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Hankel determinants, Padé approximations, and irrationality exponents

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Abstract. The irrationality exponent of an irrational number ξ, which measures the approximation rate of ξ by rationals, is in general extremely difficult to compute explicitly, unless we know the continued fraction expansion of ξ. Results obtained so far are rather fragmentary and often treated case by case. In this work, we shall unify all the known results on the subject by showing that the irrationality exponents of large classes of automatic numbers and Mahler numbers (which are transcendental) are exactly equal to 2. Our classes contain the Thue–Morse–Mahler numbers, the sum of the reciprocals of the Fermat numbers, the regular paperfolding numbers, which have been previously considered respectively by Bugeaud, Coons, and Guo, Wu and Wen, but also new classes such as the Stern numbers and so on. Among other ingredients, our proofs use results on Hankel determinants obtained recently by Han.

1. Introduction

Let ξ be an irrational real number. The irrationality exponent $\mu(\xi)$ of ξ is the supremum of the real numbers μ such that the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^\mu}$$

has infinitely many solutions in rational numbers p/q. Hence, we have

$$\mu(\xi) = 1 - \liminf_{q\to\infty} \frac{\log \|q\xi\|}{\log q}, \tag{1.1}$$

where $\|x\|$ denotes the distance between the real number x and its nearest integer. An easy covering argument shows that $\mu(\xi)$ is at most equal to 2 for almost all real numbers ξ (with respect to the Lebesgue measure). It follows from the theory of continued fractions that the irrationality

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exponent of an irrational real number is always greater than or equal to 2. More precisely, let $[a_0; a_1, a_2, \ldots]$ denote the continued fraction expansion of an irrational real number ξ and $(p_n/q_n)_{n\geq 1}$ denote the sequence of its convergents (for more about continued fractions, see for example [La95]). Then, we have

$$\mu(\xi) = 2 + \limsup_{n\to\infty} \frac{\log a_{n+1}}{\log q_n}. \tag{1.2}$$

Furthermore, Roth's theorem [Ro55] asserts that the irrationality exponent of every algebraic irrational number is equal to 2. However, it is in general a very difficult problem to determine the irrationality exponent of a given transcendental real number ξ. Apart from some numbers involving the exponential function or the Bessel function (see the end of Section 1 in [Ad10]) and apart from more or less *ad hoc* constructions (see below), it seems to us that the only known method to determine the irrationality exponent of (certain) transcendental numbers is the method developed in [Bu11]. Up to now, this method has been applied to a handful of irrational numbers [Bu11, Co13, GWW14, WW14]. The main purpose of the present work is to considerably extend these results and to exhibit infinite families of transcendental numbers with irrationality exponent equal to 2.

Let us now focus on a special class of real numbers.

A real number ξ is automatic if there exist two integers $k, b \geq 2$ such that the b-ary expansion of ξ is k-automatic. This means that, if we write $\xi = \sum_{n\geq 0} \frac{a(n)}{b^n}$ with $a(n) \in \mathbb{Z}$ $(n \geq 0)$ and $0 \leq a(n) < b$ for $n \geq 1$, then the set of subsequences

$$\left\{ \big(a(k^r n + s)\big)_{n\geq 0} \mid r \geq 0,\ 0 \leq s < k^r \right\}$$

is finite (For more on automatic sequences, see for example Allouche [Al87] and also the book of Allouche and Shallit [AS03]). For example, the case of Kmošek-Shallit numbers $f_{KS}\big(\frac{1}{b}\big) = \sum_{n\geq 0} \frac{1}{b^{2^n}}$ (studied independently by Kmošek [Km79] and Shallit [Sh79] in 1979 to give "natural" examples of real numbers with bounded partial quotients) corresponds to the characteristic function of the set $\{2^n \mid n \geq 0\}$, which is 2-automatic but not ultimately periodic. These numbers are transcendental (see [Ke16], [Ma29], and also [LVdP77]). It was long conjectured and finally has been proved by Adamczewski and Bugeaud in [AB07] that an automatic number is either rational or transcendental; see [BBC15] and [Ph15] for two recent alternative proofs. We have thus a large family of "simple" transcendental numbers, and one can then ask what are their irrationality exponents.

In 2006, Adamczewski and Cassaigne showed in [AC06] that an automatic number cannot be a Liouville number (recall that, by definition, a Liouville number is a real number whose irrationality exponent is infinite). Subsequently, Adamczewski and Rivoal [AR09] obtained in 2009 upper bounds for the irrationality exponents of some famous automatic numbers constructed from the Thue–Morse, Rudin–Shapiro, paperfolding and Baum–Sweet sequences. In 2008, Bugeaud [Bu08] constructed explicitly elements of the classical middle third Cantor set with any prescribed irrationality exponent (an analog for the function field case has been obtained very recently by Pedersen [Pe14]), and proved that there exist automatic real numbers with any prescribed rational irrationality exponent. But what is the exact value of the irrationality exponent of a given automatic irrational number (for example, the famous Thue–Morse–Mahler numbers)? This question was addressed in [BKS11], and the results obtained on this subject are rather fragmentary even until now, often treated case by case, and can be summarized as follows.

The history begun in 2011 with the paper [Bu11], in which Bugeaud developed a method to show that the irrationality exponents of the Thue–Morse–Mahler numbers are equal to 2. Recall that the famous Thue–Morse sequence $(t_n)_{n\geq 0}$ on $\{0,1\}$ is defined recursively by $t_0 = 0$, $t_{2n} = t_n$ and $t_{2n+1} = 1-t_n$ for all integers $n \geq 0$, and that the Thue–Morse–Mahler numbers take the form

$$f_{TMM}\Big(\frac{1}{b}\Big) = \sum_{n\geq 0} \frac{t_n}{b^n},$$

where $b \geq 2$ is an integer. Recall also that the Thue–Morse sequence is 2-automatic but not ultimately periodic, and Mahler [Ma29] already showed in 1929 that $f_{TMM}(1/2)$ is transcendental (see also Dekking [De77] for another proof).

In 2013, Coons considered in [Co13] the following two power series

$$\mathcal{F}(z) = \sum_{n\geq 0} \frac{z^{2^n}}{1+z^{2^n}}, \quad \mathcal{G}(z) = \sum_{n\geq 0} \frac{z^{2^n}}{1-z^{2^n}}, \tag{1.3}$$

and showed that for all integers $b \geq 2$, we have $\mu(\mathcal{F}(1/b)) = \mu(\mathcal{G}(1/b)) = 2$. Note here that the special value $\mathcal{F}(1/2)$ is the sum of the reciprocals of the Fermat numbers $F_n := 2^{2^n}+1$, and the sequence of coefficients of $\mathcal{G}(z)$ is usually called the *Gros sequence* [Gr72, HKMP13].

In 2014, Guo, Wu and Wen considered in [GWW14] the regular paperfolding numbers defined by

$$f_{RPF}\Big(\frac{1}{b}\Big) := \sum_{n\geq 0} \frac{u_n}{b^n},$$

where $b \geq 2$ is an integer, and $(u_n)_{n\geq 0}$ is the regular paperfolding sequence on $\{0,1\}$ defined recursively by $u_{4n} = 1$, $u_{4n+2} = 0$, and $u_{2n+1} = u_n$, for all integers $n \geq 0$. They proved that the irrationality exponents of these numbers are all equal to 2. For more on the regular paperfolding sequence, see for example [Al87] and [AS03].

Very recently, Wen and Wu [WW14] studied the Cantor real numbers

$$f_C\left(\frac{1}{b}\right) := \sum_{n\geq 0} \frac{v_n}{b^n},$$

where $b \geq 2$ is an integer, and $(v_n)_{n\geq 0}$ is the Cantor sequence on $\{0,1\}$ such that for all integers $n \geq 0$, we have $v_n = 1$ if and only if the ternary expansion of n does not contain the digit 1. They showed that the irrationality exponents of these numbers are also equal to 2. We point out that the Cantor sequence is 3-automatic (see for example [AS03]) and that its generating function f_C satisfies $f_C(z) = (1+z^2)f_C(z^3)$.

In the present work, we shall unify all the above results together and compute the irrationality exponent of some new families of transcendental numbers. We do not restrict our attention to automatic numbers and take a more general point of view.

Mahler's method [Ma29, Ma30a, Ma30b] is a method in transcendence theory whereby one uses a function $F(z) \in \mathbb{Q}[[z]]$ that satisfies a functional equation of the following form

$$\sum_{i=0}^{n} P_i(z)F(z^{d^i}) = 0, \tag{1.4}$$

for some integers $n \geq 1$ and $d \geq 2$, and polynomials $P_0(z), \ldots, P_n(z)$ in $\mathbb{Z}[x]$ with $P_0(z)P_n(z) \neq 0$, to give results about the nature of the numbers $F(1/b)$ with $b \geq 2$ an integer such that $1/b$ is less than the radius of convergence of $F(z)$. We refer to such numbers $F(1/b)$ as *Mahler numbers.* It is well known that automatic numbers are special cases of Mahler numbers (see [Be94, Theorem 1]). The following theorem, established in [BBC15], extends the main result of [AC06], quoted above.

