# Large Induced Forests in Graphs

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**Abstract:** In this article, we prove three theorems. The first is that every connected graph of order $n$ and size $m$ has an induced forest of order at least $(8n-2m-2)/9$ with equality if and only if such a graph is obtained from a tree by expanding every vertex to a clique of order either 4 or 5. This improves the previous lower bound $\frac{2n^2}{2m+n}$ of Alon–Kahn–Seymour for $m \leq 5n/2$, and implies that such a graph has an induced forest of order at least $n/2$ for $m < \lfloor 7n/4 \rfloor$. This latter result relates to the conjecture of Albertson and Berman that every planar graph of order $n$ has an induced forest of order at least $n/2$. The second is that every connected triangle-free graph of order $n$ and size $m$ has an induced forest of order at least $(20n-5m-5)/19$. This bound is sharp by the cube and the Wagner graph. It also improves the previous lower bound $n-m/4$ of Alon–Mubayi–Thomas for $m \leq 4n-20$, and implies that such a graph has an induced forest of order at least $5n/8$ for $m < \lfloor 13n/8 \rfloor$. This latter result relates to the conjecture of Akiyama and Watanabe that every bipartite planar graph of order $n$ has an induced forest of order at least $5n/8$. The third is that every connected planar graph of order $n$ and size $m$ with girth at least 5 has an induced forest of order at least $(8n-2m-2)/7$ with equality if and only if such a graph is obtained from a

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tree by expanding every vertex to one of five specific graphs. This implies that such a graph has an induced forest of order at least $2(n+1)/3$, where $7n/10$ was conjectured to be the best lower bound by Kowalik, Lužar, and Škrekovski. 

Keywords: *acyclic set; girth; induced forest; planar graph; triangle-free*

## 1. INTRODUCTION

For a (simple, undirected) graph $G = (V, E)$, a subset $S \subset V$ is called *acyclic* if the subgraph $G[S]$ induced by $S$ is a forest. Let $a(G)$ denote the maximum order of an acyclic set in $G$. It is known that determining this invariant is NP-hard even for planar graphs, see [12]. In 1979, Albertson and Berman [2] posed the following conjecture.

**Conjecture 1.** *[2] If G is a planar graph of order n, then* $a(G) \geq n/2$.

This conjecture implies that every planar graph of order $n$ has a stable set of order at least $n/4$. This fact is known to be true only as a consequence of the Four Color Theorem [6, 7]. The best known lower bound on $a(G)$ for a planar graph $G$ is due to Borodin [8]. A coloring of a graph $G$ is acyclic if the union of every two color classes induces a forest. Borodin proved that every planar graph has an acyclic 5-coloring, which implies that such a graph of order $n$ has an induced forest of order at least $2n/5$. In 1987, Akiyama and Watanabe [1] posed a similar conjecture on bipartite planar graphs.

**Conjecture 2.** *[1] If G is a bipartite planar graph of order n, then* $a(G) \geq 5n/8$.

In 2010, inspired by the fact that the dodecahedron has the minimal ratio of vertex to edge among all connected planar graphs of girth at least 5, Kowalik, Lužar, and Škrekovski [13] posed a sharp conjecture on such graphs.

**Conjecture 3.** *[13] If G is a planar graph of order n and girth at least 5, then* $a(G) \geq 7n/10$.

The three conjectures, if true, are sharp by $K_4$ (the clique of order 4), the cube $Q_3$ and the dodecahedron $D_{20}$ (see Fig. 1), respectively. These conjectures motivate to study this invariant for sparse graphs. Alon, Kahn, and Seymour [4] determined the minimum possible value of $a(G)$, where $G$ ranges over all graphs of order $n$ and size $m$ for every $n$ and $m$; in particular, their results imply that $a(G) \geq \frac{2n^2}{2m+n}$ for $m \geq n$. Refining results for sparse bipartite graphs in [3], Alon, Mubayi, and Thomas [5] proved that every

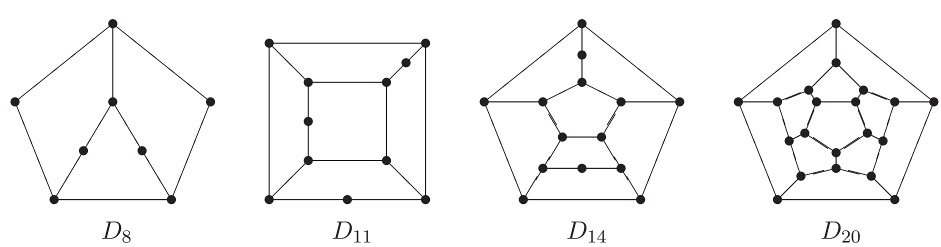


FIGURE 1. The graphs $D_n$.

triangle-free graph of order $n$ and size $m$ has an induced forest of order at least $n - m/4$. Salavatipour [16] proved that every planar triangle-free graph of order $n$ and size $m$ has an induced forest of order at least $(29n - 6m)/32$.

Let $\mathcal{G}$ be a set of connected graphs. We denote by $\mathcal{F}(\mathcal{G})$ the family of connected graphs $G$ consisting of some disjoint copies of graphs chosen from $\mathcal{G}$ such that the multigraph obtained by contracting each copy to a single vertex is a tree and if $H_1$ and $H_2$ are two copies from $\mathcal{G}$, then $G$ has at most one edge between $H_1$ and $H_2$. For example, if $G$ consists of a $K_4$ and a $K_5$ linked by an edge, then $G \in \mathcal{F}(K_4, K_5)$. In this article, the following sharp results are obtained for connected graphs.

**Theorem 1.** *If $G$ is a connected graph of order $n$ and size $m$, then $a(G) \geq (8n - 2m - 2)/9$ with equality if and only if $G \in \mathcal{F}(K_4, K_5)$.*

Theorem 1 improves the lower bound $\frac{2n^2}{2m+n}$ of Alon–Kahn–Seymour for $m \leq 5n/2$ and it yields immediately the following consequence that implies that Conjecture 1 remains open only for $m \geq \lfloor 7n/4 \rfloor$.

**Corollary 1.** *Let $G$ be a connected graph of order $n$ and size $m$. If $m < \lfloor 7n/4 \rfloor$, then $a(G) \geq n/2$ with equality if and only if $G \in \mathcal{F}(K_4)$.*

**Theorem 2.** *If $G$ is a connected triangle-free graph of order $n$ and size $m$, then $a(G) \geq (20n - 5m - 5)/19$.*

The Wagner graph, denoted by $W$, is a Möbius ladder of order 8, which consists of a cycle $u_0u_1u_2u_3v_0v_1v_2v_3$ and a perfect matching $\{u_iv_i \mid i = 0, 1, 2, 3\}$, cf. [9]. The bound in Theorem 2 is sharp by all graphs in $\mathcal{F}(Q_3, W)$ and it also improves the lower bound $n - m/4$ of Alon–Mubayi–Thomas for $m \leq 4n - 20$. The Euler formula implies that every connected planar triangle-free graph of order $n$ has size $m \leq 2n - 4$. It is easy to see that Theorem 2 even improves the lower bound $(29n - 6m)/32$ of Salavatipour for such graphs with $m \leq (89n - 160)/46 \approx 1.93n - 3.47$. Theorem 2 also yields immediately the following consequence.

**Corollary 2.** *Let $G$ be a connected triangle-free graph of order $n$ and size $m$. If $m < \lfloor 13n/8 \rfloor$, then $a(G) \geq 5n/8$.*

Corollary 2 improves a previous result of Alon et al. [5] with the same lower bound only for subcubic graphs and also confirms Conjecture 2 for sparse graphs.

Let $D_5$ denote a pentagon, that is, a cycle of length 5, let $D_{3k+2}$ for $k = 2, 3, 4$ be the graphs as depicted in Figure 1 and let $\mathcal{D} := \{D_{3k+2} \mid k = 1, 2, 3, 4, 6\}$.

**Theorem 3.** *If $G$ is a connected planar graph of order $n$ and size $m$ with girth at least 5, then $a(G) \geq (8n - 2m - 2)/7$ with equality if and only if $G \in \mathcal{F}(\mathcal{D})$.*

The Euler formula implies that such a graph $G$ in Theorem 3 satisfies $m \leq 5(n - 2)/3$. Theorem 3 readily implies Conjecture 3 for $m < \lfloor 31n/20 \rfloor$ and also yields immediately the following consequence.

**Corollary 3.** *If $G$ is a connected planar graph of order $n$ with girth at least 5, then $a(G) \geq 2(n + 1)/3$ with equality if and only if $G$ is isomorphic to a graph in $\mathcal{D}$.*

More results on the induced forests can be found in a survey of Punnim [15]. A closely related problem is to study the order of the largest induced tree in a graph. This problem was initiated by Erdős, Saks, and Sós [10] in 1986 and it was followed by Matoušek and Šámal [14], and also by Fox, Loh, and Sudakov [11].

The proofs of the three theorems are by induction on the order of the graphs and they are presented in Sections 2, 3 and 4, respectively.

## 2. GENERAL GRAPHS

In this section, we prove Theorem 1 by induction on the order $n$ of the graph $G$. Let $t(G) := (8n - 2m - 2)/9$. For $n = 1$, we have $a(G) = 1 > 6/9 = t(G)$. So assume that Theorem 1 holds for all connected graphs of order less than $n$, and we aim to prove it for such a graph $G$ of order $n$ and size $m$. Let $\mathcal{F} := \mathcal{F}(K_4, K_5)$ for brevity.

**Claim 2.1.** *If $G$ has a bridge, then $a(G) \geq t(G)$ with equality if and only if $G \in \mathcal{F}$.*

Assume that $e$ is a bridge of $G$. Then $G - e$ consists of two connected components, say $G_1$ and $G_2$. By the induction hypothesis, $G_i$ has an acyclic set $F_i$ with $|F_i| \geq t(G_i)$ with equality if and only if $G_i \in \mathcal{F}$ for $i = 1, 2$. Then $F_1 \cup F_2$ is an acyclic set in $G$ and thus $a(G) \geq |F_1 \cup F_2| \geq t(G_1) + t(G_2) = t(G)$. Observing that $F$ is an acyclic set in $G$ if and only if $F \cap V(G_i)$ is an acyclic set in $G_i$ for $i = 1, 2$, we have $a(G) = a(G_1) + a(G_2)$. Thus, $a(G) = t(G)$ if and only if $a(G_i) = t(G_i)$ for $i = 1, 2$. Note that $G \in \mathcal{F}$ if and only if $G_i \in \mathcal{F}$ for $i = 1, 2$. Hence, $a(G) = t(G)$ if and only if $G \in \mathcal{F}$.

