d\mu(y)d\mu(x)\\ &\leq \sum_{k=1}^{n}\left\{\frac{1}{n^2}\int u^2 d\Gamma_L(\phi_k)+\frac{3}{n^2}\int_{M\times M} u^2(x)\left(\phi_k(x)-\phi_k(y)\right)^2 J(x,y)d\mu(y)d\mu(x)\right\}\\ &\quad+\frac{2}{n^2}\sum_{k=1}^{n-2}\sum_{j=k+2}^{n}\int_{M\times M} u^2(x)\phi_k(x)(1-\phi_j(y))J(x,y)d\mu(y)d\mu(x)\\ &\quad+\frac{2}{n^2}\sum_{k=1}^{n-2}\sum_{j=k+2}^{n}\int_{M\times M} u^2(x)\phi_k(y)(1-\phi_j(x))J(x,y)d\mu(y)d\mu(x)\\ &:= I_1+2I_2+2I_3.\end{aligned} \tag{2.20}$$

We will estimate I_1, I_2, I_3 separately.

For the term I_1, we apply (2.16) to obtain

$$\begin{aligned} I_1 &\leq \frac{3}{n^2}\sum_{k=1}^{n}\int u^2 d\Gamma(\phi_k)\leq \frac{3}{n^2}\sum_{k=1}^{n}\left\{C_1\int d\Gamma(u)+\frac{C_2}{(r/n)^\beta}\int u^2 d\mu\right\}\\ &\leq \frac{3C_1}{n}\int d\Gamma(u)+\frac{3C_2 n^{\beta-1}}{r^\beta}\int u^2 d\mu.\end{aligned} \tag{2.21}$$

For the term I_2, observe that for any $1\leq k\leq n-2$ and $k+2\leq j\leq n$,

$$\operatorname{dist}\left(\operatorname{supp}(\phi_k),\operatorname{supp}(1-\phi_j)\right)\geq \operatorname{dist}\left(B_k,B_{j-1}^c\right)\geq r/n. \tag{2.22}$$

Recall that by $(J_\leq)$, $(V_\leq)$, we have that

$$\int_{B(x,\rho)^c} J(x,y)d\mu(y)\leq C\rho^{-\beta} \tag{2.23}$$

for μ-almost all $x\in M$ (cf. [21, Proof of Proposition 4.7]). From this and using (2.22), we have

$$\begin{aligned}&\int_{M\times M} u^2(x)\phi_k(x)(1-\phi_j(y))J(x,y)d\mu(y)d\mu(x)\\ &\qquad=\int_{B_k\times B_{j-1}^c} u^2(x)\phi_k(x)(1-\phi_j(y))J(x,y)d\mu(y)d\mu(x)\\ &\qquad\leq \int_{B_k\times B_{j-1}^c} u^2(x)J(x,y)d\mu(y)d\mu(x)\\ &\qquad\leq \frac{C'}{(r/n)^\beta}\int_{B_k} u^2(x)d\mu(x).\end{aligned}$$

Hence,

$$I_2 \le \frac{1}{n^2}\sum_{k=1}^{n-2}\sum_{j=k+2}^{n}\frac{C'}{(r/n)^\beta}\int_{B_k} u^2(x)d\mu(x) \le \frac{C'n^\beta}{r^\beta}\int u^2 d\mu. \tag{2.24}$$

Similarly, we have that, using (2.22), (2.23),

$$\begin{aligned}\int_{M\times M} u^2(x)\phi_k(y)(1-\phi_j(x))J(x,y)d\mu(y)d\mu(x) &\le \int_{B_{j-1}^c\times B_k} u^2(x)J(x,y)d\mu(y)d\mu(x)\\ &\le \frac{C'}{(r/n)^\beta}\int_{B_{j-1}^c} u^2(x)d\mu(x),\end{aligned}$$

which gives that

$$I_3 \le \frac{1}{n^2}\sum_{k=1}^{n-2}\sum_{j=k+2}^{n}\frac{C'}{(r/n)^\beta}\int_{B_{j-1}^c} u^2(x)d\mu(x) \le \frac{C'n^\beta}{r^\beta}\int u^2 d\mu. \tag{2.25}$$

Therefore, plugging (2.21), (2.24) and (2.25) into (2.20), we conclude that

$$\int u^2 d\Gamma(\phi) \le \frac{3C_1}{n}\int d\Gamma(u) + \frac{(3C_2+4C')\,n^\beta}{r^\beta}\int u^2 d\mu,$$

thus proving (2.13). The proof is complete. □

Remark 2.4. A self-improvement of condition (AB) for the jump type (nonlocal) Dirichlet form is addressed in [17, Lemma 2.9] by using a much more complicated cutoff function constructed in [1, Lemma 5.1]. The reason is that in [17, Lemma 2.9] one needs to replace the second integral in (2.13) by $\int \phi^2 d\Gamma(u)$- a smaller integral involving the function ϕ^2. See also [12, the proof of Proposition 2.4] for a self-improvement of condition CSJ(ϕ).

Remark 2.5. If the Dirichlet form is *strongly local* or if M is a *graph*, a sharper version than (2.13) was proved in [32, Lemma 2.1], [31, Lemma 2.7], but starting from a stronger assumption, called the *cutoff Sobolev inequality*, wherein the last integral on the right-hand side of (CIB) is over the *annulus*, instead of over the *whole space* M as in condition (CIB) here. To our knowledge, one can not directly obtain the same sharp version as in [32, 31] for nonlocal Dirichlet forms on metric measure spaces if one starts from condition (CIB) here.

3. Off-diagonal upper bound

In this section, we prove Theorem 1.3 and Theorem 1.4. To prove Theorem 1.3, we need to obtain upper estimate of the heat kernel $q_t^{(\rho)}(x,y)$ associated with the *truncated* Dirichlet form $(\mathcal{E}_\rho,\mathcal{F}_\rho)$ defined by (1.11) for any $0<\rho<\infty$. This can be done by carrying out Davies' perturbation method. Note that the form $(\mathcal{E},\mathcal{F})$ or $(\mathcal{E}_\rho,\mathcal{F}_\rho)$ may not be conservative at this stage.

For any regular Dirichlet form $(\mathcal{E},\mathcal{F})$, recall the following identity (cf. [14, formula (4.5.7), p.181]): for any $u,v\in\mathcal{F}$,

$$\begin{aligned}\mathcal{E}(u,v) = \lim_{t\to0+}\mathcal{E}^{(t)}(u,v) := \lim_{t\to0+}\Big\{&\frac{1}{2t}\int_{M\times M}(u(x)-u(y))(v(x)-v(y))P_t(x,dy)d\mu(x)\\ &+\frac{1}{t}\int_M uv(1-P_t1)d\mu\Big\},\end{aligned} \tag{3.1}$$

where $P_t(x,dy)$ is the transition function. Using this, we have that for any three measurable functions u,v,w such that all functions u,vw,uv,w belong to $\mathcal{F}$,

$$\begin{aligned}\mathcal{E}(u,vw) =&\mathcal{E}(uv,w)\\ &+\lim_{t\to0+}\frac{1}{2t}\int_{M\times M}(u(x)w(y)-u(y)w(x))(v(x)-v(y))P_t(x,dy)d\mu(x),\end{aligned} \tag{3.2}$$

and that, using (1.6),

$$\int u d\Gamma(v) = \lim_{t\to 0+} \int u d\Gamma^{(t)}(v) \tag{3.3}$$

for any $u, v \in \mathcal{F} \cap L^\infty$, where

$$\int u d\Gamma^{(t)}(v) := \frac{1}{2t}\left\{\int_{M\times M} u(x)(v(x)-v(y))^2 P_t(x,dy)d\mu(x) + \int_M uv^2(1-P_t 1)d\mu\right\}.$$

For any function f and any number $\rho > 0$, set

$$\operatorname{osc}(f,\rho) := \sup_{\substack{x,y\in M\\ d(x,y)\le\rho}} |f(y)-f(x)|. \tag{3.4}$$

The following lemma is motivated by [9, Theorem 3.9], [31, Lemma 3.4].

