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New Lyapunov-type inequalities for a class of even-order linear differential equations

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In this paper, we obtain some new Lyapunov-type inequalities for a class of even-order linear differential equations, the results are new and generalize and improve some early results in this field.

1 Introduction

It is well-known that the Lyapunov inequality for second-order linear differential equation

$$x''(t) + q(t)x(t) = 0 \tag{1.1}$$

states that if $q \in C[a, b]$, $x(t)$ is a nonzero solution of (1.1) such that $x(a) = x(b) = 0$, then the following inequality holds:

$$\int_a^b |q(t)|\, dt > \frac{4}{b-a} \tag{1.2}$$

and the constant 4 is sharp, which means that it cannot be replaced by a larger one.

Since this result plays an important role in the study of various properties of solutions of the differential equation (1.1) such as oscillation theory, disconjugacy and eigenvalues problems and in the application in many directions of mathematics research areas. Therefore there has been many proofs and generalizations as well as improvements in this inequality. Such as to nonlinear second order equations, to delay differential equations, to higher order differential equations, to difference equations and to differential and difference systems. See, for example, the references [1]–[14] and the references therein. In [1], Çakmark considered the following even-order linear differential equation:

$$x^{(2n)} + q(t)x = 0, \quad a \le t \le b, \tag{1.3}$$

where $q \in C[a, b]$, and $x(t)$ satisfies the following boundary conditions

$$x^{(2k)}(a) = x^{(2k)}(b) = 0, \quad k = 0, 1, 2, \ldots, n-1. \tag{1.4}$$

He obtained the following result:

If there exists a nonzero solution $x(t)$ of (1.3) satisfying (1.4), then

$$\int_a^b |q(t)|\, dt > \frac{2^{2n}}{(b-a)^{2n-1}}. \tag{1.5}$$

Recently, Zhang and He [14] established the following result:

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For $n \geq 2$, if there exists a nonzero solution $x(t)$ of (1.3) satisfying (1.4), then

$$\int_a^b |q(t)| \left[(t-a)(b-t)\right]^2 dt > \frac{3\pi^{2n-4}}{(b-a)^{2n-5}}. \tag{1.6}$$

As a direct consequence of the inequality (1.6), the following inequality holds:

$$\int_a^b |q(t)|\, dt > \frac{48\pi^{2n-4}}{(b-a)^{2n-1}}. \tag{1.7}$$

In [11], Watanabe et al., by using the best Sobolev constant method, obtained the following result: If (1.3) has a nonzero solution $x(t)$ satisfying the boundary condition (1.4), then the following inequality holds:

$$\int_a^b |q(t)|\, dt > \frac{2^{2n-1}\pi^{2n}}{(2^{2n}-1)\zeta(2n)(b-a)^{2n-1}}, \tag{1.8}$$

where $\zeta(s)$ is the Riemann zeta function: $\zeta(s) = \sum_{k=1}^{+\infty} \frac{1}{k^s},\ s > 1$. Moreover, the inequality is sharp, that is, the left-hand side of this inequality cannot be replaced by a larger one.

If we replace the boundary condition (1.4) by the following Dirichlet boundary condition:

$$x^{(k)}(a) = x^{(k)}(b) = 0, \qquad k = 0, 1, 2, \ldots, n-1. \tag{1.9}$$

Then it was first conjectured by Levin [6] that if Equation (1.3) has a nonzero solution $x(t)$ satisfying (1.9), then the following inequality holds:

$$\frac{4^{2n-1}(2n-1)[(n-1)!]^2}{(b-a)^{2n-1}} < \int_a^b |q(t)|\, dt. \tag{1.10}$$

Later, Das and Vatsala [3] proved (1.10) and showed that the inequality is sharp.

2 Main result

In this paper, we consider the following even-order linear differential equation:

$$x^{(2n)} + \sum_{k=0}^{n} p_k(t)x^{(k)} = 0, \tag{2.1}$$

where $p_k(t) \in C[a,\ b],\ k = 0, 1, 2, \ldots, n$, and obtained some new Lyapunov-type inequalities under the boundary conditions (1.4) and (9).

