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Complete structure of the Fučík spectrum of the *p*-Laplacian with integrable potentials on an interval

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To characterize the complete structure of the Fučík spectrum of the p-Laplacian on higher dimensional domains is a long-standing problem. In this paper, we study the p-Laplacian with integrable potentials on an interval under the Dirichlet or the Neumann boundary conditions. Based on the strong continuity and continuous differentiability of solutions in potentials, we will give a comprehensive characterization of the corresponding Fučík spectra: each of them is composed of two trivial lines and a double-sequence of differentiable, strictly decreasing, hyperbolic-like curves; all asymptotic lines of these spectral curves are precisely described by using eigenvalues of the p-Laplacian with potentials; and moreover, all these spectral curves have strong continuity in potentials, i.e. as potentials vary in the weak topology, these spectral curves are continuously dependent on potentials in a certain sense.

Keywords: p-Laplacian; integrable potentials; Fučík spectrum; asymptotic lines; strong continuity.

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1. Introduction

Fučík spectrum was first introduced for the Laplacian on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, by Dancer [4] and by Fučík [5] in the 1970s, in connection with the study of semilinear elliptic boundary value problems with jumping nonlinearities.

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Thereafter this important concept was generalized to the p-Laplacian Δ_p, $p > 1$. See [12] and references therein. Precisely, let $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $u_\pm = \max\{\pm u, 0\}$, then the Fučík spectrum of Δ_p is defined as the set of those points $(\lambda, \mu) \in \mathbb{R}^2$ such that

$$-\Delta_p u = \lambda(u_+)^{p-1} - \mu(u_-)^{p-1}, \quad u \in W_0^{1,p}(\Omega), \tag{1.1}$$

has a non-zero solution. Here the Dirichlet boundary condition is considered. The Fučík spectrum of the Laplacian Δ_2 corresponds to $p = 2$, while the (usual) spectrum of Δ_p corresponds to $\lambda = \mu$.

Fučík spectrum has been studied extensively for the last several decades, and it has been widely used to study boundary value problems and Lagrangian stability of semilinear equations. However, till now the complete structure of the Fučík spectrum for general cases $p > 1$ and $N > 1$ remains open, and even the usual spectrum of Δ_p is partially known ([6, 8]). In fact, it has been a long-standing question whether Δ_p has other eigenvalues besides the well-known sequence of so-called variational eigenvalues. One remarkable result is that for some examples, where non-constant weights and potentials are involved, and when the Neumann boundary conditions are considered, non-variational eigenvalues for the p-Laplacian, $p \neq 2$, do exist ([2]). On the one hand, the PDE case of (1.1) is complicated. On the other hand, for the ODE case ($N = 1$ and $p > 1$), the Fučík spectrum is quite simple and clear, because it can be obtained explicitly by finding solutions of the ODE. It is an interesting question whether one can give a complete description on the Fučík spectrum of some kind of p-Laplacian, which is neither so difficult as the PDE case nor so easy as the ODE case. Then there come problems for the scalar p-Laplacian involving potentials and/or weights and different kinds of boundary conditions. The aim of this paper is to give a positive answer to this question.

Given an exponent $p \in (1, \infty)$, let $\phi_p : \mathbb{R} \to \mathbb{R}$ be defined as $\phi_p(x) = |x|^{p-2}x$ for any $x \in \mathbb{R}$. For non-zero integrable functions a, $b \in \mathcal{L}^1 := L^1[0, 1]$, the Fučík spectrum of the scalar p-Laplacian with a pair of weights (a, b), denoted by $\Pi_p^D(a, b)$ or $\Pi_p^N(a, b)$, are the sets of those $(\lambda, \mu) \in \mathbb{R}^2$ such that the ODE

$$(\phi_p(x'))' + \lambda a(t)\phi_p(x_+) - \mu b(t)\phi_p(x_-) = 0, \quad t \in [0, 1] \tag{1.2}$$

has non-zero solutions satisfying the Dirichlet boundary condition

$$x(0) = x(1) = 0, \tag{D}$$

or, the Neumann boundary condition

$$x'(0) = x'(1) = 0, \tag{N}$$

respectively. Alif [1] studied $\Pi_p^D(a, b)$ and $\Pi_p^N(a, b)$ by means of "zero functions", where the weights a and b were assumed to be sign-changing (i.e. $a_\pm \not\equiv 0$ and $b_\pm \not\equiv 0$) continuous functions without "singular points" (which is a technical hypothesis). Their main results are as follows. Besides the trivial horizontal lines and vertical lines, restricted to each quadrant of $\mathbb{R}^2$, $\Pi_p^{D/N}(a, b)$ consists of a non-zero odd or infinite number of hyperbolic-like curves. The asymptotic behavior of the first nontrivial curves in each quadrant was also studied. It was observed that for instance the first curve of $\Pi_p^N(a,b)$ in $\mathbb{R}^+ \times \mathbb{R}^+$ is not asymptotic on any side to the trivial horizontal and vertical lines. In other words, there are always gaps between its asymptotic lines and the trivial horizontal and vertical lines. However, the exact asymptotic lines were not found in that paper.

The problem we choose to study is as follows. Given $a, b \in \mathcal{L}^1$, called potentials, we consider the ODE

$$(\phi_p(x'))' + (\lambda + a(t))\phi_p(x_+) - (\mu + b(t))\phi_p(x_-) = 0, \quad t \in [0,1]. \tag{1.3}$$

The corresponding Fučík spectrum, denoted by $\Sigma_p^D(a,b)$ or $\Sigma_p^N(a,b)$, are the sets of those $(\lambda,\mu) \in \mathbb{R}^2$ such that (1.3) has non-zero solutions satisfying (D) or (N), respectively. In this paper we will give a comprehensive study for these Fučík spectra. The main results obtained are as follows.

(i) We will give a complete characterization on these Fučík spectra: each of them is composed of one horizontal line, one vertical line and a double-sequence of differentiable, strictly decreasing, hyperbolic-like curves. We use the notations

$$\Sigma_p^D(a,b) = \bigcup_{k\geq 1} C_{k,\pm}^D(a,b), \quad \Sigma_p^N(a,b) = \bigcup_{k\geq 0} C_{k,\pm}^N(a,b),$$

where $C_{1,+}^D(a,b)$ and $C_{0,+}^N(a,b)$ denote the trivial vertical lines, $C_{1,-}^D(a,b)$ and $C_{0,-}^N(a,b)$ the trivial horizontal lines, and $\{C_{k,\pm}^D(a,b)\}_{k\geq 2}$ and $\{C_{k,\pm}^N(a,b)\}_{k\geq 1}$ the hyperbolic-like curves.

(ii) We will give all asymptotes of these Fučík spectral curves $C_{k,\pm}^D(a,b)$ and $C_{k,\pm}^N(a,b)$ by using (Sturm–Liouville) eigenvalues of scalar p-Laplacian with potentials.

(iii) Furthermore, we will study the dependence of these curves $C_{k,\pm}^{D/N}(a,b)$ on potentials $(a,b) \in \mathcal{L}^1 \times \mathcal{L}^1$. It will be shown that they have strong continuity in potentials, namely, if $a_n \xrightarrow{w_1} a_0$ and $b_n \xrightarrow{w_1} b_0$ in $\mathcal{L}^1$, where w_1 denotes the weak topology in $\mathcal{L}^1$, then $C_{k,\pm}^{D/N}(a_n,b_n)$ converge to $C_{k,\pm}^{D/N}(a_0,b_0)$ in a certain sense which will be precisely explained in the content.

The method employed in this paper to obtain the complete spectral structure is quite different from those traditional ones, such as the min–max procedure in [12] or the "zero functions" procedure in [1]. Our proofs are based on the (strong) continuity and the Fréchet differentiability of the solutions of (1.3) in potentials. Moreover, based on the comparison theorem for ODEs and quasi-monotonicity of solutions of the argument equations, we will develop some *scissors-and-paste* technique, which is used to obtain all precise asymptotic lines of Fučík spectral curves $C_{k,\pm}^{D/N}(a,b)$. Although strong continuity has not been widely studied in the literature, it gradually proves to be a powerful tool to study some problems in infinite dimensional spaces [9–11, 13, 15]. The strong continuity of the first and the second eigenvalues of the Laplacian (PDE case) on potentials/weights has been studied by Cuesta and Quoirin [3]. In a very recent paper, Salort [14] has studied the strong continuous dependence of the first nontrivial Fučík spectral curve of the p-Laplacian (PDE case) on the (positive) weights, in the sense of the weak* topology of $L^\infty(\Omega)$. In comparison, the continuous dependence of Fučík spectral curves on potentials obtained in this paper is much stronger, because it is in the sense of the weak topology of $\mathcal{L}^1$.

This paper is organized as follows. In Sec. 2, we list some preliminary results. Section 3 is devoted to the Fučík spectrum for the case with Dirichlet boundary conditions, and Sec. 4 the case with Neumann boundary conditions.

2. Preliminary Results

2.1. *The p-cosine and the p-sine*

Let $p > 1$. Denote $p^* = \frac{p}{p-1}$, the conjugate number of p. The unique solution of the initial value problem

$$\begin{cases} \dfrac{dx(\theta)}{d\theta} = -\phi_{p^*}(y(\theta)), & \dfrac{dy(\theta)}{d\theta} = \phi_p(x(\theta)), \\ x(0) = 1, \quad y(0) = 0, \end{cases}$$

is $(x(\theta), y(\theta)) = (\cos_p \theta, \sin_p \theta)$, where $\cos_p \theta$ and $\sin_p \theta$ are the so-called p-cosine and p-sine, because they possess properties similar to those of the standard cosine and sine (with $p = 2$). Some properties of p-cosine and p-sine are listed as follows (cf. [17]).

- Both $\cos_p \theta$ and $\sin_p \theta$ are $2\pi_p$-periodic, where $\pi_p = \frac{2\pi(p-1)^{1/p}}{p \sin(\pi/p)}$;
- $\cos_p \theta$ is even and $\sin_p \theta$ is odd in $\theta \in \mathbb{R}$;
- $\cos_p(\theta + \pi_p) = -\cos_p \theta$ and $\sin_p(\theta + \pi_p) = -\sin_p \theta$ for all $\theta \in \mathbb{R}$;
- $\cos_p \theta = 0$ if and only if $\theta = k\pi_p - \pi_p/2$, $k \in \mathbb{Z}$, and $\sin_p \theta = 0$ if and only if $\theta = k\pi_p$, $k \in \mathbb{Z}$; and
- $|\cos_p\theta|^p + (p-1)|\sin_p \theta|^{p^*} \equiv 1$.

For convenience, throughout this paper, we will denote

$$\Pi_k := k\pi_p - \frac{\pi_p}{2}, \quad k \in \mathbb{Z},$$

which are the zeros of $\cos_p \theta$. For example,

$$\Pi_0 = -\frac{\pi_p}{2}, \quad \Pi_1 = \frac{\pi_p}{2}.$$

2.2. *Properties of solutions of the p-Laplacian with potentials*

Let $p > 1$ be given. For an integrable potential $a \in \mathcal{L}^1$, consider the following second-order ODE

$$(\phi_p(x'))' + a(t)\phi_p(x) = 0, \quad t \in [0, 1]. \tag{2.1}$$

Let $y = -\phi_p(x')$. Then Eq. (2.1) is equivalent to the first-order planar system

$$x' = -\phi_{p^*}(y), \quad y' = a(t)\phi_p(x).$$

In the p-polar coordinates

$$x = r^{2/p}\cos_p\theta, \quad y = r^{2/p^*}\sin_p\theta, \tag{2.2}$$

the planar system is equivalent to the following system for the argument θ and the radius r

$$\theta' = a(t)|\cos_p\theta|^p + (p-1)|\sin_p\theta|^{p^*}, \tag{2.3}$$

$$(\log r)' = \frac{p}{2}(a(t)-1)\phi_p(\cos_p\theta)\phi_{p^*}(\sin_p\theta). \tag{2.4}$$

The right-hand sides of (2.3) and (2.4) are independent of r and are π_p-periodic in θ. For $\vartheta \in \mathbb{R}$, denote by $\theta(t;\vartheta,a)$ the unique solution of (2.3) satisfying the initial value condition $\theta(0) = \vartheta$. Because of the π_p-periodicity of (2.3) in θ, one has

$$\theta(t;\vartheta + k\pi_p, a) \equiv \theta(t;\vartheta,a) + k\pi_p, \quad \forall\, k \in \mathbb{Z}. \tag{2.5}$$

For a pair of potentials $a, b \in \mathcal{L}^1$, we consider the ODE

$$(\phi_p(x'))' + a(t)\phi_p(x_+) - b(t)\phi_p(x_-) = 0, \quad t \in [0,1]. \tag{2.6}$$

When $b = a$, Eq. (2.6) is reduced to (2.1). Again, let $y = -\phi_p(x')$ and use the p-polar coordinates (2.2). Then (2.6) can be written as

$$\theta' = A(t,\theta) := a(t)\,(\cos_p\theta)_+^p + b(t)\,(\cos_p\theta)_-^p + (p-1)|\sin_p\theta|^{p^*}, \tag{2.7}$$

$$(\log r)' = \frac{p}{2}((a(t)-1)\,(\cos_p\theta)_+^{p-1} - (b(t)-1)\,(\cos_p\theta)_-^{p-1})\phi_{p^*}(\sin_p\theta). \tag{2.8}$$

The right-hand sides of (2.7) and (2.8) are independent of r, but $2\pi_p$-periodic in θ in general. One may compare (2.7) and (2.8) with (2.3) and (2.4).