Theorem 1.1. *A Mahler number cannot be a Liouville number.*

By means of a suitable adaptation of the so-called Mahler's method, it is proved in [BBC15] that an irrational Mahler number is transcendental when $P_0(z)$ in (1.4) is a nonzero integer. However, the general case remains an open problem. Note that Corvaja and Zannier [CZ02] explained how, beside Mahler's method, the Schmidt Subspace Theorem can be used to prove, under quite general assumptions, the transcendence of values of power series with integer coefficients at nonzero algebraic points.

We formulate the following open question.

Problem 1.2. *To determine the set of irrationality exponents of irrational Mahler numbers.*

Actually we will consider power series $F(z)$ satisfying a functional equation of the special form

$$P_{-1}(z) + P_0(z)F(z) + P_1(z)F(z^d) = 0, \tag{1.5}$$

for some integer $d \geq 2$ and polynomials $P_{-1}(z), P_0(z), P_1(z) \in \mathbb{Z}[x]$, with $P_0(z)$ and $P_1(z)$ being nonzero. Observe that, by combining (1.5) with the equation obtained by substituting z with z^d in (1.5), we see that $F(z)$ satisfies an equation of the type (1.4). We also point out here that by a general result of Zannier [Za98, p. 18], the function $F(z)$ is either rational or transcendental over $\mathbb{Q}(z)$.

The present work is organized as follows. In Section 2, we highlight several of our results. Then, in Section 3, we recall some basic notation and results about Padé approximation, which is the starting point of our study. In Section 4, we compute with Hankel determinants the irrationality exponent of certain transcendental numbers, which are values at the inverse of integers ≥ 2 of power series satisfying a functional equation of type (1.5). Since it is extremely difficult to compute explicitly the Hankel determinants of a given sequence, we collect, in Section 5, some results about Hankel continued fractions obtained very recently by Han [H15a, H15b], and apply them in Section 6 to obtain directly (this means, without condition on Hankel determinants) the irrationality exponent of special values of some power series satisfying a special type of functional equation. Our results cover all the known results on irrationality exponent listed above, and in the final Section 7, we shall give several new applications to obtain the irrationality exponent of new families of transcendental numbers.

2. Results

Let $d \geq 2$ be an integer, and $(c_m)_{m\geq 0}$ be an integer sequence such that $f(z) = \sum_{m=0}^{+\infty} c_m z^m$ converges inside the unit disk. Suppose that there exist integer polynomials $A(z)$, $B(z)$, $C(z)$, and $D(z)$ such that

$$f(z) = \frac{A(z)}{B(z)} + \frac{C(z)}{D(z)} f(z^d). \tag{2.1}$$

Under various assumptions on these polynomials, we are able to show that, for every integer $b \geq 2$, the irrationality exponent of $f(1/b)$ is equal to 2.

One of our tools is a careful study of the sequence $(H_n(f))_{n\geq 0}$ of the Hankel determinants of f, defined by $H_0(f) = 1$ and

$$H_n(f) := \begin{vmatrix} c_0 & c_1 & \dots & c_{n-1} \\ c_1 & c_2 & \dots & c_n \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_n & \dots & c_{2n-2} \end{vmatrix}, \quad \text{for all integers } n \geq 1.$$

We state below a consequence of our Theorem 4.1, which highlights a relationship between the irrationality exponent of $f(1/b)$ and the sequence $(H_n(f))_{n\geq 0}$, and correct, improve and generalize the main result recently obtained by Guo, Wu, and Wen [GWW14].

Theorem 2.1. *Let $d \geq 2$ be an integer, and $(c_j)_{j\geq 0}$ be an integer sequence such that $f(z) = \sum_{j=0}^{+\infty} c_j z^j$ converges inside the unit disk. Suppose that there exist integer polynomials $A(z)$, $B(z)$, and $C(z)$ such that*

$$f(z) = \frac{A(z)}{B(z)} + C(z)f(z^d).$$

Let $b \geq 2$ be an integer such that $C(\frac{1}{b^{d^m}}) \neq 0$ for all integers $m \geq 0$. If there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(f) \neq 0$ for all integers $i \geq 0$ and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$, then $f(1/b)$ is transcendental and its irrationality exponent is equal to 2.

Remark. In [GWW14] the authors need to assume the existence of an infinite sequence $(n_i)_{i\geq 1}$ satisfying $\liminf_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$ and such that $H_{n_i}(f)H_{n_i+1}(f)$ is nonzero for all integers $i \geq 1$. However, in their proof, they make use of the stronger assumption that this limit inferior is actually a limit, and also use implicitly the fact that $C(\frac{1}{b^{d^m}}) \neq 0$ for all integers $m \geq 0$. Thus, our general result Theorem 4.1 considerably extends (and corrects) Theorem 1 of [GWW14].

Theorem 2.1 will be proved in Section 4.

However, the computation of the sequence $(H_n(f))_{n\geq 0}$ is not an easy task, and even to get information on its vanishing terms is difficult. Very recently, Han [H15a, H15b] has developed a new and fruitful method. As a result, we obtain in particular the following theorem.

Theorem 2.2. *Let $f(z) \in \mathbb{Z}[[z]]$ be the power series defined by*

$$f(z) = \prod_{n\geq 0} \left(1 + uz^{2^n} + 2z^{2^{n+1}} \frac{C(z^{2^n})}{D(z^{2^n})}\right), \tag{2.2}$$

where $u \in \mathbb{Z}$*, and* $C(z), D(z) \in \mathbb{Z}[z]$ *with* $D(0) = 1$*. Let* $b \geq 2$ *be an integer such that* $D\big(\frac{1}{b^{2^m}}\big) f\big(\frac{1}{b^{2^m}}\big) \neq 0$ *for all integers* $m \geq 0$*. If* $f(z) \pmod 4$ *is not a rational function, then* $f(1/b)$ *is transcendental and its irrationality exponent is equal to* 2*.*

Remark. Taking $C(z) = 0$, $D(z) = 1$, and $u = -1$ in Theorem 2.2, we recover the result of [Bu11] about Thue–Morse–Mahler numbers which states that $\mu(f_{TMM}(1/b)) = 2$, for all integers $b \geq 2$.

Theorem 2.2 will be proved in Section 6.

For all integers $\alpha, \beta \geq 0$, define

$$F_{\alpha,\beta}(z) = \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} \frac{z^{2^{n+\alpha}}}{1 + z^{2^{n+\beta}}} = \sum_{n,j \geq 0}^{\infty} (-1)^j z^{(j2^{\beta-\alpha}+1)2^{n+\alpha} - 2^\alpha},$$

$$G_{\alpha,\beta}(z) = \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} \frac{z^{2^{n+\alpha}}}{1 - z^{2^{n+\beta}}} = \sum_{n,j \geq 0}^{\infty} z^{(j2^{\beta-\alpha}+1)2^{n+\alpha} - 2^\alpha}.$$

The radius of convergence of $F_{\alpha,\beta}$ (resp. $G_{\alpha,\beta}$) is at least equal to 1. Moreover if $\beta = \alpha + 1$, then $G_{\alpha,\beta}(z)$ is a rational function, since we have

$$\begin{aligned} G_{\alpha,\alpha+1}(z) &= \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} \frac{z^{2^{n+\alpha}}}{1 - z^{2^{n+\alpha+1}}} \\ &= \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} z^{2^{n+\alpha}} \sum_{j=0}^{\infty} z^{j2^{n+\alpha+1}} \\ &= \frac{1}{z^{2^\alpha}} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (z^{2^\alpha})^{(2j+1)2^n} \\ &= \frac{1}{z^{2^\alpha}} \sum_{m=0}^{\infty} (z^{2^\alpha})^m \\ &= \frac{1}{z^{2^\alpha}(1 - z^{2^\alpha})}. \end{aligned}$$

For $\beta \neq \alpha + 1$, we have the following result.

Theorem 2.3. *Let* $\alpha, \beta \geq 0$ *be integers such that* $\beta \neq \alpha + 1$*. Let* $b \geq 2$ *be an integer. Then both* $F_{\alpha,\beta}(1/b)$ *and* $G_{\alpha,\beta}(1/b)$ *are transcendental, and their irrationality exponent are equal to* 2*.*

Remark. The case $\alpha = \beta = 0$ implies that both $\mathcal{F}(1/b)$ and $\mathcal{G}(1/b)$ are transcendental for all integers $b \geq 2$, and also the result obtained by Coons [Co13], namely that $\mu(\mathcal{F}(1/b)) = \mu(\mathcal{G}(1/b)) = 2$, for all integers

$b \geq 2$. The case $\alpha = 0$ and $\beta = 2$ shows that for all integers $b \geq 2$, the regular paperfolding numbers $f_{RPF}(1/b)$ are transcendental and their irrationality exponents are equal to 2. The latter was conjectured by Coons and Vrbik [CV12] and has recently been established by Guo, Wu and Wen [GWW14].

Recall that Stern's sequence $(a_n)_{n\geq 0}$ and its twisted version $(b_n)_{n\geq 0}$ are defined, respectively, by (see [BV13, Ba10, St58])

$$\begin{cases} a_0 = 0,\ a_1 = 1, \\ a_{2n} = a_n,\ a_{2n+1} = a_n + a_{n+1},\ (n \geq 1), \end{cases}$$

and

$$\begin{cases} b_0 = 0,\ b_1 = 1, \\ b_{2n} = -b_n,\ b_{2n+1} = -(b_n + b_{n+1}),\ (n \geq 1). \end{cases}$$

Put $S(z) = \sum\limits_{n=0}^{\infty} a_{n+1} z^n$ and $T(z) = \sum\limits_{n=0}^{\infty} b_{n+1} z^n$. Then S and T converge inside the unit disk, since $|a_n| \leq n$ and $|b_n| \leq n$ for all integers $n \geq 0$. Recently, Bundschuh and Väänänen [BV13] proved that $\mu(S(1/b)) \leq 2.929$ and $\mu(T(1/b)) \leq 3.555$ for all integers $b \geq 2$. Our next result gives the exact irrationality exponent of the Stern number and also that of the twisted Stern number, and it will be proved in Section 7.

