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Optimal lower bound for the first eigenvalue of the fourth order equation

Gang Meng [a,*], Ping Yan [b]

[a] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
[b] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

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Abstract

In this paper we will find optimal lower bound for the first eigenvalue of the fourth order equation with integrable potentials when the L^1 norm of potentials is known. We establish the minimization characterization for the first eigenvalue of the measure differential equation, which plays an important role in the extremal problem of ordinary differential equation. The conclusion of this paper will illustrate a new and very interesting phenomenon that the minimizing measures will no longer be located at the center of the interval when the norm is large enough.

MSC: 34L15; 34L40

Keywords: Eigenvalue; The fourth order equation; Integrable potential; Minimization problem; Optimal lower bound

1. Introduction

Given an integrable potential $q \in \mathcal{L}^1 := \mathcal{L}^1([0,1],\mathbb{R})$, we consider eigenvalue problem of the fourth order beam equation

$$y^{(4)}(t) + q(t)y(t) = \lambda y(t), \qquad t \in [0,1], \tag{1.1}$$

* Corresponding author.
E-mail addresses: menggang@ucas.ac.cn (G. Meng), pyan@math.tsinghua.edu.cn (P. Yan).

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with the Lidstone boundary condition

$$y(0) = y''(0) = 0 = y(1) = y''(1). \tag{1.2}$$

It is well-known that problem (1.1), (1.2) has a sequence of (real) eigenvalues

$$\lambda_1(q) < \lambda_2(q) < \cdots < \lambda_m(q) < \cdots,$$

satisfying $\lim_{m\to\infty} \lambda_m(q) = +\infty$. See [3]. For constant potentials, one has

$$\lambda_m(c) = (m\pi)^4 + c \qquad \forall m \in \mathbb{N}, \quad c \in \mathbb{R}. \tag{1.3}$$

In this paper we are concerned with the first eigenvalues $\lambda_1(q)$ and will give their optimal lower bounds when the L^1 norms $\|q\|_1 = \|q\|_{L^1([0,1])}$ are known. To this end, we will solve the following minimization problem

$$\mathbf{L}(r) := \inf\{\lambda_1(q) : q \in B_1[r]\}. \tag{1.4}$$

Here, for $r \in [0, +\infty)$,

$$B_1[r] := \left\{ q \in \mathcal{L}^1 : \|q\|_1 \le r \right\}$$

is the ball of $(\mathcal{L}^1, \|\cdot\|_1)$. Once minimization problem (1.4) is solved, one has the following lower bound for $\lambda_1(q)$

$$\lambda_1(q) \ge \mathbf{L}(\|q\|_1) \qquad \forall q \in \mathcal{L}^1, \tag{1.5}$$

which will be shown to be optimal in a certain sense.

Problems linking the coefficient of an operator to the sequence of its eigenvalues are among the most fascinating of mathematical analysis. One of the reasons which make them so attractive is that the solutions are involved of many different branches of mathematics. Moreover, they are very simple to state and generally hard to solve. For both ordinary and partial differential operators, there have evolved a lot of results [1,5–7,9,14,19,23]. In recent years, the authors and their collaborators have revealed some deep properties on the dependence of eigenvalues on potentials and completely solved the extremal value problems for eigenvalues of the second order Sturm–Liouville and p-Laplace operators with potentials varied in balls in $\mathcal{L}^1$ space. See [11,18,22,24]. These extremal value problems cannot be solved directly by variational methods, because eigenvalues $\lambda_n(q)$ are implicit functionals of potentials q, the space $\mathcal{L}^1$ is infinite dimensional, and the balls $B_1[r]$ with radius r in $\mathcal{L}^1$ are non-compact non-smooth sets. Based on some topological facts on Lebesgue spaces and strong continuity and Fréchet differentiability of eigenvalues in potentials, the authors have used an analytical method to solve these extremal value problems in two steps.

Step 1. Deal with the corresponding problems in $\mathcal{L}^p$, $p > 1$, space by using the standard variational methods, because those balls $B_p[r]$ in L^p are smooth in usual topology and compact in weak topology when $p > 1$. Obtained in this step are the critical equations, which are autonomous Hamiltonian systems of 1-degree-of-freedom.

Step 2. Employ limiting approaches $p \downarrow 1$ from the viewpoint of conservation laws to obtain the limiting systems and extremal eigenvalues in $\mathcal{L}^1$ balls.

Like the second order equation, it is desirable to develop analogous ideas to study the extremal problem (1.4) for eigenvalues of the fourth order equation. However, how to use this analytical method to solve the extremal problem of the fourth order equation is still open to the authors. The difficulty is caused by the following two aspects. Firstly, the critical equations obtained in the case are Hamiltonian systems of 2-degree-of-freedom. It is an open problem that whether these critical equations are completely integrable. Secondly, as $p \downarrow 1$, it is unknown to the authors that what the corresponding systems for these limiting solutions and eigenvalues are. For more details, see [20]. Hence, in this paper we will adopt a different technique to study the extremal problem for eigenvalues of the fourth order equation.

Since the L^1 balls $B_1[r]$ lack compactness even in the weak topology of $\mathcal{L}^1$, we usually do not know if minimization problem (1.4) can be attained by some potentials from $B_1[r]$. To overcome this, different from the approach in [11,18], here we will extend the problem to the measure case. More precisely, let $\mu : [0, 1] \to \mathbb{R}$ be a measure. Firstly, we will explain what are solutions and eigenvalues of the corresponding equation with a measure

$$\mathrm{d}y^{(3)}(t) + y(t)\,\mathrm{d}\mu(t) = \lambda y(t)\,\mathrm{d}t, \qquad t \in [0, 1]. \tag{1.6}$$

Secondly, we will establish the minimization characterization for the first eigenvalue $\lambda_1(\mu)$ of the fourth order measure differential equation (MDE) (1.6) with the corresponding Lidstone boundary condition. See Theorem 3.2. Thirdly, as did in [12] for second order linear MDE, we will show some strong continuous dependence results of solutions and the first eigenvalues of (1.6) on measures μ with the weak* topology. See Theorem 2.9 and Theorem 3.4. Finally, we will find the explicit optimal lower bound of the first eigenvalue of MDE (1.6) when the total variation of measure μ is known. See Lemma 4.2. Based on the relationship between minimization problem of ODE and of MDE, we can obtain the main result of this paper as follows.

Theorem 1.1. *Given $r \geq 0$, one has*

$$\mathbf{L}(r) = \lambda_1(-r\delta_{a_*}) = \lambda_1(-r\delta_{1-a_*}). \tag{1.7}$$

Here $a_* \in (0, 1)$ satisfies

$$\lambda_1(-r\delta_{a_*}) = \min_{a\in(0,1)} \lambda_1(-r\delta_a) = \min_{a\in(0,1)} Y_r(a) = Y_r(a_*), \tag{1.8}$$

where δ_a and Y_r are as in (2.7) and (4.14), respectively.