The main results of this paper are the following theorems:

Theorem 2.1 *If the boundary value problem (2.1) and (1.9) has a nonzero solution $x(t)$, then we have the following inequality:*

$$\frac{2^{2n-1}(n-1)!\sqrt{2n-1}}{(b-a)^{n-1/2}} < \left(\int_a^b p_n^2(t)\, dt\right)^{\frac{1}{2}} + \sum_{k=0}^{n-1} \frac{(b-a)^{n-k-1/2}}{\sqrt{2n-2k-1}[(n-k-1)!]2^{2n-2k-1}} \int_a^b |p_k(t)|\, dt. \tag{2.2}$$

Theorem 2.2 *If the boundary value problem (2.1) and (1.4) has a nonzero solution $x(t)$, then the following inequality holds:*

$$\frac{2^{n-1/2}\pi^n}{\sqrt{(2^{2n}-1)\zeta(2n)}(b-a)^{n-1/2}} < \left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} + \sum_{k=0}^{n-1}\frac{(b-a)^{n-k-1/2}\sqrt{(2^{2(n-k)}-1)\zeta(2(n-k))}}{2^{n-k-1/2}\pi^{n-k}}\int_a^b |p_k(t)|\,dt, \tag{2.3}$$

where $\zeta(s)$ is the Riemann zeta function: $\zeta(s)=\sum_{k=1}^{+\infty}\frac{1}{k^s},\ s>1$.

If $p_1(t)=p_2(t)=\cdots=p_n(t)\equiv 0$ and denote $q(t)=p_0(t)$, then (2.2) and (2.3) reduces to (1.10) and (1.8) respectively.

3 Proof of theorems

For the proof of Theorem 2.1 and Theorem 2.2, we need the following lemmas:

Lemma 3.1 (Theorem 1.2 and Corollary 1.3 [10]) *Let $M\in N$, $H=H(M)$ be a Sonolev space associated with the inner product $(\cdot,\cdot)_M$:*

$$H=\left\{u \mid u^{(M)}\in L^2(a,\ b), u^{(k)}(a)=u^{(k)}(b)=0, (0\le k\le M-1)\right\},$$

$$(u,v)_M=\int_a^b u^{(M)}(t)\overline{v}^{(M)}(t)\,dt, \qquad \|u\|_M^2=(u,u)_M.$$

Define the functional $S(u)$ as follows:

$$S(u)=\frac{\left(\sup_{a\le t\le b}|u(t)|\right)^2}{\|u\|_M^2}.$$

Then the supremum $C(M)$ of the Sobolev functional S is given by

$$C(M)=\frac{(b-a)^{2M-1}}{(2M-1)[(M-1)!]^2 4^{2M-1}}. \tag{3.1}$$

Then for any $u\in H$, the best constant of the Sobolev inequality

$$\left(\sup_{a\le t\le b}|u(t)|\right)^2\le C\int_a^b\left|u^{(M)}(t)\right|^2dt,$$

is $C(M)$.

Lemma 3.2 (Proposition 2.1[11]) *Let $M\in N$,*

$$H_D=\left\{u \mid u^{(M)}\in L^2(a,\ b), u^{(2k)}(a)=u^{(2k)}(b)=0,\ 0\le k\le[(M-1)/2]\right\}.$$

Then for any $u\in H_D$, there exists a positive constant $D=D(M)$ such that the Sobolev inequality:

$$\left(\sup_{a\le t\le b}|u(t)|\right)^2\le D\int_a^b\left|u^{(M)}(t)\right|^2dt$$

holds. Moreover, the best constant $D(M)$ is as follows:

$$D(M)=\frac{\left(2^{2M}-1\right)\zeta(2M)(b-a)^{2M-1}}{2^{2M-1}\pi^{2M}}.$$

4 Proof of theorems

4.1 Proof of Theorem 2.1

Proof. Multiplying both sides of (2.1) by $x(t)$ and integrating from a to b by parts, then by using the boundary value condition (1.9), we obtain

$$\int_a^b x^{(2n)}(t)x(t)\,dt = (-1)^n \int_a^b \left(x^{(n)}(t)\right)^2 dt = -\sum_{k=0}^{n} \int_a^b p_k(t)x^{(k)}(t)x(t)\,dt.$$

From this equation we obtain

$$\begin{aligned}\textstyle\int_a^b \left(x^{(n)}(t)\right)^2 dt &\le \sum_{k=0}^{n} \int_a^b \left|p_k(t)\right| \left|x^{(k)}(t)x(t)\right| \,dt \\ &= \int_a^b \left|p_n(t)\right| \left|x^{(n)}(t)x(t)\right| \,dt \\ &\quad + \sum_{k=0}^{n-1} \int_a^b \left|_k(t)\right| \left|x^{(k)}(t)x(t)\right| \,dt.\end{aligned} \tag{4.1}$$