Theorem 2.4. *For all integers $b \geq 2$, both $S(1/b)$ and $T(1/b)$ are transcendental and their irrationality exponents are equal to 2.*

The following theorem will be proved in Section 6.

Theorem 2.5. *Let $f(z) \in \mathbb{Z}[[z]]$ be a power series defined by*

$$f(z) = \prod_{n=0}^{\infty} \frac{C(z^{3^n})}{D(z^{3^n})}, \tag{2.3}$$

with $D(z), C(z) \in \mathbb{Z}[z]$ such that $C(0) = D(0) = 1$. Let $b \geq 2$ be an integer such that $C(\frac{1}{b^{3^m}})D(\frac{1}{b^{3^m}}) \neq 0$ for all integers $m \geq 0$. If $f(z) \pmod 3$ is not a rational function, then $f(1/b)$ is transcendental and its irrationality exponent is equal to 2.

Remark. Taking $C(z) = 1 + z^2$ and $D(z) = 1$ in Theorem 2.5 and using the fact that the Cantor sequence on $\{0, 1\}$ is not ultimately periodic, we obtain that $f_C(1/b)$ is transcendental for all integers $b \geq 2$, where the function f_C is defined in Section 1. We also recover the result of [WW14] about Cantor real numbers, namely that $\mu(f_C(1/b)) = 2$ for all integers $b \geq 2$.

For additional results, see Theorems 4.2, 6.1, 7.1, 7.2, and Corollary 6.2.

3. Hankel determinants and Padé approximation

In this section we summarize several basic facts on Padé approximation. For more details, we refer the reader for example to [Br80, BG96].

Let $\mathbb{F}$ be a field and z be an indeterminate over $\mathbb{F}$. For any sequence $\mathbf{c} = (c_m)_{m\geq 0}$ of elements in $\mathbb{F}$, we put $f = f(z) = \sum_{m=0}^{+\infty} c_m z^m$, and call it the generating function of $\mathbf{c}$. For all integers $n \geq 1$ and $k \geq 0$, the Hankel determinant of the power series f (or of the sequence $\mathbf{c}$) is defined by

$$H_n^{(k)}(f) := \begin{vmatrix} c_k & c_{k+1} & \cdots & c_{k+n-1} \\ c_{k+1} & c_{k+2} & \cdots & c_{k+n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k+n-1} & c_{k+n} & \cdots & c_{k+2n-2} \end{vmatrix} \in \mathbb{F}. \tag{3.1}$$

By convention, we put $H_0^{(k)}(f) = 1$, for all integers $k \geq 0$. For all integers $n \geq 0$, write $H_n(f) := H_n^{(0)}(f)$. The sequence $H(f) := (H_n(f))_{n\geq 0}$ is called the *sequence of the Hankel determinants* of f.

Let p and q be nonnegative integers. By definition, the Padé approximant $[p/q]_f(z)$ to f is the rational fraction $P(z)/Q(z)$ in $\mathbb{F}[[z]]$ such that

$$\deg(P) \leq p, \ \deg(Q) \leq q, \ \text{and} \ f(z) - \tfrac{P(z)}{Q(z)} = \mathcal{O}(z^{p+q+1}).$$

The pair (P, Q) has no reason to be unique, but the fraction $P(z)/Q(z)$ is unique. Moreover if we assume that P and Q are coprime, then $Q(0) \neq 0$.

If there exists an integer $k \geq 1$ such that $H_k(f)$ is nonzero, then we know that the Padé approximant $[k-1/k]_f(z)$ exists and we have

$$f(z) - [k-1/k]_f(z) = \frac{H_{k+1}(f)}{H_k(f)}\, z^{2k} + \mathcal{O}(z^{2k+1}). \tag{3.2}$$

This formula is of little help if $H_{k+1}(f) = 0$. But even in this case, we still have the following fundamental result.

Theorem 3.1. *With the notation as above, suppose that there exist two integers ℓ, k such that $\ell > k \geq 1$ and $H_\ell(f)H_k(f) \neq 0$. Then the Padé approximant $[k-1/k]_f(z)$ exists, and there exist a nonzero element h_k in $\mathbb{F}$ and an integer k' such that $k \leq k' < \ell$ and*

$$f(z) - [k-1/k]_f(z) = h_k\, z^{k+k'} + \mathcal{O}(z^{k+k'+1}). \tag{3.3}$$

Remark. It seems to us that Theorem 3.1 is new. An important point in its statement is the non-vanishing of h_k.

Proof. Since $H_\ell(f)$ is nonzero, all the column vectors in $H_\ell(f)$ are linearly independent, in particular, the rank of the $\ell \times (k+1)$ matrix

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_k \\ c_1 & c_2 & \cdots & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\ell-2} & c_{\ell-1} & \cdots & c_{\ell+k-2} \\ c_{\ell-1} & c_\ell & \cdots & c_{\ell+k-1} \end{pmatrix}$$

is equal to $k+1$. By hypothesis, we also have $H_k(f) \neq 0$, thus there exists a smallest integer k' such that $k \leq k' < \ell$ and

$$H_{k,k'}(f) := \begin{vmatrix} c_0 & c_1 & \cdots & c_k \\ c_1 & c_2 & \cdots & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-1} \\ c_{k'} & c_{k'+1} & \cdots & c_{k+k'} \end{vmatrix} \neq 0.$$

Hence for all integers $j = k, \ldots, k'-1$, we have $H_{k,j}(f) = 0$. Define

$$Q^{[k-1/k]}(z) := \begin{vmatrix} c_0 & c_1 & \cdots & c_{k-1} & c_k \\ c_1 & c_2 & \cdots & c_k & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-2} & c_{2k-1} \\ z^k & z^{k-1} & \cdots & z & 1 \end{vmatrix},$$

$$P^{[k-1/k]}(z) := \begin{vmatrix} c_0 & c_1 & \cdots & c_{k-1} & c_k \\ c_1 & c_2 & \cdots & c_k & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-2} & c_{2k-1} \\ 0 & c_0 z^{k-1} & \cdots & \sum_{i=0}^{k-2} c_i z^{i+1} & \sum_{i=0}^{k-1} c_i z^i \end{vmatrix}.$$

Then $\deg(P^{[k-1/k]}) \leq k-1$, $\deg(Q^{[k-1/k]}) \leq k$, and (see [BG96, p. 6])

$$\begin{aligned} & Q^{[k-1/k]}(z) f(z) - P^{[k-1/k]}(z) \\ &= Q^{[k-1/k]}(z) \Big(\sum_{i=0}^{+\infty} c_i z^i \Big) - P^{[k-1/k]}(z) \\ &= \begin{vmatrix} c_0 & c_1 & \cdots & c_{k-1} & c_k \\ c_1 & c_2 & \cdots & c_k & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-2} & c_{2k-1} \\ \sum_{i=0}^{+\infty} c_i z^{i+k} & \sum_{i=0}^{+\infty} c_i z^{i+k-1} & \cdots & \sum_{i=0}^{+\infty} c_i z^{i+1} & \sum_{i=0}^{+\infty} c_i z^i \end{vmatrix} - P^{[k-1/k]}(z) \end{aligned}$$

$$= \begin{vmatrix} c_0 & c_1 & \cdots & c_{k-1} & c_k \\ c_1 & c_2 & \cdots & c_k & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-2} & c_{2k-1} \\ \sum\limits_{i=k}^{+\infty} c_i z^{i+k} & \sum\limits_{i=k+1}^{+\infty} c_i z^{i+k-1} & \cdots & \sum\limits_{i=2k-1}^{+\infty} c_i z^{i+1} & \sum\limits_{i=2k}^{+\infty} c_i z^{i} \end{vmatrix}$$

$$= \sum_{i=1}^{+\infty} z^{2k+i-1} \begin{vmatrix} c_0 & c_1 & \cdots & c_{k-1} & c_k \\ c_1 & c_2 & \cdots & c_k & c_{k+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{k-1} & c_k & \cdots & c_{2k-2} & c_{2k-1} \\ c_{k+i-1} & c_{k+i} & \cdots & c_{2k+i-2} & c_{2k+i-1} \end{vmatrix},$$

where in the first determinant, we have subtracted z^k times the first row from the last one, z^{k+1} times the second row from the last one, etc., up to z^{2k-1} times the penultimate row from the last one, and then we arrive at the second determinant.