Let us pay special attention to the following phenomenon. In general, the results known for the second order case do not necessarily hold for the corresponding fourth order problem. In [11,18], the authors have solved the minimization problem of eigenvalues of the second order equation. For the second order case, the critical measure of the Dirichlet eigenvalue is symmetric with respect to $1/2$, i.e.,

$$\int_{[0,t]} \mathrm{d}\mu(s) = \int_{[1-t,1]} \mathrm{d}\mu(s) \qquad \forall t \in [0, 1]. \tag{1.9}$$

For the fourth order case, we will find in this paper that the critical measure is symmetric when the fixed norm is small. However, when the norm is large enough, the critical measure is not symmetric with respect to $1/2$. See Remark 4.6.

This paper is organized as follows. In Section 2, we will recall basic facts on measures, the Lebesgue–Stieltjes integral and the Riemann–Stieltjes integral. In Section 3, we will use the variational method to establish the basic theory for the first eigenvalue of the fourth order MDE. In section 4, based on the minimization characterization of the first eigenvalues and the relationship between minimization problem of ODE and of MDE, we will prove Theorem 1.1.

2. Measures and solutions of MDE

Let $I = [0, 1]$. For a function $\mu : I \to \mathbb{R}$, the total variation of μ (over I) is defined as

$$\mathbf{V}(\mu, I) := \sup \left\{ \sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1,\ n \in \mathbb{N} \right\}.$$

Let

$$\mathcal{M}(I, \mathbb{R}) := \{\mu : I \to \mathbb{R} : \mu(0+)\ \exists,\ \mu(t+) = \mu(t)\ \forall t \in (0, 1),\ \mathbf{V}(\mu, I) < \infty\}$$

be the space of non-normalized $\mathbb{R}$-valued measures of I. Here, for any $t \in [0, 1)$, $\mu(t+) := \lim_{s\downarrow t} \mu(s)$ is the right-limit. The space of (normalized) $\mathbb{R}$-valued measures is

$$\mathcal{M}_0(I, \mathbb{R}) := \{\mu \in \mathcal{M}(I, \mathbb{R}) : \mu(0+) = 0\}. \tag{2.1}$$

For simplicity, we write $\mathbf{V}(\mu, I)$ as $\|\mu\|_{\mathbf{V}}$. By the Riesz representation theorem [8], $(\mathcal{M}_0(I, \mathbb{R}), \|\cdot\|_{\mathbf{V}})$ is the same as the dual space of the Banach space $(\mathcal{C}(I, \mathbb{R}), \|\cdot\|_\infty)$ of continuous $\mathbb{R}$-valued functions of I with the supremum norm $\|\cdot\|_\infty$. In fact, $\mu \in (\mathcal{M}_0(I, \mathbb{R}), \|\cdot\|_{\mathbf{V}})$ defines $\mu^* \in (\mathcal{C}(I, \mathbb{R}), \|\cdot\|_\infty)^*$ by

$$\mu^*(f) = \int_I f(t)\, \mathrm{d}\mu(t), \qquad \forall f \in \mathcal{C}(I, \mathbb{R}), \tag{2.2}$$

which refers to the Riemann–Stieltjes integral, or the Lebesgue–Stieltjes integral [2]. Moreover, one has

$$\|\mu\|_{\mathbf{V}} = \mathbf{V}(\mu, I) = \sup \left\{ \textstyle\int_I f\, \mathrm{d}\mu : f \in \mathcal{C}(I, \mathbb{R}),\ \|f\|_\infty = 1 \right\}.$$

For $t \in (0, 1]$, let

$$\mathbf{V}(\mu, (0, t]) := \sup \left\{ \sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 < t_0 < t_1 < \cdots < t_{n-1} < t_n = t,\ n \in \mathbb{N} \right\}.$$

It is known that for $\mu \in \mathcal{M}_0(I, \mathbb{R})$, $\mathbf{V}(\mu, I) = |\mu(0)| + \mathbf{V}(\mu, (0, 1])$. See, e.g., [13].

Lemma 2.1. *([15]) Let* $\nu \in \mathcal{M}_0(I, \mathbb{R})$*. Define*

$$\hat{\nu}(t) := \begin{cases} -|\nu(0)| & \text{for } t = 0, \\ \mathbf{V}(\nu, (0, t]) & \text{for } t \in (0, 1]. \end{cases} \tag{2.3}$$

Then $\hat{\nu} \in \mathcal{M}_0(I, \mathbb{R})$ *satisfies* $\|\nu\|_{\mathbf{V}} = \hat{\nu}(1) - \hat{\nu}(0)$ *and*

$$\left| \int_{[a,b]} f(s)\, \mathrm{d}\nu(s) \right| \le \int_{[a,b]} |f(s)|\, \mathrm{d}\hat{\nu}(s) \qquad \forall\, f \in \mathcal{C}(I, \mathbb{R}),\ [a, b] \subset I. \tag{2.4}$$

For general theory of the Riemann–Stieltjes integral and Lebesgue–Stieltjes integral, see, e.g., [2].

Typical examples of measures are as follows.

• Let $\ell : I \to \mathbb{R}$ be $\ell(t) \equiv t$. Then ℓ yields the Lebesgue measure of I and the Lebesgue integral. More generally, any $q \in \mathcal{L}^1(I, \mathbb{R})$ induces an absolutely continuous measure defined by

$$\mu_q(t) := \int_{[0,t]} q(s)\, \mathrm{d}s, \qquad t \in I. \tag{2.5}$$

In this case, one has

$$\|\mu_q\|_{\mathbf{V}} = \|q\|_1 = \|q\|_{\mathcal{L}^1(I,\mathbb{R})}, \tag{2.6}$$

and

$$\int_{I_0} f(t)\, \mathrm{d}\mu_q(t) = \int_{I_0} f(t) q(t)\, \mathrm{d}t = \int_{\overline{I_0}} f(t)\, \mathrm{d}\mu_q(t)$$

for any $f \in \mathcal{C}(I, \mathbb{R})$ and subinterval $I_0 \subset I$.

• For $a = 0$, the unit Dirac measure at $t = 0$ is

$$\delta_0(t) = \begin{cases} -1 & \text{for } t = 0, \\ 0 & \text{for } t \in (0, 1]. \end{cases}$$

• For $a \in (0, 1]$, the unit Dirac measure at $t = a$ is

$$\delta_a(t) = \begin{cases} 0 & \text{for } t \in [0, a), \\ 1 & \text{for } t \in [a, 1]. \end{cases} \tag{2.7}$$

In the space $\mathcal{M}_0(I, \mathbb{R})$ of measures, one has the usual topology induced by the norm $\|\cdot\|_{\mathbf{V}}$. Due to duality relation (2.2), one has also the following weak* topology w^*.

Definition 2.2. Let $\mu_0,\ \mu_n \in \mathcal{M}_0(I, \mathbb{R})$, $n \in \mathbb{N}$. We say that μ_n is weakly* convergent to μ_0 iff, for each $f \in \mathcal{C}(I, \mathbb{R})$, one has

$$\lim_{n\to\infty} \int_I f\, \mathrm{d}\mu_n = \int_I f\, \mathrm{d}\mu_0.$$

We remark that in some literature, this topology is just called the weak topology for measures. For example, as $a \downarrow 0$, one has

$$\int_I f\, \mathrm{d}\delta_a = f(a) \to f(0) = \int_I f\, \mathrm{d}\delta_0$$

for each $f \in \mathcal{C}(I,\mathbb{R})$. Thus $\delta_a \to \delta_0$ in $(\mathcal{M}_0(I,\mathbb{R}), w^*)$ as $a \downarrow 0$.