Let $\theta(t;\vartheta,a,b)$ be the unique solution of (2.7) satisfying $\theta(0) = \vartheta \in \mathbb{R}$. Then

$$\theta(t;\vartheta,a) \equiv \theta(t;\vartheta,a,a). \tag{2.9}$$

Given $\vartheta \in \mathbb{R}$, Eq. (2.8) has a unique solution $r(t;\vartheta,a,b)$ satisfying $r(1;\vartheta,a,b) = 1$. Using the p-polar coordinates (2.2), we define

$$X(t;\vartheta,a,b) := (r(t;\vartheta,a,b))^{2/p}\cos_p\theta(t;\vartheta,a,b). \tag{2.10}$$

Then $X(t;\vartheta,a,b)$ is a nontrivial solution of Eq. (2.6).

One basic observation on Eq. (2.7) is that the vector field $A(t,\theta) = p-1 > 0$ at those θ such that $\cos_p\theta = 0$, i.e. $\theta = \Pi_m$, $m \in \mathbb{Z}$. Since $a(t)$ and $b(t)$ are integrable only, the derivative $\theta'(t)$ at any specific t is meaningless. However, one can still use such an observation to obtain the following property, called quasi-monotonicity in t. We refer the readers to [17, Lemma 2.3] for a detailed proof.

Lemma 2.1. *Given* $a, b \in \mathcal{L}^1$ *and* $\vartheta \in \mathbb{R}$, *let* $\theta(t) = \theta(t; \vartheta, a, b)$ *be a solution of* (2.7) *satisfying* $\theta(0) = \vartheta$. *If* $\theta(t_0) \geq \Pi_m$ *for some* $t_0 \in [0, 1)$ *and* $m \in \mathbb{Z}$, *then*

$$\theta(t) > \Pi_m, \quad \forall\, t \in (t_0, 1].$$

Remark 2.1. Let $\vartheta = \Pi_0$ in (2.10) and use the simple notations

$$X(t) = X(t; \Pi_0, a, b), \quad r(t) = r(t; \Pi_0, a, b), \quad \theta(t) = \theta(t; \Pi_0, a, b).$$

Then $X(t) = r(t)^{2/p} \cos_p \theta(t)$ and $X(0) = r(0)^{2/p} \cos_p \Pi_0 = 0$. By Lemma 2.1, we have $\theta(t) > \Pi_0$ for any $t \in (0, 1]$, and hence $X'(0) \geq 0$. Since $X(t)$ is a nontrivial solution of (2.6) and $X(0) = 0$, it is necessary that $X'(0) > 0$.

In the Lebesgue space $\mathcal{L}^1$, besides the usual topology induced by the L^1 norm $\|\cdot\|_1$, one has the so-called weak topology w_1. By $g_n \to g_0$ in $(\mathcal{L}^1, w_1)$, it means that

$$\int_0^1 g_n(t) f(t) dt \to \int_0^1 g_0(t) f(t) dt, \quad \forall\, f \in L^\infty[0, 1].$$

Let $\vartheta \in \mathbb{R}$ be given. As mappings of $(a, b) \in \mathcal{L}^1 \times \mathcal{L}^1$, it turns out that solutions $\theta(\cdot; \vartheta, a, b)$, $r(\cdot; \vartheta, a, b)$ and $X(\cdot; \vartheta, a, b)$ admit Fréchet differentiability and (strong) continuity. Some of such properties are listed in the following theorem. For detailed proof, we refer to [7, Theorems 2.1 and 2.2] and the corresponding corollaries in [7].

Theorem 2.1 ([7]). *Let* $\vartheta \in \mathbb{R}$ *be fixed. One has the following results.*

(i) *As mappings from* $(\mathcal{L}^1, w_1)^2$ *to* $(C[0, 1], \|\cdot\|_\infty)$, $\theta(\cdot; \vartheta, a, b)$ *and* $r(\cdot; \vartheta, a, b)$ *are continuous. More precisely, if* $a_n \xrightarrow{w_1} a_0$ *and* $b_n \xrightarrow{w_1} b_0$ *in* $\mathcal{L}^1$, *then*

$$\|\theta(\cdot; \vartheta, a_n, b_n) - \theta(\cdot; \vartheta, a_0, b_0)\|_\infty \to 0,$$

$$\|r(\cdot; \vartheta, a_n, b_n) - r(\cdot; \vartheta, a_0, b_0)\|_\infty \to 0,$$

$$\|X(\cdot; \vartheta, a_n, b_n) - X(\cdot; \vartheta, a_0, b_0)\|_\infty \to 0,$$

as $n \to \infty$. *Here* $\|\cdot\|_\infty$ *is the supremum norm.*

(ii) *The functional* $(\mathcal{L}^1, w_1)^2 \to \mathbb{R}$, $(a, b) \mapsto \theta(1; \vartheta, a, b)$ *is continuous. More precisely, if* $a_n \xrightarrow{w_1} a_0$ *and* $b_n \xrightarrow{w_1} b_0$ *in* $\mathcal{L}^1$, *then* $\theta(1; \vartheta, a_n, b_n) \to \theta(1; \vartheta, a_0, b_0)$, *as* $n \to \infty$.

(iii) *The functional* $(\mathcal{L}^1, \|\cdot\|_1)^2 \to \mathbb{R}$, $(a, b) \mapsto \theta(1; \vartheta, a, b)$ *is continuously differentiable in the sense of Fréchet. The differentials of* $\theta(1; \vartheta, a, b)$ *with respect to* a *and* b, *denoted respectively by* $\partial_a \theta(1; \vartheta, a, b)$ *and* $\partial_b \theta(1; \vartheta, a, b)$, *are*

$$\partial_a \theta(1; \vartheta, a, b) = (X_+(\cdot; \vartheta, a, b))^p, \tag{2.11}$$

$$\partial_b \theta(1; \vartheta, a, b) = (X_-(\cdot; \vartheta, a, b))^p, \tag{2.12}$$

where $X_\pm := \max\{\pm X, 0\} \geq 0$. *Moreover, as mappings from* $(\mathcal{L}^1, \|\cdot\|_1)^2$ *to* $(C[0, 1], \|\cdot\|_\infty)$, *both* $\partial_a \theta(1; \vartheta, a, b)$ *and* $\partial_b \theta(1; \vartheta, a, b)$ *are continuous. Here* (2.11) *and* (2.12) *are understood as the following bounded linear functionals*

$$\mathcal{L}^1 \ni h \mapsto \int_0^1 (X_\pm(t;\vartheta,a,b))^p\, h(t)dt.$$

Remark 2.2. Let $\vartheta \in \mathbb{R}$ and $a_i, b_i \in \mathcal{L}^1$, $i = 0, 1$. We take the notation $a_0 \succ a_1$ if $a_0 \geq a_1$ on $[0,1]$ and $\int_0^1 (a_0(t) - a_1(t))dt > 0$. There hold the following monotonicity results.

(i) $a_0 \geq a_1, b_0 \geq b_1 \Rightarrow \theta(1;\vartheta,a_0,b_0) \geq \theta(1;\vartheta,a_1,b_1)$;
(ii) $a_0 \succ a_1 \Rightarrow \theta(1;\vartheta,a_0) > \theta(1;\vartheta,a_1)$;
(iii) the function $\theta(1;\vartheta,\lambda + a_0)$ is continuous and strictly increasing in $\lambda \in \mathbb{R}$.

In fact, for any $\tau \in [0,1]$, let $a_\tau = a_0 + \tau(a_1 - a_0)$, $b_\tau = b_0 + \tau(b_1 - b_0)$ and

$$X_\tau(t) = X(t;\vartheta,a_\tau,b_\tau), \quad \varphi(\tau) = \theta(1;\vartheta,a_\tau,b_\tau).$$

One get from (2.11) and (2.12) that

$$\begin{aligned}\varphi'(\tau) &= \partial_a\theta(1;\vartheta,a_\tau,b_\tau) \circ (a_1 - a_0) + \partial_b\theta(1;\vartheta,a_\tau,b_\tau) \circ (b_1 - b_0)\\ &= \int_0^1 (X_\tau(t))_+^p (a_1(t) - a_0(t))dt + \int_0^1 (X_\tau(t))_-^p (b_1(t) - b_0(t))dt. \end{aligned} \tag{2.13}$$

If $a_0 \geq a_1$ and $b_0 \geq b_1$, then $\varphi'(\tau) \leq 0$ for any $\tau \in [0,1]$, and hence $\varphi(0) \geq \varphi(1)$, proving the result in (i). Let $b_i = a_i$, $i = 0, 1$, and $a_0 \succ a_1$ in (2.13). Then

$$\varphi'(\tau) = \int_0^1 |X_\tau(t)|^p (a_1(t) - a_0(t))dt < 0, \quad \forall\, \tau \in [0,1]$$

because $X_\tau(t)$ has only finite zeroes in $[0,1]$. Therefore $\varphi(0) > \varphi(1)$, and the result in (ii) is proved. The result in (iii) follows immediately from (ii).

We consider Eqs. (2.3) and (2.7) again. Let $t_0 \in [0,1]$. Use the notations $\theta_{t_0}(t;\vartheta,a)$ and $\theta_{t_0}(t;\vartheta,a,b)$ to denote the solutions of these equations satisfying initial value conditions $\theta(t_0) = \vartheta \in \mathbb{R}$. Thus

$$\theta(t;\vartheta,a) \equiv \theta_0(t;\vartheta,a), \quad \theta(t;\vartheta,a,b) \equiv \theta_0(t;\vartheta,a,b).$$

Some properties of $\theta_{t_0}(t;\vartheta,a)$ are as follows.

- Like equality (2.5), one has

$$\theta_{t_0}(t;\vartheta + k\pi_p, a) \equiv \theta_{t_0}(t;\vartheta,a) + k\pi_p, \quad k \in \mathbb{Z}. \tag{2.14}$$

- Given $a \in \mathcal{L}^1$, let $\theta_{t_0}(t;\vartheta,\lambda + a)$ be a solution of the following equation

$$\theta' = (\lambda + a(t))|\cos_p\theta|^p + (p-1)|\sin_p\theta|^{p^*}. \tag{2.15}$$

  If $0 \leq t_0 < t_1 \leq 1$, then $\theta_{t_0}(t_1;\vartheta,\lambda + a)$ is a continuous, strictly increasing function in $\lambda \in \mathbb{R}$. This follows from the comparison theorem for ODEs.

- Let a and $t_0 < t_1$ be as above. If the initial value $\vartheta \in [\Pi_k, \Pi_{k+1})$ for some $k \in \mathbb{Z}$, then

$$\lim_{\lambda \to +\infty} \theta_{t_0}(t_1; \vartheta, \lambda + a) = +\infty, \quad \lim_{\lambda \to -\infty} \theta_{t_0}(t_1; \vartheta, \lambda + a) = \Pi_k. \tag{2.16}$$

  These results can be proved by similar arguments as in the proofs of [7, Lemmas 3.2 and 3.3], respectively.
- Let $a, b \in \mathcal{L}^1$ and $\vartheta \in \mathbb{R}$. If $0 \le t_0 < t_1 \le 1$, it follows immediately from the comparison theorem for ODEs that $\theta_{t_0}(t_1; \vartheta, \lambda + a, \mu + b)$ are non-decreasing in both $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$.

The following three lemmas will play important role in studying the asymptotic behaviors of the Fučík spectrum for scalar p-Laplacian.