Lemma 3.1. *Let $(\mathcal{E}_\rho, \mathcal{F}_\rho)$ be a regular Dirichlet form defined in (1.11) for $0<\rho<\infty$ with energy measure $d\Gamma_\rho(\cdot)$. Then*

$$\mathcal{E}_\rho(e^{-\psi} f, e^{\psi} f^{2p-1}) \ge \frac{1}{2p}\mathcal{E}_\rho(f^p) - 9p\Lambda_\psi \int_M f^{2p} d\Gamma_\rho(\psi) \tag{3.5}$$

for any $\psi \in \mathcal{F}\cap L^\infty$, any non-negative $f \in \mathcal{F}\cap L^\infty$ and any $p \ge 1$, where

$$\Lambda_\psi = \begin{cases} 1, & J \equiv 0,\\ e^{2osc(\psi,\rho)}, & J \ne 0.\end{cases} \tag{3.6}$$

Proof. We note that $e^\psi - 1 \in \mathcal{F}\cap L^\infty$ by using (3.1) and the elementary inequality

$$(e^a-1)^2 \le e^{2|a|}a^2 \tag{3.7}$$

for all $a \in \mathbb{R}$. It follows that both functions $e^\psi g$ and $e^{-\psi} g$ belong to $\mathcal{F}\cap L^\infty$ if $g \in \mathcal{F}\cap L^\infty$. By symmetry $Q_t(x,dy)d\mu(x) = Q_t(y,dx)d\mu(y)$ for the transition function $Q_t(x,dy)$ associated with the form $(\mathcal{E}_\rho, \mathcal{F}_\rho)$,

$$m_t(dx,dy) := \frac{1}{2t}Q_t(x,dy)d\mu(x) = \frac{1}{2t}Q_t(y,dx)d\mu(y).$$

Applying (3.2) with $u = e^{-\psi} f$, $v = e^\psi$, $w = f^{2p-1}$ and $\mathcal{E}$ being replaced by $\mathcal{E}_\rho$, we have

$$\begin{aligned}\mathcal{E}_\rho(e^{-\psi} f, e^{\psi} f^{2p-1}) \;=\;& \mathcal{E}_\rho(f, f^{2p-1})\\ &+\lim_{t\downarrow 0+}\left\{\int_{M\times M}\left[\left(e^{-\psi}f\right)(x)f^{2p-1}(y) - \left(e^{-\psi}f\right)(y)f^{2p-1}(x)\right]\right.\\ &\left.\times\left(e^{\psi(x)}-e^{\psi(y)}\right)m_t(dx,dy)\right\} := I_1 + I_2. \end{aligned}\tag{3.8}$$

For I_1, we have

$$I_1 = \mathcal{E}_\rho(f, f^{2p-1}) \ge \frac{2p-1}{p^2}\mathcal{E}_\rho(f^p) \tag{3.9}$$

(cf. [9, formulae (3.17)] that is valid for any regular Dirichlet form by using (3.1)).

To estimate I_2, note that

$$\begin{aligned} I_2 = \lim_{t\downarrow 0+}&\left\{\int_{M\times M}\left(f^{2p}(y)-f^{2p}(x)\right)e^{-\psi(x)}\left(e^{\psi(x)}-e^{\psi(y)}\right)m_t(dx,dy)\right.\\ &+\int_{M\times M} f^{2p}(x)\left(e^{-\psi(x)}-e^{-\psi(y)}\right)\left(e^{\psi(x)}-e^{\psi(y)}\right)m_t(dx,dy)\\ &\left.+2\int_{M\times M} f^{2p-1}(y)\left(f(x)-f(y)\right)e^{\psi(y)}\left(e^{-\psi(y)}-e^{-\psi(x)}\right)m_t(dx,dy)\right\}. \end{aligned}\tag{3.10}$$

From this and using the Cauchy-Schwarz and the following elementary inequalities

$$\int f^{2p-2} d\Gamma_\rho(f) \le \mathcal{E}_\rho(f^{2p-1}, f) \le \mathcal{E}_\rho(f^p)$$

(see [9, formulas (3.16), (3.17)]), we have

$$\begin{aligned} I_2 \ge &- \sqrt{\mathcal{E}_\rho(f^p)}\left(\sqrt{\int f^{2p}e^{2\psi}d\Gamma_\rho(e^{-\psi}-1)} + \sqrt{\int f^{2p}e^{-2\psi}d\Gamma_\rho(e^{\psi}-1)}\right) \\ &- \left(\int f^{2p}e^{2\psi}d\Gamma_\rho(e^{-\psi}-1)\cdot\int f^{2p}e^{-2\psi}d\Gamma_\rho(e^{\psi}-1)\right)^{1/2} \\ &- 2\left(\mathcal{E}_\rho(f^p)\int f^{2p}e^{2\psi}d\Gamma_\rho(e^{-\psi}-1)\right)^{1/2}. \end{aligned} \tag{3.11}$$

We further estimate I_2 by using the following fact:

$$\max\left\{\int f^{2p}e^{2\psi}d\Gamma_\rho(e^{-\psi}-1), \int f^{2p}e^{-2\psi}d\Gamma_\rho(e^{\psi}-1)\right\} \le \Lambda_\psi \int f^{2p} d\Gamma_\rho(\psi), \tag{3.12}$$

and indeed, this fact can be proved by noting that

$$e^{-2\psi}d\Gamma_L(e^{\psi}-1) = d\Gamma_L(\psi) = e^{2\psi}d\Gamma_L(e^{-\psi}-1)$$

from the chain rule for the energy measure $d\Gamma_L(\cdot)$ (cf. [14, Theorem 3.2.2]), and that

$$e^{2\psi(x)}(e^{-\psi(x)} - e^{-\psi(y)})^2 \le e^{2\mathrm{osc}(\psi,\rho)}\,|\psi(x)-\psi(y)|^2$$

for any x, y with $d(x,y) \le \rho$.

Then, plugging (3.12) into (3.11) and then using the elementary inequality $4ab \le \frac{a^2}{2p} + 8pb^2$ for $a, b > 0$, we obtain

$$I_2 \ge -\frac{1}{2p}\mathcal{E}_\rho(f^p) - 9p\Lambda_\psi \int f^{2p} d\Gamma_\rho(\psi). \tag{3.13}$$

Finally, plugging (3.9), (3.13) into (3.8), we conclude that, using $\frac{2p-1}{p^2} \ge \frac{1}{p}$,

$$\mathcal{E}_\rho(e^{-\psi}f, e^{\psi}f^{2p-1}) \ge \frac{1}{2p}\mathcal{E}_\rho(f^p) - 9p\Lambda_\psi \int f^{2p} d\Gamma_\rho(\psi),$$

thus proving (3.5), as desired. □

We estimate the last term in (3.5) by using the cutoff inequality developed in Section 2. For $0 < \eta < 1$, set

$$c_1(\eta) = \begin{cases} 0, & J \equiv 0, \\ 2(\beta+1)\left(\eta + 2\eta^2\right), & J \not\equiv 0. \end{cases} \tag{3.14}$$

Lemma 3.2. *Let $B(x_0, R)$, $B(x_0, R + r)$ be two concentric balls in M with $r > 0$, $R > 0$ and let $(\mathcal{E}_\rho, \mathcal{F}_\rho)$ be the truncated Dirichlet form defined in (1.11) with $0 < \rho < r$. Assume all the conditions ($J_\le$), ($V_\le$) and (CIB) are satisfied by the form $(\mathcal{E}, \mathcal{F})$. Then for any $p \ge 1$ and any $\lambda \ge \eta^{-1}$ with $\eta := \rho/r$, and for any non-negative $f \in \mathcal{F} \cap L^\infty$, there exists some function $\phi = \phi_{p,\lambda} \in$cutoff$(B(x_0, R), B(x_0, R + r))$ such that*

$$\mathcal{E}_\rho(e^{-\lambda\phi}f, e^{\lambda\phi}f^{2p-1}) \ge \frac{1}{4p}\mathcal{E}(f^p) - TC_0 p^{2\beta+1}\lambda^{2\beta+2}\int f^{2p} d\mu, \tag{3.15}$$

and that

$$\|\phi - \Phi\|_\infty \le \frac{1}{(6\lambda p)^2} < \frac{1}{\lambda p}, \tag{3.16}$$

where C_0 is some universal constant independent of $x_0, R, r, \rho, p, \lambda$ and functions ϕ, f, and where Φ is given by (2.15), and

$$T = \begin{cases} 1/r^\beta, & J \equiv 0, \\ e^{c_1(\eta)\lambda}/\rho^\beta, & J \neq 0, \end{cases} \tag{3.17}$$

with $c_1(\eta)$ given by (3.14).