By Claim 2.1, we can assume that $G$ has neither a bridge nor a vertex of degree 1.

**Claim 2.2.** *If $G$ has maximum degree greater than 4, then $a(G) > t(G)$.*

For a vertex $v$ of $G$, we denote by $N(v)$ the neighborhood of $v$ and by $d(v)$ the degree of $v$, that is, $d(v) := |N(v)|$. Assume that $d(v) > 4$ and $G - v$ consists of $k$ connected components $G_i$ for $i = 1, 2, \ldots, k$. Since $G$ has no bridge, we have $|N(v) \cap V(G_i)| \geq 2$ for $i = 1, 2, \ldots, k$. Then $d(v) \geq 2k$ and thus $d(v) > (4 + 2k)/2 = 2 + k$. By the induction hypothesis, $G_i$ has an acyclic set $F_i$ with $|F_i| \geq t(G_i)$ for $i = 1, 2, \ldots, k$. Then $\cup_{i=1}^{k} F_i$ is an acyclic set in $G$ and thus

$$a(G) \geq \left| \bigcup_{i=1}^{k} F_i \right| \geq \sum_{i=1}^{k} t(G_i) = \frac{1}{9}\{8(n-1) - 2[m - d(v)] - 2k\} \geq t(G). \qquad (1)$$

The equality holds in equation (1) only if $d(v) = 3 + k$ and $|F_i| = t(G_i)$ for $i = 1, 2, \ldots, k$. Since $d(v) \geq \max\{5, 2k\}$, we obtain that $k = 2$ or 3.

**Case 1** $k = 2$.

In this case, $d(v) = 5$. Since $G$ has no bridge, $2 \leq |N(v) \cap V(G_i)| \leq 3$ for $i = 1, 2$. By the induction hypothesis, $G - G_i$ has an acyclic set $H_{3-i}$ with $|H_{3-i}| \geq t(G - G_i) \geq t(G_{3-i}) + 2/9 > |F_{3-i}|$ for $i = 1, 2$. Then $H_1 \cap H_2 = \{v\}$ and $H_1 \cup H_2$ is an acyclic set in $G$. Thus $a(G) \geq |H_1 \cup H_2| > |F_1| + |F_2| = t(G)$.

**Case 2** $k = 3$.

In this case, $d(v) = 6$ and $|N(v) \cap V(G_i)| = 2$ for $i = 1, 2, 3$. By the induction hypothesis, $G - G_1 - G_2$ has an acyclic set $H_3$ with $|H_3| \geq t(G - G_1 - G_2) = t(G_3) + 4/9 > |F_3|$. Similarly, $G - G_1 - G_3$ has an acyclic set $H_2$ with $|H_2| > |F_2|$ and $G - G_2 - G_3$ has an acyclic set $H_1$ with $|H_1| > |F_1|$. Then $H_1 \cap H_2 \cap H_3 = \{v\}$ and $H_1 \cup H_2 \cup H_3$ is an acyclic set in $G$. Thus $a(G) \geq |H_1 \cup H_2 \cup H_3| > |F_1| + |F_2| + |F_3| = t(G)$.

**Claim 2.3.** *If G has maximum degree at most 3, then $a(G) \geq t(G)$ with equality if and only if G is a clique of order 4.*

We only need to consider two cases that $G$ is 3-regular or not.

**Case 1** $G$ is 3-regular.

Let $v$ be a vertex of $G$ and $N(v) = \{u, w, x\}$. If $uw \notin E(G)$, then $H := G - \{v, x\} + uw$ is connected, for otherwise $x$ would be incident to a bridge in $G$, and by the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [8(n-2) - 2(m-4) - 2]/9 > t(G) - 1$. Thus $F \cup \{v\}$ is an acyclic set in $G$ of order greater than $t(G)$. So we have $uw \in E(G)$ and similarly, $ux, wx \in E(G)$, which implies that $G$ is a clique of order 4 and $a(G) = t(G) = 2$.

**Case 2** $G$ is not 3-regular.

Let $v$ be a vertex of degree 2 in $G$ with $N(v) = \{u, w\}$. If $uw \notin E(G)$, then $G - v + uw$ is connected, and by the induction hypothesis, it has an acyclic set $F$ with $|F| \geq [8(n-1) - 2(m-1) - 2]/9 > t(G) - 1$. Hence, $F \cup \{v\}$ is an acyclic set in $G$ of order greater than $t(G)$. So we can assume $uw \in E(G)$. If $V(G) = \{u, v, w\}$, then $G$ is a triangle and $a(G) = 2 > 16/9 = t(G)$. So we may assume that $G$ is of order greater than 3. Then $H := G - \{u, v, w\}$ is connected, for otherwise either $u$ or $w$ would be incident to a bridge in $G$. By the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [8(n-3) - 2(m-5) - 2]/9 > t(G) - 2$. Thus $F \cup \{u, v\}$ is an acyclic set in $G$ of order greater than $t(G)$.

By Claims 2.2 and 2.3, we can assume that $G$ has maximum degree 4. Let $v$ be a vertex of degree 4 in $G$ and $G - v$ consists of $k$ connected components $G_i$ for $i = 1, 2, \ldots, k$. Since $G$ has no bridge, we have $d(v) \geq 2k$, which gives that $k \leq 2$.

Consider $k = 1$. By the induction hypothesis, $G_1$ (and thus $G$) has an acyclic set of order at least $[8(n-1) - 2(m-4) - 2]/9 = t(G)$. If $a(G) = t(G)$, then $a(G_1) = t(G_1)$ and $G_1 \in \mathcal{F}$. Since every vertex is of degree at most 4 in $G$, the vertex $v$ cannot be adjacent to a clique of order 5 in $G_1$. We show that the vertex $v$ and its neighborhood $N(v)$ induce a clique of order 5. Assume to contrary that $N(v)$ intersects at least two distinct cliques of order 4 in $G_1$. Take any vertex $u \in N(v)$. It is clear that $d(u) = 4$, $G - u$ is connected, $d_{G-u}(v) = 3$, and $v$ is not in a 4-clique of $G - u$. Thus $G - u \notin \mathcal{F}$, which contradicts the above argument by replacing $v$ with $u$. This contradiction shows that $G$ is a clique of order 5.

Thus we can assume that $k = 2$. Again by the induction hypothesis, $G_i$ has an acyclic set $F_i$ with $|F_i| \geq t(G_i)$ for $i = 1, 2$. Note that $F_1 \cup F_2$ is an acyclic set in $G$ and

$$|F_1 \cup F_2| \geq t(G_1) + t(G_2) = [8(n-1) - 2(m-4) - 4]/9 = t(G) - 2/9.$$

Therefore, $|F_i| \leq t(G_i) + 2/9$ for $i = 1, 2$. Since $d(v) = 4$, $k = 2$ and $G$ has no bridge, $|N(v) \cap V(G_i)| = 2$ for $i = 1, 2$. By the induction hypothesis, $G - G_i$ has an acyclic set $H_{3-i}$ with $|H_{3-i}| \geq t(G - G_i) = t(G_{3-i}) + 4/9 > |F_{3-i}|$ for $i = 1, 2$. Then $H_1 \cap H_2 = \{v\}$ and $H_1 \cup H_2$ is an acyclic set in $G$. Thus $a(G) \geq |H_1 \cup H_2| \geq |F_1| + |F_2| + 1 \geq t(G) + 7/9 > t(G)$.

## 3. TRIANGLE-FREE GRAPHS

In this section, we prove Theorem 2 also using induction on the order $n$ of the graph $G$. Let $t(G) := (20n - 5m - 5)/19$. For $n = 1$, we have $a(G) = 1 > 15/19 = t(G)$. So assume that Theorem 2 holds for all connected triangle-free graphs of order less than $n$, and we aim to prove it for such a graph $G$ of order $n$ and size $m$. Analogous to Claims 2.1 and 2.2, one can show that $G$ has no bridge and every vertex of $G$ is of degree 2, 3, or 4. If $G$ is a cycle, then we are done. We start by showing that the maximum degree of $G$ is 3.

**Claim 3.1.** *If the maximum degree of $G$ is 4, then $a(G) \geq t(G)$.*

Indeed, let $v$ be a vertex of degree 4 in $G$ and $G - v$ consists of $k$ connected components $G_i$ for $i = 1, 2, \ldots, k$. Since $G$ has no bridge, we have $d(v) \geq 2k$, which gives that $k \leq 2$. If $k = 1$, then by the induction hypothesis, $G_1$ (and thus $G$) has an acyclic set of order at least $[20(n-1) - 5(m-4) - 5]/19 = t(G)$. Thus we can assume that $k = 2$. Again by the induction hypothesis, $G_i$ has an acyclic set $F_i$ with $|F_i| \geq t(G_i)$ for $i = 1, 2$. Note that $F_1 \cup F_2$ is an acyclic set in $G$ and

$$|F_1 \cup F_2| \geq t(G_1) + t(G_2) = [20(n-1) - 5(m-4) - 10]/19 = t(G) - 5/19.$$

Therefore, $|F_i| < t(G_i) + 5/19$ for $i = 1, 2$. Since $d(v) = 4$, $k = 2$, and $G$ has no bridge, $|N(v) \cap V(G_i)| = 2$ for $i = 1, 2$. By the induction hypothesis, $G - G_i$ has an acyclic set $H_{3-i}$ with $|H_{3-i}| \geq t(G - G_i) = t(G_{3-i}) + 10/19 > |F_{3-i}|$ for $i = 1, 2$. Then $H_1 \cap H_2 = \{v\}$ and $H_1 \cup H_2$ is an acyclic set in $G$. Thus $a(G) \geq |H_1 \cup H_2| \geq |F_1| + |F_2| + 1 \geq t(G) + 14/19 > t(G)$.

**Claim 3.2.** *If $G$ has a vertex of degree 2 adjacent to a vertex of degree 2 and the other of degree 3, then $a(G) > t(G)$.*

Assume that $v$ is such a vertex of $G$ with $N(v) = \{u, w\}$ and $d(u) = 2, d(w) = 3$. Then $H := G - \{u, v, w\}$ is connected, for otherwise $w$ would be incident to a bridge in $G$. Thus by the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [20(n-3) - 5(m-5) - 5]/19 > t(G) - 2$. Then $F \cup \{u, v\}$ is an acyclic set in $G$ of order greater than $t(G)$.