Lemma 2.2. *Let $a \in \mathcal{L}^1$, $0 \le t_1 < t_2 < t_3 \le 1$, $k \in \mathbb{Z}$, $\vartheta \in \mathbb{R}$ and $\lambda^* \in \mathbb{R}$. If*

$$\theta_{t_1}(t_2; \vartheta, \lambda^* + a) = \Pi_k,$$

then there exists $\lambda_* < \lambda^*$ such that

$$\theta_{t_1}(t_3; \vartheta, \lambda_* + a) = \Pi_k.$$

Proof. Since $\theta_{t_1}(t_2; \vartheta, \lambda^* + a) = \Pi_k$ and $t_1 < t_2$, the quasi-monotonicity result in Lemma 2.1 shows that $\vartheta \in [\Pi_m, \Pi_{m+1})$ for some $m < k$. By the second equality in (2.16), for $t_3 > t_1$, we have

$$\lim_{\lambda \to -\infty} \theta_{t_1}(t_3; \vartheta, \lambda + a) = \Pi_m < \Pi_k. \tag{2.17}$$

On the other hand, by Lemma 2.1 again, we have

$$\theta_{t_1}(t_3; \vartheta, \lambda^* + a) > \Pi_k, \tag{2.18}$$

because $\theta_{t_1}(t_2; \vartheta, \lambda^* + a) = \Pi_k$ and $t_3 > t_2$. Now the lemma can be proved by (2.17), (2.18) and the continuity of $\theta_{t_1}(t_3; \vartheta, \lambda + a)$ in λ. □

Lemma 2.3. *Let $a, b \in \mathcal{L}^1$, $0 \le t_1 < t_2 \le 1$, $k \in \mathbb{Z}$, $\vartheta \in [\Pi_{2k}, \Pi_{2k+1})$ and $\lambda_* \in \mathbb{R}$. If*

$$\theta_{t_1}(t_2; \vartheta, \lambda_* + a) = \Pi_{2k+1},$$

then for any $\lambda^* > \lambda_*$, there exists $\mu^* \in \mathbb{R}$ such that

$$\theta_{t_1}(t_2; \vartheta, \lambda^* + a, \mu^* + b) = \Pi_{2k+2}.$$

Proof. Since $\theta_{t_1}(t_2; \vartheta, \lambda_* + a) = \Pi_{2k+1}$ and $\lambda^* > \lambda_*$, by the comparison theorem for ODEs one has $\theta_{t_1}(t_2; \vartheta, \lambda^* + a) > \Pi_{2k+1}$. Since the initial value $\vartheta < \Pi_{2k+1}$, there exists $t_3 \in (t_1, t_2)$ such that

$$\theta_{t_1}(t_3; \vartheta, \lambda^* + a) = \Pi_{2k+1}. \tag{2.19}$$

By the equalities in (2.16), there exists $\mu^* \in \mathbb{R}$ such that

$$\theta_{t_3}(t_2; \Pi_{2k+1}, \mu^* + b) = \Pi_{2k+2}. \tag{2.20}$$

By Lemma 2.1, it is necessary that

$$\theta_{t_1}(t;\vartheta,\lambda^*+a) \in [\Pi_{2k},\Pi_{2k+1}], \qquad \forall\, t\in[t_1,t_3],$$
$$\theta_{t_3}(t;\Pi_{2k+1},\mu^*+b) \in [\Pi_{2k+1},\Pi_{2k+2}], \quad \forall\, t\in[t_3,t_2],$$

and hence

$$\cos_p\theta_{t_1}(t;\vartheta,\lambda^*+a)\geq 0, \quad \forall\, t\in[t_1,t_3], \tag{2.21}$$

$$\cos_p\theta_{t_3}(t;\Pi_{2k+1},\mu^*+b)\leq 0, \quad \forall\, t\in[t_3,t_2]. \tag{2.22}$$

Notice that Eq. (2.7) turns to be (2.3) if $\cos_p\theta\geq 0$. Then it follows from (2.21) that

$$\theta_{t_1}(t;\vartheta,\lambda^*+a,\mu^*+b)=\theta_{t_1}(t;\vartheta,\lambda^*+a), \quad \forall\, t\in[t_1,t_3].$$

Setting $t=t_3$ and using the condition (2.19), we have

$$\theta_{t_1}(t_3;\vartheta,\lambda^*+a,\mu^*+b)=\Pi_{2k+1}.$$

Then it follows from (2.22) that

$$\theta_{t_1}(t;\vartheta,\lambda^*+a,\mu^*+b)=\theta_{t_3}(t;\Pi_{2k+1},\mu^*+b), \quad \forall\, t\in[t_3,t_2].$$

Setting $t=t_2$ and using the condition (2.20), we have

$$\theta_{t_1}(t_2;\vartheta,\lambda^*+a,\mu^*+b)=\Pi_{2k+2},$$

completing the proof of this lemma. □

Lemma 2.4. *Let* $a\in\mathcal{L}^1$, $0\leq t_0<t_1\leq 1$, $\vartheta_0<\vartheta_1$ *and* $\lambda_*\in\mathbb{R}$. *If*

$$\theta_{t_0}(t_1;\vartheta_0,\lambda_*+a)=\vartheta_1,$$

then for any $\lambda^*>\lambda_*$ *there exists* $s_0,s_1\in\mathbb{R}$, *such that* $t_0<s_0<s_1<t_1$ *and*

$$\theta_{s_0}(s_1;\vartheta_0,\lambda^*+a)=\vartheta_1.$$

Proof. By the comparison theorem for ODEs, one has

$$\theta_{t_0}(t_1;\vartheta_0,\overline{\lambda}+a)>\theta_{t_0}(t_1;\vartheta_0,\lambda_*+a)=\vartheta_1\,(>\vartheta_0),$$

where $\overline{\lambda}:=(\lambda^*+\lambda_*)/2>\lambda_*$. Then there exists $s_1\in(t_0,t_1)$ such that

$$\theta_{t_0}(s_1;\vartheta_0,\overline{\lambda}+a)=\vartheta_1.$$

Since $\lambda^*>\overline{\lambda}$, we get

$$\theta_{t_0}(s_1;\vartheta_0,\lambda^*+a)>\theta_{t_0}(s_1;\vartheta_0,\overline{\lambda}+a)=\vartheta_1.$$

The function

$$\varphi(s):=\theta_s(s_1;\vartheta_0,\lambda^*+a), \quad s\in[t_0,s_1]$$

is continuous in s and $\varphi(t_0)>\vartheta_1>\vartheta_0=\varphi(s_1)$. Therefore there exists $s_0\in(t_0,s_1)$ such that $\varphi(s_0)=\vartheta_1$, completing the proof of this lemma. □

2.3. Eigenvalues of the p-Laplacian with an integrable potential

Let $a \in \mathcal{L}^1$ be an integrable potential. We consider eigenvalue problem of the p-Laplacian

$$(\phi_p(x'))' + (\lambda + a(t))\phi_p(x) = 0, \quad t \in [0,1]. \tag{2.23}$$

Notice that Eq. (2.23) is the same as (2.1) with $a(t)$ replaced by $\lambda + a(t)$. Eigenvalues of problem (2.23) with two-point boundary conditions are completely known [17]. Besides Dirichlet boundary conditions (D) and the Neumann boundary conditions (N), we consider also the Dirichlet–Neumann boundary conditions

$$x(0) = x'(1) = 0, \tag{DN}$$

and the Neumann–Dirichlet boundary conditions

$$x'(0) = x(1) = 0. \tag{ND}$$

Denote eigenvalues of these problems by $\{\lambda_m^D(a)\}_{m\in\mathbb{N}}$, $\{\lambda_m^N(a)\}_{m\in\mathbb{Z}^+}$, $\{\lambda_m^{DN} \times (a)\}_{m\in\mathbb{Z}^+}$, and $\{\lambda_m^{ND}(a)\}_{m\in\mathbb{N}}$, respectively. Each of them is an increasing sequence tending to $+\infty$. Using the solutions $\theta(t;\vartheta,a)$, these eigenvalues are characterized by the following equations

$$\theta\left(1;\Pi_0,\lambda_m^D(a)+a\right) = \Pi_m, \quad \forall\, m \in \mathbb{N}, \tag{2.24}$$

$$\theta\left(1;0,\lambda_m^N(a)+a\right) = m\pi_p, \quad \forall\, m \in \mathbb{Z}^+, \tag{2.25}$$

$$\theta\left(1;\Pi_0,\lambda_m^{DN}(a)+a\right) = m\pi_p, \quad \forall\, m \in \mathbb{Z}^+, \tag{2.26}$$

$$\theta\left(1;0,\lambda_m^{ND}(a)+a\right) = \Pi_m, \quad \forall\, m \in \mathbb{N}. \tag{2.27}$$

Eigenfunctions associated with $\lambda_m^D(a)$ have precisely $(m-1)$ zeros in $(0,1)$. The nodal structure of other eigenfunctions are also known. Moreover, from (2.24)–(2.27) and the comparison theorem for ODEs, one can check that

$$\lambda_0^N(a) < \lambda_1^{ND}(a) < \lambda_1^N(a) < \cdots < \lambda_m^N(a) < \lambda_{m+1}^{ND}(a) < \lambda_{m+1}^N(a) < \cdots,$$

$$\lambda_0^N(a) < \lambda_0^{DN}(a) < \lambda_1^D(a) < \cdots < \lambda_m^D(a) < \lambda_m^{DN}(a) < \lambda_{m+1}^D(a) < \cdots.$$

These eigenvalues will be used to characterize the asymptotes of those hyperbolic-like curves in $\Sigma_p^D(a,b)$ and $\Sigma_p^N(a,b)$. See Figs. 1 and 2.

Changing t to $1-t$ in (2.23) and denoting $\widetilde{a}(t) := a(1-t)$, one can check that

$$\lambda_m^D(\widetilde{a}) = \lambda_m^D(a), \quad \lambda_m^N(\widetilde{a}) = \lambda_m^N(a), \quad \lambda_m^{ND}(\widetilde{a}) = \lambda_{m-1}^{DN}(a), \tag{2.28}$$

for any admissible $m \in \mathbb{Z}$.

3. Fučík Spectrum for the Dirichlet Boundary Problems

3.1. Structure of Fučík spectrum $\Sigma_p^D(a,b)$

Given potentials $a, b \in \mathcal{L}^1$, the Fučík spectrum $\Sigma_p^D(a,b)$ is defined as the set of $(\lambda,\mu) \in \mathbb{R}^2$ such that Eq. (1.3) has a non-zero solution satisfying the Dirichlet boundary conditions (D).

If we replace a and b in Eq. (2.6) by $\lambda + a$ and $\mu + b$, respectively, it becomes (1.3). Using the p-polar coordinates (2.2) and the notations above, we see that $\theta(t) = \theta(t; \vartheta, \lambda + a, \mu + b)$ is the unique solution of

$$\theta' = (\lambda + a(t))(\cos_p \theta)_+^p + (\mu + b(t))(\cos_p \theta)_-^p + (p-1)|\sin_p \theta|^{p^*}, \tag{3.1}$$

satisfying $\theta(0) = \vartheta \in \mathbb{R}$.

In order to characterize the Fučík spectrum $\Sigma_p^D(a,b)$, we need the following functions

$$\Theta(\lambda, \mu) = \Theta_{a,b}(\lambda, \mu) := \theta(1; \Pi_0, \lambda + a, \mu + b), \quad (\lambda, \mu) \in \mathbb{R}^2, \tag{3.2}$$

$$\widetilde{\Theta}(\lambda, \mu) = \widetilde{\Theta}_{a,b}(\lambda, \mu) := \theta(1; \Pi_1, \lambda + a, \mu + b), \quad (\lambda, \mu) \in \mathbb{R}^2. \tag{3.3}$$

One can check that

$$\theta(t; \Pi_1, \lambda + a, \mu + b) = \theta(t; \Pi_0, \mu + b, \lambda + a) + \pi_p.$$

Let $t = 1$ and we get

$$\widetilde{\Theta}_{a,b}(\lambda, \mu) = \Theta_{b,a}(\mu, \lambda) + \pi_p. \tag{3.4}$$

Lemma 3.1. *Let $a, b \in \mathcal{L}^1$ be given. Then the function $\Theta(\lambda, \mu) = \Theta_{a,b}(\lambda, \mu)$ possesses the following properties.*

(i) *$\Theta(\lambda, \mu)$ is continuously differentiable in $(\lambda, \mu) \in \mathbb{R}^2$.*
(ii) *$\Theta(\lambda, \mu) > \Pi_0$ for any $(\lambda, \mu) \in \mathbb{R}^2$.*
(iii) *There hold*

$$\partial_\lambda \Theta(\lambda, \mu) > 0, \quad \forall\, (\lambda, \mu) \in \mathbb{R}^2, \tag{3.5}$$

$$\partial_\mu \Theta(\lambda, \mu) \geq 0, \quad \forall\, (\lambda, \mu) \in \mathbb{R}^2, \tag{3.6}$$

$$\partial_\mu \Theta(\lambda, \mu) > 0, \quad \forall\, \lambda > \lambda_1^D(a), \;\; \mu \in \mathbb{R}. \tag{3.7}$$

In particular, $\Theta(\lambda, \mu)$ is strictly increasing in $\lambda \in \mathbb{R}$ and non-decreasing in $\mu \in \mathbb{R}$, and, when restricted to the following region in the λ–μ plane

$$Q_{a,b} := \left\{(\lambda, \mu) : \lambda > \lambda_1^D(a) \text{ and } \mu > \lambda_1^D(b)\right\} \subset \mathbb{R}^2, \tag{3.8}$$

$\Theta(\lambda, \mu)$ is strictly increasing in both λ and μ.

(iv) *There hold*

$$\lim_{\lambda \to +\infty, \mu \to +\infty} \Theta(\lambda, \mu) = +\infty, \qquad \lim_{\lambda \to -\infty, \mu \to -\infty} \Theta(\lambda, \mu) = \Pi_0. \tag{3.9}$$

Consequently, the range of the function $\Theta(\lambda, \mu)$ is the interval $(\Pi_0, +\infty)$.

Proof. (i) This property follows immediately from Theorem 2.1(iii).