Remark 3.3. *We will see from the proof below that the energy $\mathcal{E}_\rho(e^{\lambda\phi}f, e^{-\lambda\phi}f^{2p-1})$ has the same lower bound as in (3.15).*

Proof. Applying Lemma 3.1 with $\psi = \lambda\phi$, we have

$$\mathcal{E}_\rho(e^{-\lambda\phi}f, e^{\lambda\phi}f^{2p-1}) \geq \frac{1}{2p}\mathcal{E}_\rho(f^p, f^p) - 9p\lambda^2\Lambda_\phi^\lambda \int f^{2p} d\Gamma_\rho(\phi), \tag{3.18}$$

for any $\phi \in \text{cutoff}(B(x_0, R), B(x_0, R+r))$, any $0 \leq f \in \mathcal{F} \cap L^\infty$ and any $p \geq 1$, $\lambda > 0$, where Λ_ϕ is given by (3.6) with ψ being replaced by ϕ. Using (2.23) and [21, Proposition 4.1], we have

$$\mathcal{E}(f^p) - \mathcal{E}_\rho(f^p) \leq 4\int f^{2p} d\mu \cdot \sup_{x\in M}\left\{\int_{B(x,\rho)^c} J(x,y)d\mu(y)\right\} \leq \frac{C_5}{\rho^\beta}\int f^{2p}d\mu, \tag{3.19}$$

where $C_5 \geq 0$ is some universal constant independent of f, p, ρ (noting that $C_5 = 0$ if $J \equiv 0$).

Plugging this into (3.18) we obtain

$$\mathcal{E}_\rho(e^{-\lambda\phi}f, e^{\lambda\phi}f^{2p-1}) \geq \frac{1}{2p}\left\{\mathcal{E}(f^p) - \frac{C_5}{\rho^\beta}\int f^{2p}d\mu\right\} - 9p\lambda^2\Lambda_\phi^\lambda \int f^{2p}d\Gamma_\rho(\phi). \tag{3.20}$$

We further estimate the energy $\mathcal{E}_\rho(e^{-\lambda\phi}f, e^{\lambda\phi}f^{2p-1})$ starting from (3.20) by using a self-improvement (2.13) of condition (CIB).

To do this, we claim that

$$\begin{aligned}\mathcal{E}_\rho(e^{-\lambda\phi}f, e^{\lambda\phi}f^{2p-1}) \quad &\geq \quad \frac{1}{4p}\mathcal{E}(f^p) - \frac{C_5}{2p\rho^\beta}\int f^{2p}d\mu \\ &\quad -9p\lambda^2 e^{2\lambda c_2(\eta)}\frac{C_4 n^{2\beta}}{r^\beta}\int f^{2p}d\mu,\end{aligned} \tag{3.21}$$

where $c_2(\eta)$ is defined by

$$c_2(\eta) = \begin{cases} 0, & J \equiv 0, \\ \eta + 2\eta^2, & J \neq 0. \end{cases} \tag{3.22}$$

We distinguish two cases.

Case $J \neq 0$. Applying (2.13) with $u = f^p$ and n being replaced by n^2, we have that for each integer $n \geq 1$, there exists $\phi := \phi_n \in \text{cutoff}(B(x_0, R), B(x_0, R+r))$ satisfying that

$$\int f^{2p}d\Gamma_\rho(\phi) \leq \int f^{2p}d\Gamma(\phi) \leq \frac{C_3}{n^2}\mathcal{E}(f^p) + \frac{C_4 n^{2\beta}}{r^\beta}\int_M f^{2p}d\mu, \tag{3.23}$$

and that

$$\|\phi - \Phi\|_\infty \leq \frac{1}{n^2}. \tag{3.24}$$

By definition (2.15) of Φ, we see that

$$\text{osc}(\Phi, \rho) \leq \sup_{d(x,y)\leq\rho}\frac{d(x,y)}{r} \leq \frac{\rho}{r} = \eta. \tag{3.25}$$

Thus, we see from (3.24), (3.25), (3.22) that

$$\text{osc}(\phi, \rho) \leq \text{osc}(\Phi, \rho) + \frac{2}{n^2} \leq \eta + 2\eta^2 = c_2(\eta)$$

provided that

$$n \geq \frac{1}{\eta}. \tag{3.26}$$

This implies by (3.6) that

$$\Lambda_\phi^\lambda = e^{2\lambda \mathrm{osc}(\phi,\rho)} \leq e^{2\lambda c_2(\eta)}.$$

Therefore, using this, we obtain from (3.20), (3.23) that under (3.26),

$$\begin{aligned}
\mathcal{E}_\rho(e^{-\lambda\phi}f, e^{\lambda\phi}f^{2p-1}) &\geq \frac{1}{2p}\left\{\mathcal{E}(f^p) - \frac{C_5}{\rho^\beta}\int f^{2p}d\mu\right\} \\
&\quad -9p\lambda^2 e^{2\lambda c_2(\eta)}\left\{\frac{C_3}{n^2}\mathcal{E}(f^p) + \frac{C_4 n^{2\beta}}{r^\beta}\int_M f^{2p}d\mu\right\} \\
&= \left\{\frac{1}{2p} - 9p\lambda^2 e^{2\lambda c_2(\eta)}\frac{C_3}{n^2}\right\}\mathcal{E}(f^p) \\
&\quad -\left\{\frac{C_5}{2p\rho^\beta} + 9p\lambda^2 e^{2\lambda c_2(\eta)}\frac{C_4 n^{2\beta}}{r^\beta}\right\}\int f^{2p}d\mu.
\end{aligned}\tag{3.27}$$

Choose the least integer $n \geq 1$ such that

$$\frac{1}{2p} - 9p\lambda^2 e^{2\lambda c_2(\eta)}\frac{C_3}{n^2} \geq \frac{1}{4p},$$

that is,

$$n = \left\lceil 6p\lambda \exp(\lambda c_2(\eta))\sqrt{C_3}\right\rceil. \tag{3.28}$$

With such choice of n, condition (3.26) is satisfied by using the assumption that $\lambda \geq \eta^{-1}$, since $C_3 \geq 1$ and

$$n = \left\lceil 6p\lambda \exp(\lambda c_2(\eta))\sqrt{C_3}\right\rceil \geq 6p\lambda > \frac{1}{\eta}.$$

From this and using (3.27), we obtain that (3.21) holds for $J \neq 0$.

Case $J \equiv 0$. It is not difficult to see from above that (3.21) also follows from (3.20) with $C_5 = 0$ and $c_2(\eta) = 0$ if $J \equiv 0$, since $\Lambda_\phi \equiv 1$.

Therefore, inequality (3.21) holds, and our claim is true.

Noting that by (3.28)

$$n \leq 6p\lambda \exp(\lambda c_2(\eta))\sqrt{C_3} + 1 \leq 12p\lambda \exp(\lambda c_2(\eta))\sqrt{C_3},$$

we have that, using the fact that $c_1(\eta) = 2(\beta+1)c_2(\eta)$ by (3.14), (3.22),

$$\begin{aligned}
9p\lambda^2 e^{2\lambda c_2(\eta)}\frac{C_4 n^{2\beta}}{r^\beta} &\leq 9p\lambda^2 e^{2\lambda c_2(\eta)}\frac{C_4\left(12p\lambda\exp(\lambda c_2(\eta))\sqrt{C_3}\right)^{2\beta}}{r^\beta} \\
&= C_6 p^{2\beta+1}\lambda^{2\beta+2}\frac{\exp(2(\beta+1)c_2(\eta)\lambda)}{r^\beta} \\
&= C_6 p^{2\beta+1}\lambda^{2\beta+2}\frac{\exp(c_1(\eta)\lambda)}{r^\beta},
\end{aligned}\tag{3.29}$$

where $C_6 = 9 \times 12^{2\beta}C_4C_3^\beta$. Plugging (3.29) into (3.21), we see that

$$\begin{aligned}
\mathcal{E}_\rho(e^{-\lambda\phi}f, e^{\lambda\phi}f^{2p-1}) &\geq \frac{1}{4p}\mathcal{E}(f^p) - \frac{C_5}{2p\rho^\beta}\int f^{2p}d\mu \\
&\quad -C_6 p^{2\beta+1}\lambda^{2\beta+2}\frac{\exp(c_1(\eta)\lambda)}{r^\beta}\int f^{2p}d\mu,
\end{aligned}\tag{3.30}$$

which gives that

$$\mathcal{E}_\rho(e^{-\lambda\phi}f, e^{\lambda\phi}f^{2p-1}) \geq \frac{1}{4p}\mathcal{E}(f^p) - p^{2\beta+1}\lambda^{2\beta+2}\left[\frac{C_5}{\rho^\beta} + \frac{C_6\exp(c_1(\eta)\lambda)}{r^\beta}\right]\int f^{2p}d\mu,$$

thus, proving (3.15) by setting $C_0 = C_5 + C_6$, with T given by (3.17).