In general, a measure cannot be a limit of smooth measures in the norm $\|\cdot\|_{\mathbf{V}}$. However, in the w^* topology, the following conclusion holds.

Lemma 2.3. *([9]) Given $\mu_0 \in \mathcal{M}_0(I,\mathbb{R})$, there exists a sequence of measures $\{\mu_n\} \subset C^\infty(I,\mathbb{R}) \cap \mathcal{M}_0(I,\mathbb{R})$ such that*

$$\mu_n \to \mu_0 \text{ in } (\mathcal{M}_0(I,\mathbb{R}), w^*).$$

Moreover, if μ_0 is increasing (decreasing) on I, then the sequence $\{\mu_n\}$ above can be chosen such that for each $n \in \mathbb{N}$, μ_n is increasing (deceasing) on I and $\|\mu_n\|_{\mathbf{V}} = \|\mu_0\|_{\mathbf{V}}$.

Considering $q \in \mathcal{L}^1(I,\mathbb{R})$ as a density, one has the measure or distribution given by (2.5). Since $\|\mu_q\|_{\mathbf{V}} = \|q\|_1$,

$$(\mathcal{L}^1(I,\mathbb{R}), \|\cdot\|_1) \hookrightarrow (\mathcal{M}_0(I,\mathbb{R}), \|\cdot\|_{\mathbf{V}}) \text{ is an isometric embedding.} \tag{2.8}$$

For $r \in [0,\infty)$, denote

$$B_0[r] := \{\mu \in \mathcal{M}_0(I,\mathbb{R}) : \|\mu\|_{\mathbf{V}} \le r\}.$$

Via (2.5), by the Hölder inequality and the isometrical embedding (2.8), one has the following result on these balls.

Lemma 2.4. *Let $r > 0$. The following inclusion is proper*

$$B_1[r] \subset B_0[r].$$

As for the compactness of these balls in weak* topology, we have the following result.

Lemma 2.5. *([8]) Let $r > 0$. Then $B_0[r] \subset (\mathcal{M}_0(I,\mathbb{R}), w^*)$ is sequentially compact.*

Given a measure $\mu \in \mathcal{M}_0 := \mathcal{M}_0(I,\mathbb{R})$, we will write the fourth order linear MDE with the measure μ as

$$\mathrm{d}y^{(3)}(t) + y(t)\,\mathrm{d}\mu(t) = 0, \qquad t \in [0,1]. \tag{2.9}$$

Definition 2.6. A function $y : I \to \mathbb{R}$ is called a solution to the equation (2.9) on the interval I if

- $y \in \mathcal{C}(I,\mathbb{R})$, and
- there exist $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4$ and functions $y^{(1)}, y^{(2)}, y^{(3)} : [0,1] \to \mathbb{R}$ such that the following are satisfied

$$y(t) = y_0 + \int_{[0,t]} y^{(1)}(s)\,\mathrm{d}s, \qquad t \in [0,1], \tag{2.10}$$

$$y^{(1)}(t) = y_1 + \int_{[0,t]} y^{(2)}(s)\,\mathrm{d}s, \qquad t \in [0,1], \tag{2.11}$$

$$y^{(2)}(t) = y_2 + \int_{[0,t]} y^{(3)}(s)\,\mathrm{d}s, \qquad t \in [0,1], \tag{2.12}$$

$$y^{(3)}(t) = \begin{cases} y_3, & t = 0, \\ y_3 - \int_{[0,t]} y(s)\,\mathrm{d}\mu(s), & t \in (0,1]. \end{cases} \tag{2.13}$$

The initial condition of MDE (2.9) can be written as

$$(y(0), y^{(1)}(0), y^{(2)}(0), y^{(3)}(0)) = (y_0, y_1, y_2, y_3). \tag{2.14}$$

Since we have assumed that $y \in \mathcal{C} := \mathcal{C}([0,1], \mathbb{R})$, the right-hand sides of (2.10), (2.11), (2.12) are the Lebesgue integral and (2.13) Lebesgue–Stieltjes integral respectively.

Because solutions of (2.9)–(2.14) are defined via fixed point equations, there are many methods to prove the existence and uniqueness of solutions. For example, one can find a proof from [4,16,17] based on the Kurzweil–Stieltjes integral, which applies also to the first order linear MDE.

Lemma 2.7. *For each $(y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, problem (2.9)–(2.14) has the unique solution $y(t)$ defined on $[0,1]$.*

For $p \in [1, \infty]$, let $\mathcal{L}^p := L^p([0,1], \mathbb{R})$ be the Lebesgue space of real-valued functions with the L^p norm $\|\cdot\|_p$. For $n \in \mathbb{N}$, let $\mathcal{W}^{n,p} := W^{n,p}([0,1], \mathbb{R})$ and

$$\mathcal{W}_0^{n,p} := W_0^{n,p}([0,1], \mathbb{R}) = \{y \in \mathcal{W}^{n,p} : y(0) = y(1) = 0\}$$

be the usual Sobolev spaces with the norm $\|\cdot\|_{\mathcal{W}^{n,p}}$. For $p = 2$, $\mathcal{W}^{n,2}$ and $\mathcal{W}_0^{n,2}$ are denoted simply by $\mathcal{H}^n$ and $\mathcal{H}_0^n$, respectively, with the norm $\|\cdot\|_{\mathcal{H}^n}$.

By the properties of Lebesgue integral and Lebesgue–Stieltjes integral, some regularity results for solutions $y(t)$ are as follows.

Corollary 2.8. *Let $y(t)$ be the solution of (2.9). Then $y \in \mathcal{H}^3$ and $y^{(3)} \in \mathcal{M} := \mathcal{M}([0,1], \mathbb{R})$. Hence,*

$$y^{(1)}(t) = y'(t) \in \mathcal{C}^1 := \mathcal{C}^1([0,1], \mathbb{R}), \qquad y^{(2)}(t) = y''(t) \in \mathcal{AC} := \mathcal{AC}([0,1], \mathbb{R}),$$

and $y^{(3)}(t) = y'''(t)$ a.e. $t \in [0,1]$. Here $'$ denotes the derivative with respect to t.