(ii) By the quasi-monotonicity of $\theta(t) = \theta(t; \Pi_0, \lambda + a, \mu + b)$ in t (see Lemma 2.1), there holds $\Theta(\lambda, \mu) = \theta(1) > \Pi_0$.

(iii) Let $X(t) = X(t; \Pi_0, \lambda + a, \mu + b)$ be defined by (2.10). Then $X(0) = 0$ and $X'(0) > 0$. Therefore $X_+(t)$ is not identically zero. By (2.11), there holds

$$\partial_\lambda \Theta(\lambda, \mu) = \int_0^1 X_+^p(t)dt > 0.$$

This proves (3.5). Similarly, by (2.12), we have

$$\partial_\mu \Theta(\lambda, \mu) = \int_0^1 X_-^p(t)dt \geq 0. \tag{3.10}$$

This gives (3.6).

Suppose that $\lambda > \lambda_1^D(a)$. In order to obtain (3.7), it follows from (3.10) that it suffices to show that $X_-(t) \not\equiv 0$. If this is false, then we have $X_-(t) \equiv 0$ and $X(t) \geq 0$ for all $t \in [0, 1]$. By (2.10), one has $\cos_p \theta(t) \geq 0$ on $[0, 1]$, where $\theta(t) = \theta(t; \Pi_0, \lambda + a, \mu + b)$. Since $\theta(0) = \Pi_0$, we have necessarily $\Pi_0 \leq \theta(t) \leq \Pi_1$ for all $t \in [0, 1]$. Now Eq. (3.1) for $\theta(t)$ is the same as (2.3) with a being replaced by $\lambda + a$. Thus

$$\theta(t) \equiv \theta\left(t; \Pi_0, \lambda + a\right) \leq \Pi_1, \quad \forall\, t \in [0, 1]. \tag{3.11}$$

From Remark 2.2(iii), $\theta\left(1; \Pi_0, \zeta + a\right)$ is strictly increasing in $\zeta \in \mathbb{R}$. Since $\lambda_1^D(a) < \lambda$, we have

$$\theta\left(1; \Pi_0, \lambda_1^D(a) + a\right) < \theta\left(1; \Pi_0, \lambda + a\right) \leq \Pi_1,$$

where (3.11) is used. This is a contradiction with the characterization (2.24) for $\lambda_1^D(a)$.

(iv) It has been proved in [7, Lemma 3.2] that

$$\lim_{\lambda \to +\infty} \left(\theta(1; \vartheta_0, \lambda + a, \lambda + b) - \vartheta_0\right) = +\infty$$

uniformly in $\vartheta_0 \in \mathbb{R}$. By using the monotonicity of $\Theta(\lambda, \mu)$ in λ and μ, as stated in (iii), when $\xi := \min(\lambda, \mu) \to +\infty$, one has

$$\Theta(\lambda, \mu) \geq \Theta(\xi, \xi) = \theta(1, \Pi_0, \xi + a, \xi + b) \to +\infty.$$

Thus one has the first result of (3.9).

It has been proved in [7, Lemma 3.3] that $\lim_{\lambda \to -\infty} \Theta\left(\lambda, \lambda\right) = \Pi_0$. By using the monotonicity of $\Theta(\lambda, \mu)$ in λ and μ, when $\eta := \max(\lambda, \mu) \to -\infty$, one has

$$\Theta(\lambda, \mu) \leq \Theta(\eta, \eta) \to \Pi_0.$$

Combining with (ii), one has the second result of (3.9). □

Now we are going to construct the Fučík spectrum $\Sigma_p^D(a, b)$ by using the functions $\Theta(\lambda, \mu)$ and $\widetilde{\Theta}(\lambda, \mu)$ in (3.2) and (3.3). Suppose that $(\lambda, \mu) \in \Sigma_p^D(a, b)$. Then Eq. (1.3) has a non-zero solution $x(t)$, called a spectral function, satisfying boundary conditions (D). In the p-polar coordinates (r, θ), one has the corresponding solution $\theta(t)$ of Eq. (3.1). Because of the $2\pi_p$-periodicity of the right-hand side of Eq. (3.1) in θ, one can restrict

$$\theta(0) \in [\Pi_0, \Pi_2). \tag{3.12}$$

Notice that (D) corresponds to the following conditions for $\theta(t)$

$$\theta(0) = \Pi_l \quad \text{and} \quad \theta(1) = \Pi_{k+l} \quad \text{for some } l,\, k \in \mathbb{Z}. \tag{3.13}$$

The spectral function $x(t)$ satisfies $x'(0) \neq 0$. We distinguish the following two cases.

Case 1. $x(t)$ satisfies $x'(0) > 0$. In this case, one has $\theta(0) = \Pi_0$ in (3.12) and $l = 0$ in (3.13). Making use of the function $\Theta(\lambda, \mu) = \Theta_{a,b}(\lambda, \mu)$ in (3.2), the second condition of (3.13) determines, for $k \in \mathbb{Z}$, a set

$$C^D_{k,+}(a, b) := \{(\lambda, \mu) \in \mathbb{R}^2 : \Theta_{a,b}(\lambda, \mu) = \Pi_k\}. \tag{3.14}$$

One sees that $C^D_{k,+}(a, b) = \emptyset$ for all $k \le 0$ by Lemma 3.1(ii). On the other hand, we have $C^D_{k,+}(a, b) \neq \emptyset$ for all $k \ge 1$ by Lemma 3.1(iv).

Case 2. $x(t)$ satisfies $x'(0) < 0$. In this case, one has $\theta(0) = \Pi_1$ in (3.12) and $l = 1$ in (3.13). The second condition of (3.13) also determines, for $k \in \mathbb{Z}$, a set

$$C^D_{k,-}(a, b) := \{(\lambda, \mu) \in \mathbb{R}^2 : \widetilde{\Theta}_{a,b}(\lambda, \mu) = \Pi_{k+1}\}. \tag{3.15}$$

By using (3.4), this set can also be written as

$$C^D_{k,-}(a, b) = \{(\lambda, \mu) \in \mathbb{R}^2 : \Theta_{b,a}(\mu, \lambda) = \Pi_k\}.$$

Therefore a basic relation between $C^D_{k,-}(a, b)$ and $C^D_{k,+}(b, a)$ is

$$(\lambda, \mu) \in C^D_{k,-}(a, b) \Leftrightarrow (\mu, \lambda) \in C^D_{k,+}(b, a). \tag{3.16}$$

In other words, $C^D_{k,-}(a, b)$ is a reflection of $C^D_{k,+}(b, a)$ with respect to the line $\lambda = \mu$. Consequently, $C^D_{k,-}(a, b) = \emptyset$ for all $k \le 0$ and $C^D_{k,-}(a, b) \neq \emptyset$ for all $k \ge 1$.

Now the set $\Sigma^D_p(a, b)$ can be decomposed as

$$\Sigma^D_p(a, b) = \bigcup_{k \in \mathbb{N}} C^D_{k,\pm}(a, b). \tag{3.17}$$

Suppose $(\lambda, \mu) \in C^D_{k,+}(a, b)$ for some $k \in \mathbb{N}$. By (3.14), $\theta(t) = \theta(t; \Pi_0, \lambda + a, \mu + b)$ satisfies $\theta(0) = \Pi_0$ and $\theta(1) = \Pi_k$. It follows from the quasi-monotonicity in Lemma 2.1 that equations

$$\theta(t) = \Pi_i, \quad i \in \mathbb{Z},$$

has precisely $(k - 1)$ solutions for t in the interval $(0, 1)$. This means that the corresponding spectral function $x(t)$ has exactly $(k-1)$ zeroes in $(0, 1)$. Furthermore, we have $x'(0) > 0$ because $\theta(0) = \Pi_0$.

Similarly, if $(\lambda, \mu) \in C^D_{k,-}(a, b)$ for some $k \in \mathbb{N}$, then the corresponding spectral function $x(t)$ has exactly $(k-1)$ zeroes in $(0, 1)$ and satisfies $x'(0) < 0$. In conclusion we have the following characterization on $(\lambda, \mu) \in \Sigma^D_p(a, b)$.

Property 3.1. *Given $(\lambda,\mu)\in\Sigma_p^D(a,b)$, there hold the following.*

(i) $(\lambda,\mu)\in C_{k,+}^D(a,b)$ *for some* $k\in\mathbb{N}\Leftrightarrow$ *any spectral function* $x(t)$ *associated with* (λ,μ) *satisfies* $x'(0)>0$ *and* $x(t)$ *has exactly* $(k-1)$ *zeroes in* $(0,1)$.

(ii) $(\lambda,\mu)\in C_{k,-}^D(a,b)$ *for some* $k\in\mathbb{N}\Leftrightarrow$ *any spectral function* $x(t)$ *associated with* (λ,μ) *satisfies* $x'(0)<0$ *and* $x(t)$ *has exactly* $(k-1)$ *zeroes in* $(0,1)$.

Relation (3.16) can also be checked easily by Property 3.1 and the change of variables $x\to -x$ in Eq. (1.3).

Because of relation (3.16), we need only to concentrate on analyzing subsets $C_{k,+}^D(a,b)$, $k\in\mathbb{N}$.

For even-order sets $C_{2k,+}^D(a,b)$, we have the following relation.

Property 3.2. *There holds*

$$(\lambda,\mu)\in C_{2k,+}^D(a,b)\Leftrightarrow(\mu,\lambda)\in C_{2k,+}^D(\widetilde{b},\widetilde{a})\quad\forall\, k\in\mathbb{N},\tag{3.18}$$

where $\widetilde{a}(t):=a(1-t)$ *and* $\widetilde{b}(t):=b(1-t)$.

Proof. Take the change of variables $t\to 1-t$ in (1.3). Then $\widetilde{x}(t):=-x(1-t)$ satisfies

$$(\phi_p(\widetilde{x}'))'+(\mu+\widetilde{b}(t))\phi_p(\widetilde{x}_+)-(\lambda+\widetilde{a}(t))\phi_p(\widetilde{x}_-)=0.\tag{3.19}$$

If $x(t)$ is a spectral function associated with $(\lambda,\mu)\in C_{2k,+}^D(a,b)$, then $x'(0)>0$ and $x'(1)>0$. Therefore $\widetilde{x}(t)$ satisfies (D) and $\widetilde{x}'(0)=x'(1)>0$. Moreover, $\widetilde{x}(t)$ and $x(t)$ have the same number of zeros in $(0,1)$. By Property 3.1 and Eq. (3.19), one has $(\mu,\lambda)\in C_{2k,+}^D(\widetilde{b},\widetilde{a})$.

Conversely, since $\widetilde{\widetilde{a}}=a$ and $\widetilde{\widetilde{b}}=b$, we have (3.18). □

Property 3.3. *One has* $C_{1,+}^D(a,b)=\lambda_1^D(a)\times\mathbb{R}$ *and* $C_{1,-}^D(a,b)=\mathbb{R}\times\lambda_1^D(b)$.

Proof. It is well-known that the first eigenvalue $\lambda_1^D(a)$ of (2.23) has some eigenfunction which is positive in $(0,1)$. Then Eq. (2.23) is the same as (1.3) with $\lambda=\lambda_1^D(a)$ and an arbitrary $\mu\in\mathbb{R}$. By the definition of Fučík spectrum, one has $\lambda_1^D(a)\times\mathbb{R}\subset C_{1,+}^D(a,b)$.

Conversely, if $(\lambda,\mu)\in C_{1,+}^D(a,b)$, then $\theta(t):=\theta(t;\Pi_0,\lambda+a,\mu+b)$ satisfies $\theta(0)=\Pi_0$ and $\theta(1)=\Pi_1$. By the quasi-monotonicity in Lemma 2.1, one has $\Pi_0<\theta(t)<\Pi_1$ for all $t\in(0,1)$. Therefore, the corresponding spectral function $x(t)$ is positive on $(0,1)$. Hence Eq. (1.3) is reduced to (2.23) with the same λ. Since $x(t)$ is non-negative and satisfies (D), λ must be $\lambda_1^D(a)$, i.e. $C_{1,+}^D(a,b)\subset\lambda_1^D(a)\times\mathbb{R}$. Now we can conclude that $C_{1,+}^D(a,b)=\lambda_1^D(a)\times\mathbb{R}$.

Similar arguments show that $C_{1,-}^D(a,b)=\mathbb{R}\times\lambda_1^D(b)$. □

Now we study the set $C_{2,+}^D(a,b)$.

Property 3.4. *One has* $C_{2,+}^D(a,b)\subset Q_{a,b}$, *where* $Q_{a,b}$ *is defined as in* (3.8).

Proof. Let $(\lambda, \mu) \in C^D_{2,+}(a, b)$. Then

$$\Theta(\lambda, \mu) = \Pi_2 > \Pi_1 = \Theta(\lambda^D_1(a), \mu),$$

where the last equality holds because $C^D_{1,+}(a, b) = \lambda^D_1(a) \times \mathbb{R}$. Therefore one has $\lambda > \lambda^D_1(a)$ by (3.5).