**Claim 3.3.** *If $G$ has a vertex of degree 3 adjacent to two vertices of degree 2, then $a(G) > t(G)$.*

The proof is analogous to that of Claim 3.2. Assume that $v$ is such a vertex of $G$ with $d(v) = 3$, $\{u, w\} \subset N(v)$ and $d(u) = d(w) = 2$. Then $H := G - \{u, v, w\}$ is connected, for otherwise $v$ would be incident to a bridge in $G$. Thus by the induction hypothesis, $H$ has an acyclic set $F$ with $|F| > t(G) - 2$. Then $F \cup \{u, w\}$ is an acyclic set in $G$ of order greater than $t(G)$.

**Claim 3.4.** *If $G$ has a vertex of degree 2 not in any quadrilateral, then $a(G) > t(G)$.*

Assume that $v$ is such a vertex in $G$ and $N(v) = \{u, w\}$. Then $G - v + uw$ is both connected and triangle-free, and by the induction hypothesis, it has an acyclic set $F$ with $|F| \geq [20(n-1) - 5(m-1) - 5]/19 > t(G) - 1$. Hence, $F \cup \{v\}$ is an acyclic set in $G$ of order greater than $t(G)$.

**Claim 3.5.** *If $G$ has a vertex of degree 2, then $a(G) > t(G)$.*

Assume that $v$ is such a vertex of $G$ with $N(v) = \{u, w\}$. Claim 3.4 implies that the pair of vertices $u$ and $w$ has another common neighbor, say $x$. Since $G$ is not a cycle, we can assume that $G$ is of order at least 5. By Claims 3.1–3.3, we have $d(u) = d(w) = d(x) = 3$. Now we consider two cases according to the neighbors of $u$ and $w$.

**Case 1** The two vertices $u$ and $w$ have a third common neighbor, say $y$, besides $v$ and $x$.

In this case, $d(y) = 3$ by Claim 3.3. Suppose that the pair of vertices $x$ and $y$ has a third common neighbor, say $z$, besides $u$ and $w$. If $d(z) = 2$, then $G$ is of size 8 on the vertex set $\{u, v, w, x, y, z\}$ and the set $F := \{u, v, w, z\}$ for instance, is acyclic with $|F| = 4 > 75/19 = t(G)$. If $d(z) = 3$, then $z$ is incident to a bridge in $G$, which is impossible. So the pair of vertices $x$ and $y$ has only two common neighbors $u$ and $w$. But then $H := G - \{u, v, w\} + xy$ is connected and triangle-free, and by the induction hypothesis, it has an acyclic set $F$ with $|F| \geq [20(n-3) - 5(m-5) - 2]/19 > t(G) - 2$. Then $F \cup \{u, v\}$ is acyclic in $G$ of order greater than $t(G)$.

**Case 2** The two vertices $u$ and $w$ have only two common neighbors $v$ and $x$.

Let $y$ and $z$ be the third neighbors of $u$ and $w$, respectively. By Claim 3.3, we have $d(y) = d(z) = 3$. Suppose that $yz \notin E(G)$ and let $H := G - \{u, w, x\} + vy + vz$. Then it is clear that $H$ is both connected and triangle-free. By the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [20(n-3) - 5(m-5) - 5]/19 > t(G) - 2$. Then it is easy to see that $F \cup \{u, w\}$ is an acyclic set in $G$ of order greater than $t(G)$. So we have $yz \in E(G)$, but then $H := G - \{u, v, w, y, z\}$ is connected and triangle-free, for otherwise one of the three vertices $x$, $y$, and $z$ would be incident to a bridge in $G$; and by the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [20(n-5) - 5(m-9) - 2]/19 > t(G) - 3$. Then $F \cup \{u, v, z\}$ is acyclic in $G$ of order greater than $t(G)$.

By Claim 3.5, we can further assume that $G$ is 3-regular. Let $v$ be a vertex of $G$ with $N(v) = \{u, w, x\}$.

**Claim 3.6.** *If the vertex $v$ is the only common neighbor of the pair of vertices $u$ and $w$, then $a(G) \geq t(G)$.*

Indeed, let $H := G - \{v, x\} + uw$. By assumption, $H$ is triangle-free, and meanwhile $H$ is also connected, for otherwise the vertex $x$ would be incident to a bridge in $G$. Note that $H$ has two vertices of degree 2. By the induction hypothesis and Claim 3.5, $H$ has an acyclic set $F$ with $|F| > [20(n-2) - 5(m-4) - 5]/19 = (20n - 5m - 25)/19$. Thus $|F| \geq (20n - 5m - 24)/19 = t(G) - 1$ and $F \cup \{v\}$ is an acyclic set in $G$ of order at least $t(G)$.

**Claim 3.7.** *If a pair of vertices has three common neighbors in $G$, then $a(G) > t(G)$.*

Assume that $u$ and $v$ are such a pair of vertices with three common neighbors $x$, $y$, and $z$ in $G$. Then $H := G - \{u, v, x, y, z\}$ is connected, for otherwise one of the three vertices $x$, $y$, and $z$ would be incident to a bridge in $G$. By the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [20(n-5) - 5(m-9) - 5]/19 > t(G) - 3$. Thus $F \cup \{u, v, x\}$ is an acyclic set in $G$ of order greater than $t(G)$.

By Claim 3.6, we can assume that every vertex of $G$ lies in at least three quadrilaterals. So $G$ has a quadrilateral, say $u_1u_2v_2v_1$. By Claim 3.7 and the fact that $G$ is triangle-free and 3-regular, there are four distinct vertices, say $u_i$ and $v_i$ for $i = 0, 3$, such that $u_iu_{i+1}, v_iv_{i+1} \in E(G)$ for $i = 0, 1, 2$. Then Claim 3.6 with the regularity of $G$ implies that $G$ must be the cube with $a(G) = t(G) = 5$.

## 4. PLANAR GRAPHS

In this section, we prove Theorem 3. For clarity, we split the proof into two parts, one is the proof of the inequality and the other is of the necessary and sufficient condition for the equality. Let $\mathcal{F} := \mathcal{F}(\mathcal{D})$ for brevity.

### A. The First Part: The Inequality

We still use induction on the order $n$ of the graph $G$. Let $t(G) := (8n - 2m - 2)/7$. For $n = 1$, we have $a(G) = 1 > 6/7 = t(G)$. Assume that the inequality in Theorem 3 holds for all connected planar graphs of order less than $n$ with girth at least 5, and we aim to prove it for such a graph $G$ of order $n$ and size $m$.

It is easy to verify that Claims 2.1 and 2.2 hold in this context, so the analogous proofs are omitted. As before, we can assume that $G$ is bridgeless with maximum degree at most 4.

**Claim 4.1.** *If $G$ has a vertex of degree 4, then $a(G) \geq t(G)$.*

Assume that $v$ is such a vertex of $G$ with $d(v) = 4$ and $G - v$ consists of $k$ connected components $G_i$ for $i = 1, 2, \ldots, k$. Since $G$ has no bridge, we have $d(v) \geq 2k$, which gives that $k \leq 2$. If $k = 1$, then by the induction hypothesis, $G_1$ (and thus $G$) has an acyclic set of order at least $[8(n-1) - 2(m-4) - 2]/7 = t(G)$. Thus we can assume that $k = 2$. Again by the induction hypothesis, $G_i$ has an acyclic set $F_i$ with $|F_i| \geq t(G_i)$ for $i = 1, 2$. Note that $F_1 \cup F_2$ is an acyclic set in $G$ and $|F_1 \cup F_2| \geq t(G_1) + t(G_2) = [8(n - 1) - 2(m-4) - 4]/7 = t(G) - 2/7$. Therefore, $|F_i| < t(G_i) + 2/7$ for $i = 1, 2$. Since $d(v) = 4$, $k = 2$, and $G$ has no bridge, $|N(v) \cap V(G_i)| = 2$ for $i = 1, 2$. By the induction hypothesis, $G - G_i$ has an acyclic set $H_{3-i}$ with $|H_{3-i}| \geq t(G - G_i) = t(G_{3-i}) + 4/7 > |F_{3-i}|$ for $i = 1, 2$. Then $H_1 \cup H_2$ is an acyclic set in $G$ and thus $a(G) \geq |H_1 \cup H_2| \geq |F_1| + |F_2| + 1 \geq t(G) + 5/7 > t(G)$.

By Claim 4.1, we can further assume that every vertex of $G$ is of degree either 2 or 3.

**Claim 4.2.** *If $G$ has a pair of adjacent vertices both of degree 2, then $a(G) \geq t(G)$.*

Assume that $u$ and $v$ are such a pair of vertices in $G$. Let $w$ be the other neighbor of $v$ besides $u$. Suppose $d(w) = 3$. Then $H := G - \{u, v, w\}$ is connected, for otherwise $w$ would be incident to a bridge in $G$. Thus by the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [8(n-3) - 2(m-5) - 2]/7 = t(G) - 2$. Then $F \cup \{u, v\}$ is an acyclic set in $G$ of order at least $t(G)$. So we have $d(w) = 2$ and this implies that $G$ must be a cycle of order $n \geq 5$ and $a(G) = n - 1 \geq (8n - 2m - 2)/7 = t(G)$.

**Claim 4.3.** *If $G$ has a vertex of degree 3 adjacent to at least two vertices of degree 2, then $a(G) \geq t(G)$.*

The proof is analogous to that of Claim 4.2. Assume that $v$ is such a vertex of $G$ with $d(v) = 3$, $\{u, w\} \subset N(v)$, and $d(u) = d(w) = 2$. Then $H := G - \{u, v, w\}$ is connected, for otherwise $v$ would be incident to a bridge in $G$. Thus by the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq t(G) - 2$. Then $F \cup \{u, w\}$ is an acyclic set in $G$ of order at least $t(G)$.

**Claim 4.4.** *If $G$ has a vertex of degree 2 not in any pentagon, then $a(G) > t(G)$.*

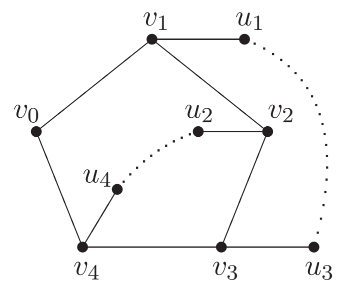


FIGURE 2. Part of a planar embedding of $G$.