Finally, inequality (3.16) follows directly from (3.24), (3.28) by noting that $\frac{1}{n^2} \le \frac{1}{(6p\lambda)^2}$. The proof is complete. ◻

To prove the off-diagonal upper bound (UE), we need the following two lemmas. We begin with the first one, Lemma 3.4 below, which is a slight improvement of [31, Lemma 3.7], see also [9, Lemma 3.21].

Lemma 3.4. *Let $w : (0,\infty) \to (0,\infty)$ be a non-decreasing function and suppose that $u \in C^1([0,\infty);(0,\infty))$ satisfies that for all $t \ge 0$,*

$$u'(t) \le -b\frac{t^{p-2}}{w^\theta(t)}u^{1+\theta}(t) + Ku(t) \tag{3.31}$$

for some $b > 0$, $p > 1$, $\theta > 0$ and $K > 0$. Then

$$u(t) \le \left(\frac{2p^\nu}{\theta b}\right)^{1/\theta} t^{-(p-1)/\theta} e^{Kp^{-\nu}t} w(t) \tag{3.32}$$

for any $\nu \ge 1$.

Proof. Set $h(t) := e^{-Kt}u(t)$. Then by (3.31)

$$h'(t) = e^{-Kt}\left(u'(t) - Ku(t)\right) \le e^{-Kt}\left\{-\frac{bt^{p-2}}{w^\theta(t)}u^{1+\theta}(t)\right\} = \left\{-e^{\theta Kt}\frac{bt^{p-2}}{w^\theta(t)}\right\}h^{1+\theta}(t).$$

From this, we have

$$\frac{d}{dt}\left(h^{-\theta}(t)\right) = -\theta\cdot\frac{h'(t)}{h^{1+\theta}(t)} \ge \theta e^{\theta Kt}\frac{bt^{p-2}}{w^\theta(t)}. \tag{3.33}$$

Integrating (3.33) over $(0,t)$ and using the facts that $h(0) = u(0) > 0$ and that the function w is non-decreasing, we obtain

$$h^{-\theta}(t) > h^{-\theta}(t) - h^{-\theta}(0) \ge \int_0^t \theta e^{\theta Ks}\frac{bs^{p-2}}{w^\theta(s)}ds \ge \frac{\theta b}{w^\theta(t)}\int_0^t e^{\theta Ks}s^{p-2}ds. \tag{3.34}$$

For any $p > 1$ and any $\nu \ge 1$, the following inequality is elementary:

$$\frac{1-(1-p^{-\nu})^{p-1}}{p-1} \ge \frac{1}{2p^\nu}. \tag{3.35}$$

Indeed, for any $2 \le p \le 3$ and $\nu \ge 0$, we have

$$\frac{1-(1-p^{-\nu})^{p-1}}{p-1} = \int_{1-p^{-\nu}}^1 s^{p-2}ds \ge \frac{1+(1-p^{-\nu})^{p-2}}{2}\frac{1}{p^\nu} > \frac{1}{2p^\nu}$$

by using the concavity of s^{p-2}. Whilst for any $p \in (1,2)\cup(3,+\infty)$ and $\nu \ge 1$ we have by Jensen's inequality

$$\frac{1-(1-p^{-\nu})^{p-1}}{p-1} = \int_{1-p^{-\nu}}^1 s^{p-2}ds \ge (1-p^{-\nu}/2)^{p-2}\frac{1}{p^\nu}.$$

On one hand, if $p > 3$ and $\nu \ge 1$, as $\nu \mapsto (1-p^{-\nu}/2)^{p-2}$ is non-decreasing, we see that

$$\begin{aligned}(1-p^{-\nu}/2)^{p-2} &\ge \left(1-p^{-1}/2\right)^{p-2} \ge \lim_{p\to\infty}\left(1-p^{-1}/2\right)^{p-2}\\ &= e^{-1/2} > \frac{1}{2},\end{aligned}$$

where we have used the fact that $p \mapsto \left(1-p^{-1}/2\right)^{p-2}$ is non-increasing. On the other hand, if $1 < p < 2$ and $\nu \ge 1$, it is trivial to see that

$$(1-p^{-\nu}/2)^{p-2} > \frac{1}{2}.$$

Thus, inequality (3.35) is always true.

It follows from (3.35) that

$$\begin{aligned}\int_0^t e^{\theta K s} s^{p-2} ds &= \frac{t}{\theta K}\int_0^{\theta K} e^{yt}\left(\frac{yt}{\theta K}\right)^{p-2} dy \geq \left(\frac{t}{\theta K}\right)^{p-1}\int_{\theta K(1-p^{-\nu})}^{\theta K} e^{yt} y^{p-2} dy \\ &\geq \left(\frac{t}{\theta K}\right)^{p-1} e^{\theta K(1-p^{-\nu})t}\int_{\theta K(1-p^{-\nu})}^{\theta K} y^{p-2} dy \\ &= t^{p-1} e^{\theta K(1-p^{-\nu})t}\frac{1-(1-p^{-\nu})^{p-1}}{p-1} \geq \frac{t^{p-1} e^{\theta K(1-p^{-\nu})t}}{2p^{\nu}}.\end{aligned}$$

Therefore, plugging this inequality into (3.34), we obtain that

$$e^{\theta K t} u^{-\theta}(t) = h^{-\theta}(t) \geq \frac{\theta b}{w^{\theta}(t)}\frac{t^{p-1} e^{\theta K(1-p^{-\nu})t}}{2p^{\nu}},$$

that is,

$$u^{-\theta}(t) \geq \frac{\theta b}{w^{\theta}(t)}\frac{t^{p-1} e^{-\theta K p^{-\nu} t}}{2p^{\nu}},$$

thus proving (3.32), as desired. □

We give the following second lemma that is of independent interest.

Lemma 3.5. *Assume that condition (V$_{\leq}$) holds. If the heat kernel $p_t(x,y)$ satisfies*

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}}\exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta'-1}}\right) \tag{3.36}$$

for μ-almost all $x,y \in M$ and for all $t>0$, for some $\beta' > \beta > 1$, then it also satisfies (UE$_{loc}$) (that is, estimate (3.36) also holds with β' being replaced by β).

Remark 3.6. *Lemma 3.5 is a self-improvement of the heat kernel estimate, raising some power $\frac{\beta}{\beta'-1}$ to the power $\frac{\beta}{\beta-1}$, the best one possible. The smaller β' is, the sharper (3.36).*

The proof below is inspired by [19, proof of Theorem 5.7, pp. 542-544] wherein $\beta' = \beta+1$. We will see that $\beta' = 2\beta+2$ in our application.

Proof. We claim that if $\beta' \geq \beta+1$, then (3.36) also holds with β' being replaced by $\beta'-1$. The proof is quite long.

Let

$$\theta := \beta/(\beta'-1). \tag{3.37}$$

Clearly, $0<\theta\leq 1$. Fix $t>0$, $x\in M$. Set $B := B(x,r)$, $B_k = kB$ $(k\geq 1)$, $B_0 = \emptyset$ and

$$r = 2t^{1/\beta}/\delta \tag{3.38}$$

with $\delta>0$ to be chosen later.

Let us show that for any $0<\varepsilon<1$, there exists some $\delta=\delta(\varepsilon)>0$ such that

$$P_t 1_{B_k^c} \leq k^{\alpha}\varepsilon^{k^{\theta}} \text{ in } \frac{1}{4}B \tag{3.39}$$

for any integer $k\geq 1$.

Indeed, using (V$_{\leq}$) and (3.38), we have from (3.36) that for μ-almost all $y\in\frac{1}{4}B$ and any $k\geq 1$,

$$\begin{aligned}P_t 1_{B_k^c}(y) &\leq \int_{M\setminus B(x,kr)}\frac{C}{t^{\alpha/\beta}}\exp\left(-c\left(\frac{d(y,z)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta'-1}}\right)d\mu(z) \\ &\leq \int_{M\setminus B(x,kr)}\frac{C}{t^{\alpha/\beta}}\exp\left(-c'\left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta'-1}}\right)d\mu(z)\end{aligned}$$

$$\leq \quad C' \int_{k/\delta}^{\infty} s^{\alpha-1} \exp(-c' s^{\theta}) ds = k^{\alpha} C' \int_{1/\delta}^{\infty} s^{\alpha-1} \exp(-c' (ks)^{\theta}) ds, \tag{3.40}$$

see [20, formula (3.7)].