On the other hand, $(\lambda, \mu) \in C^D_{2,+}(a, b)$ implies $(\mu, \lambda) \in C^D_{2,+}(\widetilde{b}, \widetilde{a})$ by relation (3.18) with $k = 1$. Applying the results proved above to $C^D_{2,+}(\widetilde{b}, \widetilde{a})$, one obtains $\mu > \lambda^D_1(\widetilde{b}) = \lambda^D_1(b)$, where the last equality holds by (2.28). □

Due to Property 3.4, the determining equation for $C^D_{2,+}(a, b)$ can be rewritten as

$$\Theta(\lambda, \mu) = \Pi_2, \quad (\lambda, \mu) \in Q_{a,b}. \tag{3.20}$$

It has been proved in Lemma 3.1(iii) that $\partial_\lambda \Theta(\lambda, \mu) > 0$ and $\partial_\mu \Theta(\lambda, \mu) > 0$ on $Q_{a,b}$.

Property 3.5. *The set $C^D_2(a, b)$ is a strictly decreasing differentiable curve in the λ–μ plane*

$$\mu = M^D_2(\lambda), \quad \lambda \in I^D_2, \tag{3.21}$$

where I^D_2 is an open interval of the form $(\alpha, +\infty)$ and $\alpha \in \mathbb{R}$. Moreover, one has

$$\lim_{\lambda \to \alpha+} M^D_2(\lambda) = +\infty. \tag{3.22}$$

Proof. Because of inequality (3.7), the Implicit Function Theorem (IFT) is applicable to Eq. (3.20). At any point $(\lambda_0, \mu_0) \in C^D_{2,+}(a, b)$, there is some neighborhood of λ_0 in which the solution μ of Eq. (3.20) is given by (3.21). Moreover, the IFT implies that $M^D_2(\lambda)$ is continuously differentiable and

$$\frac{dM^D_2(\lambda)}{d\lambda} = -\frac{\partial_\lambda \Theta(\lambda, M^D_2(\lambda))}{\partial_\mu \Theta(\lambda, M^D_2(\lambda))} < 0.$$

See (3.5) and (3.7). Thus the curve in (3.21) is locally strictly decreasing.

Let us show that the domain I^D_2 in (3.21) is an interval. Assume that $(\lambda_i, \mu_i) \in C^D_{2,+}(a, b)$, $i = 1, 2$, are two different points. By Lemma 3.1(iii) and Property 3.4, one sees that every horizontal line and every vertical line in the λ–μ plane can only intersect $C^D_2(a, b)$ at one point at most. Then $\lambda_1 \neq \lambda_2$. Assume that $\lambda_1 < \lambda_2$. We need only to show that for any $\lambda \in (\lambda_1, \lambda_2)$, there holds (3.20) for some $\mu \in \mathbb{R}$. If this is not true for some $\lambda^* \in (\lambda_1, \lambda_2)$, by the continuity of $\Theta(\lambda, \mu)$, one has either

$$\Theta(\lambda^*, \mu) < \Pi_2, \quad \forall\, \mu \in \mathbb{R}$$

or

$$\Theta(\lambda^*, \mu) > \Pi_2, \quad \forall\, \mu \in \mathbb{R}.$$

For the former case, since $\lambda_1 < \lambda^*$, one has

$$\Theta(\lambda_1, \mu_1) < \Theta(\lambda^*, \mu_1) < \Pi_2,$$

which is impossible because $(\lambda_1, \mu_1) \in C^D_{2,+}(a,b)$. Similarly, the latter case is also impossible, and hence I^D_2 in (3.21) is an interval.

Now we show that I^D_2 has the form $(\alpha, +\infty)$. Recall from Property 3.4 that $I^D_2 \subset (\lambda^D_1(a), +\infty)$ and $\lambda^D_1(b) < M^D_2(\lambda) < +\infty$ for all $\lambda \in I^D_2$.

Let α be the (finite) left end-point of I^D_2. We assert that $\alpha \notin I^D_2$. Otherwise, one has $\alpha \in I^D_2$. Applying the IFT to (3.20) at the point $(\alpha, M^D_2(\alpha)) \in C^D_{2,+}(a,b)$, one sees that α cannot be the left end-point of I^D_2.

Let β be the right end-point of I^D_2. Since $M^D_2(\lambda)$ is decreasing and has lower bound $\lambda^D_1(b)$, we know that the limit

$$\mu_* := \lim_{\lambda \to \beta^-} M^D_2(\lambda) \tag{3.23}$$

does exist and $\mu_* \geq \lambda^D_1(b)$. We assert that $\beta = +\infty$. If $\beta < +\infty$, then $(\beta, \mu_*) \in C^D_{2,+}(a,b)$, because

$$\Theta(\beta, \mu_*) = \lim_{\lambda \to \beta^-} \Theta(\lambda, M^D_2(\lambda)) = \Pi_2.$$

One can apply the IFT to (3.20) at the point (β, μ_*) to obtain that β cannot be the right end-point of I^D_2.

Finally, let us prove (3.22). If it is false, one has $\mu^* := \lim_{\lambda \to \alpha^+} M^D_2(\lambda) \in [\lambda^D_1(b), +\infty)$. Since $\Theta(\alpha, \mu^*) = \lim_{\lambda \to \alpha^+} \Theta(\lambda, M^D_2(\lambda)) = \Pi_2$, we have $(\alpha, \mu^*) \in C^D_{2,+}(a,b)$, a contradiction to the fact that $\alpha \notin I^D_2$. □

Remark 3.1. We conclude from (3.22) and (3.23) (with $\beta = +\infty$) that

$$\lambda = \alpha \quad \text{and} \quad \mu = \mu_*$$

are respectively the vertical and the horizontal asymptotic lines of the Fučík curve $C^D_{2,+}(a,b)$. In Theorem 3.2 below, we will show that $\alpha = \lambda^D_1(a)$ and $\mu_* = \lambda^D_1(b)$.

Based on the structure of $C^D_{2,+}(a,b)$ in Property 3.5 and those properties of $\Theta(\lambda, \mu)$ in Lemma 3.1, we can characterize $C^D_{3,+}(a,b)$. By Remark 3.1, the curve $C^D_{2,+}(a,b)$ decomposes the λ–μ plane into two regions. From Lemma 3.1(iii), one knows that $\Theta(\lambda,\mu) < \Pi_2$ if and only if (λ,μ) lies below $C^D_{2,+}(a,b)$, and $\Theta(\lambda,\mu) > \Pi_2$ if and only if (λ,μ) lies above $C^D_{2,+}(a,b)$. For $(\lambda,\mu) \in C^D_{3,+}(a,b)$, one has $\Theta(\lambda,\mu) = \Pi_3 > \Pi_2$. Thus (λ,μ) is above $C^D_{2,+}(a,b)$. Now we see that

$$C^D_{3,+}(a,b) \subset \left\{(\lambda,\mu) \in \mathbb{R}^2 : \lambda > \alpha, \mu > M^D_2(\lambda)\right\} \subset Q_{a,b}.$$

Thus the determining equation for $C^D_{3,+}(a,b)$ is

$$\Theta(\lambda,\mu) = \Pi_3, \quad (\lambda,\mu) \in Q_{a,b},$$

for which the IFT is applicable. By similar arguments as in the proof of Property 3.5, we can show that $C^D_{3,+}(a,b)$ is also a strictly decreasing differentiable curve. Moreover, $C^D_{3,+}(a,b)$ has also vertical and horizontal asymptotic lines. Inductively, we have the following results.

Theorem 3.1. *Let $a \in \mathcal{L}^1$ and $b \in \mathcal{L}^1$ be given. Then the Fučík spectrum $\Sigma^D_p(a,b)$ is decomposed into* (3.17), *where $C^D_{k,+}(a,b)$ and $C^D_{k,-}(a,b)$ are defined by* (3.14) *and* (3.15) *respectively. There holds the relation* (3.16), *which means that* $C^D_{k,-}(a,b)$ *is a reflection of* $C^D_{k,+}(b,a)$ *with respect to the line* $\lambda = \mu$. *Moreover,*

(i) $C^D_{1,+}(a,b) = \lambda^D_1(a) \times \mathbb{R}$ *and* $C^D_{1,-}(a,b) = \mathbb{R} \times \lambda^D_1(b)$;
(ii) *for any* $k \geq 2$, $C^D_{k,+}(a,b) \subset Q_{a,b}$ *and* $C^D_{k,-}(a,b) \subset Q_{b,a}$ *are strictly decreasing differentiable curves of the form* (3.21), *defined on open intervals of the form* $(\alpha, +\infty)$;
(iii) *for any* $k \geq 2$, $C^D_{k,\pm}(a,b)$ *has a vertical asymptotic line and a horizontal asymptotic line*; *and*
(iv) *for any* $k \geq 2$, $C^D_{k+1,+}(a,b)$ *lies above* $C^D_{k,+}(a,b)$, *and* $C^D_{k+1,-}(a,b)$ *lies above* $C^D_{k,-}(a,b)$.

For $k \geq 1$, $C^D_{k,+}(a,b)$ and $C^D_{k,-}(a,b)$ are called respectively the kth *Fučík spectral curves* of the first and the second type of problem (1.3)–(D). The precise vertical and horizontal asymptotic lines of all these curves will be given in Theorem 3.2.

It is well-known that if $a(t) = b(t) \equiv c$ is constant, then the Fučík spectral curves are

$$C^D_{2k,+}(c,c) = C^D_{2k,-}(c,c) = \left\{(\lambda,\mu) \in \mathbb{R}^2 : \frac{k}{\sqrt[p]{\lambda + c}} + \frac{k}{\sqrt[p]{\mu + c}} = \frac{1}{\pi_p}\right\}, \quad k \geq 1$$

and

$$C^D_{2k+1,+}(c,c) = \left\{(\lambda,\mu) \in \mathbb{R}^2 : \frac{k+1}{\sqrt[p]{\lambda + c}} + \frac{k}{\sqrt[p]{\mu + c}} = \frac{1}{\pi_p}\right\}, \quad k \geq 0,$$

$$C^D_{2k+1,-}(c,c) = \left\{(\lambda,\mu) \in \mathbb{R}^2 : \frac{k}{\sqrt[p]{\lambda + c}} + \frac{k+1}{\sqrt[p]{\mu + c}} = \frac{1}{\pi_p}\right\}, \quad k \geq 0.$$

In this case, the Fučík spectral curves $C^D_{2k,+}(c,c)$ and $C^D_{2k,-}(c,c)$, $k \geq 1$ coincide. In fact, we have more general results.

Property 3.6. *If* $a \in \mathcal{L}^1$ *satisfies* $\widetilde{a} = a$, *namely* $a(t) = a(1-t)$ *for a.e.* $t \in [0,1]$, *then the even-order Fučík spectral curves* $C^D_{2k,-}(a,a)$ *and* $C^D_{2k,+}(a,a)$ *coincide, for any* $k \geq 1$.

Proof. Since $\widetilde{a} = a$, relation (3.18) shows that $C^D_{2k,+}(a,a)$ is symmetric with respect to the diagonal line $\lambda = \mu$. Furthermore, (3.16) means that $C^D_{2k,-}(a,a)$ is the reflection of $C^D_{2k,+}(a,a)$ with respect to $\lambda = \mu$. Combining these two facts together, we can conclude that $C^D_{2k,-}(a,a) = C^D_{2k,+}(a,a)$. □

3.2. *Asymptotic lines of Fučík spectral curves* $C^D_{k,\pm}(a,b)$

The aim of this subsection is to determine all asymptotic lines of all nontrivial Fučík spectral curves $C^D_{k,+}(a,b)$, $k \geq 2$. Those asymptotic lines for $C^D_{k,-}(a,b)$, $k \geq 2$, can be deduced accordingly by using relation (3.16).

At first, we consider those even-order curves $C^D_{2k,+}(a,b)$.

Property 3.7. *Given* $a,b \in \mathcal{L}^1$, *if* $(\lambda_*,\mu_*) \in C^D_{2k,+}(a,b)$ *for some* $k \in \mathbb{N}$, *then* $\lambda_* > \lambda^D_k(a)$ *and* $\mu_* > \lambda^D_k(b)$.