Assume that $v$ is such a vertex of $G$ and $N(v) = \{u, w\}$. Then $G - v + uw$ is both connected, planar, and of girth at least 5, and thus by the induction hypothesis, it has an acyclic set $F$ with $|F| \geq [8(n-1) - 2(m-1) - 2]/7 > t(G) - 1$. Hence, $F \cup \{v\}$ is an acyclic set in $G$ of order greater than $t(G)$.

**Claim 4.5.** *If $G$ has a vertex of degree 2, then $a(G) \geq t(G)$.*

Let $v_0$ be such a vertex of $G$. Then by Claim 4.4, $v_0$ is on some pentagon, say $v_0v_1v_2v_3v_4$. By Claims 4.2 and 4.3, $d(v_i) = 3$ for $i = 1, 2, 3, 4$. For a pair of vertices $u$ and $v$ in $G$, we denote by $d(u, v)$ the distance (the length of a shortest path) between them. Let $u_i$ be the third neighbor of $v_i$ for $i = 1, 2, 3, 4$, respectively. Then all $u_i$ for $i = 1, 2, 3, 4$ are distinct from each other and $d(u_i, v_j) \geq 2$ for $i \neq j$ as $G$ has girth at least 5. The following statement holds:

*If both $d(u_1, u_3) \leq 2$ and $d(u_2, u_4) \leq 2$, then $a(G) > t(G)$.*

Indeed, considering a planar embedding of $G$, by the Jordan curve theorem and without loss of generality, we may assume that a path of length at most two links $u_1$ and $u_3$ outside the pentagon $v_0v_1v_2v_3v_4$, while another path also of length at most two links $u_2$ and $u_4$ inside, see Figure 2. Let $H := G - \{v_0, v_2, v_3, v_4\} + v_1u_2 + u_3u_4$ and draw the edges $v_1u_2$ and $u_3u_4$ along the paths $v_1v_2u_2$ and $u_3v_3v_4u_4$, respectively, in the embedding of $G$. Also note that the five vertices $v_1$ and $u_i$ for $i = 1, 2, 3, 4$ lie on a common cycle of length at least 5 in $H$. Therefore, $H$ is a connected planar graph with girth at least 5, and by the induction hypothesis, it has an acyclic set $F$ with $|F| \geq [8(n-4) - 2(m-6) - 2]/7 > t(G) - 3$. Since $F$ is acyclic in $H$, the two pairs of vertices $\{v_1, u_3\}$ and $\{u_2, u_4\}$ cannot both lie in common connected components in $G[F]$, the subgraph of $G$ induced by $F$. If the pair of vertices $v_1$ and $u_3$ is not in a common component of $G[F]$, then $F \cup \{v_0, v_3, v_4\}$ is acyclic in $G$ of order greater than $t(G)$. Else if the pair of vertices $u_2$ and $u_4$ is not in a common component of $G[F]$, then $F \cup \{v_0, v_2, v_4\}$ is acyclic in $G$ of order greater than $t(G)$.

By the above statement and without loss of generality, we may assume that $d(u_1, u_3) > 2$. Now let $H := G - \{v_0, v_3, v_4\} + v_1u_3$ and draw the edge $v_1u_3$ along the path $v_1v_0v_4v_3u_3$ in a planar embedding of $G$. Then $H$ is planar with girth at least 5 and it is also connected, for otherwise $u_4v_4$ would be a bridge in $G$. By the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [8(n-3) - 2(m-5) - 2]/7 = t(G) - 2$. If $v_1 \in F$, then neither $\{u_3, v_1\}$ nor $\{u_3, v_2\}$ belong to a component of $G[F]$ thus $F \cup \{v_0, v_3\}$ is acyclic in $G$ of order at least $t(G)$. If $v_1 \notin F$, then $F \cup \{v_0, v_4\}$ is acyclic in $G$ of order at least $t(G)$.

By Claim 4.5, we can assume that $G$ is 3-regular. This implies that $G$ has no cut vertex, for otherwise this cut vertex would be incident to a bridge in $G$. Then by the Whitney theorem [17], each face in a planar embedding of $G$ is bounded by a cycle. The *degree*

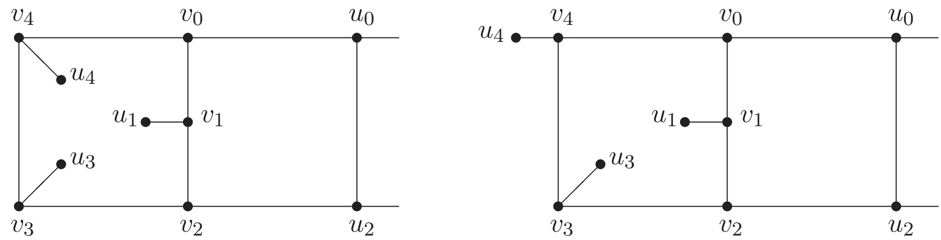

FIGURE 3. Two possible cases of embedding.

of a face is the length of its boundary (the length of its boundary cycle in case of $G$). Let $\ell$ be the number of faces of $G$. The Euler formula is $n - m + \ell = 2$. The handshake theorem gives $3n = 2m$. Thus $\ell = 2 + n/2$ and the average face degree is

$$\frac{2m}{\ell} = \frac{3n}{2 + n/2} < 6.$$

Thus $G$ is of girth 5 with some pentagon, say $v_0v_1v_2v_3v_4$. Let $u_i$ be the third neighbor of $v_i$ for $i = 0, 1, 2, 3, 4$, respectively. Then all $u_i$ for $i = 0, 1, 2, 3, 4$ are distinct from each other.

**Claim 4.6.** *If $u_0u_2 \in E(G)$, then $a(G) \geq t(G)$.*

Indeed, assume that $u_0u_2 \in E(G)$ and consider a planar embedding of $G$ such that $u_0u_2$ lies outside the pentagon $C := v_0v_1v_2v_3v_4$. The edge $u_1v_1$ lies inside either $C$ or the pentagon $u_0u_2v_2v_1v_0$. Without loss of generality, assume that it is inside $C$, then either $u_3v_3$ or $u_4v_4$ also lies inside $C$, for otherwise $u_1v_1$ would be a bridge in $G$. Without loss of generality, assume that $u_3v_3$ is also inside $C$. Now the proof splits into two cases according to the position of the edge $u_4v_4$ in the embedding, see Figure 3.

**Case 1** The edge $u_4v_4$ lies inside $C$.

In this case, $G - C$ has exactly two components, say $G_1$ and $G_2$, where $G_1$ lies inside $C$ and $G_2$ lies outside (for otherwise one of the edges $u_1v_1$, $u_3v_3$, $u_4v_4$ would be a bridge in $G$). Applying induction to both $G_1$ and $G_2$, we obtain that $G - C$ has an acyclic set $F$ with

$$\begin{aligned}|F| = a(G_1) + a(G_2) \geq t(G_1) + t(G_2) = [8(n-5)\\ -2(m-10) - 4]/7 = (8n - 2m - 24)/7.\end{aligned}$$

Then $F \cup \{v_0, v_2, v_3\}$ is acyclic in $G$ of order at least $(8n - 2m - 3)/7$. If $t(G_1)$ is not an integer, then we are done. So we may assume that it is an integer. Observe that $G - u_0 - u_2$ has exactly two components, say $H_1 \supset G_1$ and $H_2 \subset G_2$, for otherwise either $u_0$ or $u_2$ would be incident to a bridge in $G$. Note that $H_1$ has five vertices and eight edges more than $G_1$. By the induction hypothesis, $H_1$ has an acyclic set $F_1$ with $|F_1| \geq \lceil t(H_1) \rceil = \lceil t(G_1) + 24/7 \rceil = t(G_1) + 4$ and $H_2$ has an acyclic set $F_2$ with $|F_2| \geq t(H_2)$. Then $F_1 \cup F_2 \cup \{u_0\}$ is acyclic in $G$ of order

$$\begin{aligned}|F_1| + |F_2| + 1 &\geq t(G_1) + t(H_2) + 5\\ &= [8(n-7) - 2(m-13) - 4]/7 + 5\\ &= (8n - 2m + 1)/7 > t(G).\end{aligned}$$

**Case 2** The edge $u_4v_4$ lies outside $C$.

In this case, $G-C$ also has exactly two components, say $G_1$ and $G_2$, where $G_1$ lies inside $C$ and $G_2$ lies outside, for otherwise both $u_1v_1$ and $u_3v_3$ would be two bridges or $u_4v_4$ would be a bridge in $G$. As in Case 1 applying induction to both $G_1$ and $G_2$, we get that $G-C$ has an acyclic set $F$ with $|F| \geq (8n-2m-24)/7$. Then $F \cup \{v_0, v_2, v_3\}$ is acyclic in $G$ of order at least $(8n-2m-3)/7$. If $t(G_2)$ is not an integer, then we are done. So we may assume that it is an integer. Observe that $G-u_1-u_3$ consists of two disjoint subgraphs, say $H_1$ ($\subset G_1$) inside $C$ and the other $H_2$ ($\supset G_2$) containing $C$ as a subgraph. It is clear that $H_2$ is connected. Note that $H_1$ has at most two connected components, for otherwise either the vertex $u_1$ or the vertex $u_3$ would be incident to a bridge in $G$. Also note that $H_2$ has five vertices and eight edges more than $G_2$. By the induction hypothesis, $H_1$ has an acyclic set $F_1$ with $|F_1| \geq t(H_1) - 2/7$ and $H_2$ has an acyclic set $F_2$ with $|F_2| \geq \lceil t(H_2) \rceil = \lceil t(G_2) + 24/7 \rceil = t(G_2) + 4$. Then $F_1 \cup F_2 \cup \{u_1\}$ is acyclic in $G$ of order

$$\begin{aligned} |F_1| + |F_2| + 1 &\geq t(H_1) - 2/7 + t(G_2) + 5 \\ &\geq [8(n-7) - 2(m-13) - 4]/7 + 33/7 \\ &= (8n-2m-1)/7 > t(G). \end{aligned}$$

**Claim 4.7.** *If neither $u_0, u_1$ nor $u_2, u_3$ have a common neighbor in $G$, then $a(G) \geq t(G)$.*

Indeed, let $H := G - \{v_0, v_3, v_4\} + u_0v_1 + v_2u_3$ and draw the edges $u_0v_1$ and $v_2u_3$ along the paths $u_0v_0v_1$ and $v_2v_3u_3$, respectively, in a planar embedding of $G$. Then $H$ is planar with girth at least 5 by Claim 4.6 and it is also connected, for otherwise $u_4v_4$ would be a bridge in $G$. By the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [8(n-3) - 2(m-5) - 2]/7 = t(G) - 2$, then $F \cup \{v_0, v_3\}$ is acyclic in $G$ of order at least $t(G)$.