We use $y(t, y_0, y_1, y_2, y_3)$ to denote the unique solution of (2.9)-(2.14). Let

$$\varphi_1(t) := y(t, 1, 0, 0, 0), \qquad \varphi_2(t) := y(t, 0, 1, 0, 0),$$
$$\varphi_3(t) := y(t, 0, 0, 1, 0), \qquad \varphi_4(t) := y(t, 0, 0, 0, 1),$$

called the fundamental solutions of (2.9). By the linearity of (2.9) and the uniqueness of solution, one has that, for $t \in [0, 1]$,

$$\begin{pmatrix} y(t, y_0, y_1, y_2, y_3) \\ y^{(1)}(t, y_0, y_1, y_2, y_3) \\ y^{(2)}(t, y_0, y_1, y_2, y_3) \\ y^{(3)}(t, y_0, y_1, y_2, y_3) \end{pmatrix} = \begin{pmatrix} \varphi_1(t) & \varphi_2(t) & \varphi_3(t) & \varphi_4(t) \\ \varphi_1^{(1)}(t) & \varphi_2^{(1)}(t) & \varphi_3^{(1)}(t) & \varphi_4^{(1)}(t) \\ \varphi_1^{(2)}(t) & \varphi_2^{(2)}(t) & \varphi_3^{(2)}(t) & \varphi_4^{(2)}(t) \\ \varphi_1^{(3)}(t) & \varphi_2^{(3)}(t) & \varphi_3^{(3)}(t) & \varphi_4^{(3)}(t) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$=: N_\mu(t) \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

In [12,21], the authors have obtained the continuity of solutions in measures for the second order linear MDE. For the fourth order linear MDE, we can argue in a similar way to prove the following conclusion.

Theorem 2.9. *Let $y(t, \mu)$ be the solution of (2.9)-(2.14). Then the following solution mappings are continuous*

$$(\mathcal{M}_0, w^*) \to (\mathcal{C}, \|\cdot\|_\infty), \qquad \mu \to y(\cdot, \mu), \tag{2.15}$$

$$(\mathcal{M}_0, w^*) \to (\mathcal{C}, \|\cdot\|_\infty), \qquad \mu \to y^{(1)}(\cdot, \mu), \tag{2.16}$$

$$(\mathcal{M}_0, w^*) \to (\mathcal{C}, \|\cdot\|_\infty), \qquad \mu \to y^{(2)}(\cdot, \mu), \tag{2.17}$$

$$(\mathcal{M}_0, w^*) \to (\mathcal{M}, w^*), \qquad \mu \to y^{(3)}(\cdot, \mu). \tag{2.18}$$

By Corollary 2.8, we have the following corollary.

Corollary 2.10. *The following solution mapping*

$$(\mathcal{M}_0, w^*) \to (\mathcal{C}^2, \|\cdot\|_{\mathcal{H}^2}), \qquad \mu \to y(\cdot, \mu), \tag{2.19}$$

is continuous, where $\mathcal{C}^2 := \mathcal{C}^2([0, 1], \mathbb{R})$.

3. The first eigenvalue of MDE

We consider eigenvalue problem of the fourth order equation (1.6) with the Lidstone boundary condition (1.2).

Definition 3.1. Given $\mu \in \mathcal{M}_0$, we say that $\lambda \in \mathbb{R}$ is an eigenvalue of the Lidstone problem (1.6), (1.2), if MDE (1.6) with such a parameter λ has non-zero solutions $y(t)$ satisfying (1.2). The corresponding solutions $y(t)$ are called eigenfunctions associated with λ.

Besides the Sobolev spaces $\mathcal{H}_0^2$ and $\mathcal{H}_0^3$, let us introduce

$$\mathcal{H}_{00}^3 := \{y \in \mathcal{H}^3 : y \text{ satisfies (1.2)}\} = \{y \in \mathcal{H}^3 : y(0) = y(1) = y''(0) = y''(1) = 0\}.$$

One has the proper inclusions $\mathcal{H}_{00}^3 \subset \mathcal{H}_0^3 \subset \mathcal{H}_0^2$.

It remains open to us what is the complete structure of eigenvalues of problem (1.6), (1.2). Let us introduce the Rayleigh form for problem (1.6)

$$R(u) = R_\mu(u) := \frac{\int_{[0,1]} (u'')^2\,\mathrm{d}t + \int_{[0,1]} u^2\,\mathrm{d}\mu(t)}{\int_{[0,1]} u^2\,\mathrm{d}t} \qquad \text{for } u \in \mathcal{H}_0^2\backslash\{0\}. \tag{3.1}$$

It is a standard result that any (possible) eigenvalue $\lambda \in \mathbb{R}$ of problem (1.6), (1.2) with eigenfunction $u \in \mathcal{H}_{00}^3$ must satisfy

$$R(u) = \lambda. \tag{3.2}$$

In the following theorem, we show that problem (1.6), (1.2) does admit the smallest (first) eigenvalue. In fact, we have the minimization characterizations of the first eigenvalue using $R(u)$.

Theorem 3.2. *Given $\mu \in \mathcal{M}_0$, problem (1.6), (1.2) admits a first (smallest) eigenvalue $\lambda_1(\mu)$, which has the following minimization characterizations*

$$\lambda_1(\mu) = \min_{u\in\mathcal{H}_0^2\backslash\{0\}} R(u) = \min_{u\in\mathcal{H}_0^3\backslash\{0\}} R(u) = \min_{u\in\mathcal{H}_{00}^3\backslash\{0\}} R(u). \tag{3.3}$$

Proof. Let $u \in \mathcal{H}_0^2$ be non-zero. We have

$$\begin{aligned}
\|u\|_\infty^2 &\le \left(\int_{[0,1]} |u'|\,\mathrm{d}t\right)^2 \le \int_{[0,1]} u'u'\,\mathrm{d}t \\
&= uu'\Big|_0^1 - \int_{[0,1]} uu''\,\mathrm{d}t \le \int_{[0,1]} |uu''|\,\mathrm{d}t \\
&\le \|u\|_2\|u''\|_2,
\end{aligned}$$

i.e.,

$$\|u''\|_2 \ge \|u\|_\infty^2/\|u\|_2. \tag{3.4}$$

Since

$$\begin{aligned}
\int_{[0,1]} u^2\,\mathrm{d}\mu(t) &\ge -\|\mu\|_{\mathbf{V}}\|u\|_\infty^2 = -\frac{\|\mu\|_{\mathbf{V}}\|u\|_2}{\sqrt{2}}\cdot\frac{\sqrt{2}\|u\|_\infty^2}{\|u\|_2} \\
&\ge -\frac{1}{2}\left(\frac{\|\mu\|_{\mathbf{V}}^2\|u\|_2^2}{2} + \frac{2\|u\|_\infty^4}{\|u\|_2^2}\right),
\end{aligned} \tag{3.5}$$

we have from (3.4) and (3.5)

$$\int_{[0,1]} (u'')^2\,\mathrm{d}t + \int_{[0,1]} u^2\,\mathrm{d}\mu(t) \ge \frac{\|u\|_\infty^4}{\|u\|_2^2} - \frac{1}{2}\left(\frac{\|\mu\|_{\mathbf{V}}^2\|u\|_2^2}{2} + \frac{2\|u\|_\infty^4}{\|u\|_2^2}\right)$$
$$= -\frac{\|\mu\|_{\mathbf{V}}^2\|u\|_2^2}{4}.$$

Thus

$$R(u) \ge -\frac{\|\mu\|_{\mathbf{V}}^2}{4} \qquad \forall 0 \ne u \in \mathcal{H}_0^2. \tag{3.6}$$

Due to (3.6),

$$\lambda_1 := \inf_{u\in\mathcal{H}_0^2\setminus\{0\}} R(u) > -\infty. \tag{3.7}$$