Proof. The condition $(\lambda_*,\mu_*) \in C^D_{2k,+}(a,b)$ means that

$$\theta(1;\Pi_0,\lambda_*+a,\mu_*+b) = \Pi_{2k}.$$

Moreover, there exist $0 = t_0 < t_1 < t_2 \cdots < t_{2k-1} < t_{2k} = 1$ such that

$$\theta(t_i;\Pi_0,\lambda_*+a,\mu_*+b) = \Pi_i, \quad i = 0,1,2,\ldots,2k. \tag{3.24}$$

For each $i = 1,2,\ldots,k$, denote $J_i := [t_{2i-2},t_{2i-1}]$ and $K_i := [t_{2i-2},t_{2i}]$. By Lemma 2.1, the quasi-monotonicity property, if $t \in J_i$, then $\theta(t) := \theta(t;\Pi_0,\lambda_*+a,\mu_*+b) \in [\Pi_{2i-2},\Pi_{2i-1}]$ and $\cos_p\theta(t) \geq 0$. Noticing that (2.7) turns to be (2.3) when $\cos_p\theta \geq 0$, we have

$$\theta(t;\Pi_0,\lambda_*+a,\mu_*+b) = \theta_{t_{2i-2}}(t;\Pi_{2i-2},\lambda_*+a), \quad \forall\, t \in J_i,\ i = 1,2,\ldots,k. \tag{3.25}$$

Combining (2.14) and (3.25), we have

$$\begin{aligned}\theta_{t_{2i-2}}(t_{2i-1};\Pi_{i-1},\lambda_*+a) &= \theta_{t_{2i-2}}(t_{2i-1};\Pi_{2i-2},\lambda_*+a) - (i-1)\pi_p \\ &= \theta(t_{2i-1};\Pi_0,\lambda_*+a,\mu_*+b) - (i-1)\pi_p,\end{aligned}$$

for each $i = 1,2,\ldots,k$. Then it follows from (3.24) that

$$\theta_{t_{2i-2}}(t_{2i-1};\Pi_{i-1},\lambda_*+a) = \Pi_i, \quad i = 1,2,\ldots,k. \tag{3.26}$$

Applying Lemma 2.2 on each interval $K_i (\supset J_i)$, there exist $\lambda^*_i < \lambda_*$, such that

$$\theta_{t_{2i-2}}(t_{2i};\Pi_{i-1},\lambda^*_i+a) = \Pi_i, \quad i = 1,2,\ldots,k. \tag{3.27}$$

Define a potential q on the interval $[0,1] = \bigcup_{i=1}^k K_i$ as

$$q(t) = \lambda^*_i + a(t), \quad \forall\, t \in K_i,\ i = 1,2,\ldots,k.$$

Then (3.27) gives

$$\theta(1;\Pi_0,q) = \Pi_k.$$

Since $\lambda_* > \lambda^*_i$ for all $i = 1,2,\ldots,k$, one has $\lambda_* + a(t) > q(t)$ on $[0,1]$, and hence

$$\theta(1;\Pi_0,\lambda_*+a) > \theta(1;\Pi_0,q) = \Pi_k. \tag{3.28}$$

Comparing this inequality with (2.24), we get $\lambda_* > \lambda^D_k(a)$.

Due to relation (3.18), there holds $(\mu_*,\lambda_*) \in \widetilde{C}_{2k}(\widetilde{b},\widetilde{a})$. By the results proved above, we get $\mu_* > \lambda^D_k(\widetilde{b}) = \lambda^D_k(b)$, where the last equality follows from (2.28). □

Property 3.8. *Given* $a,b \in \mathcal{L}^1$ *and* $k \in \mathbb{N}$, *then*

(i) *any vertical line* $\lambda^* \times \mathbb{R}$ *intersects* $C^D_{2k,+}(a,b)$ *if* $\lambda^* > \lambda^D_k(a)$; *and*
(ii) *any horizontal line* $\mathbb{R} \times \mu^*$ *intersects* $C^D_{2k,+}(a,b)$ *if* $\mu^* > \lambda^D_k(b)$.

Proof. (i) The eigenvalue $\lambda_k^D(a)$ satisfies $\theta(1; \Pi_0, \lambda_k^D(a) + a) = \Pi_k$. Moreover, one has $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ such that

$$\theta(t_i; \Pi_0, \lambda_k^D(a) + a) = \Pi_i, \quad i = 0, 1, 2, \ldots, k.$$

These equalities can also be rewritten as

$$\theta_{t_{i-1}}(t_i; \Pi_{i-1}, \lambda_k^D(a) + a) = \Pi_i, \quad i = 1, 2, \ldots, k.$$

By (2.14), we have

$$\theta_{t_{i-1}}(t_i; \Pi_{2i-2}, \lambda_k^D(a) + a) = \Pi_{2i-1}, \quad i = 1, 2, \ldots, k.$$

Let $\lambda^* > \lambda_k^D(a)$ be given. Apply Lemma 2.3 on the interval $[t_{i-1}, t_i]$. There exist $\mu_i \in \mathbb{R}$, such that

$$\theta_{t_{i-1}}(t_i; \Pi_{2i-2}, \lambda^* + a, \mu_i + b) = \Pi_{2i}, \quad i = 1, 2, \ldots, k. \tag{3.29}$$

Define a potential q on the interval $[0, 1]$ as

$$q(t) = \mu_i + b(t), \quad \forall\, t \in [t_{i-1}, t_i], \ \ i = 1, 2, \ldots, k.$$

Then it follows from (3.29) that

$$\theta(1; \Pi_0, \lambda^* + a, q) = \Pi_{2k}.$$

Let $\overline{\mu} := \max_{1 \le i \le k}\{\mu_i\}$ and $\underline{\mu} := \min_{1 \le i \le k}\{\mu_i\}$. One has $\overline{\mu} + b \ge q \ge \underline{\mu} + b$ on $[0, 1]$. Now the monotonicity of $\theta(1; \vartheta, a, b)$ in $b \in \mathcal{L}^1$ (see Remark 2.2(i)) implies that

$$\Theta(\lambda^*, \overline{\mu}) = \theta(1; \Pi_0, \lambda^* + a, \overline{\mu} + b) \ge \theta(1; \Pi_0, \lambda^* + a, q) = \Pi_{2k},$$

$$\Theta(\lambda^*, \underline{\mu}) = \theta(1; \Pi_0, \lambda^* + a, \underline{\mu} + b) \le \theta(1; \Pi_0, \lambda^* + a, q) = \Pi_{2k}.$$

Thus there exists $\mu^* \in [\,\underline{\mu}, \overline{\mu}\,]$ such that $\Theta(\lambda^*, \mu^*) = \Pi_{2k}$, and hence $(\lambda^*, \mu^*) \in C_{2k,+}^D(a, b)$, proving the result in (i).

(ii) If $\mu^* > \lambda_k^D(b) = \lambda_k^D(\widetilde{b})$ (see (2.28)), then it follows from (i) that there exists $\lambda^* \in \mathbb{R}$ such that $(\mu^*, \lambda^*) \in C_{2k,+}^D(\widetilde{b}, \widetilde{a})$. By (3.18), this can be translated into $(\lambda^*, \mu^*) \in C_{2k,+}^D(a, b)$, completing the proof. □

It has been proved in Theorem 3.1 that $C_{k,+}^D(a, b)$ has a vertical asymptotic line and a horizontal asymptotic line. Properties 3.7 and 3.8 mean that, for the even order curve $C_{2k,+}^D(a, b)$, the vertical asymptotic line is just $\lambda = \lambda_k^D(a)$, and the horizontal asymptotic line is just $\mu = \lambda_k^D(b)$. These are stated as result (i) in Theorem 3.2 below. See Fig. 1.

Theorem 3.2. *Suppose $a, b \in \mathcal{L}^1$ and $k \ge 1$. Then*

(i) *the vertical and the horizontal asymptotic lines of $C_{2k,+}^D(a, b)$ are respectively $\lambda = \lambda_k^D(a)$ and $\mu = \lambda_k^D(b)$; and*
(ii) *the vertical and the horizontal asymptotic lines of $C_{2k+1,+}^D(a, b)$ are respectively $\lambda = \lambda_{k+1}^D(a)$ and $\mu = \lambda_k^D(b)$.*

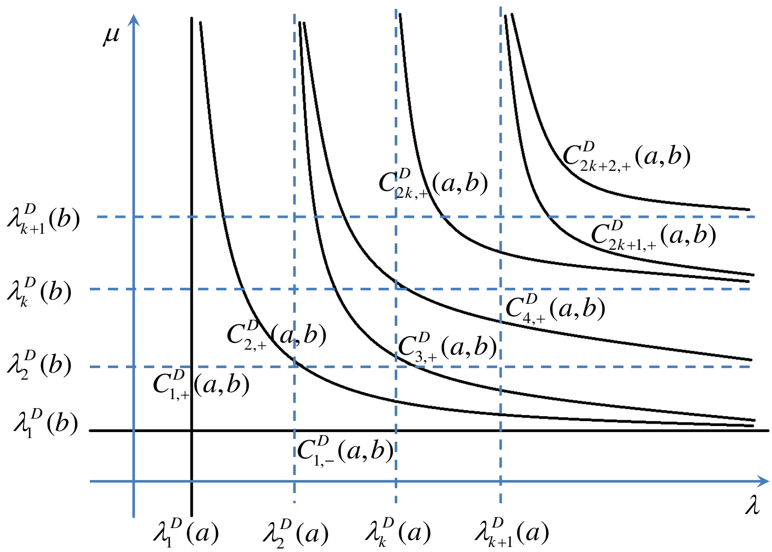

Fig. 1. Asymptotic lines of $C^D_{k,+}(a,b)$, $k \geq 2$.

Proof. We only need to prove (ii).

At first, let us find the vertical asymptotic line of $C^D_{2k+1,+}(a,b)$.

On the one hand, suppose that $\lambda^* > \lambda^D_{k+1}(a)(> \lambda^D_k(a))$. By Property 3.8(i), there exist $\mu_1, \mu_2 \in \mathbb{R}$ such that $(\lambda^*, \mu_1) \in C^D_{2k,+}(a,b)$ and $(\lambda^*, \mu_2) \in C^D_{2k+2,+}(a,b)$. Therefore

$$\Theta(\lambda^*, \mu_1) = \Pi_{2k} < \Pi_{2k+2} = \Theta(\lambda^*, \mu_2).$$

Now the continuity of $\Theta(\lambda, \mu)$ guarantees the existences of $\mu^* \in (\mu_1, \mu_2)$ such that $\Theta(\lambda^*, \mu^*) = \Pi_{2k+1}$, and hence $(\lambda^*, \mu^*) \in C^D_{2k+1,+}(a,b)$.

On the other hand, suppose that $(\lambda_*, \mu_*) \in C^D_{2k+1,+}(a,b)$. Then

$$\Theta(\lambda_*, \mu_*) = \theta(1; \Pi_0, \lambda_* + a, \mu_* + b) = \Pi_{2k+1},$$

and there exist $0 = t_0 < t_1 < t_2 \cdots < t_{2k} < t_{2k+1} = 1$ such that

$$\theta(t_i; \Pi_0, \lambda_* + a, \mu_* + b) = \Pi_i, \quad i = 0, 1, 2, \ldots, 2k+1.$$

Denote $J_i := [t_{2i-2}, t_{2i-1}]$, $i = 1, 2, \ldots, k+1$ and $K_i = [t_{2i-2}, t_{2i}]$, $i = 1, 2, \ldots, k$. Similar arguments as in the proof of Property 3.7 show that

$$\theta_{t_{2k}}(1; \Pi_k, \lambda_* + a) = \Pi_{k+1}, \tag{3.30}$$

$$\vartheta_{2k} := \theta(t_{2k}; \Pi_0, \lambda_* + a) > \Pi_k. \tag{3.31}$$

One can compare (3.30) and (3.31) with (3.26) and (3.28), respectively. Then we have

$$\theta(1; \Pi_0, \lambda_* + a) = \theta_{t_{2k}}(1; \vartheta_{2k}, \lambda_* + a) > \theta_{t_{2k}}(1; \Pi_k, \lambda_* + a) = \Pi_{k+1},$$

and hence $\lambda_* > \lambda^D_{k+1}(a)$ by (2.24).

Now we can conclude that the vertical asymptotic line of $C^D_{2k+1,+}(a,b)$ is $\lambda = \lambda^D_{k+1}(a)$.

Then let us consider the horizontal asymptotic line of $C^D_{2k+1,+}(a,b)$. Notice that this cannot be deduced directly from relation (3.16), because the vertical asymptote of $\widetilde{C}^D_{2k+1}(\widetilde{b},\widetilde{a})$ is unknown at the moment.

On the one hand, since the curve $C^D_{2k+1,+}(a,b)$ lies above $C^D_{2k,+}(a,b)$, while $C^D_{2k,+}(a,b)$ lies above the horizontal line $\mu = \lambda^D_k(b)$ by Property 3.7, we have necessarily $\mu > \lambda^D_k(b)$ whenever $(\lambda,\mu) \in C^D_{2k+1,+}(a,b)$.