Now, by using Lemma 3.1, we get for any $t \in [a,\ b]$,

$$|x(t)| \le \sqrt{C(n)} \left(\int_a^b (x^{(n)}(t))^2 dt\right)^{\frac{1}{2}} \tag{4.2}$$

and

$$\left|x^{(k)}(t)\right| \le \sqrt{C(n-k)} \left(\int_a^b (x^{(n)}(t))^2 dt\right)^{\frac{1}{2}}. \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), we obtain

$$\begin{aligned}\int_a^b \left(x^{(n)}(t)\right)^2 dt \le \sqrt{C(n)} \int_a^b \left|p_n(t)\right| \left|x^{(n)}(t)\right| \,dt \left(\int_a^b (x^{(n)}(t))^2 dt\right)^{\frac{1}{2}} \\ + \sum_{k=0}^{n-1} \sqrt{C(n)C(n-k)} \int_a^b \left|p_k(t)\right| \,dt \int_a^b \left(x^{(n)}(t)\right)^2 dt.\end{aligned} \tag{4.4}$$

Now by applying Hölder's inequality, we get

$$\int_a^b \left|p_n(t)x^{(n)}(t)\right| \,dt \le \left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} \left(\int_a^b \left(x^{(n)}(t)\right)^2 dt\right)^{\frac{1}{2}}. \tag{4.5}$$

Substituting (4.5) into (4.4) and by using the fact that $x(t)$ is not a constant function, we obtain the following strict inequality:

$$\begin{aligned}\int_a^b \left(x^{(n)}(t)\right)^2 dt < \sqrt{C(n)} \left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} \int_a^b \left(x^{(n)}(t)\right)^2 dt \\ + \sum_{k=0}^{n-1} \sqrt{C(n)C(n-k)} \int_a^b |p_k(t)|\,dt \int_a^b \left(x^{(n)}(t)\right)^2 dt.\end{aligned} \tag{4.6}$$

Dividing both sides of (4.6) by $\int_a^b (x^{(n)}(t))^2 dt$, which can be proved to be positive by using the boundary value condition (1.9) and the assumption that $x(t) \not\equiv 0$, we obtain

$$1 < \sqrt{C(n)} \left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} + \sum_{k=0}^{n-1} \sqrt{C(n)C(n-k)} \int_a^b |p_k(t)|\,dt,$$

which is equivalent to (2.2). Thus we finished the proof of Theorem 2.1.

4.2 Proof of Theorem 2.2

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1 (instead of using Lemma 3.1, we need to use Lemma 3.2), so we omit it for simplicity.

Remark 4.1 The results of Theorem 2.1 and Theorem 2.2 are new and natural generalization of the well-known Lyapunov inequality for second order equation. Moreover, inequalities (2.2) and (2.3) are natural generalization of (1.10) and (1.8) respectively.

From the expression of $D(n)$ and the property $\zeta(2n) \to 1$, if $n \to +\infty$, we define $\frac{1}{D(n)} = \frac{C_n}{(b-a)^{2n-1}}$, then

$$C_n = \frac{\pi^{2n}}{2\left(1 - \frac{1}{2^{2n}}\right)\zeta(2n)} \longrightarrow \frac{\pi^{2n}}{2}, \quad \text{if } n \longrightarrow +\infty.$$

Hence, for $n \gg 1$, we can use $\frac{\pi^{2n}}{2}$ to approximate the value C_n.

We give in the next table the first 6 values of the numerators of the right-hand sides of (1.5), (1.7) and (1.8)[1].

n	1	2	3	4	5	6	7	8
(1.5)	4	16	64	256	1024	4096	16384	65536
(1.7)		48	473.7	4675.6	46146.7	455449.5	4495105.4	44364912.1
(1.8)	4	48	480	4743.5	46823.2	462133.7	4561089.6	45016023.7

Table 1

This table shows that the result of (1.7) improves the result of (1.5) significantly and the result of (1.8) improves (1.7). Since (1.8) is sharp, it cannot be improved anymore.

The next table gives the first 8 values of $\zeta(2n)$ and $\delta_n = \frac{1}{\left(1-\frac{1}{2^{2n}}\right)\zeta(2n)}$.

n	1	2	3	4	5	6	7	8
$\zeta(2n)$	1.65	1.08	1.02	1.004	1.001	1.0002	1.00006	1.00001
δ_n	0.81	0.98	0.99	0.999	0.99998	1.000004	1.000001	1.0000002

Table 2

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