Take a sequence $\{u_n\} \subset \mathcal{H}_0^2$ such that

$$\|u_n\|_\infty = 1 \text{ and } \lim_{n\to+\infty} R(u_n) = \lambda_1. \tag{3.8}$$

Then

$$\int_{[0,1]} (u_n'')^2\,\mathrm{d}t = R(u_n)\int_{[0,1]} u_n^2\,\mathrm{d}t - \int_{[0,1]} u_n^2\,\mathrm{d}\mu(t) \le |R(u_n)| + \|\mu\|_{\mathbf{V}}$$

is bounded. See (3.8). Combining with the assumption $\|u_n\|_\infty = 1$, it is easy to see that $\{u_n\} \subset \mathcal{H}_0^2$ is bounded. As $\mathcal{H}_0^2$ is a Hilbert space and is compactly embedded into $\mathcal{C}^1$, there exists a non-zero $u_0 \in \mathcal{H}_0^2$ such that

$$u_n \to u_0 \text{ in } (\mathcal{H}_0^2, w) \text{ and } u_n \to u_0 \text{ in } (\mathcal{C}^1, \|\cdot\|_{\mathcal{C}^1}),$$

going to a subsequence if necessary. Thus

$$\int_{[0,1]} (u_0'')^2\,\mathrm{d}t = \lim_{n\to+\infty}\int_{[0,1]} u_0''u_n''\,\mathrm{d}t \le \liminf_{n\to+\infty}\left(\int_{[0,1]} (u_0'')^2\,\mathrm{d}t\right)^{1/2}\left(\int_{[0,1]} (u_n'')^2\,\mathrm{d}t\right)^{1/2}.$$

This implies that

$$\int_{[0,1]} (u_0'')^2\,\mathrm{d}t \le \liminf_{n\to+\infty}\int_{[0,1]} (u_n'')^2\,\mathrm{d}t.$$

Hence

$$
\begin{aligned}
R(u_0) &\le \frac{\liminf_{n\to+\infty} \int_{[0,1]} (u_n'')^2 \,\mathrm{d}t + \int_{[0,1]} u_0^2 \,\mathrm{d}\mu(t)}{\int_{[0,1]} u_0^2 \,\mathrm{d}t} \\
&= \liminf_{n\to+\infty} \frac{\int_{[0,1]} (u_n'')^2 \,\mathrm{d}t + \int_{[0,1]} u_n^2 \,\mathrm{d}\mu(t)}{\int_{[0,1]} u_n^2 \,\mathrm{d}t} \\
&= \liminf_{n\to+\infty} R(u_n) = \lambda_1.
\end{aligned}
$$

Combining with (3.7), one has

$$
R(u_0) = \lambda_1 = \min_{u \in \mathcal{H}_0^2 \setminus \{0\}} R(u). \tag{3.9}
$$

Take any

$$
\phi \in \mathcal{C}_c^\infty := \{\phi \in C^\infty([0,1]) : \operatorname{supp}\phi \subset (0,1)\}.
$$

Then $u_0 + s\phi \in \mathcal{H}_0^2 \setminus \{0\}$ for all $s \in \mathbb{R}$ with $|s|$ small enough.

As a function of s, it follows from (3.9) that $R(u_0 + s\phi)$ takes a minimum at $s = 0$. Thus

$$
\begin{aligned}
0 &= \left.\frac{\mathrm{d}R(u_0 + s\phi)}{\mathrm{d}s}\right|_{s=0} \\
&= \frac{2}{\left(\int_{[0,1]} u_0^2 \,\mathrm{d}t\right)^2} \left(\int_{[0,1]} u_0^2 \,\mathrm{d}t \left(\int_{[0,1]} u_0''\phi'' \,\mathrm{d}t + \int_{[0,1]} u_0\phi \,\mathrm{d}\mu(t) \right) \right. \\
&\quad \left. - \left(\int_{[0,1]} (u_0'')^2 \,\mathrm{d}t + \int_{[0,1]} u_0^2 \,\mathrm{d}\mu(t) \right) \int_{[0,1]} u_0\phi \,\mathrm{d}t \right) \\
&= \frac{2}{\int_{[0,1]} u_0^2 \,\mathrm{d}t} \left(\int_{[0,1]} u_0''\phi'' \,\mathrm{d}t + \int_{[0,1]} u_0\phi \,\mathrm{d}\mu(t) - \lambda_1 \int_{[0,1]} u_0\phi \,\mathrm{d}t \right).
\end{aligned} \tag{3.10}
$$

Here (3.9) is used and the derivative is found using definition (3.1) for $R(u)$. By introducing the measure $\hat{\mu} : [0,1] \to \mathbb{R}$

$$
\hat{\mu}(t) := \mu(t) - \lambda_1 t, \qquad t \in [0,1],
$$

equation (3.10) is

$$
\int_{[0,1]} u_0''(t)\phi''(t) \,\mathrm{d}t + \int_{[0,1]} u_0(t)\phi(t) \,\mathrm{d}\hat{\mu}(t) = 0 \qquad \forall \phi \in \mathcal{C}_c^\infty. \tag{3.11}
$$

Since $\phi \in \mathcal{C}_c^\infty$, one has

$$\phi(t) = \int_{[0,t]} \left(\int_{[0,s]} \phi''(\tau)\,\mathrm{d}\tau \right) \mathrm{d}s = \int_{[0,t]} (t-s)\phi''(s)\,\mathrm{d}s.$$

Then

$$\begin{aligned}
\int_{[0,1]} u_0(t)\phi(t)\,\mathrm{d}\hat{\mu}(t) &= \int_{[0,1]} \left(\int_{[0,t]} (t-s)u_0(t)\phi''(s)ds \right) \mathrm{d}\hat{\mu}(t) \\
&= \int_{[0,1]} \left(\int_{(s,1]} (t-s)u_0(t)\,\mathrm{d}\hat{\mu}(t) \right) \phi''(s)\,\mathrm{d}s \\
&= \int_{[0,1]} \left(\int_{(t,1]} (s-t)u_0(s)\,\mathrm{d}\hat{\mu}(s) \right) \phi''(t)\,\mathrm{d}t.
\end{aligned}$$