On the other hand, suppose that $\mu^* > \lambda^D_k(b)$. By Property 3.8, we get

$$\Theta(\underline{\lambda}, \mu^*) = \Pi_{2k} \tag{3.32}$$

for some $\underline{\lambda} > \lambda^D_k(a)$. The strict monotonicity of $\Theta(\lambda,\mu)$ in $\lambda \in \mathbb{R}$ (see (3.5)) implies

$$\Theta(\underline{\lambda}+1, \mu^*) = \theta(1; \Pi_0, \underline{\lambda}+1+a, \mu^*+b) > \Pi_{2k}.$$

Then there exist $s \in (0,1)$ such that

$$\theta(s; \Pi_0, \underline{\lambda}+1+a, \mu^*+b) = \Pi_{2k}. \tag{3.33}$$

By the first equality in (2.16), there exists $\widetilde{\lambda} \in \mathbb{R}$ such that

$$\theta_s(1; \Pi_{2k}, \widetilde{\lambda}+a) = \Pi_{2k+1}. \tag{3.34}$$

Moreover, by Lemma 2.1, we have $\theta_s(t; \Pi_{2k}, \widetilde{\lambda}+a) \in [\Pi_{2k}, \Pi_{2k+1}]$ for any $t \in [s,1]$, and hence

$$\cos_p \theta_s(t; \Pi_{2k}, \widetilde{\lambda}+a) \geq 0, \quad \forall\, t \in [s,1]. \tag{3.35}$$

Define a potential q on $[0,1]$ as

$$q(t) = \begin{cases} \underline{\lambda}+1+a(t), & t \in [0,s], \\ \widetilde{\lambda}+a(t), & t \in (s,1]. \end{cases}$$

Then it follows from (3.33)–(3.35), that

$$\theta(1; \Pi_0, q, \mu^*+b) = \Pi_{2k+1}.$$

Let $\overline{\lambda} = \max\{\underline{\lambda}+1, \widetilde{\lambda}\}$. Then $\overline{\lambda}+a(t) \geq q(t)$ on $[0,1]$, and hence

$$\Theta(\overline{\lambda}, \mu^*) = \theta(1; \Pi_0, \overline{\lambda}+a, \mu^*+b) \geq \theta(1; \Pi_0, q, \mu^*+b) = \Pi_{2k+1}. \tag{3.36}$$

It follows from (3.32), (3.36) and the continuity of $\Theta(\lambda,\mu)$ in λ, that there exists $\lambda^* \in (\underline{\lambda}, \overline{\lambda}]$ such that $\Theta(\lambda^*, \mu^*) = \Pi_{2k+1}$.

Now we can conclude that the horizontal asymptotic line of $C^D_{2k+1,+}(a,b)$ is $\mu = \lambda^D_k(b)$. □

Remark 3.2. The asymptotes are related with the nodal property of spectral functions. For $k \geq 2$, denote $k_+ := [(k+1)/2] \in \mathbb{N}$ and $k_- := [k/2] = k - k_+ \in \mathbb{N}$. For any $(\lambda,\mu) \in C^D_{k,+}(a,b)$, the corresponding spectral function $x(t)$ has precisely k_+ maximal open sub-intervals of $(0,1)$ on which $x(t)$ is positive, and precisely

k_- maximal open sub-intervals of $(0,1)$ on which $x(t)$ is negative. The vertical and the horizontal asymptotes of $C^D_{k,+}(a,b)$ are respectively the $\lambda = \lambda^D_{k_+}(a)$ and $\mu = \lambda^D_{k_-}(b)$. The asymptotes of $C^D_{k,-}(a,b)$ can be deduced from Theorem 3.2 by using relation (3.16) and can be explained similarly using the nodal property of spectral functions.

3.3. *Strong continuous dependence of Fučík spectral curves on potentials*

It is known that eigenvalues of (2.23) have strong continuous dependence on potentials (see [16, Theorem 1.1]). Namely, if $a_n \xrightarrow{w_1} a_0$ in $\mathcal{L}^1$, then $\lambda^D_1(a_n) \to \lambda^D_1(a_0)$. By Theorem 3.1(i), the first-order Fučík spectral curves also have strong continuity in potentials, i.e. if $a_n \xrightarrow{w_1} a_0$ in $\mathcal{L}^1$, then straight lines $C^D_{1,\pm}(a_n,b_n)$ converge to $C^D_{1,\pm}(a_0,b_0)$, respectively.

Now comes a natural question whether higher-order Fučík spectral curves also have strong continuous dependence on potentials. To answer this question, we first parameterize the curve $C^D_{k,+}(a,b)$, $k \geq 2$. Introduce the parameter

$$s = \frac{\mu - \lambda^D_1(b)}{\lambda - \lambda^D_1(a)}, \quad (\lambda,\mu) \in C^D_{k,+}(a,b).$$

By Theorem 3.1, the hyperbolic-like curve $C^D_{k,+}(a,b)$ lies above the horizontal line $\mathbb{R} \times \lambda^D_1(b)$ and on the right-hand side of the vertical line $\lambda^D_1(a) \times \mathbb{R}$, and it does have a horizontal and a vertical asymptotic line. Although these asymptotes are not $\mathbb{R} \times \lambda^D_1(b)$ or $\lambda^D_1(a) \times \mathbb{R}$ for general $k \geq 2$, we still have $s \in (0,+\infty)$, and $C^D_{k,+}(a,b)$ can be parameterized as

$$\begin{cases} \lambda = \lambda^D_k(s) = \lambda^D_k(s;a,b), \\ \mu = \mu^D_k(s) = \mu^D_k(s;a,b) = \lambda^D_1(b) + s[\lambda^D_k(s;a,b) - \lambda^D_1(a)]. \end{cases} \tag{3.37}$$

The following theorem shows that these curves $C^D_{k,+}(a,b)$ have strong continuity in potentials (a,b).

Theorem 3.3. *Given any integer $k \geq 2$ and suppose that $a_n \xrightarrow{w_1} a_0$ and $b_n \to w_1 b_0$ in $\mathcal{L}^1$, then the curves $C^D_{k,+}(a_n,b_n)$ converge to $C^D_{k,+}(a_0,b_0)$ as $n \to +\infty$, in the sense that for any given $s \in (0,\infty)$, there hold*

$$\lim_{n\to+\infty} \lambda^D_k(s;a_n,b_n) = \lambda^D_k(s;a_0,b_0) \quad \text{and} \quad \lim_{n\to+\infty} \mu^D_k(s;a_n,b_n) = \mu^D_k(s;a_0,b_0).$$

Proof. Fix $s \in (0,+\infty)$ and $k \geq 2$. For simplicity, let us write $\lambda_n = \lambda^D_k(s,a_n,b_n)$, $\mu_n = \mu^D_k(s,a_n,b_n)$ and $\nu_n = \lambda^D_1(b_n) - s\lambda^D_1(a_n)$. Suppose $a_n \xrightarrow{w_1} a_0$, $b_n \to w_1 b_0$ in $\mathcal{L}^1$. Then $\lambda^D_1(a_n) \to \lambda^D_1(a_0)$, $\lambda^D_1(b_n) \to \lambda^D_1(b_0)$ (see [16, Theorem 1.1]), and hence $\nu_n \to \nu_0$.

If $\lambda_n \to \lambda_0$, then it follows from (3.37) that $\mu_n = s\lambda_n + \nu_n \to s\lambda_0 + \nu_0 = \mu_0$, completing the proof of the theorem.

If $\lambda_n \not\to \lambda_0$, without loss of generality, we may assume (passing to a subsequence if necessary) that there exist $\varepsilon_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$\lambda_n > \lambda_0 + \varepsilon_0, \quad \forall\, n \geq n_0.$$

Thus $\mu_n > s\lambda_0 + s\varepsilon_0 + \nu_n$ for any $n > n_0$. Then it follows from (3.5) and (3.6) that

$$\begin{aligned}\Pi_k &= \Theta(\lambda_n + a_n, \mu_n + b_n) \\ &\geq \Theta(\lambda_0 + \varepsilon_0 + a_n, s\lambda_0 + s\varepsilon_0 + \nu_n + b_n)\end{aligned}$$

for any $n > n_0$. Let $n \to +\infty$. Notice that $a_n \xrightarrow{w_1} a_0$ and $\nu_n + b_n \xrightarrow{w_1} \nu_0 + b_0$. By Theorem 2.1(ii), and again (3.5) and (3.6), one has

$$\begin{aligned}\Pi_k &\geq \Theta(\lambda_0 + \varepsilon_0 + a_0, s\lambda_0 + s\varepsilon_0 + \nu_0 + b_0) \\ &> \Theta(\lambda_0 + a_0, s\lambda_0 + \nu_0 + b_0) \\ &= \Pi_k,\end{aligned}$$

a contradiction. □

4. Fučík Spectrum for the Neumann Boundary Problems

Given $a, b \in \mathcal{L}^1$, the Fučík spectrum $\Sigma_p^N(a,b)$ is defined as the set of all $(\lambda,\mu) \in \mathbb{R}^2$ such that (1.3) has a non-zero solution satisfying the Neumann boundary conditions (N). The complete structure of $\Sigma_p^N(a,b)$ are characterized in the following theorem.

Theorem 4.1. *Given $a, b \in \mathcal{L}^1$, then*

$$\Sigma_p^N(a,b) = \bigcup_{k\geq 0} C_{k,\pm}^N(a,b),$$

where $C_{k,\pm}^N(a,b)\,(\neq \emptyset)$ are defined as

$$\begin{aligned}C_{k,+}^N(a,b) &:= \{(\lambda,\mu) \in \mathbb{R}^2 : \theta(1; 0, \lambda + a, \mu + b) = k\pi_p\}, \\ C_{k,-}^N(a,b) &:= \{(\lambda,\mu) \in \mathbb{R}^2 : \theta(1; \pi_p, \lambda + a, \mu + b) = \pi_p + k\pi_p\}.\end{aligned}$$

For any $k \geq 0$, there holds the relations

$$(\lambda,\mu) \in C_{k,-}^N(a,b) \Leftrightarrow (\mu,\lambda) \in C_{k,+}^N(b,a), \tag{4.1}$$

$$(\lambda,\mu) \in C_{2k+1,+}^N(a,b) \Leftrightarrow (\mu,\lambda) \in C_{2k+1,+}^N(\widetilde{b},\widetilde{a}). \tag{4.2}$$

For any $(\lambda,\mu) \in C_{k,+}^N(a,b)$, any associated Neumann spectral function $x(t)$ has exactly k zeroes in $(0,1)$ and satisfies $x(0) > 0$. For any $(\lambda,\mu) \in C_{k,-}^N(a,b)$, any associated Neumann spectral function $x(t)$ has exactly k zeroes in $(0,1)$ and satisfies $x(0) < 0$. Moreover,

- (i) $C_{0,+}^N(a,b) = \lambda_0^N(a) \times \mathbb{R}$ *and* $C_{0,-}^N(a,b) = \mathbb{R} \times \lambda_0^N(b)$;
- (ii) *for any $k \geq 1$, the set $C_{2k-1,+}^N(a,b)$ is a strictly decreasing differentiable curve lying above the horizontal asymptotic line $\mu = \lambda_{k-1}^{DN}(b)$ and on the right-hand side of the vertical asymptotic line $\lambda = \lambda_k^{ND}(a)$;*

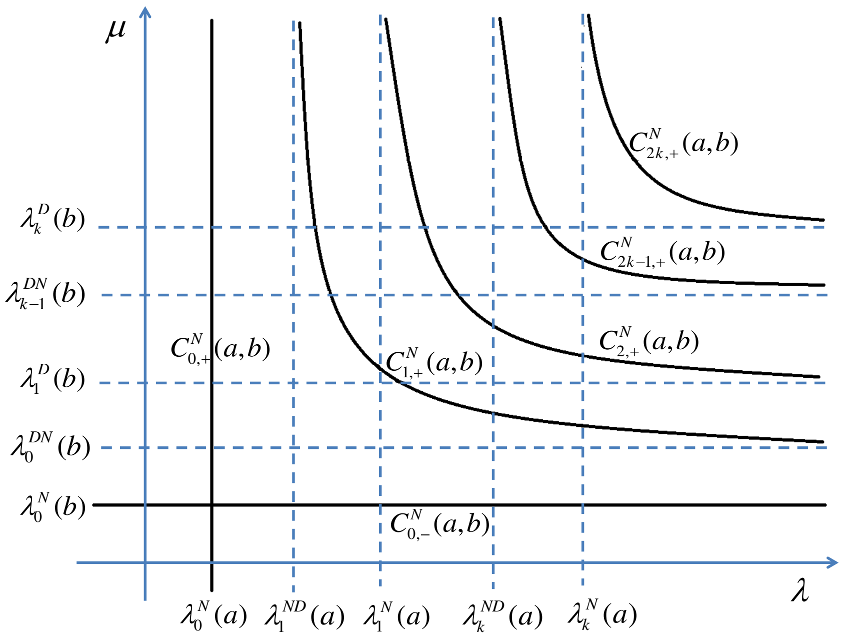

Fig. 2. Asymptotic lines of $C_{k,+}^N(a,b)$, $k \geq 1$.

(iii) *for any* $k \geq 1$, *the set* $C_{2k,+}^N(a,b)$ *is a strictly decreasing differentiable curve lying above the horizontal asymptotic line* $\mu = \lambda_k^D(b)$ *and on the right-hand side of the vertical asymptotic line* $\lambda = \lambda_k^N(a)$; *and*

(iv) *for any* $k \geq 2$, *the curve* $C_{k,+}^N(a,b)$ *lies above the former curve* $C_{k-1,+}^N(a,b)$.