By the definition of the integer k', we obtain

$$Q^{[k-1/k]}(z)f(z) - P^{[k-1/k]}(z) = H_{k,k'}(f)z^{k+k'} + \mathcal{O}(z^{k+k'+1}).$$

Note that $Q^{[k-1/k]}(0) = H_k(f) \neq 0$, thus we have

$$\begin{aligned} f(z) - \frac{P^{[k-1/k]}(z)}{Q^{[k-1/k]}(z)} &= \frac{H_{k,k'}(f)}{Q^{[k-1/k]}(z)} z^{k+k'} + \mathcal{O}(z^{k+k'+1}) \\ &= \frac{H_{k,k'}(f)}{H_k(f)} z^{k+k'} + \mathcal{O}(z^{k+k'+1}). \end{aligned}$$

Finally it suffices to put

$$[k-1/k]_f(z) := \frac{P^{[k-1/k]}(z)}{Q^{[k-1/k]}(z)},\ h_k := \frac{H_{k,k'}(f)}{H_k(f)} \neq 0,$$

and we obtain at once the desired result. □

To conclude this section, we recall some properties of rational functions in $\mathbb{Z}[[z]]$, which are related to Hankel determinants. Let $(c_m)_{m\geq 0}$ be an integer sequence such that the power series $f(z) = \sum\limits_{m=0}^{+\infty} c_m z^m$ converges inside the unit disk. By Fatou's theorem (see [Fa06]), we know that the power series $f(z)$ is either rational or transcendental over $\mathbb{Q}(z)$. Moreover, by Kronecker's theorem (see for example [Sa63, p. 5]), we know also that the power series $f(z)$ is rational if and only if there exists an integer $n_0 \geq 0$ such that $H_n(f) = 0$ for all integers n larger than n_0. Equivalently, $f(z)$ is not rational if and only if there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(f) \neq 0$, for all integers $i \geq 0$.

Finally we point out that since the power series $f(z)$ has only integer coefficients, thus it is transcendental over $\mathbb{Q}(z)$ if and only if it is transcendental over $\mathbb{C}(z)$ (see for example [SW88]).

4. Irrationality exponent with Hankel determinants

In this section, we compute with Hankel determinants the irrationality exponent of transcendental numbers, which are special values at the inverse of integers ≥ 2 of power series satisfying a special type of functional equation.

Theorem 4.1. *Let $d \geq 2$ be an integer, and $(c_j)_{j\geq 0}$ be an integer sequence such that $f(z) = \sum_{j=0}^{+\infty} c_j z^j$ converges inside the unit disk. Suppose that there exist integer polynomials $A(z)$, $B(z)$, $C(z)$, and $D(z)$ such that*

$$f(z) = \frac{A(z)}{B(z)} + \frac{C(z)}{D(z)} f(z^d). \tag{4.1}$$

Let $b \geq 2$ be an integer such that $B(\frac{1}{b^{d^m}})C(\frac{1}{b^{d^m}})D(\frac{1}{b^{d^m}}) \neq 0$, for all integers $m \geq 0$. If there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(f) \neq 0$ for all integers $i \geq 0$ and $\limsup_{i\to+\infty} \frac{n_{i+1}}{n_i} = \rho$, then $f(1/b)$ is transcendental, and we have

$$\mu\Big(f\Big(\frac{1}{b}\Big)\Big) \leq (1+\rho)\min\{\rho^2, d\}.$$

In particular, the irrationality exponent of $f(1/b)$ is equal to 2 if $\rho = 1$.

Proof. From the equation (4.1), we deduce immediately that

$$\begin{pmatrix} 1 \\ f(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{A(z)}{B(z)} & \frac{C(z)}{D(z)} \end{pmatrix} \begin{pmatrix} 1 \\ f(z^d) \end{pmatrix}.$$

Since $B(\frac{1}{b^{d^m}})C(\frac{1}{b^{d^m}})D(\frac{1}{b^{d^m}}) \neq 0$ for all integers $m \geq 0$, then by a result due to Nishioka (see [Ni90, Corollary 2]), we obtain

$$\mathrm{tr.deg}_{\mathbb{Q}}\mathbb{Q}\big(1, f(1/b)\big) = \mathrm{tr.deg}_{\mathbb{C}(z)}\mathbb{C}(z)\big(1, f(z)\big).$$

Now that there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(f) \neq 0$ for all integers $i \geq 0$, the power series $f(z)$ is not a rational function. Thus, it is transcendental over $\mathbb{C}(z)$ by Fatou's theorem, hence $f(1/b)$ is transcendental.

By iteration of Formula (4.1), we have, for all integers $m \geq 1$,

$$f(z) = \frac{A_m(z)}{B_m(z)} + \frac{C_m(z)}{D_m(z)} f(z^{d^m}), \tag{4.2}$$

where $C_m(z) = \prod_{j=0}^{m-1} C(z^{d^j})$, $D_m(z) = \prod_{j=0}^{m-1} D(z^{d^j})$, and

$$B_m(z) = D_{m-1}(z) \prod_{j=0}^{m-1} B(z^{d^j}),\ \ A_m(z) = \sum_{j=0}^{m-1} C_j(z) A(z^{d^j}) \cdot \frac{B_m(z)}{D_j(z) B(z^{d^j})},$$

where we have put $C_0(z) = D_0(z) = 1$.

Put $\alpha = \deg(A(z))$, $\beta = \deg(B(z))$, $\gamma = \deg(C(z))$, and $\delta = \deg(D(z))$. Then,

$$\begin{aligned}
\deg(C_m(z)) &= \sum_{j=0}^{m-1} \deg(C(z^{d^j})) = \sum_{j=0}^{m-1} \gamma d^j = \frac{\gamma(d^m - 1)}{d - 1} \leq \gamma d^m, \\
\deg(D_m(z)) &= \sum_{j=0}^{m-1} \deg(D(z^{d^j})) = \sum_{j=0}^{m-1} \delta d^j = \frac{\delta(d^m - 1)}{d - 1} \leq \delta d^m, \\
\deg(B_m(z)) &= \deg(D_{m-1}(z)) + \sum_{j=0}^{m-1} \deg(B(z^{d^j})) \\
&= \frac{\delta(d^{m-1} - 1)}{d - 1} + \frac{\beta(d^m - 1)}{d - 1} \leq (\beta + \delta) d^m, \\
\deg(A_m(z)) &\leq \max_{0 \leq j \leq m-1} \Big(\deg(C_j(z)) + \deg(A(z^{d^j})) + \deg(B_m(z)) \Big) \\
&\leq \max_{0 \leq j \leq m-1} \Big(\frac{\gamma(d^j - 1)}{d - 1} + \alpha d^j + \frac{\delta(d^{m-1} - 1)}{d - 1} + \frac{\beta(d^m - 1)}{d - 1} \Big) \\
&\leq (\alpha + \beta + \gamma + \delta) d^m.
\end{aligned}$$

Let $i \geq 0$ be an integer. As in the proof of Theorem 3.1, we denote by n_i' the smallest integer such that $n_i \leq n_i' < n_{i+1}$ and $H_{n_i, n_i'}(f) \neq 0$. Then we can find $h_i \in \mathbb{Q} \setminus \{0\}$, and $P_i(z), Q_i(z) \in \mathbb{Z}[z]$ with $\deg(P_i(z)) \leq n_i - 1$, $\deg(Q_i(z)) \leq n_i$, and $Q_i(0) \neq 0$ such that

$$f(z) - \frac{P_i(z)}{Q_i(z)} = h_i z^{n_i + n_i'} + \mathcal{O}(z^{n_i + n_i' + 1}) = h_i z^{n_i + n_i'} \big(1 + \mathcal{O}(z)\big).$$

Thus, for all integers $m \geq 1$, we obtain

$$f(z^{d^m}) - \frac{P_i(z^{d^m})}{Q_i(z^{d^m})} = h_i z^{(n_i+n_i')d^m}\big(1+\mathcal{O}(z^{d^m})\big).$$

Combined with Formula (4.2), this gives

$$f(z) - \frac{A_m(z)}{B_m(z)} - \frac{C_m(z)}{D_m(z)} \cdot \frac{P_i(z^{d^m})}{Q_i(z^{d^m})} = h_i z^{(n_i+n_i')d^m} \frac{C_m(z)}{D_m(z)}\big(1+\mathcal{O}(z^{d^m})\big).$$

To simplify the notation, we define

$$\begin{aligned} P_{i,m}(z) &= A_m(z)D_m(z)Q_i(z^{d^m}) + B_m(z)C_m(z)P_i(z^{d^m}), \\ Q_{i,m}(z) &= B_m(z)D_m(z)Q_i(z^{d^m}). \end{aligned}$$