Substituting into (3.11), we obtain

$$\int_{[0,1]} \left(u_0''(t) + \int_{(t,1]} (s-t)u_0(s)\,\mathrm{d}\hat{\mu}(s) \right) \phi''(t)\,\mathrm{d}t = 0$$

for all $\phi \in \mathcal{C}_c^\infty$. Hence $u_0(t)$ satisfies

$$u_0''(t) + \int_{(t,1]} (s-t)u_0(s)\,\mathrm{d}\hat{\mu}(s) = ct + \hat{c} \qquad \text{a.e. } t \in [0,1], \tag{3.12}$$

where $c,\ \hat{c}$ are some constants. Note that

$$\begin{aligned}
\int_{[0,t]} \left(\int_{[0,\tau]} u_0(s)\,\mathrm{d}\hat{\mu}(s) \right) \mathrm{d}\tau &= \int_{[0,t]} (t-s)u_0(s)\,\mathrm{d}\hat{\mu}(s) \\
&= \int_{(t,1]} (s-t)u_0(s)\,\mathrm{d}\hat{\mu}(s) + \int_{[0,1]} (t-s)u_0(s)\,\mathrm{d}\hat{\mu}(s) \\
&= \int_{(t,1]} (s-t)u_0(s)\,\mathrm{d}\hat{\mu}(s) + c_1 t + \hat{c}_1,
\end{aligned}$$

where c_1 and $\hat{c}_1$ are constants. Hence equation (3.12) can be rewritten as

$$u_0''(t) + \int_{[0,t]} \left(\int_{[0,\tau]} u_0(s)\,\mathrm{d}\hat{\mu}(s) \right) \mathrm{d}\tau = c_2 t + \hat{c}_2 \qquad \text{a.e. } t \in [0,1], \tag{3.13}$$

where $c_2 = c - c_1$ and $\hat{c}_2 = \hat{c} - \hat{c}_1$. By the properties of Lebesgue integral and Lebesgue–Stieltjes integral, one knows from (3.13) that $u_0''(t)$ is absolutely continuous and satisfies

$$u_0'''(t) + \int_{[0,t]} u_0(s)\,\mathrm{d}\mu(s) - \lambda_1 \int_{[0,t]} u_0(s)\,\mathrm{d}s = c_2 \qquad \text{a.e. } t \in [0,1]. \tag{3.14}$$

Equation (3.14) implies that $u_0 \in \mathcal{H}^3$ and therefore $u_0 \in \mathcal{H}_0^3$. With the explanation to solutions of MDE, (3.14) shows that $u_0(t)$ is a non-zero solution of the MDE (1.6) with the choice $\lambda = \lambda_1$. Moreover, it is standard to verify that $u_0(t)$ also satisfies the boundary condition $u_0''(0) = u_0''(1) = 0$ (see [3, p. 208]). Thus $u_0 \in \mathcal{H}_{00}^3$ and λ_1 is necessarily an eigenvalue of problem (1.6), (1.2) with the eigenfunction u_0. Because of (3.9) and the fact that $u_0 \in \mathcal{H}_{00}^3$, $\lambda_1 = \lambda_1(\mu)$ which is characterized as in (3.3). Finally, due to result (3.2) for general eigenvalues, we know that $\lambda_1(\mu)$ must be the smallest eigenvalue of problem (1.6), (1.2). □

Let us introduce the following ordering for measures. We say that measures $\mu_2 \geq \mu_1$ if

$$\int_{[0,1]} f(t)\,\mathrm{d}\mu_2(t) \geq \int_{[0,1]} f(t)\,\mathrm{d}\mu_1(t) \qquad \text{for all } f \in \mathcal{C}_+ := \{f \in \mathcal{C} : f(t) \geq 0,\ t \in [0,1]\}.$$

As a consequence of (3.3) in Theorem 3.2, we can obtain the following result.

Corollary 3.3. *Let $\mu_1, \mu_2 \in \mathcal{M}_0$. Then*

$$\mu_2 \geq \mu_1 \Rightarrow \lambda_1(\mu_2) \geq \lambda_1(\mu_1).$$

Now the continuity of the first eigenvalue in measures with the weak* topology can be proved by the same arguments as those in [10].

Theorem 3.4. *As a nonlinear functional, $\lambda_1(\mu)$ is continuous in $\mu \in (\mathcal{M}_0, w^*)$.*

4. The optimal lower bound of the first eigenvalue

Let us explicitly find the first eigenvalues for Dirac measures $-r\delta_a$, where $a \in (0,1)$ and $r \geq 0$. To this end, we need to solve the following equation

$$\mathrm{d}y^{(3)}(t) - r y(t)\,\mathrm{d}\delta_a(t) = \lambda y(t)\,\mathrm{d}t, \qquad t \in [0,1]. \tag{4.1}$$

From the explanation to solutions of MDE, one knows that solutions $y(t)$ of (4.1) satisfies the classical ODE

$$y^{(4)}(t) = \lambda y(t) \tag{4.2}$$

for t on the intervals $[0, a)$ and $(a, 1]$. At $t = a$, one has the following relations

$$\begin{cases} y(a+) = y(a-), & y'(a+) = y'(a-), \\ y''(a+) = y''(a-), & y'''(a+) = y'''(a-) + ry(a-), \end{cases} \tag{4.3}$$

or,

$$\begin{pmatrix} y(a+) \\ y'(a+) \\ y''(a+) \\ y'''(a+) \end{pmatrix} = A_r \begin{pmatrix} y(a-) \\ y'(a-) \\ y''(a-) \\ y'''(a-) \end{pmatrix}, \qquad \text{where } A_r := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r & 0 & 0 & 1 \end{pmatrix}. \tag{4.4}$$

From the first two conditions of (1.2), let us consider the initial value

$$(y(0), y'(0), y''(0), y'''(0)) = (0, c, 0, \hat{c}) \neq 0$$

at $t = 0$. From ODE (4.2) on $[0, a)$, we obtain

$$y(t) = c_1 \sin \omega t + c_2 \sinh \omega t, \qquad t \in [0, a), \tag{4.5}$$

for some $(c_1, c_2) \neq 0$. Here,

$$\omega := \begin{cases} \sqrt[4]{\lambda} \in \mathbb{R} & \text{for } \lambda \geq 0, \\ \sqrt[4]{|\lambda|} e^{\frac{\pi i}{4}} \in \mathbb{C} & \text{for } \lambda < 0. \end{cases} \tag{4.6}$$

By (4.3), we have

$$\begin{aligned} y(a+) &= z_0 := c_1 \sin \omega a + c_2 \sinh \omega a, \\ y'(a+) &= z_1 := c_1 \omega \cos \omega a + c_2 \omega \cosh \omega a, \\ y''(a+) &= z_2 := -c_1 \omega^2 \sin \omega a + c_2 \omega^2 \sinh \omega a, \\ y'''(a+) &= z_3 := c_1 \left(-\omega^3 \cos \omega a + r \sin \omega a \right) + c_2 \left(\omega^3 \cosh \omega a + r \sinh \omega a \right). \end{aligned}$$

By using this as the initial value at $t = a$, we obtain from ODE (4.2) that

$$\begin{aligned} y(t) = {} & \frac{z_0 - \frac{z_2}{\omega^2}}{2} \cos \omega(t-a) + \frac{\frac{z_1}{\omega} - \frac{z_3}{\omega^3}}{2} \sin \omega(t-a) \\ & + \frac{z_0 + \frac{z_2}{\omega^2}}{2} \cosh \omega(t-a) + \frac{\frac{z_1}{\omega} + \frac{z_3}{\omega^3}}{2} \sinh \omega(t-a) \\ = {} & c_1 \sin \omega a \cos \omega(t-a) \\ & + \left(c_1 \cos \omega a - r \frac{c_1 \sin \omega a + c_2 \sinh \omega a}{2\omega^3} \right) \sin \omega(t-a) \\ & + c_2 \sinh \omega a \cosh \omega(t-a) \\ & + \left(c_2 \cosh \omega a + r \frac{c_1 \sin \omega a + c_2 \sinh \omega a}{2\omega^3} \right) \sinh \omega(t-a) \end{aligned}$$

$$= c_1 \sin \omega t + c_2 \sinh \omega t \\ + r \frac{c_1 \sin \omega a + c_2 \sinh \omega a}{2\omega^3} (\sinh \omega(t-a) - \sin \omega(t-a)) \tag{4.7}$$

for $t \in (a, 1]$. Now the last two conditions $y(1) = y''(1) = 0$ of (1.2) are the following linear system for (c_1, c_2)

$$\begin{cases} c_1 \sin \omega + c_2 \sinh \omega + r \frac{\sinh \omega(1-a) - \sin \omega(1-a)}{2\omega^3} (c_1 \sin \omega a + c_2 \sinh \omega a) = 0, \\ \omega^2 (-c_1 \sin \omega + c_2 \sinh \omega) + r \frac{\sinh \omega(1-a) + \sin \omega(1-a)}{2\omega} (c_1 \sin \omega a + c_2 \sinh \omega a) = 0. \end{cases} \tag{4.8}$$