Claim 4.7 implies that every pentagon of $G$ has a vertex with three pentagons around.

**Claim 4.8.** *Let $v$ be a vertex of $G$ with $N(v) = \{u, w, x\}$ and $N(u) = \{v, u_1, u_2\}$, $N(w) = \{v, w_1, w_2\}$. If neither the path $u_1uu_2$ nor the path $w_1ww_2$ is a part of any pentagon, then $a(G) \geq t(G)$.*

Indeed, let $H := G - \{u, v, w\} + u_1u_2 + w_1w_2$ and draw the edges $u_1u_2$ and $w_1w_2$ along the paths $u_1uu_2$ and $w_1ww_2$, respectively, in a planar embedding of $G$. Then $H$ is clearly planar, and it is of girth at least 5 since the path $uvw$ is not a part of any pentagon by Claim 4.7 and the assumption that neither the path $u_1uu_2$ nor the path $w_1ww_2$ is a part of any pentagon. $H$ is also connected, for otherwise the vertex $v$ would be incident to a bridge in $G$. By the induction hypothesis, it has an acyclic set $F$ with $|F| \geq [8(n-3) - 2(m-5) - 2]/7 = t(G) - 2$. Thus $F \cup \{u, w\}$ is an acyclic set in $G$ of order at least $t(G)$.

Claim 4.8 implies that every vertex of $G$ has at least two neighbors on some pentagons. Let $f$ be a face with maximum degree in a planar embedding of $G$ and let $\partial f := v_0, v_1, \ldots, v_k$ ($k \geq 4$) be the boundary of $f$. For convenience, let $v_{k+1} := v_0$. If $k = 4$, then $f$ is of degree 5 and so are all faces in the planar embedding, which together with the Euler formula implies that $G$ is in fact a dodecahedron with $a(G) = 14 = t(G)$. So we can assume that $k \geq 5$ and thus $\partial f$ is not a pentagon. Note that $G$ is 3-regular. We denote by $u_i$ the third neighbor of $v_i$ for $i = 0, 1, \ldots, k$. We will show that all vertices $u_i$ are not on $\partial f$ and that they are distinct from each other.

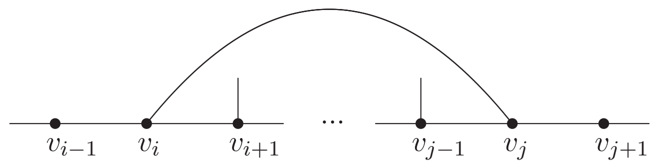


FIGURE 4. Part of the embedding for $u_i = v_j$.

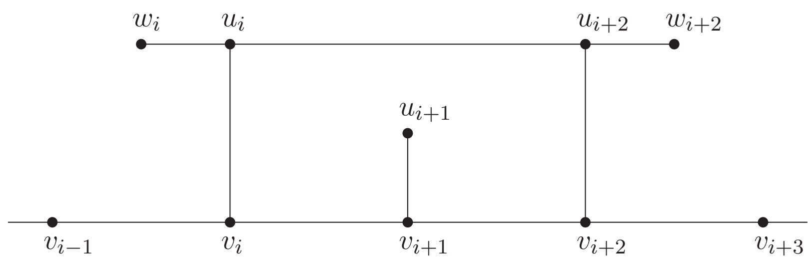


FIGURE 5. Part of the embedding for $u_iu_{i+2} \in E(G)$.

**Claim 4.9.** *If a vertex $u_i$ coincides with a vertex $v_j$ for some indices i and j, then $a(G) \geq t(G)$.*

Indeed, by the Jordan curve theorem, $G - v_i - v_j$ has exactly two components, say $G_1$ and $G_2$, see Figure 4. Applying induction to both $G_1$ and $G_2$, we get that $G_l$ has an acyclic set $F_l$ with $|F_l| \geq t(G_l)$ for $l = 1, 2$. Then $F_1 \cup F_2 \cup \{v_i\}$ is acyclic in $G$ of order

$$|F_1| + |F_2| + 1 \geq t(G_1) + t(G_2) + 1 = [8(n-2) \\ -2(m-5) - 4]/7 + 1 = (8n - 2m - 3)/7.$$

If $t(G_1)$ or $t(G_2)$ is not an integer, then we are done. So we may assume that both of them are integers and $a(G_l) = t(G_l)$ for $l = 1, 2$. Applying induction to each $G - G_l$ for $l = 1, 2$, we obtain that $G - G_l$ has an acyclic set $F'_l$ with $|F'_l| \geq \lceil t(G - G_l) \rceil = \lceil t(G_{3-l}) + 10/7 \rceil = t(G_{3-l}) + 2$. Then $v_i, v_j \in F'_1 \cap F'_2$ and $F'_1 \cup F'_2$ is acyclic in $G$ of order $|F'_1| + |F'_2| - 2 \geq t(G_1) + t(G_2) + 2 > t(G)$.

**Claim 4.10.** *If $u_i = u_j$ for some indices i and j, then $a(G) \geq t(G)$.*

This follows readily from Claim 4.8 applied to the vertex $u_i = u_j$ and its two neighbors $v_i$ and $v_j$ and the fact that $\partial f$ is not a pentagon.

**Claim 4.11.** *If $u_iu_{i+2} \in E(G)$ for some index i, then $a(G) \geq t(G)$.*

Indeed, let $w_i$ be the third neighbor of $u_i$ distinct from $u_{i+2}$ and $v_i$, and $w_{i+2}$ the third neighbor of $u_{i+2}$ distinct from $u_i$ and $v_{i+2}$, respectively. By the Jordan curve theorem, the pentagon $C' := u_iu_{i+2}v_{i+2}v_{i+1}v_i$ cuts the plane into two regions. Applying Claim 4.7 to the pentagon $C'$, we obtain that the vertex $u_{i+1}$ lies inside $C'$ and the vertices $v_{i-1}, v_{i+3}, w_i, w_{i+2}$ all lie outside $C'$, see Figure 5. This implies that $u_{i+1}v_{i+1}$ is a bridge in $G$, a contradiction.

Claims 4.9 and 4.10 confirm that all vertices $u_i$ for $i = 0, 1, \ldots, k$ are not on $\partial f$ and that they are distinct from each other. Applying Claim 4.8 to the vertex $v_3$, we know that either the path $u_2v_2v_1$ or the path $u_4v_4v_5$ is part of a pentagon. Without loss of

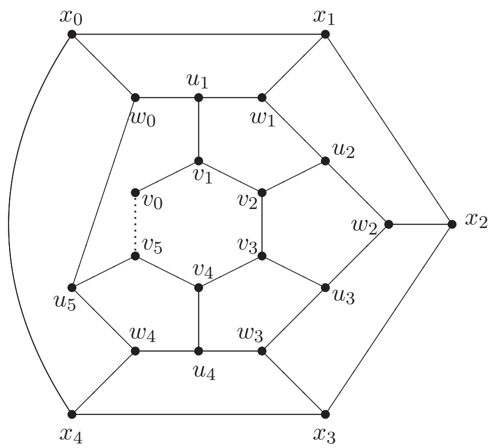


FIGURE 6. The planar embedding of $G$ for $x_0 = x_5$.

generality, we assume the former. Then Claims 4.10 and 4.11 imply that this pentagon must be $u_1v_1v_2u_2w_1$, where $w_1 \notin \partial f$ is a common neighbor of $u_1$ and $u_2$. The vertex $w_1$ is also distinct from all vertices $u_i$ for $i = 0, 1, \ldots, k$. Indeed, assume to the contrary that $u_i = w_1$ for some index $i$. Then applying Claim 4.7 to the pentagon $u_1v_1v_2u_2w_1$ with the fact that $\partial f$ is not a pentagon, we know that $u_2u_{i-1}v_{i-1}v_iu_i$ is a pentagon. The same token shows that $u_2u_{i-1}u_3v_3v_2$ is also a pentagon. Along this way sooner or later, we arrive at a pentagon with an edge $u_ju_{j+1}$ for some $j < i$, contrary to the girth constraint of $G$.

Let $w_0$ be the third neighbor of $u_1$ distinct from $v_1$ and $w_1$, and $w_2$ the third neighbor of $u_2$ distinct from $v_2$ and $w_1$, respectively. Clearly $w_0 \neq w_2$ by the girth constraint of $G$. Applying Claim 4.7 to the pentagon $u_1v_1v_2u_2w_1$, we know that either $u_0$ is a common neighbor of $v_0$ and $w_0$ or $u_3$ is a common neighbor of $v_3$ and $w_2$. Without loss of generality, we assume the latter and so $u_2v_2v_3u_3w_2$ is a pentagon. As for the vertex $w_1$, one can also show that $w_2$ is distinct from all $u_i$ and $v_i$ for $i = 0, 1, \ldots, k$. Applying Claim 4.8 to the vertex $v_2$ and its two neighbors $v_1$ and $v_3$, we know that either the path $u_1v_1v_0$ or the path $u_3v_3v_4$ is part of a pentagon. Without loss of generality, we assume the latter. Then $u_3v_3v_4u_4w_3$ is a pentagon where $w_3$ is a common neighbor of $u_3$ and $u_4$. As the vertex $w_2$, the vertex $w_3$ is also distinct from $w_1, w_2$ and all vertices $u_i, v_i$ for $i = 0, 1, \ldots, k$. We also have $w_3 \neq w_0$, for otherwise Claim 4.7 applied to the pentagon $u_3v_3v_4u_4w_3$ with the fact that $\partial f$ is not a pentagon would imply that $w_1$ is a common neighbor of $u_1$ and $w_2$ and then $u_2w_1w_2$ is a triangle that is impossible. Let $w_4$ be the third neighbor of $u_4$ distinct from $v_4$ and $w_3$, and let $x_i$ be the third neighbor of $w_i$ for $i = 0, 1, 2, 3, 4$. Recall that $\partial f$ is not a pentagon. Applying Claim 4.7 to the three pentagons $u_iv_iv_{i+1}u_{i+1}w_i$ for $i = 1, 2, 3$ one by one, we obtain four more pentagons, namely $u_jw_{j-1}x_{j-1}x_jw_j$ for $j = 1, 2, 3, 4$, where all vertices $u_i, v_i$ for $i = 0, 1, \ldots, k$ and $w_j, x_j$ for $j = 0, 1, 2, 3, 4$ are distinct from each other, see Figure 6.