Since $B(z)C(z)D(z) \neq 0$, then we can write

$$B(z) = b_\kappa z^\kappa(1+z\tilde{B}(z)),\ C(z) = c_\eta z^\eta(1+z\tilde{C}(z)),\ D(z) = d_\iota z^\iota(1+z\tilde{D}(z))$$

with $\kappa, \eta, \iota \geq 0$ integers, $b_\kappa, c_\eta, d_\iota \in \mathbb{Z} \setminus \{0\}$, and $\tilde{B}(z), \tilde{C}(z), \tilde{D}(z) \in \mathbb{Q}[z]$. Note that $\tilde{C}(z), \tilde{D}(z)$ are bounded on the unit disk, thus both $\sum_{j=0}^{\infty} \frac{1}{b^{d^j}} |\tilde{C}(\frac{1}{b^{d^j}})|$ and $\sum_{j=0}^{\infty} \frac{1}{b^{d^j}} |\tilde{D}(\frac{1}{b^{d^j}})|$ converge. Note also that $C(\frac{1}{b^{d^m}})D(\frac{1}{b^{d^m}}) \neq 0$ for all integers $m \geq 0$, thus the following two limits

$$\begin{aligned} \sigma &= \lim_{m\to+\infty} \frac{C_m(\frac{1}{b})}{c_\eta^m b^{-\frac{\eta(d^m-1)}{d-1}}} = \prod_{j=0}^{\infty}\Big(1+\frac{1}{b^{d^j}}\tilde{C}(\frac{1}{b^{d^j}})\Big), \\ \tau &= \lim_{m\to+\infty} \frac{D_m(\frac{1}{b})}{d_\iota^m b^{-\frac{\iota(d^m-1)}{d-1}}} = \prod_{j=0}^{\infty}\Big(1+\frac{1}{b^{d^j}}\tilde{D}(\frac{1}{b^{d^j}})\Big) \end{aligned}$$

do exist and are different from zero. Hence, for m tending to $+\infty$, we have

$$\begin{aligned} f\Big(\frac{1}{b}\Big) - \frac{P_{i,m}(\frac{1}{b})}{Q_{i,m}(\frac{1}{b})} &= \frac{h_i}{b^{(n_i+n_i')d^m}} \frac{C_m(\frac{1}{b})}{D_m(\frac{1}{b})}\Big(1+\mathcal{O}(\frac{1}{b^{d^m}})\Big) \\ &\sim \frac{h_i\sigma}{\tau}\Big(\frac{c_\eta}{d_\iota}\Big)^m \frac{1}{b^{(n_i+n_i')d^m + \frac{(\eta-\iota)(d^m-1)}{d-1}}}. \end{aligned} \tag{4.3}$$

Moreover, we also have

$$\begin{aligned} \deg(P_{i,m}(z)) &\leq \max\Big(\deg(A_m(z)) + \deg(D_m(z)) + \deg(Q_i(z^{d^m})), \\ &\qquad \deg(B_m(z)) + \deg(C_m(z)) + \deg(P_i(z^{d^m}))\Big) \\ &\leq \max\Big((\alpha+\beta+\gamma+2\delta+n_i)d^m, (\beta+\delta+\gamma+n_i-1)d^m))\Big) \\ &\leq (\alpha+\beta+\gamma+2\delta+n_i)d^m, \\ \deg(Q_{i,m}(z)) &\leq \deg(B_m(z)) + \deg(D_m(z)) + \deg(Q_i(z^{d^m})) \\ &\leq (\beta+2\delta+n_i)d^m. \end{aligned}$$

Put $e_i = \alpha + \beta + \gamma + 2\delta + n_i$. Recall that, by assumption, we have $B(\frac{1}{b^{d^m}})D(\frac{1}{b^{d^m}}) \neq 0$ for all integers $m \geq 0$. Moreover $Q_i(0) \neq 0$. Thus we can find an integer $N_{0,i} > 0$ and two constants $\alpha_{1,i}, \alpha_{2,i} > 0$ (which depend only on i) such that we have

$$\alpha_{1,i} \leq b^{\kappa d^m} \Big| B\Big(\frac{1}{b^{d^m}}\Big)\Big|,\ b^{\iota d^m} \Big| D\Big(\frac{1}{b^{d^m}}\Big)\Big| \leq \alpha_{2,i}, \text{ for all integers } m \geq 0,$$

and $\alpha_{1,i} \leq \big|Q_i\big(\frac{1}{b^{d^m}}\big)\big| \leq \alpha_{2,i}$, for all integers $m \geq N_{0,i}$. Set

$$q_{i,m} = b^{e_i d^m} |Q_{i,m}(\frac{1}{b})|, \text{ and } p_{i,m} = b^{e_i d^m} P_{i,m}(\frac{1}{b}) \mathrm{sgn}(Q_{i,m}(\frac{1}{b})).$$

Then $q_{i,m}, p_{i,m}$ are integers, and for all integers $m \geq N_{0,i}$, we have

$$\alpha_{1,i}^{3m} b^{e_i d^m - g_m} \leq q_{i,m} \leq \alpha_{2,i}^{3m} b^{e_i d^m - g_m}, \tag{4.4}$$

with $g_m = \frac{(\kappa+\iota)(d^m-1)+\iota(d^{m-1}-1)}{d-1}$, from which we deduce immediately

$$\frac{\alpha_{1,i}^{3(m+1)}}{\alpha_{2,i}^{3m}} b^{e_i d^m(d-1)-(d\kappa+d\iota+\iota)d^{m-1}} q_{i,m} \leq q_{i,m+1} \leq \frac{\alpha_{2,i}^{3(m+1)}}{\alpha_{1,i}^{3md}} q_{i,m}^d. \tag{4.5}$$

Let ε be a sufficiently small positive real number. Since $\lim_{i\to+\infty} n_i = +\infty$, there exists an integer $N_1 > 1$ (independent of m) such that for $i > N_1$ and $m \geq 2$, we have

$$\begin{cases} (1-\varepsilon)n_i d^m \leq e_i d^m - g_m \leq (1+\varepsilon)n_i d^m, \\ e_i d^m(d-1) - (d\kappa + d\iota + \iota)d^{m-1} > (1-\varepsilon)n_i d^m. \end{cases}$$

Then it follows from Formulas (4.4) and (4.5) that there exists an integer $N_{1,i} > N_{0,i}$ such that for all integers $m \geq N_{1,i}$, we have

$$b^{n_i d^m(1-2\varepsilon)} \leq q_{i,m} \leq b^{n_i d^m(1+2\varepsilon)}, \tag{4.6}$$

$$q_{i,m} < q_{i,m+1} \leq q_{i,m}^{d(1+\varepsilon)}. \tag{4.7}$$

Similarly it follows from Formula (4.3) that there exists an integer $N_{2,i} > N_{1,i}$ such that for all integers $m \geq N_{2,i}$, we have

$$\frac{1}{b^{(n_i+n_i'+\eta+\iota)(1+\varepsilon)d^m}} \leq \Big|f\Big(\frac{1}{b}\Big) - \frac{p_{i,m}}{q_{i,m}}\Big| \leq \frac{1}{b^{(n_i+n_i')(1-\varepsilon)d^m}} \tag{4.8}$$

and by Formula (4.6), we obtain also

$$\frac{1}{q_{i,m}^{(n_i+n_i'+\eta+\iota)(1+4\varepsilon)/n_i}} \leq \Big|f\Big(\frac{1}{b}\Big) - \frac{p_{i,m}}{q_{i,m}}\Big| \leq \frac{1}{q_{i,m}^{(n_i+n_i')(1-4\varepsilon)/n_i}}. \tag{4.9}$$

By hypothesis, we have $\limsup_{i\to\infty} \frac{n_{i+1}}{n_i} = \rho$, then we can find an integer $i_0 > N_1$ such that for all integers $i \geq i_0$, we have $\frac{n_{i+1}}{n_i} < \rho + \varepsilon$ and

$$\begin{cases} \dfrac{(n_i + n_i' + \eta + \iota)(1+4\varepsilon)}{n_i} \leq (1+\rho)(1+6\varepsilon), \\ \dfrac{(n_i + n_i')(1-4\varepsilon)}{n_i} \geq 2(1-6\varepsilon), \end{cases} \tag{4.10}$$

from which we deduce at once

$$\frac{1}{q_{i,m}^{(1+\rho)(1+6\varepsilon)}} \leq \left| f\Big(\frac{1}{b}\Big) - \frac{p_{i,m}}{q_{i,m}} \right| \leq \frac{1}{q_{i,m}^{2(1-6\varepsilon)}}. \tag{4.11}$$

Applying Lemma 4.1 from [AR09, p. 668] with (4.7) and (4.11), we obtain

$$\mu\Big(f\Big(\frac{1}{b}\Big)\Big) \leq \frac{(1+\rho)(1+6\varepsilon)}{2(1-6\varepsilon)-1} d(1+\varepsilon).$$

Since ε is positive and can be chosen arbitrarily small, we get

$$\mu\Big(f\Big(\frac{1}{b}\Big)\Big) \leq (1+\rho)d. \tag{4.12}$$

Fix $\ell > 1$ an integer such that $d^{\ell-1} > n_{i_0}$. Let $\mathcal{A}_\ell$ be the set of integers $i > i_0$ such that $n_i \in [d^{\ell-1}, d^\ell - 1]$. Assume that $\mathcal{A}_\ell$ is non-empty (it could be empty when ρ is large, but it is certainly non-empty for infinitely many ℓ), and denote its elements as $n_{i_1} < n_{i_2} < \cdots < n_{i_t}$. Then $t \geq 1$, $n_{i_j} = n_{i_1+j-1}$ $(1 \leq j \leq t)$, $n_{i_1} < (\rho+\varepsilon)n_{i_1-1} \leq (\rho+\varepsilon)(d^{\ell-1}-1)$, and $d^\ell \leq n_{i_t+1} < (\rho+\varepsilon)n_{i_t}$. Put

$$M_\ell = \max_{1\leq i\leq i_t} N_{2,i}.$$

Arrange the integers $q_{i_l,m}$ $(1 \leq l \leq t$ and $m \geq M_\ell)$ as an increasing sequence, which we denote by $(r_{\ell,j})_{j\geq 0}$.