In order that system (4.8) has non-zero solutions (c_1, c_2), the corresponding determinant of (4.8) is necessarily zero. This yields the following equation

$$G(\lambda, a) = r, \tag{4.9}$$

where $G : (-\infty, \pi^4] \times (0, 1) \to [0, +\infty)$ is defined as

$$G(\lambda, a) := \begin{cases} \frac{2\omega^3 \sinh \omega \sin \omega}{\sin(\omega a) \sin(\omega(1-a)) \sinh \omega - \sinh(\omega a) \sinh(\omega(1-a)) \sin \omega} & \text{for } \lambda \neq 0. \\ \frac{3}{a^2(1-a)^2} & \text{for } \lambda = 0. \end{cases} \tag{4.10}$$

Then, by the existence of the first eigenvalue, we conclude

$$\lambda_1(-r\delta_a) = \min\{\lambda \in \mathbb{R} : G(\lambda, a) - r = 0\}.$$

It is easy to check that $G(\lambda, a)$ is a well-defined real function of $(\lambda, a) \in (-\infty, \pi^4] \times (0, 1)$ with $G(\pi^4, a) = 0$ and $G(\lambda, a) = G(\lambda, 1-a)$. Moreover, the following properties of $G(\lambda, a)$ can be proved.

Lemma 4.1. *Let $G(\lambda, a)$ be defined as in (4.10). One has*

(i) *When $a \in (0, 1)$ is fixed, $G(\lambda, a)$ is decreasing in $\lambda \in (-\infty, \pi^4]$.*
(ii) *When $\lambda \in (-\infty, \pi^4]$ is fixed, there exists $a_\lambda \in (0, 1)$ such that*

$$G(\lambda, a_\lambda) = \min_{a \in (0,1)} G(\lambda, a),$$

and $E(\lambda, a_\lambda) = 0$, where $E : (-\infty, \pi^4] \times (0, 1) \to \mathbb{R}$ is defined as

$$E(\lambda, a) = \sin(\omega(1-2a)) \sinh \omega - \sinh(\omega(1-2a)) \sin \omega. \tag{4.11}$$

(iii) *When $\lambda \in [\Lambda_1, \pi^4]$ is fixed, one has $a_\lambda = 1/2$, i.e.,*

$$G(\lambda, 1/2) = \min_{a \in (0,1)} G(\lambda, a). \tag{4.12}$$

Here $\Lambda_1 (\approx -950.8843)$ is the unique root of $H(\lambda) = 0$ with

$$H : [-(3\pi/\sqrt{2})^4, 0) \to \mathbb{R}, \qquad \lambda \to H(\lambda) = w(\sin w - \sinh w). \tag{4.13}$$

Proof. Since these results are concerned with elementary functions, we only give the basic ideas of the proof of this lemma.

(i) When $a \in (0, 1)$ is fixed, one can check that

$$\frac{\partial}{\partial \lambda} G(\lambda, a) \le 0, \qquad \forall \lambda \in (-\infty, \pi^4].$$

(ii) Since

$$\lim_{a \to 0} G(\lambda, a) = \lim_{a \to 1} G(\lambda, a) = +\infty,$$

there exists $a_\lambda \in (0, 1)$ such that

$$G(\lambda, a_\lambda) = \min_{a \in (0,1)} G(\lambda, a),$$

which is equivalent to

$$\frac{1}{G(\lambda, a_\lambda)} = \max_{a \in (0,1)} \frac{1}{G(\lambda, a)}.$$

Thus one has

$$0 = \frac{\partial}{\partial a} \frac{1}{G(\lambda, a)}\bigg|_{a=a_\lambda} = \frac{\sin(\omega(1-2a_\lambda))\sinh\omega - \sinh(\omega(1-2a_\lambda))\sin\omega}{2\omega^2 \sinh\omega \sin\omega},$$

which implies $E(\lambda, a_\lambda) = 0$.

(iii) When $\lambda \in [\Lambda_1, \pi^4]$ is fixed, one has

$$\frac{\partial^2}{\partial^2 a} \frac{1}{G}(\lambda, a) = \frac{\sin\omega \cosh(\omega(1-2a)) - \sinh\omega \cos(\omega(1-2a))}{\omega \sin\omega \sinh\omega} \le 0 \qquad \forall a \in (0, 1/2],$$

and

$$\frac{\partial}{\partial a} \frac{1}{G(\lambda, a)}\bigg|_{a=1/2} = 0.$$

Hence

$$\frac{\partial}{\partial a} \frac{1}{G(\lambda, a)} \ge 0 \qquad \forall a \in (0, 1/2],$$

which means that $\frac{1}{G(\lambda,a)}$ is increasing and then $G(\lambda, a)$ is decreasing in $a \in (0, 1/2]$. Thus (4.12) follows from $G(\lambda, a) = G(\lambda, 1-a)$. □

By Lemma 4.1, one has that for fixed $a \in (0, 1)$, $r \ge 0$,

$$\lambda_1(-r\delta_a) = Y_r(a).$$

Here $Y_r : (0, 1) \to (-\infty, \pi^4]$ is defined as

$$Y_r(a) = \lambda_a, \tag{4.14}$$

where $\lambda_a \in (-\infty, \pi^4]$ is the unique root of

$$G(\lambda_a, a) - r = 0.$$

Now, we are ready to find the explicit optimal lower bounds of the first eigenvalues of MDE and the relationship between minimization problem of ODE and of MDE.

Firstly, we study the following minimization problem

$$\tilde{\mathbf{L}}(r) := \inf\{\lambda_1(\mu) : \mu \in B_0[r]\} = \min\{\lambda_1(\mu) : \mu \in B_0[r]\}, \tag{4.15}$$

because $B_0[r]$ is sequentially compact in $(\mathcal{M}_0, w^*)$. See Theorem 3.4.

Lemma 4.2. *Given $r \geq 0$, one has*

$$\tilde{\mathbf{L}}(r) = \lambda_1(-r\delta_{a_*}) = \lambda_1(-r\delta_{1-a_*}),$$

where $a_* \in (0, 1)$ satisfies (1.8).