For any $0 < \varepsilon < 1$, choose $\delta > 0$ to be so small that both of what follows are satisfied:

$$\begin{aligned} C' \int_{1/\delta}^{\infty} s^{\alpha-1} \exp\left(-\frac{c'}{2} s^{\theta}\right) ds &\leq 1, \\ \exp\left(-\frac{c'}{2} \delta^{-\theta}\right) &\leq \varepsilon. \end{aligned}$$

Therefore, by (3.40),

$$\begin{aligned} P_t 1_{B_k^c}(y) &\leq k^{\alpha} C' \int_{1/\delta}^{\infty} s^{\alpha-1} \exp(-c' (ks)^{\theta}) ds \\ &= k^{\alpha} C' \int_{1/\delta}^{\infty} \exp\left(-\frac{c'}{2} k^{\theta} s^{\theta}\right) \cdot s^{\alpha-1} \exp\left(-\frac{c'}{2} k^{\theta} s^{\theta}\right) ds \\ &\leq k^{\alpha} \exp\left(-\frac{c'}{2} k^{\theta} \delta^{-\theta}\right) \cdot C' \int_{1/\delta}^{\infty} s^{\alpha-1} \exp\left(-\frac{c'}{2} s^{\theta}\right) ds \\ &\leq k^{\alpha} \left\{\exp\left(-\frac{c'}{2} \delta^{-\theta}\right)\right\}^{k^{\theta}} \leq k^{\alpha} \varepsilon^{k^{\theta}} \end{aligned}$$

for μ-almost all $y \in \frac{1}{4}B$ and any $k \geq 1$, thus proving (3.39).

Define the function $E_{t,x}$ by

$$E_{t,x}(\cdot) = \exp\left(a \left(\frac{d(x, \cdot)}{t^{1/\beta}}\right)^{\theta}\right) \tag{3.41}$$

for some constant $a > 0$ to be determined later. Let us show that

$$P_t E_{t,x} \leq A_1 \text{ a.a. in } B(x, r/4), \tag{3.42}$$

where A_1 is some constant depending on ε, δ only.

Indeed, by (3.41) and (3.39), (3.38), we have that in $\frac{1}{4}B$,

$$\begin{aligned} P_t E_{t,x} = \sum_{k=0}^{\infty} P_t \left(1_{B_{k+1} \setminus B_k} E_{t,x}\right) &\leq \sum_{k=0}^{\infty} \left\|E_{t,x}\right\|_{L^{\infty}(B_{k+1})} P_t 1_{B_{k+1} \setminus B_k} \\ &\leq \sum_{k=0}^{\infty} \exp\left(a \left(\frac{(k+1) r}{t^{1/\beta}}\right)^{\theta}\right) \cdot P_t 1_{B_k^c} \leq \sum_{k=0}^{\infty} \exp\left(a 2^{\theta} \left(\frac{k+1}{\delta}\right)^{\theta}\right) \cdot k^{\alpha} \varepsilon^{k^{\theta}}. \end{aligned}$$

Choose $a < \frac{1}{3} (\delta/2)^{\theta} \log(1/\varepsilon)$ such that this series converges, proving (3.42).

Let us show that for all $t > 0$ and all $x \in M$

$$P_t E_{t,x} \leq A_2 E_{t,x} \text{ in } M, \tag{3.43}$$

for some constant $A_2 = A_2(\varepsilon, \delta)$.

Indeed, using the elementary inequality that $(a+b)^{\theta} \leq a^{\theta} + b^{\theta}$ for any $a, b \geq 0$ and any $0 \leq \theta \leq 1$, we have that for any $x, y \in M$

$$\begin{aligned} E_{t,x}(y) &= \exp\left(a \left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\theta}\right) \\ &\leq \exp\left(a \left(\frac{d(x,z)}{t^{1/\beta}}\right)^{\theta}\right) \exp\left(a \left(\frac{d(z,y)}{t^{1/\beta}}\right)^{\theta}\right) = E_{t,x}(z) E_{t,z}(y), \end{aligned}$$

that is, $E_{t,x} \le E_{t,x}(z) E_{t,z}$, and thus

$$P_t E_{t,x} \le E_{t,x}(z) P_t E_{t,z}. \tag{3.44}$$

Note that by (3.42)

$$P_t E_{t,z} \le A_1 \text{ a.a. in } B(z, r/4). \tag{3.45}$$

For all $y \in B(z, r/4)$, by (3.38),

$$E_{t,y}(z) \le \exp\left(a \left(\frac{r}{4t^{1/\beta}}\right)^{\theta}\right) = \exp\left(a\,(2\delta)^{-\theta}\right) := A_3,$$

and hence,

$$E_{t,x}(z) \le E_{t,x}(y) E_{t,y}(z) \le A_3 E_{t,x}(y).$$

It follows from (3.44), (3.45) that

$$P_t E_{t,x} \le A_1 A_3 E_{t,x} \text{ a.a. in } B(z, r/4).$$

Since the point z is arbitrary, we cover M by a countable sequence of balls like $B(z, r)$, and obtain that (3.43) is true with $A_2 = A_1 A_3$.

Let us show that for all $t > 0$, $x \in M$, and for any integer $k \ge 1$,

$$P_{kt} E_{t,x} \le A_2^k \text{ a.a. in } B(x, r/4). \tag{3.46}$$

Indeed, by (3.43)

$$P_{kt} E_{t,x} = P_{(k-1)t} P_t E_{t,x} \le A_2 P_{(k-1)t} E_{t,x} \le \ldots \le A_2^{k-1} P_t E_{t,x},$$

which together with (3.42) gives (3.46), where we have used $A_2 \ge A_1$.

Fix $B_R := B(x_0, R)$ for $R > 0$. We show that for any $t, \lambda > 0$,

$$P_t 1_{B_R^c} \le A_0 \exp\left(a' \lambda t - a \left(R\lambda^{1/\beta}\right)^{\theta}\right) \text{ in } \frac{1}{2} B_R, \tag{3.47}$$

for some constants A_0, a' depending on ε, δ only.

Indeed, observe that for any $x \in \frac{1}{2} B_R$,

$$P_t 1_{B_R^c} \le P_t 1_{B(x,R/2)^c}.$$

It suffices to show that for any $x \in \frac{1}{2} B_R$,

$$P_t 1_{B(x,R/2)^c} \le A_0 \exp\left(a' \lambda t - a \left(R\lambda^{1/\beta}\right)^{\theta}\right), \tag{3.48}$$

in a (small) ball containing x. Then covering $\frac{1}{2} B_R$ by a countable family of such balls, we obtain (3.47).

To see this, replacing t by t/k in (3.46), we have that

$$P_t E_{t/k,x} \le A_2^k \text{ in } B(x, r_k),$$

where $r_k = (t/k)^{1/\beta}/(2\delta)$. Since

$$E_{t/k,x} \ge \exp\left(a \left(\frac{R}{(t/k)^{1/\beta}}\right)^{\theta}\right) \text{ in } B(x, R)^c,$$

we have that

$$1_{B(x,R)^c} \le \exp\left(-a \left(\frac{R}{(t/k)^{1/\beta}}\right)^{\theta}\right) E_{t/k,x}.$$

It follows that in $B(x, r_k)$

$$P_t 1_{B(x,R)^c} \le \exp\left(-a \left(\frac{R}{(t/k)^{1/\beta}}\right)^{\theta}\right) P_t E_{t/k,x} \le \exp\left(a' k - a \left(\frac{R}{(t/k)^{1/\beta}}\right)^{\theta}\right),$$

where $a' = \log A_2$. Given $\lambda > 0$, choose an integer $k \geq 1$ such that

$$\frac{k-1}{t} < \lambda \leq \frac{k}{t}.$$

With such choice of k, we conclude that in $B(x, r_k)$

$$P_t 1_{B(x,R)^c} \leq \exp\left(a'(\lambda t + 1) - a\left(\frac{R}{(1/\lambda)^{1/\beta}}\right)^{\theta} \right),$$

which finishes the proof of (3.48), and also of (3.47).

Choosing λ in (3.47) such that $a'\lambda t = a\left(R\lambda^{1/\beta}\right)^{\theta}/2$, that is,

$$\lambda = \left(\frac{aR^{\theta}}{2a't}\right)^{\frac{\beta}{\beta-\theta}},$$

we conclude by (3.37) that in $B(x_0, R/2)$,

$$P_t 1_{B_R^c} \leq A_0 \exp(-a'\lambda t) = A_0 \exp\left(-c\left(\frac{R}{t^{1/\beta}}\right)^{\frac{\beta}{\beta/\theta - 1}}\right) \tag{3.49}$$

for some universal constant $c > 0$. From this and (DUE) and using the semigroup property of $p_t(x, y)$, we obtain that (3.36) holds with β' being replaced by $\beta' - 1 = \beta/\theta$ (cf. [22, pp. 183-184]), thus proving our claim.