Fix $j \geq 0$, and write $r_{\ell,j} = q_{i_l,m}$ with $1 \leq l \leq t$. By (4.6), we have

$$b^{n_{i_l} d^m (1-2\varepsilon)} \leq q_{i_l,m} \leq b^{n_{i_l} d^m (1+2\varepsilon)}.$$

We distinguish below two cases:

Case I: $n_{i_t} > n_{i_l}(1+2\varepsilon)/(1-2\varepsilon)$. Then $i_t > i_l$, and thus there exists a smallest integer v such that $l < v \leq t$ such that

$$n_{i_v} > n_{i_l}(1+2\varepsilon)/(1-2\varepsilon).$$

Consequently we have $q_{i_l,m} < q_{i_v,m}$ and

$$\frac{\log q_{i_v,m}}{\log q_{i_l,m}} \le \frac{n_{i_v}(1+2\varepsilon)}{n_{i_l}(1-2\varepsilon)}.$$

By the minimality v, we have

$$n_{i_v} < (\rho+\varepsilon)n_{i_{v-1}} \le (\rho+\varepsilon)n_{i_l}(1+2\varepsilon)/(1-2\varepsilon),$$

from which we deduce directly

$$1 < \frac{\log r_{\ell,j+1}}{\log r_{\ell,j}} \le \frac{\log q_{i_v,m}}{\log q_{i_l,m}} < \frac{(\rho+\varepsilon)(1+2\varepsilon)^2}{(1-2\varepsilon)^2}.$$

Case II: $n_{i_t} \le n_{i_l}(1+2\varepsilon)/(1-2\varepsilon)$. Since $n_{i_t} < d^\ell \le dn_{i_1}$, we have

$$n_{i_t}\frac{1+2\varepsilon}{1-2\varepsilon} < dn_{i_1},$$

for all $\varepsilon > 0$ small enough. Then we get

$$\frac{\log q_{i_1,m+1}}{\log q_{i_l,m}} \ge \frac{n_{i_1}d(1-2\varepsilon)}{n_{i_l}(1+2\varepsilon)} > \frac{n_{i_t}}{n_{i_l}} \ge 1.$$

Moreover, from $n_{i_t} \le n_{i_l}(1+2\varepsilon)/(1-2\varepsilon)$, we obtain also

$$\frac{\log q_{i_1,m+1}}{\log q_{i_l,m}} \le \frac{dn_{i_1}(1+2\varepsilon)}{n_{i_l}(1-2\varepsilon)} \le \frac{dn_{i_1}(1+2\varepsilon)^2}{n_{i_t}(1-2\varepsilon)^2}.$$

Note that $n_{i_1} < (\rho+\varepsilon)(d^{\ell-1}-1)$ and $n_{i_t} > \frac{d^\ell}{\rho+\varepsilon}$, hence

$$\frac{dn_{i_1}}{n_{i_t}} < (\rho+\varepsilon)^2\frac{d(d^{\ell-1}-1)}{d^\ell} < (\rho+\varepsilon)^2,$$

and then we obtain

$$1 < \frac{\log r_{\ell,j+1}}{\log r_{\ell,j}} \le \frac{\log q_{i_1,m+1}}{\log q_{i_l,m}} < \frac{(\rho+\varepsilon)^2(1+2\varepsilon)^2}{(1-2\varepsilon)^2}.$$

In conclusion, since $\rho \ge 1$, we have established in both cases that

$$1 < \frac{\log r_{\ell,j+1}}{\log r_{\ell,j}} < \frac{(\rho+\varepsilon)^2(1+2\varepsilon)^2}{(1-2\varepsilon)^2}, \tag{4.13}$$

for all integers $j \ge 0$.

Once again applying Lemma 4.1 from [AR09, p. 668] with (4.11) and (4.13), we get

$$\mu\Big(f\Big(\frac{1}{b}\Big)\Big) \le \frac{(1+\rho)(1+6\varepsilon)}{2(1-6\varepsilon)-1}\cdot\frac{(\rho+\varepsilon)^2(1+2\varepsilon)^2}{(1-2\varepsilon)^2}.$$

Since ε is positive and can be chosen arbitrarily small, we obtain

$$\mu\Big(f\Big(\frac{1}{b}\Big)\Big) \le (1+\rho)\rho^2.$$

Combined with (4.12), this gives

$$\mu\Big(f\Big(\frac{1}{b}\Big)\Big) \le (1+\rho)\min\{\rho^2, d\},$$

as asserted. In particular, if $\rho = 1$, then $f(1/b) \le 2$. But $f(1/b)$ is transcendental, thus its irrationality exponent is equal to 2. $\square$

Remarks. (1) Note that Nishioka's result (quoted at the beginning of the proof of Theorem 4.1) may fail if we remove the condition that $C(\frac{1}{b^{d^m}}) \neq 0$ for all integers $m \ge 0$. Consider the power series

$$f(z) = \prod_{n\ge 0}(1-2z^{2^n}).$$

Then $f(z) = (1-2z)f(z^2)$, and f is analytic inside the unit disk. It is also a transcendental function for it has infinitely many zeros. However $f(1/2) = 0$. For more detail on this example, see [Be94, p. 283].

(2) In the statement of Theorem 4.2, if we replace $1/b$ by a/b with a an integer satisfying $0 < |a| < b$, then the same proof yields that $f(a/b)$ is transcendental. If we suppose further $0 < |a| < \sqrt{b}$, then with slight modifications, we can show the upper bound (see [Du14] for the case of Thue–Morse)

$$\mu\Big(f\Big(\frac{a}{b}\Big)\Big) \le \frac{\log b - \log|a|}{\log b - 2\log|a|}(1+\rho)\min\{\rho^2, d\}.$$

We are now in position to establish Theorem 2.1.

Proof of Theorem 2.1. Without loss of generality, we can suppose that the polynomials $A(z)$ and $B(z)$ are coprime. From the functional equation, we obtain that $\frac{A(z)}{B(z)} = f(z) - C(z)f(z^d)$ is analytic inside the unit disk, so $B(\frac{1}{b^{d^m}}) \neq 0$ for all integers $m \ge 0$ and $b \ge 2$. Then by Theorem 4.1, the desired result holds. $\square$

We display another application of Theorem 4.1.

Theorem 4.2. *Let $d \geq 2$ be an integer, and $(c_j)_{j\geq 0}$ be an integer sequence taking only finitely many values. Put $f(z) = \sum_{j=0}^{+\infty} c_j z^j$. Suppose that there exist integer polynomials $A(z)$, $B(z)$, $C(z)$, and $D(z)$ such that*

$$f(z) = \frac{A(z)}{B(z)} + \frac{C(z)}{D(z)} f(z^d).$$

Let $b \geq 2$ be an integer such that $C\big(\frac{1}{b^{d^m}}\big) \neq 0$ for all integers $m \geq 0$. If there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(f) \neq 0$ for all integers $i \geq 0$ and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$, then $f(1/b)$ is transcendental and its irrationality exponent is equal to 2.

Proof. Since the sequence $(c_j)_{j\geq 0}$ is bounded, the function $f(z)$ converges inside the unit disk, and for all integers $b \geq 2$, we can find an integer $\ell > 2$ such that $|c_j| < b^{d^\ell - 1}$, for all integers $j \geq 0$. Note also that $f(z)$ is not rational, for there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(f) \neq 0$ for $i \geq 0$. As in the proof of Theorem 4.1, for any integer $\ell > 2$, we can find $A_\ell(z)$, $B_\ell(z), C_\ell(z)$, and $D_\ell(z)$ in $\mathbb{Z}[z]$ such that

$$f(z) = \frac{A_\ell(z)}{B_\ell(z)} + \frac{C_\ell(z)}{D_\ell(z)} f(z^{d^\ell}). \tag{4.14}$$

Without loss of generality, we can also suppose that

$$\gcd(A_\ell(z), B_\ell(z)) = 1 \text{ and } \gcd(C_\ell(z), D_\ell(z)) = 1.$$

We argue by contradiction. Suppose that there is an integer $m \geq 0$ such that $B_\ell\big(\frac{1}{b^{d^m}}\big) D_\ell\big(\frac{1}{b^{d^m}}\big) = 0$. Then we can write

$$B_\ell(z) = \Big(z - \frac{1}{b^{d^m}}\Big)^s E(z),\ D_\ell(z) = \Big(z - \frac{1}{b^{d^m}}\Big)^t F(z),$$

where $E(z), F(z) \in \mathbb{Q}[z]$ are not equal to zero at $z = \frac{1}{b^{d^m}}$, and $s, t \geq 0$ are integers such that $\max\{s, t\} \geq 1$.

If $s > t$, then from Formula (4.14), we obtain

$$\Big(z - \frac{1}{b^{d^m}}\Big)^t f(z) - \frac{C_\ell(z)}{F(z)} f(z^{d^\ell}) = \frac{A_\ell(z)}{(z - \frac{1}{b^{d^m}})^{s-t} E(z)}.$$

The left hand side is regular at $z = \frac{1}{b^{d^m}}$, while the right side is not, giving us the required contradiction.

If $s \le t$, then from Formula (4.14), we have

$$\Big(z - \frac{1}{b^{d^m}}\Big)^t f(z) - \frac{(z - \frac{1}{b^{d^m}})^{t-s} A_\ell(z)}{E(z)} = \frac{C_\ell(z)}{F(z)} f(z^{d^\ell}).$$

Hence $f(1/b^{d^{m+\ell}})$ is a rational number. But $(c_j)_{j\ge 0}$ is the sequence of coefficients of this rational number in its base-$b^{d^{m+\ell}}$ expansion and it is bounded by $b^{d^\ell - 1}$. Thus, the sequence $(c_j)_{j\ge 0}$ is ultimately periodic. This gives again a contradiction since $f(z)$ is not rational.