Proof. Given $\mu \in B_0[r]$, we take an eigenfunction $y(t)$ associated with $\lambda_1(\mu)$ which satisfies the normalization condition $\|y\|_2 = 1$. There exists $a \in (0, 1)$ such that

$$\|y\|_\infty = \max_{t\in[0,1]} |y(t)| = |y(a)|.$$

We have that

$$\begin{aligned} \lambda_1(\mu) &= \int_{[0,1]} (y'')^2 \,\mathrm{d}t + \int_{[0,1]} y^2 \,\mathrm{d}\mu(t) \\ &\geq \int_{[0,1]} (y'')^2 \,\mathrm{d}t - \|\mu\|_{\mathbf{V}} \|y\|_\infty^2 \\ &\geq \int_{[0,1]} (y'')^2 \,\mathrm{d}t - r y^2(a) \\ &= \int_{[0,1]} (y'')^2 \,\mathrm{d}t + \int_{[0,1]} y^2 \,\mathrm{d}(-r\delta_a(t)) \\ &\geq \lambda_1(-r\delta_a). \end{aligned} \tag{4.16}$$

Here the last inequality in (4.16) follows from characterization (3.3) for $\lambda_1(-r\delta_a)$ since $\|y\|_2 = 1$. Hence

$$\tilde{\mathbf{L}}(r) = \inf_{a\in(0,1)} \lambda_1(-r\delta_a).$$

By Lemma 4.1 and the definition of Y_r in (4.14), there exists $a_* \in (0, 1)$ satisfying (1.8). □

Secondly, we can obtain the relationship between minimization problem of ODE and of MDE as follows.

Lemma 4.3. *Given $r \geq 0$, one has that*

$$\mathbf{L}(r) = \tilde{\mathbf{L}}(r). \tag{4.17}$$

Proof. Given $q \in B_1[r]$, the measure $\mu_q \in \mathcal{M}_0$ is defined as (2.5). By (2.8), we have that $\mu_q \in B_0[r]$ is absolutely continuous with respect to the Lebesgue measure.

So for any $q \in B_1[r]$,

$$\tilde{\mathbf{L}}(r) \leq \lambda_1(\mu_q) = \lambda_1(q),$$

which implies that

$$\tilde{\mathbf{L}}(r) \leq \mathbf{L}(r). \tag{4.18}$$

On the other hand, there exists $\bar{\mu} \in B_0[r]$ such that $\lambda_1(\bar{\mu}) = \tilde{\mathbf{L}}(r)$. By the property of measures in Lemma 2.1 and the monotonicity of $\lambda_1(\mu)$ in Corollary 3.3, without loss of generality, we can assume that $\bar{\mu} = -\hat{\bar{\mu}}$ is decreasing. By Lemma 2.3, there exists a sequence of measures $\{\bar{\mu}_n\} \subset C^\infty \cap \mathcal{M}_0$ such that

$$\begin{aligned} \frac{\mathrm{d}\bar{\mu}_n(t)}{\mathrm{d}t} &= \bar{q}_n(t), \\ \|\bar{\mu}_n\|_{\mathbf{V}} &= \|\bar{q}_n\|_1 = \|\bar{\mu}\|_{\mathbf{V}} \leq r, \\ \bar{\mu}_n &\to \bar{\mu} \text{ in } (\mathcal{M}_0, w^*). \end{aligned}$$

Therefore, by Theorem 3.4, we have

$$\tilde{\mathbf{L}}(r) = \lambda_1(\bar{\mu}) = \lim_{n\to\infty} \lambda_1(\bar{\mu}_n) = \lim_{n\to\infty} \lambda_1(\bar{q}_n) \geq \lim_{n\to\infty} \mathbf{L}(r) = \mathbf{L}(r). \tag{4.19}$$

Now (4.18) and (4.19) imply that $\mathbf{L}(r) = \tilde{\mathbf{L}}(r)$. □

The proof of Theorem 1.1. By Lemma 4.2 and Lemma 4.3, the conclusion holds directly. □

Remark 4.4. To compute $\mathbf{L}(r)$, it suffices to solve the following optimization problem

$$\min f(\lambda, a) = \lambda$$

subject to the constraints

$$G(\lambda, a) - r = 0, \qquad 0 < a \leq 1/2,$$

where $G(\lambda, a)$ as in (4.10).

In Fig. 1, we have plotted $\mathbf{L}(r)$ as functions of r.

By Lemma 4.1 (iii), we have the following conclusion.

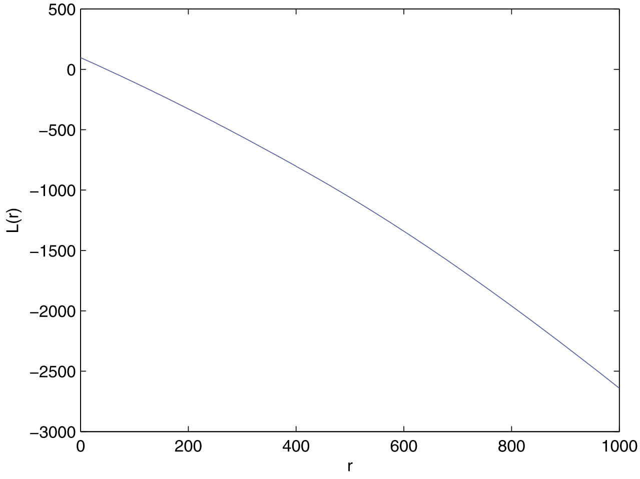

Fig. 1. Function $\mathbf{L}(r)$ of r.

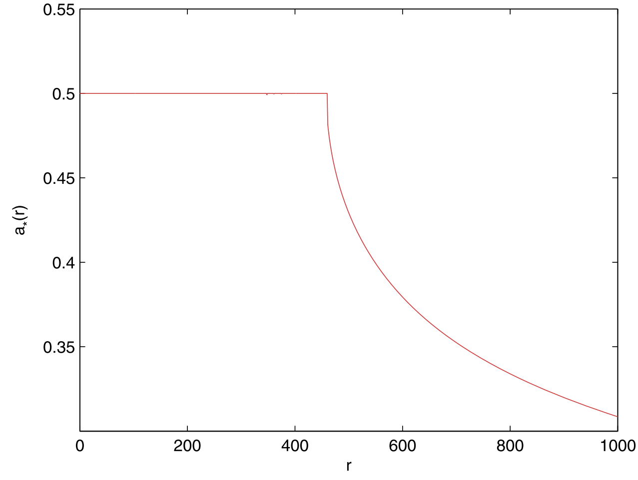

Fig. 2. Function $a_*(r)$ of r.

Corollary 4.5. *If* $0 \le r \le G(\Lambda_1, 1/2)$ (≈ 458.4163), *then*

$$a_* = a_*(r) = 1/2 \textit{ and } \mathbf{L}(r) = \lambda_1(-r\delta_{1/2}) \in [\Lambda_1, \pi^4].$$

Remark 4.6. In Fig. 2, we have plotted $a_* \in (0, 1/2]$ as functions of r. One can see that $a_* < 1/2$ when $r > G(\Lambda_1, 1/2)$. This means that if r is large enough, then the minimal measures are not symmetric with respect to $t = 1/2$, which is different from the case of the second order equation (see [11,18]).

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