Finally, repeat our claim k times until the integer k satisfies

$$\beta < \beta' - k \leq \beta + 1,$$

that is, $1 \leq \frac{\beta}{\beta'-k-1} < \frac{\beta}{\beta-1}$. Then (3.36) holds with β' being replaced by $\beta' - k$, which also implies that (3.36) holds with $\beta' = \beta + 1$ by reducing the value of $\frac{\beta}{\beta'-k-1}$ to 1. From this, we repeat the claim one more time (where $\theta = 1$), and obtain ($\mathrm{UE_{loc}}$), as desired. □

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Fix $x_0 \in M$. For $r > 0$, set

$$\rho := \eta r, \tag{3.50}$$

where $0 < \eta < 1$ will be specified later. For any integer $k \geq 0$, set $p_k = 2^k$ and

$$\psi_k := \lambda \phi_{p_k,\lambda} \tag{3.51}$$

where $\phi_{p_k,\lambda} \in \mathrm{cutoff}(B(x_0, r), B(x_0, 2r))$ is given by Lemma 3.2 with $p = p_k$, and $\lambda \geq \eta^{-1}$ will be chosen later. Clearly, for μ-almost all $x \in B(x_0, r), y \in M \setminus B(x_0, 2r)$

$$\psi_k(y) - \psi_k(x) = \lambda \cdot 0 - \lambda \cdot 1 = -\lambda. \tag{3.52}$$

Let $(\mathcal{E}_\rho, \mathcal{F}_\rho)$ be the truncated Dirichlet form given by (1.11). Denote by $q_t^{(\rho)}(x, y)$, $\{Q_t\}_{t\geq 0}$ the heat kernel and heat semigroup associated with $(\mathcal{E}_\rho, \mathcal{F}_\rho)$ respectively. We define the "perturbed semigroup" by

$$Q_t^{\psi_k} f = e^{\psi_k}\left(Q_t\left(e^{-\psi_k} f\right)\right).$$

Then we pick any nonnegative $f \in \mathcal{F} \cap L^\infty$ with $\|f\|_2 = 1$ and set

$$f_{t,k} := Q_t^{\psi_k} f, \tag{3.53}$$

for any integer $k \geq 0$. Clearly, the function $f_{t,k} \in \mathcal{F} \cap L^\infty \subset \mathcal{F}_\rho$.

By applying (3.15) with $p = p_k, R = r$ and f, ϕ being replaced by $f_{t,k}, \phi_{p_k,\lambda}$ respectively and by setting

$$K_0 := TC_0\lambda^{2\beta+2} \tag{3.54}$$

with T given by (3.17), we obtain that for any $k \geq 0$,

$$\mathcal{E}_\rho(e^{-\psi_k} f_{t,k}, e^{\psi_k} f_{t,k}^{2p_k-1}) \geq \frac{1}{4p_k}\mathcal{E}(f_{t,k}^{p_k}) - K_0 p_k^{2\beta+1} \int f_{t,k}^{2p_k} d\mu.$$

From this, we derive that

$$\begin{aligned}\frac{d}{dt}\left\|f_{t,k}\right\|_{2p_k}^{2p_k} &= -2p_k\mathcal{E}_\rho(e^{-\psi_k} f_{t,k}, e^{\psi_k} f_{t,k}^{2p_k-1}) \\ &\leq -2p_k\left\{\frac{1}{4p_k}\mathcal{E}(f_{t,k}^{p_k}) - K_0 p_k^{2\beta+1}\left\|f_{t,k}\right\|_{2p_k}^{2p_k}\right\} \\ &= -\frac{1}{2}\mathcal{E}(f_{t,k}^{p_k}) + 2K_0 p_k^{2\beta+2}\left\|f_{t,k}\right\|_{2p_k}^{2p_k}. \end{aligned} \tag{3.55}$$

In particular, for $k = 0$ ($p_0 = 1$),

$$\frac{d}{dt}\left\|f_{t,0}\right\|_2^2 \leq 2K_0\left\|f_{t,0}\right\|_2^2,$$

which gives that, using $\left\|f_{0,0}\right\|_2 = \|f\|_2 = 1$,

$$\left\|f_{t,0}\right\|_2 = \left\|f_{t,0}\right\|_{p_1} \leq e^{K_0 t}\|f\|_2 = e^{K_0 t}. \tag{3.56}$$

Since condition (DUE) implies the Nash inequality (cf. [9, Theorem 2.1]):

$$\|u\|_2^{2(1+\frac{\beta}{\alpha})} \leq C_N\mathcal{E}(u)\|u\|_1^{2\beta/\alpha}$$

for all $u \in \mathcal{F} \cap L^1$, we apply this inequality to function $f_{t,k}^{p_k} \in \mathcal{F} \cap L^1$ with $k \geq 1$

$$\mathcal{E}(f_{t,k}^{p_k}) \geq \frac{1}{C_N}\left\|f_{t,k}\right\|_{2p_k}^{2p_k(1+\frac{\beta}{\alpha})} \cdot \left\|f_{t,k}\right\|_{p_k}^{-2p_k\beta/\alpha}.$$

Plugging this into (3.55), we have

$$\begin{aligned} 2p_k\left\|f_{t,k}\right\|_{2p_k}^{2p_k-1}\frac{d}{dt}\left\|f_{t,k}\right\|_{2p_k} &= \frac{d}{dt}\left\|f_{t,k}\right\|_{2p_k}^{2p_k} \\ &\leq -\frac{1}{2C_N}\left\|f_{t,k}\right\|_{2p_k}^{2p_k(1+\frac{\beta}{\alpha})} \cdot \left\|f_{t,k}\right\|_{p_k}^{-2p_k\beta/\alpha} + 2K_0 p_k^{2\beta+2}\left\|f_{t,k}\right\|_{2p_k}^{2p_k}, \end{aligned}$$

which implies that

$$\frac{d}{dt}\left\|f_{t,k}\right\|_{2p_k} \leq -\frac{1}{4C_N p_k}\left\|f_{t,k}\right\|_{2p_k}^{1+\frac{2p_k\beta}{\alpha}}\left\|f_{t,k}\right\|_{p_k}^{-\frac{2p_k\beta}{\alpha}} + K_0 p_k^{2\beta+1}\left\|f_{t,k}\right\|_{2p_k} \tag{3.57}$$

for all $k \geq 1$.

On the other hand, we claim that for any $k \geq 0$,

$$\exp(-3/p_k) f_{t,k+1} \leq f_{t,k} \leq \exp(3/p_k) f_{t,k+1}. \tag{3.58}$$

Indeed, observe from (3.51), (3.16) and $p_{k+1} = 2p_k$,

$$\begin{aligned}\left\|\psi_{k+1} - \psi_k\right\|_\infty &= \lambda\left\|\phi_{p_{k+1},\lambda} - \phi_{p_k,\lambda}\right\|_\infty \\ &\leq \lambda\left\|\phi_{p_{k+1},\lambda} - \Phi\right\|_\infty + \lambda\left\|\phi_{p_k,\lambda} - \Phi\right\|_\infty \\ &\leq \lambda\left(\frac{1}{2\lambda p_k} + \frac{1}{\lambda p_k}\right) = \frac{3}{2p_k}. \end{aligned} \tag{3.59}$$

From this and using the Markovian property of $\{Q_t\}_{t\geq 0}$, we have

$$f_{t,k} = e^{\psi_k}\left(Q_t\left(e^{-\psi_k} f\right)\right) \leq e^{\psi_{k+1} + \frac{3}{2p_k}}\left(Q_t\left(e^{-\psi_{k+1} + \frac{3}{2p_k}} f\right)\right) = e^{3/p_k} f_{t,k+1}.$$

Similarly,

$$f_{t,k+1} \leq e^{3/p_k} f_{t,k}.$$

Thus, we obtain (3.58), proving our claim.

Therefore, we conclude from (3.57) that, using the fact that $f_{t,k} \le e^{6/p_k} f_{t,k-1}$ by (3.58),

$$\frac{d}{dt}\left\|f_{t,k}\right\|_{2p_k} \le -\frac{1}{C'_N p_k}\left\|f_{t,k}\right\|_{2p_k}^{1+\frac{2p_k\beta}{\alpha}}\left\|f_{t,k-1}\right\|_{p_k}^{-\frac{2p_k\beta}{\alpha}} + K_0 p_k^{2\beta+1}\left\|f_{t,k}\right\|_{2p_k} \tag{3.60}$$

for all $k \ge 1$, where $C'_N = 4C_N \exp(12\beta/\alpha)$.