To conclude, it suffices to apply Theorem 4.1 to the equation (4.14). ∎

The above theorems have many applications, but they also have an inconvenient: in general it is not at all easy to check the conditions about Hankel determinants, and indeed it is often extremely technical to compute explicitly Hankel determinants (see for example [APWW98, GWW14]). Later we shall compute the irrationality exponent only with information on the functional equation satisfied by the related power series. For this, we need recall some basic results about J-fractions in the following section.

5. Hankel continued fraction

For proving Theorem 2.2, we need the *grafting* technique, which has been introduced in [H15a] for the Jacobi continued fraction, and extended for the Hankel continued fraction in [H15b].

For all integers $\delta \ge 1$, a *super continued fraction* associated with δ, called *super δ-fraction* for short, is defined to be a continued fraction of the following form (see [H15b]):

$$f(z) = \cfrac{v_0 z^{k_0}}{1 + u_1(z) z - \cfrac{v_1 z^{k_0+k_1+\delta}}{1 + u_2(z) z - \cfrac{v_2 z^{k_1+k_2+\delta}}{1 + u_3(z) z - \ddots}}} \tag{5.1}$$

where $v_j \ne 0$ are constants, k_j are nonnegative integers and $u_j(z)$ are polynomials of degree less than or equal to $k_{j-1} + \delta - 2$. By convention, we set $\deg 0 = -1$.

A super 2-fraction is called an *Hankel continued fraction.* The following two results about Hankel continued fractions are established in [H15b].

Theorem 5.1. *(i) Each Hankel continued fraction defines a power series, and conversely, for each power series $f(z)$, the Hankel continued fraction expansion of $f(z)$ exists and is unique.*

(ii) Let $f(z)$ be a power series such that its Hankel continued fraction is given by (5.1) with $\delta = 2$. Then, for all integers $j \geq 0$, all non-vanishing Hankel determinants of $f(z)$ are given by

$$H_{s_j}(f) = (-1)^{\epsilon} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}}, \tag{5.2}$$

where $\epsilon = \sum_{i=0}^{j-1} k_i(k_i+1)/2$ and $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$.

For any prime number p, let $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ denote the finite field with p elements.

Theorem 5.2. *Let p be a prime number and $F(z) \in \mathbb{F}_p[[z]]$ be a power series satisfying the following quadratic equation*

$$A(z) + B(z)F(z) + C(z)F(z)^2 = 0, \tag{5.3}$$

where $A(z), B(z), C(z) \in \mathbb{F}_p[z]$ are three polynomials satisfying one of the following four conditions:

(i) $B(0) = 1,\ C(0) = 0,\ C(z) \neq 0$;

(ii) $B(0) = 1,\ C(z) = 0$;

(iii) $A(0) = 0,\ B(0) = 1,\ C(0) \neq 0$;

(iv) $p \geq 3,\ B(z) = 0,\ C(0) = 1$, and there exist an integer $k \geq 0$, a_k in $\mathbb{F}_p \setminus \{0\}$, and $\tilde{A}(z)$ in $\mathbb{F}_p[z]$ such that $A(z) = -(a_k z^k)^2(1 + z\tilde{A}(z))$.
Then, the Hankel continued fraction expansion of $F(z)$ exists and is ultimately periodic. Also, the sequence of the Hankel determinants of F is ultimately periodic.

6. Irrationality exponent without Hankel determinants

In this section, based on the information of the functional equation satisfied by the power series and applying the results of the previous section, we shall present several results about irrationality exponents without explicit conditions on Hankel determinants.

Theorem 6.1. *Let $f(z) \in \mathbb{Z}[[z]]$ be a power series analytic in the unit disk and such that*

$$A(z) + B(z)f(z) + C(z)f(z^2) = 0, \tag{6.1}$$

where $A(z)$, $B(z)$, and $C(z)$ are integer polynomials satisfying one of the following conditions:

(i) $B(0) \equiv 1,\ C(0) \equiv 0 \pmod 2$,

(ii) $A(0) \equiv 0,\ B(0) \equiv 1,\ C(0) \not\equiv 0 \pmod 2$.

Let $b \geq 2$ be an integer such that $B(\frac{1}{b^{2^m}})C(\frac{1}{b^{2^m}}) \neq 0$ for all integers $m \geq 0$. If $f(z) \pmod 2$ is not a rational function, then $f(1/b)$ is transcendental and its irrationality exponent is equal to 2.

Proof. Put $F(z) = f(z) \pmod 2 \in \mathbb{F}_2[[z]]$. By Formula (6.1), we obtain

$$A(z) + B(z)F(z) + C(z)F(z)^2 = 0.$$

By Theorem 5.2 (with conditions (i) and (iii), respectively) the sequence $H(F)$ of Hankel determinants is ultimately periodic over the field $\mathbb{F}_2$. Since $F(z)$ is not a rational function in $\mathbb{F}_2[[z]]$, there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(F) \neq 0$ for all integers $i \geq 0$ and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. Let $b \geq 2$ be an integer such that $B(\frac{1}{b^{2^m}})C(\frac{1}{b^{2^m}}) \neq 0$ for all integers $m \geq 0$. Then it follows from Theorem 4.1 that $f(1/b)$ is transcendental and its irrationality exponent is equal to 2. $\square$

Proof of Theorem 2.5. Directly from the definition and the fact that $C(0) = D(0) = 1$, we obtain that $f(z)$ converges in the unit disk, its coefficients in power series expansion are integers, and $f(z) = \frac{C(z)}{D(z)} f(z^3)$. Over the field $\mathbb{F}_3$, the power series $F(z) = f(z) \pmod 3$ satisfies the quadratic equation $-D(z) + C(z)F(z)^2 = 0$. So by Theorem 5.2 (iv), the sequence $H(F)$ of Hankel determinants is ultimately periodic over the field $\mathbb{F}_3$. Since $F(z)$ is not a rational function in $\mathbb{F}_3[[z]]$, there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(F) \neq 0$ for all integers $i \geq 0$ and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. Let $b \geq 2$ be an integer such that $C(\frac{1}{b^{3^m}})D(\frac{1}{b^{3^m}}) \neq 0$ for all integers $m \geq 0$. It follows from Theorem 4.1 that $f(1/b)$ is transcendental and its irrationality exponent is equal to 2. $\square$

Letting $C(z) = 1 - z$ (resp. $C(z) = 1 \pm z - z^2$) and $D(z) = 1$ in Theorem 2.5, we obtain at once the following corollary. The underlying Hankel determinants are evaluated in [H15a].

Corollary 6.2. *For all integers $b \geq 2$, both*

$$\prod_{k\geq 0}(1 - b^{-3^k}) \quad \textit{and} \quad \prod_{k\geq 0}(1 \pm b^{-3^k} - b^{-2\cdot 3^k})$$

are transcendental and their irrationality exponents are equal to 2.

We are now in position to establish Theorem 2.2.

Proof of Theorem 2.2. From formula (2.2) and $D(0) = 1$, we obtain directly that the power series $f(z)$ converges in the unit disk, its coefficients in power series expansion are integers, and $f(z) = \big(1+uz+2z^2\frac{C(z)}{D(z)}\big)f(z^2)$,

$$\frac{1}{f(z)} = 1 - uz + \big(- 2C(0) + u^2 - u\big)z^2 + \cdots$$

Since $u(u-1)$ is even, we can define $g(z) \in \mathbb{Z}[[z]]$ by

$$f(z) = \frac{1}{1 - uz + 2z^2 g(z)}. \tag{6.2}$$

By Theorem 5.1 (ii) (or Lemma 2.2 in [H15b]), the Hankel determinants of f and those of g are tightly related by

$$H_n(f) = (-2)^{n-1} H_{n-1}(g). \tag{6.3}$$

By the functional equation satisfied by $f(z)$ and Formula (6.2), we obtain

$$1 - uz^2 + 2z^4 g(z^2) = \Big(1 + uz + 2z^2\frac{C(z)}{D(z)}\Big)(1 - uz + 2z^2 g(z)),$$

or $A^*(z) + B^*(z)g(z) + C^*(z)g(z^2) = 0$, where

$$\begin{aligned} A^*(z) &= (1-uz)C(z) - \frac{u(u-1)}{2}D(z),\\ B^*(z) &= (1+uz)D(z) + 2z^2 C(z),\\ C^*(z) &= -z^2 D(z). \end{aligned}$$

Since $D(0) = 1$, we have $B^*(0) = 1$, $C^*(0) = 0$, and $C^*(z) \neq 0$. So the power series $g(z) \pmod 2$ satisfies the equation (5.3) with condition (i). By Theorem 5.2 (i), the sequence $H\big(g \pmod 2\big) = H(g) \pmod 2$ of Hankel determinants is ultimately periodic over the field $\mathbb{F}_2$. On the other hand, Identity (6.2) can be rewritten as:

$$\frac{1}{f(z)} - 1 + uz = 2z^2 g(z).$$

Consequently we obtain

$$\frac{1}{f(z)} - 1 + uz \pmod 4 = 2z^2 \times \big(g(z) \pmod 2\big). \tag{6.4}$$