**Claim 4.12.** *If $u_5w_4 \in E(G)$, then $a(G) \geq t(G)$.*

Indeed, assume that $u_5w_4 \in E(G)$. Then $u_4v_4v_5u_5w_4$ is a pentagon. Applying Claim 4.7 to this pentagon with the fact that $\partial f$ is not a pentagon, we obtain that the path $u_5w_4x_4$ must be part of a pentagon, say $u_5w_4x_4x_5w_5$. Now we consider two cases according to the vertex $x_0$ coinciding with $x_5$ or not.

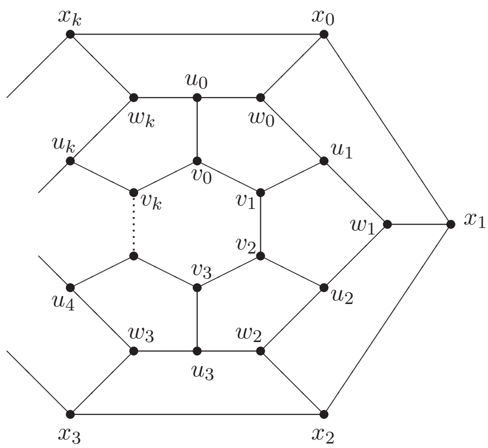


FIGURE 7. The planar embedding of $G$ for $x_0 \neq x_5$.

**Case 1** $x_0 = x_5$.

In this case, we also have $w_0 = w_5$, see Figure 6. Let $U := \{u_{i+1}, v_{i+1}, w_i, x_i \mid i = 0, 1, 2, 3, 4\}$. Then clearly $G[U] + v_1v_5$ is a dodecahedron and $G - U$ is connected and planar with girth at least 5. By the induction hypothesis, $G - U$ has an acyclic set $F$ with $|F| \geq [8(n-20) - 2(m-31) - 2]/7 = t(G) - 14$. Taking an acyclic set $F'$ of $G[U]$ with $|F'| = 14$ and avoiding one of the two vertices $v_1$ and $v_5$, we get that $F \cup F'$ is acyclic in $G$ of order at least $t(G)$.

**Case 2** $x_0 \neq x_5$.

In this case, the path $x_3x_4x_5$ is not a part of any pentagon. Then applying Claim 4.7 to the pentagon $u_5w_4x_4x_5w_5$, we have $u_6w_5 \in E(G)$. Along this process, we obtain that every pair of vertices $u_i$ and $u_{i+1}$ has a common neighbor $w_i$, where all $w_i$ are also distinct from each other and from $u_j$ and $v_j$ for $j = 0, 1, \ldots, k$ and $x_0x_1 \ldots x_k$ is a cycle in $G$, and thus the graph $G$ is actually defined on the set of vertices $\{u_i, v_i, w_i, x_i \mid i = 0, 1, \ldots, k\}$, see Figure 7.

It is clear that the graph $G$ is of order $4k + 4$. If $k \equiv 0$ or $1 \bmod 4$, then let $F := V(G) \setminus \{x_0, v_0, u_1, w_2, x_3, v_4, u_5, \ldots\}$. If $k \equiv 2$ or $3 \bmod 4$, then let $F := V(G) \setminus \{v_0, u_1, w_2, x_3, v_4, u_5, \ldots\}$. In each case, $F$ is acyclic in $G$ with

$$|F| \geq 3k + 2 \geq (20k + 18)/7 = t(G) \text{ for } k \geq 4. \qquad (2)$$

Now we can assume that $u_5w_4 \notin E(G)$. By Claim 4.11, we have $u_6 \neq w_4$. It follows that the path $v_4u_4w_4$ is not a part of any pentagon. Then applying Claim 4.7 to the pentagon $u_4w_3x_3x_4w_4$, we have $x_0x_4 \in E(G)$, see Figure 8. Applying Claim 4.8 to the vertex $u_1$ and its two neighbors $v_1$ and $w_0$ and noting that $\partial f$ is not a pentagon, we have that $w_0$ and $w_4$ have a common neighbor, say $y$. Let $z$ be the third neighbor of $y$ distinct from $w_0$ and $w_4$. Then $z \neq v_5$, for otherwise $y = u_5$ and $u_5w_4 = yw_4 \in E(G)$. By symmetry, we also have $z \neq v_0$. Since $G$ is of girth 5, one of the three edges $v_0v_5$, $zv_0$, and $zv_5$ is not in $E(G)$. Without loss of generality, we assume that $v_0v_5 \notin E(G)$. Let $X := \{w_0, x_0, y\} \cup \{u_i, v_i, w_i, x_i \mid i = 1, 2, 3, 4\}$. Then $G[X]$ is a dodecahedron missing a vertex. Now let $H := G - X + v_0vv'v_5$ and draw the additional path $v_0vv'v_5$ inside the

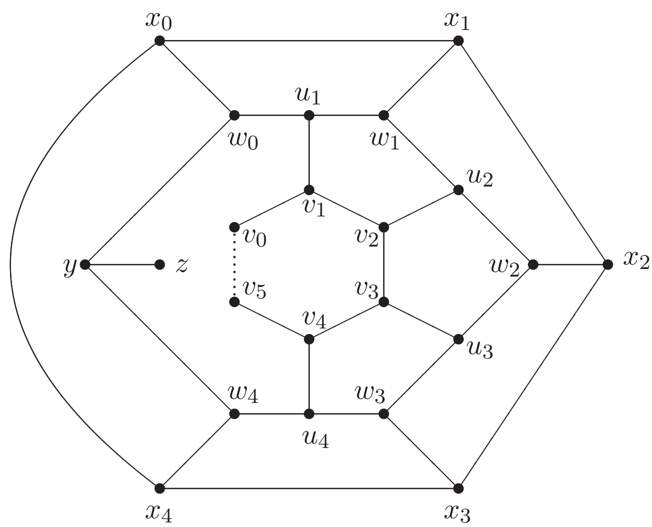


FIGURE 8. The planar embedding of $G$ for $u_5w_4 \notin E(G)$.

face $f$. It is clear that $H$ is planar with girth at least 5 and moreover it is connected, for otherwise the edge $yz$ would be a bridge in $G$. By the induction hypothesis, $H$ has an acyclic set $F$ with $|F| \geq [8(n-17) - 2(m-27) - 2]/7 \geq t(G) - 82/7$. The set $F$ can be chosen so that $v, v' \in F$. Indeed, if $v \notin F$, then adding $v$ to $F$ creates a cycle and deleting any vertex distinct from $v$ and $v'$ in the cycle gives another acyclic set of the same order as $F$. Taking an acyclic set $F'$ in $G[X]$ with $|F'| = 14$ and avoiding the vertex $y$, we get that $F \cup F' \setminus \{v, v'\}$ is acyclic in $G$ of order $|F| + |F'| - 2 \geq t(G) + 2/7 > t(G)$. This completes the proof of the first part.

## B. The Second Part: The Equality

Let $G$ be a connected planar graph of order $n$ with girth at least 5. We need to check the proof of the first part and show that $a(G) = t(G)$ if and only if $G \in \mathcal{F}$. Recall that Claims 2.1 and 2.2 still hold in this context. So the following result is immediate.

**Claim 4.13.** *If $G$ is bridgeless with $a(G) = t(G)$, then every vertex of $G$ is of degree 2, 3, or 4.*

The proofs of Claims 4.1–4.3 produce immediately the following results.

**Claim 4.14.** *If $G$ is bridgeless with $a(G) = t(G)$ and $v$ is a vertex of degree 4 in $G$, then $G - v$ is connected and $a(G - v) = t(G - v)$.*

**Claim 4.15.** *If $G$ is a cycle with $a(G) = t(G)$, then $G$ is a pentagon.*

**Claim 4.16.** *Let $v \in V(G)$, $N(v) = \{u, w\}$ with $d(u) = 2, d(w) = 3$ and let $H := G - \{u, v, w\}$. If $a(G) = t(G)$, then $H$ is connected with $a(H) = t(H)$.*

**Claim 4.17.** *Let $v \in V(G)$ with $d(v) = 3$ and $\{u, w\} \subset N(v)$ with $d(u) = d(w) = 2$, and let $H := G - \{u, v, w\}$. If $a(G) = t(G)$, then $H$ is connected with $a(H) = t(H)$.*

As in the proof of Claim 4.5, let $v_0v_1v_2v_3v_4$ be a pentagon in $G$, where $d(v_0) = 2$ and $d(v_i) = 3$ for $i = 1, 2, 3, 4$. Let $u_i$ be the third neighbor of $v_i$, respectively, for $i = 1, 2, 3, 4$. If $a(G) = t(G)$, then either $d(u_1, u_3) > 2$ or $d(u_2, u_4) > 2$. Without loss of generality, assume that $d(u_1, u_3) > 2$, then the following result is clear from the proof of Claim 4.5.

**Claim 4.18.** *Let $v_0, v_i$, and $u_i$ be as above for $i = 1, 2, 3, 4$ and let $H := G - \{v_0, v_3, v_4\} + v_1u_3$. If $G$ is bridgeless with $a(G) = t(G)$, then $H$ is connected and planar with girth at least 5 and $a(H) = t(H)$.*

Claim 4.14 concerns graphs with a vertex of degree 4, and Claims 4.15–4.17 concern graphs with some vertex of degree 2. The following claim is for 3-regular graphs and it follows from Claims 4.6–4.12.