Define $u_k(t) := \left\|f_{t,k-1}\right\|_{p_k}$ and

$$w_k(t) := \sup_{s\in(0,t]}\left\{s^{\alpha(p_k-2)/(2\beta p_k)}u_k(s)\right\}. \tag{3.61}$$

Then by (3.56)

$$w_1(t) = \sup_{s\in(0,t]}\{u_1(s)\} = \sup_{s\in(0,t]}\left\{\left\|f_{t,0}\right\|_2\right\} \le e^{K_0 t}. \tag{3.62}$$

On the other hand, we have from (3.60) that, using $u_k^{-2\beta p_k/\alpha}(t) \ge t^{p_k-2}w_k^{-2\beta p_k/\alpha}(t)$,

$$\begin{aligned} u'_{k+1}(t) &= \frac{d}{dt}\left\|f_{t,k}\right\|_{2p_k} \le -\frac{1}{C'_N p_k}u_{k+1}^{1+2\beta p_k/\alpha}(t)u_k^{-2\beta p_k/\alpha}(t) + K_0 p_k^{2\beta+1}u_{k+1}(t) \\ &\le -\frac{1}{C'_N p_k}\cdot\frac{t^{p_k-2}}{w_k^{2\beta p_k/\alpha}(t)}u_{k+1}^{1+2\beta p_k/\alpha} + K_0 p_k^{2\beta+1}u_{k+1}(t) \end{aligned}$$

for $k \ge 1$. Then the condition (3.31) is satisfied with $u(t) = u_{k+1}(t)$, $b = 1/\left(C'_N p_k\right)$, $p = p_k \ge 2$, $\theta = 2\beta p_k/\alpha$, $w = w_k$ and $K = K_0 p_k^{2\beta+1}$. Thus, applying Lemma 3.4 with $\nu = 2\beta + 2$, we obtain

$$u_{k+1}(t) \le \left(C'_N\alpha p_k^{2\beta+2}/\beta\right)^{\alpha/(2\beta p_k)} t^{-\alpha(p_k-1)/(2\beta p_k)}e^{K_0 p_k^{-1}t}w_k(t),$$

that is,

$$t^{\alpha(p_{k+1}-2)/(2\beta p_{k+1})}u_{k+1}(t) \le \left(C'_N\alpha p_k^{2\beta+2}/\beta\right)^{\alpha/(2\beta p_k)} e^{K_0 p_k^{-1}t}w_k(t), \tag{3.63}$$

for all $t > 0$. From this, we derive that

$$w_{k+1}(t) = \sup_{s\in(0,t]}\left\{s^{\alpha(p_{k+1}-2)/(2\beta p_{k+1})}u_{k+1}(s)\right\} \le \left(C'_N\alpha p_k^{2\beta+2}/\beta\right)^{\alpha/(2\beta p_k)} e^{K_0 p_k^{-1}t}w_k(t),$$

which gives that

$$\begin{aligned} w_{k+1}(t)/w_k(t) &\le \left(C'_N\alpha p_k^{2\beta+2}/\beta\right)^{\alpha/(2\beta p_k)} e^{K_0 t p_k^{-1}} \\ &= \left(2^{k(2\beta+2)}\cdot C'_N\alpha/\beta\right)^{\alpha/(\beta 2^{k+1})} e^{K_0 t 2^{-k}} \\ &= \left\{\left(C'_N\alpha/\beta\right)^{\alpha/(2\beta)} e^{K_0 t}\cdot\left(2^{\alpha(\beta+1)/\beta}\right)^k\right\}^{2^{-k}} := (Da^k)^{2^{-k}}, \end{aligned}$$

where $D := \left(C'_N\alpha/\beta\right)^{\alpha/(2\beta)} e^{K_0 t}$ and $a := 2^{\alpha(\beta+1)/\beta}$. This implies by iteration and using (3.62) that for any $k \ge 1$,

$$\begin{aligned} w_{k+1}(t) &\le (Da^k)^{1/2^k}w_k(t) \\ &\le (Da^k)^{1/2^k}\left\{(Da^{k-1})^{1/2^{k-1}}w_{k-1}(t)\right\} \le \dots \\ &\le D^{\frac{1}{2^k}+\frac{1}{2^{k-1}}+\dots+\frac{1}{2}}a^{\frac{k}{2^k}+\frac{k-1}{2^{k-1}}+\dots+\frac{1}{2}}w_1(t) \\ &\le Da^2 w_1(t) \le Da^2 e^{K_0 t} = C_7\exp(2K_0 t), \end{aligned} \tag{3.64}$$

where $C_7 = \left(C'_N\alpha/\beta\right)^{\alpha/(2\beta)} 2^{2\alpha(\beta+1)/\beta}$. Thus, we have from (3.61), (3.64) that for any $k \ge 1$, $t > 0$

$$t^{\alpha(p_{k+1}-2)/(2\beta p_{k+1})}\left\|Q_t^{\psi_k}f\right\|_{2p_k} \le w_{k+1}(t) \le C_7\exp(2K_0 t) \tag{3.65}$$

for any $0 \le f \in \mathcal{F}\cap L^\infty$ with $\|f\|_2 = 1$.

Since ψ_k is a Cauchy sequence in L^∞ by (3.59), the sequence $\{\psi_k\}$ converges uniformly to ψ_∞ as $k\to\infty$ with

$$\psi_\infty := \lambda\psi \in L^\infty$$

by using (3.16), where $\psi(y) = \left(\frac{2r-d(x_0,y)}{r}\right)_+ \wedge 1$ for $y\in M$. Clearly,

$$\psi_\infty(y) - \psi_\infty(x) = -\lambda \tag{3.66}$$

for any $x\in B(x_0,r)$ and any $y\in M\setminus B(x_0,2r)$. Set

$$f_{t,\infty} := e^{\psi_\infty}\left(Q_t\left(e^{-\psi_\infty} f\right)\right).$$

The sequence $\{f_{t,k}\}_{k\geq 1}$ converges uniformly to $f_{t,\infty}$ as $k\to\infty$, and thus

$$\left\|Q_t^{\psi_k} f\right\|_{p_k} = \left\|f_{t,k}\right\|_{p_k} \to \left\|Q_t^{\psi_\infty} f\right\|_\infty .$$

Therefore, letting $k\to\infty$ in (3.65), we obtain that

$$\left\|Q_t^{\psi_\infty} f\right\|_\infty \leq \frac{C_7}{t^{\alpha/(2\beta)}} \exp\left(2K_0 t\right).$$

for any $0\leq f\in\mathcal{F}\cap L^\infty$ with $\|f\|_2 = 1$, that is,

$$\left\|Q_t^{\psi_\infty}\right\|_{2\to\infty} := \sup_{\|f\|_2=1}\left\|Q_t^{\psi_\infty} f\right\|_\infty \leq \frac{C_7}{t^{\alpha/(2\beta)}}\exp\left(2K_0 t\right).$$

This inequality is also true for $-\psi_\infty$ by Remark 3.3 and repeating the above procedure. Since $Q_t^{-\psi_\infty}$ is the adjoint of operator $Q_t^{\psi_\infty}$, we see that

$$\left\|Q_t^{\psi_\infty}\right\|_{1\to 2} := \sup_{\|f\|_1=1}\left\|Q_t^{\psi_\infty} f\right\|_2 = \left\|Q_t^{-\psi_\infty}\right\|_{2\to\infty} \leq \frac{C_7}{t^{\alpha/(2\beta)}}\exp\left(2K_0 t\right),$$

and thus,

$$\left\|Q_t^{\psi_\infty}\right\|_{1\to\infty} \leq \left\|Q_{t/2}^{\psi_\infty}\right\|_{1\to 2}\left\|Q_{t/2}^{\psi_\infty}\right\|_{2\to\infty} \leq \frac{C_8}{t^{\alpha/\beta}}\exp\left(2K_0 t\right)$$

where $C_8 = 2^{\alpha/\beta}\left(C_7\right)^2$. From this and using (3.66), (3.54),

$$\begin{aligned} q_t^{(\rho)}(x,y) &\leq \frac{C_8}{t^{\alpha/\beta}}\exp\left(2K_0 t + \psi_\infty(y) - \psi_\infty(x)\right) \\ &= \frac{C_8}{t^{\alpha/\beta}}\exp\left(2C_0\lambda^{2\beta+2} T t - \lambda\right) \end{aligned} \tag{3.67}$$

for all $t,r>0$, μ-almost all $x\in B(x_0,r)$, $y\in M\setminus B(x_0,2r)$ and for all $\lambda\geq\eta^{-1}$ and $0<\eta<1$, where $\rho=\eta r$.