Since $f(z)$ is not a rational function modulo 4 and $f(0) = 1$, the power series $1/f(z)$ is not a rational function modulo 4. Then by relation (6.4), we know that the power series $g(z) \pmod 2$ is not a rational function. Combining this result with the fact that the sequence $H(g) \pmod 2$ of Hankel determinants is ultimately periodic over the field $\mathbb{F}_2$, we deduce at once that there exists an increasing sequence of positive integers $(n_i)_{i\geq 0}$ such that $H_{n_i}(g) \neq 0$ for all integers $i \geq 0$ and $\lim_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. By Relation (6.3), we have also $H_{n_i+1}(f) \neq 0$ for all integers $i \geq 0$ and $\lim_{i\to\infty} \frac{n_{i+1}+1}{n_i+1} = 1$. Let $b \geq 2$ be an integer such that for all integers $m \geq 0$, we have $D(\frac{1}{b^{2^m}})f(\frac{1}{b^{2^m}}) \neq 0$, then $D(\frac{1}{b^{2^m}}) \neq 0$, and

$$\Big(1 + \frac{u}{b^{2^m}}\Big)D\Big(\frac{1}{b^{2^m}}\Big) + \frac{2}{b^{2^{m+1}}}C\Big(\frac{1}{b^{2^m}}\Big) \neq 0,$$

hence it follows from Theorem 4.1 that $f(1/b)$ is transcendental and its irrationality exponent is equal to 2. □

7. Some applications

Proof of Theorem 2.3. Assume $\beta \neq \alpha + 1$. From the definition, we know directly that the power series $F_{\alpha,\beta}(z)$ and $G_{\alpha,\beta}(z)$ converge in the unit disk, and their coefficients in power series expansion are integers. Moreover we have also

$$-1 + (1 + z^{2^\beta})F_{\alpha,\beta}(z) - z^{2^\alpha}(1 + z^{2^\beta})F_{\alpha,\beta}(z^2) = 0, \tag{7.1}$$

$$-1 + (1 - z^{2^\beta})G_{\alpha,\beta}(z) - z^{2^\alpha}(1 - z^{2^\beta})G_{\alpha,\beta}(z^2) = 0. \tag{7.2}$$

The above equations are of type (6.1). By Theorem 6.1 (i), to conclude, it suffices to show that $F(z) := F_{\alpha,\beta}(z) \pmod 2 = G_{\alpha,\beta}(z) \pmod 2$ is not rational over $\mathbb{F}_2$. Put

$$P(t) = z^{2^\alpha}(1 + z^{2^\beta})t^2 + (1 + z^{2^\beta})t + 1 \in \mathbb{F}_2(z)[t].$$

We have $P(F(z)) = 0$ by (7.1). By contradiction, suppose that $F(z)$ is rational over $\mathbb{F}_2$. Then $P(t)$ is reducible over $\mathbb{F}_2(z)$. As a result, we can find $A(z)$, $B(z)$, and $C(z)$, $D(z)$ in $\mathbb{F}_2[z]$ such that

$$P(t) = (A(z)t + B(z))(C(z)t + D(z)).$$

Then $B(z)D(z) = 1$, $A(z)C(z) = z^{2^\alpha}(1 + z)^{2^\beta}$, and

$$A(z)D(z) + B(z)C(z) = 1 + z^{2^\beta} = (1 + z)^{2^\beta}, \tag{7.3}$$

thus $B(z) = D(z) = 1$. From the fact that both z and $1+z$ are irreducible over $\mathbb{F}_2$, we can find two integers m, n such that $0 \leq m \leq 2^\alpha$, $0 \leq n \leq 2^\beta$, and $A(z) = z^m(1 + z)^n$, $C(z) = z^{2^\alpha - m}(1 + z)^{2^\beta - n}$. By (7.3), we obtain

$$z^m(1 + z)^n + z^{2^\alpha - m}(1 + z)^{2^\beta - n} = (1 + z)^{2^\beta},$$

from which we deduce necessarily

$$\begin{aligned}
&z^m + z^{2^\alpha - m}(1+z)^{2^\beta - 2n} = (1+z)^{2^\beta - n}, && (\text{if } 0 \le n \le 2^{\beta-1}),\\
&z^m(1+z)^{2n-2^\beta} + z^{2^\alpha - m} = (1+z)^n, && (\text{if } 2^{\beta-1} < n \le 2^\beta).
\end{aligned}$$

Put $z = 1$ in any one of the above two formulas, we get $2^\beta = 2n$, otherwise the left hand side gives 1 while the right hand side yields 0. Hence, $\beta \ge 1$ and $z^m + z^{2^\alpha - m} = 1 + z^{2^{\beta-1}}$. We have either $m = 0,\ 2^\alpha - m = 2^{\beta-1}$ or $2^\alpha - m = 0,\ m = 2^{\beta-1}$. We deduce at once $\alpha = \beta - 1$ in both cases. This situation has already been excluded, so the desired result holds. ∎

Proof of Theorem 2.4. It is well known (see [BV13]) that

$$\begin{aligned}
S(z) &= (1+z+z^2)S(z^2),\\
T(z) &= 2 - (1+z+z^2)T(z^2).
\end{aligned}$$

On the other hand, Han has recently shown in [H15b] that the Hankel determinants of $S(z)$ and $T(z)$ satisfy, for all integers $n \ge 2$,

$$\frac{H_n(S)}{2^{n-2}} \equiv \frac{H_n(T)}{2^{n-2}} \equiv \begin{cases} 0, & \text{if } n \equiv 0,1 \bmod 4,\\ 1, & \text{if } n \equiv 2,3 \bmod 4. \end{cases}$$

Hence there exists an increasing sequence of positive integers $(n_i)_{i\ge 0}$ such that $H_{n_i}(S) \ne 0$, $H_{n_i}(T) \ne 0$ for all integers $i \ge 0$ and $\lim\limits_{i\to\infty} \frac{n_{i+1}}{n_i} = 1$. It follows from Theorem 4.1 that, for all integers $b \ge 2$, both $S(1/b)$, $T(1/b)$ are transcendental, and their irrationality exponents are equal to 2. ∎

We give further concrete examples of transcendental numbers with irrationality exponent equal to 2.

In [Va15], Väänänen studied the following two power series

$$L(z) = \sum_{j=0}^{\infty} \frac{z^{2^j}}{\prod_{i=0}^{j-1}(1 - z^{2^i})}, \quad M(z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2^j}}{\prod_{i=0}^{j-1}(1 - z^{2^i})},$$

which converge in the unit disk with integer coefficients in power series expansion, and satisfy, respectively, the functional equations

$$\begin{aligned}
z(z-1) + (1-z)L(z) - L(z^2) &= 0,\\
z(z-1) + (1-z)M(z) + M(z^2) &= 0.
\end{aligned}$$

One can check directly that neither $L(z)$ nor $M(z)$ is a rational function modulo 2. By Theorem 6.1 (ii), we obtain the following result, of which the second part was proved firstly by Väänänen [Va15].

Theorem 7.1. *For all integers $b \geq 2$, both $L(1/b)$ and $M(1/b)$ are transcendental and their irrationality exponents are equal to* 2.

In a forthcoming paper [FH15], the Hankel determinants of the following power series F_5, F_{11}, F_{13}, F_{17a} and F_{17b}, satisfying the equations

$$\begin{aligned}
F_5(z) &= (1 - z - z^2 - z^3 + z^4)\, F_5(z^5),\\
F_{11}(z) &= (1 - z - z^2 + z^3 - z^4 + z^5 + z^6 + z^7 + z^8 - z^9 - z^{10})\, F_{11}(z^{11}),\\
F_{13}(z) &= (1 - z - z^2 + z^3 - z^4 - z^5 - z^6 - z^7 - z^8\\
&\qquad + z^9 - z^{10} - z^{11} + z^{12})\, F_{13}(z^{13})\\
F_{17a}(z) &= (1 - z - z^2 + z^3 - z^4 + z^5 + z^6 + z^7 + z^8 + z^9\\
&\qquad + z^{10} + z^{11} - z^{12} + z^{13} - z^{14} - z^{15} + z^{16})\, F_{17a}(z^{17}),\\
F_{17b}(z) &= (1 - z - z^2 - z^3 + z^4 + z^5 - z^6 + z^7 + z^8 + z^9\\
&\qquad - z^{10} + z^{11} + z^{12} - z^{13} - z^{14} - z^{15} + z^{16})\, F_{17b}(z^{17})
\end{aligned}$$

are studied and are shown to verify the following relations

$$H_n(F_5)/2^{n-1} \equiv H_n(F_{11})/2^{n-1} \equiv H_n(F_{13})/2^{n-1} \equiv 1 \pmod 2,$$
$$H_n(F_{17a})/2^{n-1} \equiv H_n(F_{17b})/2^{n-1} \equiv 1 \pmod 2.$$

All these power series converge in the unit disk with integer coefficients in power series expansion, and satisfy the conditions of Theorem 4.1 for all integers $b \geq 2$, thus we obtain

Theorem 7.2. *For all integers $b \geq 2$, all the $F_5(1/b)$, $F_{11}(1/b)$, $F_{13}(1/b)$, $F_{17a}(1/b)$, $F_{17b}(1/b)$ are transcendental and their irrationality exponents are equal to* 2.

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