**Claim 4.19.** *If $G$ is both bridgeless and 3-regular with $a(G) = t(G)$, then either $G$ is a dodecahedron, or deleting its three vertices and inserting two edges as in the proofs of Claims 4.7 and 4.8 result in a connected planar graph $H$ with girth at least 5 satisfying $a(H) = t(H)$ and all its other vertices are of degree 3 except only one vertex of degree 2, or deleting a dodecahedron missing an edge from $G$ results in a connected planar graph $G'$ with girth at least 5 satisfying $a(G') = t(G')$ and all its other vertices are of degree 3 except exactly two vertices of degree 2 sharing a common face.*

In Claim 4.19, the statement on $H_1$ comes from Claims 4.7 and 4.8, and the statement on $H_2$ comes from Claim 4.12. In order to prove Claim 4.19, it suffices to prove the strict inequalities in Claims 4.6 and 4.9 under the condition of $G$ being bridgeless and 3-regular, since the equality holds in equation (2) if and only if $k = 4$, and the dodecahedron satisfies the equality. Checking the proof of Claim 4.6, it is easy to see that the only possible equalities occur at Case $i$ when $t(G_i)$ is not an integer, but $t(G_i) + 1/7$ is, for $i = 1, 2$. Checking the proof of Claim 4.9, it is also easy to see that the only possible equalities occur at the case of existing some $i \in \{1, 2\}$ such that both $t(G_i)$ and $t(G_{3-i}) + 1/7$ are integers. In all these cases, we can show that the strict inequalities hold true. Indeed, assume that $t(G_1) + 1/7$ is an integer in Case 1 of Claim 4.6. We argue as if $t(G_1)$ were an integer in the proof of Claim 4.6. The subgraph $G - u_0 - u_2$ has exactly two components, say $H_1 \supset G_1$ and $H_2 \subset G_2$, for otherwise either $u_0$ or $u_2$ would be incident to a bridge in $G$. Note that $H_1$ has five vertices and eight edges more than $G_1$. By the induction hypothesis, $H_1$ has an acyclic set $F_1$ with $|F_1| \geq \lceil t(H_1) \rceil = \lceil t(G_1) + 24/7 \rceil = t(G_1) + 29/7$ and $H_2$ has an acyclic set $F_2$ with $|F_2| \geq t(H_2)$. Then $F_1 \cup F_2 \cup \{u_0\}$ is acyclic in $G$ of order

$$\begin{aligned} |F_1| + |F_2| + 1 &\geq t(G_1) + t(H_2) + 36/7 \\ &= [8(n-7) - 2(m-13) - 4 + 36]/7 \\ &= (8n - 2m + 2)/7 > t(G). \end{aligned}$$

The same token shows the strict inequality in Case 2 of Claim 4.6 as well. Now consider Claim 4.9 and without loss of generality assume that both $t(G_1)$ and $t(G_2) + 1/7$ are integers. Applying induction to each $G - G_i$ for $i = 1, 2$, we obtain that $G - G_i$ has an acyclic set $F_i$ with

$$\begin{aligned} |F_1| &\geq \lceil t(G - G_2) \rceil = \lceil t(G_1) + 10/7 \rceil = t(G_1) + 2, \\ |F_2| &\geq \lceil t(G - G_1) \rceil = \lceil t(G_2) + 10/7 \rceil = t(G_2) + 15/7. \end{aligned}$$

Then $v_1, v_2 \in F_1 \cap F_2$ and $F_1 \cup F_2$ is acyclic in $G$ of order

$$|F_1| + |F_2| - 2 \geq t(G_1) + t(G_2) + 15/7 = (8n - 2m + 5)/7 > t(G).$$

This completes the proof of Claim 4.19.

It is easy to verify that $a(D_{3k+2}) = t(D_{3k+2}) = 2k + 2$ for $k = 1, 2, 3, 4, 6$ and thus $a(G) = t(G)$ if $G \in \mathcal{F}$ by Claim 2.1. In the following, we will use induction on the order $n$ of $G$ to prove the converse, which will complete the proof of this part and Theorem 3.

**Claim 4.20.** *If $a(G) = t(G)$, then $G \in \mathcal{F}$.*

Indeed, let $G$ be such a graph of order $n$. If $n < 5$, then $G$ must be a tree with $a(G) = n > 6n/7 = t(G)$ by the girth constraint of $G$. If $n = 5$, then it is clear that $G$ is a pentagon. Now we claim that $n \neq 6$. Indeed, suppose to the contrary that $G$ is of order $n = 6$ with $a(G) = t(G)$. Since the pentagon is the only graph of order at most 5 in $\mathcal{F}$, Claim 2.1 implies that $G$ must be bridgeless. The graph $G$ has no vertex of degree 4, for otherwise deleting such a vertex would result in a pentagon by Claim 4.14 and the induction hypothesis, which implies that $G$ has a triangle contradicting the girth constraint of $G$. Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3, and Claim 4.15 implies that $G$ is not a cycle. Thus if $G$ has a vertex of degree 2, then one of the three Claims 4.16–4.18 must occur; in each of them, however, the concerned subgraph $H$ is of order 3 with $a(H) = t(H)$, which is impossible. Hence, $G$ must be 3-regular, which is contrary to Claim 4.19. This completes the proof of the claim that $n \neq 6$.

By the same token, one can also prove that $n \neq 7$. Now consider $n = 8$. Analogous to the above, we have that $G$ must be bridgeless and every vertex of it has degree either 2 or 3. Claims 4.15 and 4.19 imply that $G$ can be neither 2- nor 3-regular. Then one of the three Claims 4.16–4.18 must occur. Claim 4.16 cannot occur to the graph $G$, for otherwise the subgraph $H$ of $G$ would be a pentagon by the induction hypothesis and thus the vertex $w$ of degree 3 would have two neighbors in $H$ and it together with these two neighbors would lie on a cycle of length at most 4 in $G$, contradicting the girth constraint of $G$. Since $d_H(v_1) = 3$ in Claim 4.18, this claim cannot occur to $G$ as well. Thus the only possible case occurs when Claim 4.17 applies, and it is easy to verify that in this case $G$ is isomorphic to $D_8$. This proves Claim 4.20 for $n \leq 8$.

Analogous argument as for $n \neq 6$ shows that $n \neq 9$. Since $\mathcal{F}$ contains no graph of order 7, it is easy to see that the graph $G$ of order 10 is unique and consists of two distinct pentagons linked by an edge. Now consider $n = 11$. Then Claim 2.1 implies that $G$ must be bridgeless since $\mathcal{F}$ contains no graph of order 6. The graph $G$ has no vertex of degree 4, for otherwise deleting such a vertex would result in a copy of the unique graph of order 10 in $\mathcal{F}$ by Claim 4.14 and the induction hypothesis, which together with the pigeonhole principle implies that this deleted vertex has at least two neighbors on a pentagon and thus it lies on a cycle of length at most 4, contradicting the girth constraint of $G$. Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3, and Claim 4.15 implies that $G$ is not a cycle. Since the graph $D_8$ has four vertices of degree 2, Claim 4.19 implies that $G$ cannot be 3-regular. So we may assume that $G$ has a vertex of degree 2. Then one of Claims 4.16–4.18 must occur and the concerned subgraph $H$ in each of them is isomorphic to $D_8$ by the induction hypothesis. Note that $D_8$ is 3-connected and thus has a unique planar embedding up to isomorphism by the Whitney theorem [18], in which the boundary of each face is a pentagon with two vertices of degree 2 and three vertices of degree 3. Claim 4.16 cannot occur to $G$, for otherwise the vertex $w$ of degree 3 would have two neighbors on a pentagon in $H$ and thus it together with these two neighbors would lie on a cycle of length at most 4, contradicting the girth constraint of $G$. Nor can Claim 4.17, for otherwise the three vertices $u$, $v$, and $w$ would send three edges to a pentagon in $H$ and thus this pentagon would have a vertex of degree at least 4 in $G$, a contradiction. Thus the only possible case occurs when Claim 4.18 applies, and by symmetry, it is easy to verify that in this case $G$ is isomorphic to $D_{11}$.

Analogous argument as for $n \neq 6$ also shows that $n \neq 12$. Now consider $n = 13$. Then it suffices to prove that $G$ has a bridge. Suppose to the contrary that $G$ is bridgeless. The graph $G$ has no vertex of degree 4, for otherwise deleting such a vertex would result in a graph of order 12 in $\mathcal{F}$ by Claim 4.14 and the induction hypothesis, which is impossible. Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3, and Claim 4.15 implies that $G$ is not a cycle. Since the unique graph of order 10 in $\mathcal{F}$ has eight vertices of degree 2, Claim 4.19 implies that $G$ cannot be 3-regular. So we may assume that $G$ has a vertex of degree 2. Then one of Claims 4.16–4.18 must occur and the concerned subgraph $H$ of $G$ is isomorphic to the unique graph of order 10 in $\mathcal{F}$ by the induction hypothesis. Since $G$ is bridgeless, the path $uvw$ must be adjacent to both pentagons of $H$ in both Claims 4.16 and 4.17. Note that given a pair of vertices on a pentagon there is a path of order 4 avoiding one of them. Then Claims 4.16 and 4.17 cannot occur to $G$, for otherwise $G$ would have an acyclic set $F$ consisting of $u$, $v$, $w$ and eight proper vertices of $H$ with $|F| = 11 > 10 = t(G)$, which contradicts the assumption of $G$. Also note that in Claim 4.18 the subgraph $H$ contains exactly two vertices with degree one less in $H$ than in $G$: one is the vertex $v_2$ and the other is the vertex $u_4$ in $H$. So Claim 4.18 can neither occur to $G$, for otherwise $H$ would still keep six vertices of degree 2 in $G$ and then Claim 4.16 would occur to $G$, which is impossible as just proven. This contradiction shows that $G$ indeed has a bridge and thus $G \in \mathcal{F}$ by Claim 2.1 and the induction hypothesis.

Now consider $n = 14$. Then Claim 2.1 implies that $G$ must be bridgeless since the only possible order of graphs in $\mathcal{F}$ is in the set $\{5, 8, 10, 11, 13\}$ for $n < 14$. The graph $G$ has no vertex of degree 4. Indeed, suppose to the contrary that $v$ is such a vertex of degree 4, then $G - v$ is a graph of order 13 in $\mathcal{F}$ by Claim 4.14 and the induction hypothesis. Thus $G - v$ consists of a pentagon and a $D_8$ linked by an edge, which together with the pigeonhole principle implies that $v$ has at least two neighbors on the pentagon or $D_8$ in $G$ and thus it lies on a cycle of length at most 4, contradicting the girth constraint of $G$. Then Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3, and Claim 4.15 implies that $G$ is not a cycle. Since $D_{11}$ is the only graph of order 11 in $\mathcal{F}$ and it has three vertices of degree 2, Claim 4.19 implies that $G$ cannot be 3-regular. So we may assume that $G$ has a vertex of degree 2. Then one of Claims 4.16–4.18 must occur and the concerned subgraph $H$ in each of them is isomorphic to $D_{11}$ by the induction hypothesis. Note that $D_{11}$ is 3-connected and thus has a unique planar embedding up to isomorphism, in which the boundary of each face is a pentagon with one vertex of degree 2 and four vertices of degree 3. Claim 4.16 cannot occur to $G$, for otherwise the vertex $w$ of degree 3 would have two neighbors on a pentagon in $H$ and thus it together with these two neighbors would lie on a cycle of length at most 4, contradicting the girth constraint of $G$. Nor can Claim 4.17, for otherwise the three vertices $u$, $v$, and $w$ would send three edges to a pentagon in $H$ and thus this pentagon would have a vertex of degree at least 4 in $G$, a contradiction. Thus the only possible case occurs when Claim 4.18 applies, and by symmetry, it is easy to verify that in this case $G$ is isomorphic to $D_{14}$.