We distinguish two cases depending on $J\neq 0$ or $J\equiv 0$.

Case $J\neq 0$. By (3.17), (3.67) and using $\rho=\eta r$, we have

$$\begin{aligned} q_t^{(\rho)}(x,y) &\leq \frac{C_8}{t^{\alpha/\beta}}\exp\left(2C_0\lambda^{2\beta+2} T t - \lambda\right) \\ &= \frac{C_8}{t^{\alpha/\beta}}\exp\left(2C_0\lambda^{2\beta+2} e^{c_1(\eta)\lambda}\frac{t}{\rho^\beta} - \lambda\right) \\ &\leq \frac{C_8}{t^{\alpha/\beta}}\exp\left(C_9(\eta) e^{2c_1(\eta)\lambda}\frac{t}{r^\beta} - \lambda\right) \end{aligned} \tag{3.68}$$

where $c_1(\eta) = 2(\beta+1)\left(\eta+2\eta^2\right)$ by (3.14), and $C_9(\eta)$ is given by

$$C_9(\eta) = 2C_0\eta^{-\beta}\left[2\left(\beta+1\right)/c_1(\eta)\right]^{2\beta+2} = 2C_0\eta^{-\beta}\left(\eta+2\eta^2\right)^{-2(\beta+1)},$$

where in the last inequality we have used the following:

$$\lambda^{2\beta+2} \leq \left\{2\left(\beta+1\right)/c_1(\eta)\right\}^{2\beta+2} e^{c_1(\eta)\lambda}$$

by the elementary inequality $a \le e^a$ for any $a \ge 0$, with $a = \frac{c_1(\eta)}{2(\beta+1)}\lambda = \left(\eta + 2\eta^2\right)\lambda$.

We first choose λ and then choose η. Choose λ such that $e^{-\lambda} = \left(\frac{r}{t^{1/\beta}}\right)^{-(\alpha+\beta)}$, that is,

$$\lambda = \frac{\alpha+\beta}{\beta}\log\left(r^\beta/t\right), \tag{3.69}$$

but we need to ensure the condition $\lambda \ge \eta^{-1}$ is satisfied, namely

$$\log\left(r^\beta/t\right) \ge \frac{\beta}{\alpha+\beta}\eta^{-1}. \tag{3.70}$$

With such choice of λ, we then choose $\eta \in (0,1)$ such that

$$e^{2c_1(\eta)\lambda}\frac{t}{r^\beta} = 1, \tag{3.71}$$

that is,

$$4(\beta+1)\left(\eta + 2\eta^2\right) = 2c_1(\eta) = \frac{\beta}{\alpha+\beta}.$$

(Clearly this can be achieved. Actually we have $\eta + 2\eta^2 \le \frac{1}{4}$, implying $0 < \eta < \frac{\sqrt{3}-1}{2}$). Once η is chosen by (3.71), then the condition (3.70) is satisfied if

$$r^\beta/t \ge c_2 \tag{3.72}$$

for some universal constant $c_2 > 0$.

Therefore, we conclude from (3.68), (3.71), (3.69) that

$$q_t^{(\rho)}(x,y) \le \frac{C_8}{t^{\alpha/\beta}}\exp\left(C_9(\eta)\right)\cdot e^{-\lambda} = C_{10}\frac{t}{r^{\alpha+\beta}} \tag{3.73}$$

for all $t, r > 0$ with $r^\beta \ge c_2 t$ and all $\rho = \eta r$, for μ-almost all $x \in B(x_0, r), y \in B(x_0, 2r)^c$, where C_{10} is a universal constant independent of x_0, t, r, x, y.

Note that

$$p_t(x,y) \le q_t^{(\rho)}(x,y) + 2t \sup_{x\in M,\ y\in B(x,\rho)^c} J(x,y),$$

see [5, Lemma 3.1 (c)], or [21, (4.13) p.6412]. It follows that, using $\rho = \eta r$,

$$p_t(x,y) \le C_{10}\frac{t}{r^{\alpha+\beta}} + C\frac{2t}{\rho^{\alpha+\beta}} \le C_{11}\frac{t}{r^{\alpha+\beta}}$$

for all $t, r > 0$ with $r^\beta \ge c_2 t$, for μ-almost all $x \in B(x_0, r), y \in B(x_0, 2r)^c$, where C_{11} is a universal constant independent of x_0, t, r, x, y.

With a certain amount of effort, we can say that

$$p_t(x,y) \le C_{12}\frac{t}{d(x,y)^{\alpha+\beta}} \tag{3.74}$$

for all $t > 0$ and μ-almost all $x, y \in M$, if $d(x,y) \ge c_3 t^{1/\beta}$, for some universal constants $C_{12} > 0$ and $c_3 > 0$ (say, $c_3 = 4c_2^{1/\beta}$), thus showing that (UE) is true.

Finally, if $d(x,y) < c_3 t^{1/\beta}$ then (UE) follows directly from (DUE).

Case $J \equiv 0$. By (3.17), (3.67) and setting $\rho = \frac{1}{2}r$ with $\eta = \frac{1}{2}$, we have

$$\begin{aligned} p_t(x,y) = q_t^{(\rho)}(x,y) &\le \frac{C_8}{t^{\alpha/\beta}}\exp\left(2C_0\lambda^{2\beta+2}Tt - \lambda\right) \\ &= \frac{C_8}{t^{\alpha/\beta}}\exp\left(2C_0\lambda^{2\beta+2}\frac{t}{r^\beta} - \lambda\right) \end{aligned}$$

for all $t, r > 0$ and μ-almost all $x \in B(x_0, r), y \in B(x_0, 2r)^c$, for all $\lambda \ge \eta^{-1} = 2$. Choosing λ such that

$$2C_0\lambda^{2\beta+2}\frac{t}{r^\beta} = \frac{\lambda}{2},$$

that is, $\lambda = \left(\frac{1}{4C_0}\frac{r^\beta}{t}\right)^{1/(2\beta+1)}$. But we need ensure that $\lambda \geq \eta^{-1} = 2$; this can be achieved if $r^\beta \geq c_4 t$ for some universal constant $c_4 > 0$. Therefore, we obtain

$$p_t(x,y) \leq \frac{C_8}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{r^\beta}{t}\right)^{1/(2\beta+1)}\right)$$

for all $t, r > 0$ with $r^\beta \geq c_4 t$ and μ-almost all $x \in B(x_0, r), y \in B(x_0, 2r)^c$, for some universal constant c.

Therefore, we conclude that

$$p_t(x,y) \leq \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\frac{\beta}{\beta'-1}}\right) \tag{3.75}$$

for all $t > 0$ and μ-almost all $x, y \in M$, where

$$\beta' := 2\beta + 2.$$

Finally, we obtain ($\mathrm{UE}_{\mathrm{loc}}$) by applying Lemma 3.5. The proof is complete. □

We finish this section by proving Theorem 1.4.

Proof of Theorem 1.4. Indeed, by Theorem 1.3, it suffices to show the following implications

$$(UE) \quad \Rightarrow \quad (DUE) + (CIB) + (J_\leq), \tag{3.76}$$
$$(UE_{\mathrm{loc}}) \quad \Rightarrow \quad (DUE) + (CIB) + (J \equiv 0). \tag{3.77}$$

However, we have that (cf. [21, Theorem 2.3])

$$(UE) \Leftrightarrow (DUE) + (S) + (J_\leq),$$

and that (cf. [19, Theorem 2.1])

$$(UE_{\mathrm{loc}}) \Rightarrow (DUE) + (S)$$

if $(\mathcal{E}, \mathcal{F})$ is conservative and condition ($\mathrm{V}_\leq$) holds. Using the fact that

$$J(x,y) = \lim_{t\to 0} \frac{1}{2t} p_t(x,y) \text{ for } \mu\text{-a.a. } (x,y) \in M \times M \setminus \text{diag},$$

we see that condition (UE_{loc}) implies $J \equiv 0$. (Alternatively $J \equiv 0$ follows from [24, Theorem 3.4].) Thus, implications (3.76), (3.77) will follow if

$$(S) \Rightarrow (CIB),$$

which, however, has been proved in Lemma 2.2 in Section 2. □

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