Proof. All those asymptotic lines of $C_{k,+}^N(a,b)$, $k \geq 1$, will be located in Properties 4.1–4.4 in the following. See Fig. 2. All other results can be obtained by reproducing the arguments and techniques applied for the Dirichlet type Fučík spectrum $\Sigma_p^D(a,b)$, and hence we omit the proof. □

Property 4.1. *The vertical asymptotic line of* $C_{2k-1,+}^N(a,b)$, $k \geq 1$, *is* $\lambda = \lambda_k^{ND}(a)$.

Proof. We need only to verify the following two claims.

Claim I. $(\lambda,\mu) \in C_{2k-1,+}^N(a,b) \Rightarrow \lambda > \lambda_k^{ND}(a)$.

Claim II. $\lambda^* > \lambda_k^{ND}(a) \Rightarrow$ there exists $\mu^* \in \mathbb{R}$ such that $(\lambda^*,\mu^*) \in C_{2k-1,+}^N(a,b)$.

First let us prove Claim I. Suppose $(\lambda,\mu) \in C_{2k-1,+}^N(a,b)$. Then

$$\theta(1;0,\lambda+a,\mu+b) = (2k-1)\pi_p.$$

Moreover, there exist $(0 = t_0 <)\, t_1 < t_2 \cdots < t_{2k-1}\, (< t_{2k} = 1)$ such that

$$\theta(t_i;0,\lambda+a,\mu+b) = \Pi_i, \quad i = 1,2,\ldots,2k-1. \tag{4.3}$$

Similar arguments as in the proof of Property 3.7 show that

$$\theta_{t_{2k-2}}(t_{2k-1};\Pi_{k-1},\lambda+a)=\Pi_k, \tag{4.4}$$

$$\vartheta_{2k-2}:=\theta(t_{2k-2};0,\lambda+a)>\Pi_{k-1}. \tag{4.5}$$

One can compare (4.4) and (4.5) with (3.26) and (3.28), respectively. Then we have

$$\begin{aligned}\theta(t_{2k-1};0,\lambda+a)&=\theta_{t_{2k-2}}(t_{2k-1};\vartheta_{2k-2},\lambda+a)\\&>\theta_{t_{2k-2}}(t_{2k-1};\Pi_{k-1},\lambda+a)=\Pi_k.\end{aligned}$$

Now Lemma 2.1 implies $\theta(1;0,\lambda+a)>\Pi_k$, and hence $\lambda>\lambda_k^{ND}(a)$ by (2.27).

Then we prove Claim II. Suppose $\lambda^*>\lambda_k^{ND}(a)$. One has

$$\theta(1;0,\lambda_k^{ND}(a)+a)=\Pi_k,$$

and there exist $(0=t_0<)\,t_1<t_2<\cdots<t_k=1$ such that

$$\theta(t_i;0,\lambda_k^{ND}(a)+a)=\Pi_i,\quad i=1,2,\dots,k.$$

Similar arguments as in the proof of Property 3.8 show that

$$\theta_{t_{k-1}}(1;\Pi_{2k-2},\lambda_k^{ND}(a)+a)=\Pi_{2k-1}, \tag{4.6}$$

and there exist $\mu_1,\mu_2\in\mathbb{R}$ such that

$$\theta(t_{k-1};0,\lambda^*+a,\mu_1+b)=\Pi_{2k-2}, \tag{4.7}$$

$$\theta(1;0,\lambda^*+a,\mu_2+b)=\Pi_{2k}>(2k-1)\pi_p. \tag{4.8}$$

By (4.6) and (4.7), we have

$$\theta(1;0,\lambda^*+a,\mu_1+b)=\Pi_{2k-1}<(2k-1)\pi_p.$$

Combining this with (4.8), we see that there exists $\mu^*\in(\mu_1,\mu_2)$ such that

$$\theta(1;0,\lambda^*+a,\mu^*+b)=(2k-1)\pi_p,$$

which means $(\lambda^*,\mu^*)\in C^N_{2k-1,+}(a,b)$. □

Property 4.2. *The horizontal asymptotic line of $C^N_{2k-1,+}(a,b)$, $k\ge 1$, is $\mu=\lambda^{DN}_{k-1}(b)$.*

Proof. Noticing that $\lambda_k^{ND}(\widetilde{b})=\lambda^{DN}_{k-1}(b)$, this result can be obtained by using (4.2) and Property 4.1. □

Property 4.3. *The vertical asymptotic line of $C^N_{2k,+}(a,b)$, $k\ge 1$, is $\lambda=\lambda_k^N(a)$.*

Proof. We omit the proof, because it is just a slight modification to the proof of Property 4.1. □

Property 4.4. *The horizontal asymptotic line of $C^N_{2k,+}(a,b)$, $k\ge 1$, is $\mu=\lambda_k^D(b)$.*

Proof. We need only to verify the following two claims.

Claim I. $(\lambda, \mu) \in C^N_{2k,+}(a, b), k \geq 1 \Rightarrow \mu > \lambda^D_k(b)$.

Claim II. $\mu^* > \lambda^D_k(b) \Rightarrow$ there exists $\lambda^* \in \mathbb{R}$ such that $(\lambda^*, \mu^*) \in C^N_{2k,+}(a, b)$.

Let us prove Claim I. Suppose that $(\lambda, , \mu) \in C^N_{2k,+}(a, b)$. Then

$$\theta(1; 0, \lambda + a, \mu + b) = \Pi_{2k},$$

and there exists $(0 = t_0 <)\, t_1 < t_2 < \cdots < t_{2k}\, (< t_{2k+1} = 1)$ such that

$$\theta(t_i; 0, \lambda + a, \mu + b) = \Pi_i, \quad i = 1, 2, \ldots, 2k.$$

Employing similar arguments as in Property 3.7 on the interval $[t_1, 1]$, we have

$$\theta_{t_1}(1; \Pi_0, \mu + b) > \Pi_k.$$

Let $\vartheta_1 := \theta(t_1; \Pi_0, \mu + b)$. Then $\vartheta_1 > \Pi_0$ by Lemma 2.1. Therefore

$$\theta(1; \Pi_0, \mu + b) = \theta_{t_1}(1; \vartheta_1, \mu + b) > \theta_{t_1}(1; \Pi_0, \mu + b) > \Pi_k,$$

and hence $\mu > \lambda^D_k(b)$.

Let us prove Claim II. There exist $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1$ such that

$$\theta(t_i; \Pi_0, \lambda^D_k(b) + b) = \Pi_i, \quad i = 0, 1, 2, \ldots, k,$$

and hence

$$\theta_{t_{i-1}}(t_i; \Pi_{i-1}, \lambda^D_k(b) + b) = \Pi_i, \quad i = 1, 2, \ldots, k.$$

By (2.14), we get

$$\theta_{t_{i-1}}(t_i; \Pi_{2i-1}, \lambda^D_k(b) + b) = \Pi_{2i}, \quad i = 1, 2, \ldots, k.$$

Let $\mu^* > \lambda^D_k(b)$ be fixed. Now Lemma 2.4 shows that there exist s_i, τ_i, $1 \leq i \leq k$, such that $t_{i-1} < s_i < \tau_i < t_i$ and

$$\theta_{s_i}(\tau_i; \Pi_{2i-1}, \mu^* + b) = \Pi_{2i}, \quad i = 1, 2, \ldots, k. \tag{4.9}$$

Denote $J^-_i = [s_i, \tau_i]$, $1 \leq i \leq k$. Lemma 2.1 indicates that

$$\psi^-_i(t) := \theta_{s_i}(t; \Pi_{2i-1}, \mu^* + b) \in [\Pi_{2i-1}, \Pi_{2i}], \quad \forall\, t \in J^-_i, 1 \leq i \leq k,$$

and hence

$$\cos_p \psi^-_i(t) \leq 0, \quad \forall\, t \in J^-_i, 1 \leq i \leq k.$$

Denote $J^+_0 = [0, s_1]$, $J^+_i = [\tau_i, s_{i+1}]$, $1 \leq i \leq k-1$, and $J^+_k = [\tau_k, 1]$. By (2.16), there are λ^*_i, $i = 0, 1, \ldots, k$, such that

$$\theta(s_1; 0, \lambda^*_0 + a) = \Pi_1; \tag{4.10}$$

$$\theta_{\tau_i}(s_{i+1}; \Pi_{2i}, \lambda^*_i + a) = \Pi_{2i+1}, \quad i = 1, 2, \ldots, k-1; \tag{4.11}$$

$$\theta_{\tau_k}(1; \Pi_{2k}, \lambda^*_k + a) = 2k\pi_p. \tag{4.12}$$

Again Lemma 2.1 indicates that

$$\psi_0^+(t) := \theta(t; 0, \lambda_0^* + a) \in [\Pi_0, \Pi_1], \quad \forall\, t \in J_0^+;$$
$$\psi_i^+(t) := \theta(t; \Pi_{2i}, \lambda_i^* + a) \in [\Pi_{2i}, \Pi_{2i+1}], \quad \forall\, t \in J_i^+,\ 1 \le i \le k.$$

Hence $\cos_p \psi_i^+(t) \ge 0$ for any $t \in J_i^+$, $0 \le i \le k$. Define a potential q on $[0,1]$ as

$$q(t) = \begin{cases} \lambda_i^* + a(t), & t \in J_i^+,\ \ 0 \le i \le k; \\ a(t), & t \in J_i^-,\ \ 1 \le i \le k. \end{cases}$$

The conditions (4.9)–(4.12) and the analysis above show that

$$\theta(1; 0, q, \mu^* + b) = 2k\pi_p.$$

Let $\overline{\lambda} = \max\{\lambda_i^*; 0 \le i \le k\}$ and $\underline{\lambda} = \min\{\lambda_i^*; 0 \le i \le k\}$. Then $\overline{\lambda} + a \ge q \ge \underline{\lambda} + a$ and

$$\theta(1; 0, \overline{\lambda} + a, \mu^* + b) \ge \theta(1; 0, q, \mu^* + b) = 2k\pi_p,$$
$$\theta(1; 0, \underline{\lambda} + a, \mu^* + b) \le \theta(1; 0, q, \mu^* + b) = 2k\pi_p.$$

Therefore, there exists $\lambda^* \in [\underline{\lambda}, \overline{\lambda}]$ such that

$$\theta(1; 0, \lambda^* + a, \mu^* + b) = 2k\pi_p,$$

which means $(\lambda^*, \mu^*) \in C_{2k,+}^N(a,b)$. □

Remark 4.1. There is always a gap between $C_{0,+}^N(a,b)$ and the vertical asymptotic line of the first nontrivial spectral curve $C_{1,+}^N(a,b)$, because $\lambda_0^N(a) < \lambda_1^{ND}(a)$. See Fig. 2. There is always a gap between $C_{0,-}^N(a,b)$ and the horizontal asymptotic line of the first nontrivial spectral curve $C_{1,+}^N(a,b)$, because $\lambda_0^N(b) < \lambda_0^{DN}(b)$.

If $a_n \xrightarrow{w_1} a_0$ and $b_n \xrightarrow{w_1} b_0$ in $\mathcal{L}^1$, then $C_{0,+}^N(a_n,b_n) = \lambda_0^N(a_n) \times \mathbb{R}$ and $C_{0,-}^N(a_n,b_n) = \mathbb{R} \times \lambda_0^N(b_n)$ converge to $C_{0,+}^N(a_0,b_0) = \lambda_0^N(a_0) \times \mathbb{R}$ and $C_{0,-}^N(a_0,b_0) = \mathbb{R} \times \lambda_0^N(b_0)$, respectively.

For any hyperbolic-like curve $C_{k,+}^N(a,b)$, $k \ge 1$, introduce the parameter

$$s = \frac{\mu - \lambda_0^N(b)}{\lambda - \lambda_0^N(a)}, \quad (\lambda, \mu) \in C_{k,+}^N(a,b).$$

Then $s \in (0, +\infty)$ and $C_{k,+}^N(a,b)$ can be parameterized as

$$\begin{cases} \lambda = \lambda_k^N(s) = \lambda_k^N(s; a, b), \\ \mu = \mu_k^N(s) = \mu_k^N(s; a, b) = \lambda_0^N(b) + s[\lambda_k^N(s; a, b) - \lambda_0^N(a)]. \end{cases} \tag{4.13}$$

The curve $C_{k,+}^N(a,b)$ has strong continuous dependence on $(a,b) \in \mathcal{L}^1 \times \mathcal{L}^1$, as stated in the following theorem. We omit the proof, because it is similar to that of Theorem 3.3.

Theorem 4.2. *Given any integer $k \geq 1$, suppose that $a_n \xrightarrow{w_1} a_0$ and $b_n \xrightarrow{w_1} b_0$ in $\mathcal{L}^1$. Then $C^N_{k,+}(a_n, b_n)$ converge to $C^N_{k,+}(a_0, b_0)$ as $n \to +\infty$, in the sense that for any $s \in (0, \infty)$, there hold*

$$\lim_{n\to+\infty} \lambda^N_k(s; a_n, b_n) = \lambda^N_k(s; a_0, b_0) \quad \textit{and} \quad \lim_{n\to+\infty} \mu^N_k(s; a_n, b_n) = \mu^N_k(s; a_0, b_0).$$

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