Now consider $n = 15$. Then it suffices to prove that $G$ has a bridge. Suppose to the contrary that $G$ is bridgeless. Note that $D_{14}$ is 3-connected and has a unique planar embedding up to isomorphism in which the boundary of each face is a pentagon. The graph $G$ has no vertex of degree 4, for otherwise deleting such a vertex would result in a $D_{14} \in \mathcal{F}$ by Claim 4.14 and the induction hypothesis, which implies that this deleted vertex is adjacent to a pentagon and thus lies on a triangle, contradicting the girth constraint of $G$. Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3, and Claim 4.15 implies that $G$ is not a cycle. If $G$ has a vertex of degree 2, then one of Claims 4.16–4.18 must occur; and the concerned subgraph $H$ of $G$ is isomorphic to a graph of order 12 in $\mathcal{F}$ by the induction hypothesis, which is impossible. So the graph $G$ must be 3-regular, but this also contradicts Claim 4.19 since there is no graph of order 12 in $\mathcal{F}$. This contradiction shows that $G$ indeed has a bridge and thus $G \in \mathcal{F}$ by Claim 2.1 and the induction hypothesis.

Now consider $n = 16$. Then it also suffices to prove that $G$ has a bridge. Suppose to the contrary that $G$ is bridgeless. Assume $G$ has a vertex $v$ of degree 4. Then deleting $v$ from $G$ results in a subgraph $G'$ of order 15 in $\mathcal{F}$ by Claim 4.14 and the induction hypothesis. Note that $G' \in \mathcal{F}$ consists of three pentagons linked by two edges, say $e_1$ and $e_2$. The vertex $v$ is adjacent to each of the three pentagons of $G'$, for otherwise either $e_1$ or $e_2$ would be a bridge of $G$ contradicting the assumption of $G$. By the pigeonhole principle, the vertex $v$ has at least two neighbors on some pentagon, and thus $v$ and its two neighbors on such a pentagon would share a cycle of length at most 4, contradicting the girth constraint of $G$. This contradiction with Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3, and Claim 4.15 implies that $G$ is not a cycle. Since each graph of order 13 in $\mathcal{F}$ consists of a pentagon and a $D_8$ linked by an edge, it has more than two vertices of degree 2 and Claim 4.19 implies that $G$ cannot be 3-regular. So we may assume that $G$ has a vertex of degree 2. Then one of Claims 4.16–4.18 must occur and the concerned subgraph $H$ of $G$ is isomorphic to a graph of order 13 in $\mathcal{F}$ by the induction hypothesis. Thus $H$ consists of a pentagon and a copy of $D_8$ linked by an edge. Note that given a pair of vertices in a pentagon there is a path of order 4 avoiding one of them, and also given a pair of vertices in $D_8$ there is an induced tree of order 6 avoiding one of them. Then Claims 4.16 and 4.17 cannot occur to $G$, for otherwise $G$ would have an acyclic set $F$ consisting of $u$, $v$, $w$ and four proper vertices in the pentagon and six proper vertices in $D_8$ with $|F| = 13 > 12 = t(G)$, which contradicts the assumption of $G$. Also recall that in Claim 4.18 the subgraph $H$ contains exactly two vertices with degree one less in $H$ than in $G$. Claim 4.18 can neither occur to $G$, for otherwise $H$ would still keep at least five vertices of degree 2 in $G$ and then Claim 4.16 would occur to $G$, which is impossible as just proven. This contradiction shows that $G$ indeed has a bridge and thus $G \in \mathcal{F}$ by Claim 2.1 and the induction hypothesis.

We now show that $n \neq 17$. Suppose to the contrary that $n = 17$. Then Claim 2.1 implies that $G$ must be bridgeless since the only possible order of graphs in $\mathcal{F}$ is in the set $\{5, 8, 10, 11, 13, 14, 15, 16\}$ for $n < 17$. We claim that $G$ has no vertex of degree 4. Indeed, suppose to the contrary that $v$ is such a vertex of degree 4 in $G$. Then $G - v$ consists of a pentagon and a $D_{11}$ linked by an edge by Claim 4.14 and the induction hypothesis, which implies that the vertex $v$ has at least two neighbors either on the pentagon or on $D_{11}$ since $G$ is bridgeless. Recall that $D_{11}$ has a unique planar embedding up to isomorphism in which the boundary of each face is a pentagon. So the vertex $v$ lies on a cycle of length at most 4 in $G$, contradicting the girth constraint of $G$. This together with Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3. Claim 4.15 implies that $G$ is not a cycle, and Claim 4.19 implies that $G$ cannot be 3-regular, since $D_{14}$ is the only graph of order 14 in $\mathcal{F}$ and it has exactly two vertices of degree 2. Thus $G$ has a vertex of degree 2. Then one of Claims 4.16–4.18 must occur and in each of them, the concerned subgraph $H$ is of order 14 with $a(H) = t(H)$. By the induction hypothesis, $H$ is isomorphic to $D_{14}$, and thus $H$ has exactly two vertices of degree 2, which are of distance 5 in $H$. Neither Claim 4.16 nor Claim 4.17 can occur to $G$, for otherwise the three vertices $u$, $v$, and $w$ of $G$ would send three edges to a pentagon in $H$ and thus this pentagon would have a vertex of degree at least 4 in $G$, a contradiction; nor can Claim 4.18 occur, by the fact that the two vertices $u_4$ and $v_2$ are of degree 2 and of distance 3 in $H$. This contradiction completes the proof of $n \neq 17$.

A proof analogous to that for $n = 13, 16$ easily shows that Claim 4.20 also holds for $n = 18, 19$. The detail is omitted. Now consider $n = 20$. Then it suffices to show that if $G$ is bridgeless, then it is a dodecahedron. Indeed, assume that $G$ is bridgeless. Note that a graph of order 19 in $\mathcal{F}$ consists either of a pentagon and a $D_{14}$ or of a $D_8$ and a $D_{11}$ linked by an edge. The graph $G$ has no vertex of degree 4, for otherwise such a vertex would have at least two neighbors on a pentagon and thus would lie on a cycle of length at most 4, contradicting the girth constraint of $G$. Then Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3 and Claim 4.15 implies that $G$ is not a cycle. The graph $G$ also has no vertex of degree 2, for otherwise one of Claims 4.16–4.18 would occur and the concerned subgraph $H$ would be of order 17 in $\mathcal{F}$, which is impossible. It follows that $G$ must be 3-regular and indeed a dodecahedron by Claim 4.19.

Up to now, we have proved Claim 4.20 for $n \leq 20$. Assuming that Claim 4.20 holds for $n \geq 20$, we will show it for $n + 1$, which will complete the induction and the whole proof as well. It suffices to prove that $G$ has a bridge. Suppose to the contrary that $G$ is bridgeless. We use that every $D_{3k+2}$ for $k = 1, 2, 3, 4, 6$ has a unique planar embedding in which the boundary of each face is a pentagon. It is easy to verify that for any pair of vertices in $D_{3k+2}$ for $k = 1, 2, 3, 4, 6$, there is an acyclic set of order $2k + 2$ avoiding one of them. We claim that $G$ has no vertex of degree 4. Indeed, suppose to the contrary that $v$ is such a vertex of degree 4. Then $G - v$ is a graph in $\mathcal{F}$ by Claim 4.14 and the induction hypothesis. Thus $a(G - v) = t(G - v) = t(G)$. If $v$ has at least two neighbors on a $D_{3k+2}$ for some $k$, then $v$ must lie on a cycle of length at most 4, contradicting the girth constraint of $G$; else the vertex $v$ together with a proper acyclic set of $G - v$ forms an acyclic set in $G$ of order $a(G - v) + 1 > t(G)$, contradicting the assumption $a(G) = t(G)$. This contradiction proves that $G$ indeed has no vertex of degree 4. Then Claim 4.13 implies that every vertex of $G$ is of degree 2 or 3, and Claim 4.15 implies that $G$ is not a cycle. It is clear that if a graph is in $\mathcal{F}$ with each vertex of degree 2 or 3, then it can neither have only one vertex of degree 2 nor have exactly two vertices of degree 2, which share a common face in a planar embedding. Then Claim 4.19 implies that $G$ cannot be 3-regular. So we may assume that $G$ has a vertex of degree 2. Then one of Claims 4.16–4.18 must occur and the concerned subgraph $H$ of $G$ is isomorphic to a graph in $\mathcal{F}$ by the induction hypothesis and thus $a(H) = t(H) = t(G) - 2$. In both Claims 4.16 and 4.17, the three vertices $u$, $v$, and $w$ send exactly three edges to $H$ in $G$. Since $G$ is bridgeless, the three vertices $u$, $v$, and $w$ must send at most two edges to every possible $D_{3k+2}$ for $k = 1, 2, 3, 4, 6$ in $H$ in both Claims 4.16 and 4.17. Then Claims 4.16 and 4.17 cannot occur to $G$, for otherwise $G$ would have an acyclic set $F$ consisting of $u$, $v$, $w$ and a proper acyclic set of $H$ with $|F| = a(H) + 3 > t(G)$, which contradicts the assumption of $G$. Recall that in Claim 4.18 the subgraph $H$ contains exactly two vertices with degree one less in $H$ than in $G$. Claim 4.18 can neither occur to $G$, for otherwise $H$ would still keep enough vertices of degree 2 in $G$ and then either Claim 4.16 or Claim 4.17 would occur to $G$, which is impossible as just proven. This contradiction shows that $G$ indeed has a bridge and thus $G \in \mathcal{F}$ by

Claim 2.1 and the induction hypothesis. This completes the induction and the proof of Theorem 3.

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