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Entanglement classification and invariant-based entanglement measures

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We propose a method of classifying n-qubit states into stochastic local operations and classical communication inequivalent families in terms of the rank of the square matrix $C(i\sigma_y)^{\otimes k}C^T$, where C is the rectangular coefficient matrix of the state and σ_y is the Pauli operator. The rank of the square matrix $C(i\sigma_y)^{\otimes k}C^T$ is capable of distinguishing between n-qubit Greenberger-Horne-Zeilinger and W states. The determinant of the matrix gives rise to a family of polynomial invariants for n qubits which include as special cases well-known polynomial invariants in the literature. In addition, explicit expressions can be given for these polynomial invariants and this allows us to investigate the properties of entanglement measures built upon the absolute values of polynomial invariants for product states.

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I. INTRODUCTION

Quantum entanglement is an essential resource in quantum teleportation, quantum cryptography, and quantum computation [1]. A crucial task in entanglement theory is the classification of entanglement. To classify entangled states, some equivalence relation has to be introduced. Of particular importance is the equivalence under stochastic local operations and classical communication (SLOCC). If two states are SLOCC equivalent then they can perform the same tasks in quantum information theory [2]. As equivalence under SLOCC induces a natural partition of quantum states, the key point of SLOCC classification is to classify quantum states according to a criterion that is invariant under SLOCC. While entanglement classification of two and three qubits is well understood, the task of classifying entanglement beyond three qubits becomes increasingly difficult. For four or more qubits, there exists an infinite number of inequivalent SLOCC classes. It is highly desirable to partition the infinite SLOCC classes into a finite number of families such that states belonging to the same family possess similar properties, according to some criteria for determining to which family a given state belongs. Considerable efforts have been undertaken over the last decade for SLOCC entanglement classification of four-qubit states, resulting in a finite number of families or classes [3–10]. For more than four qubits, a few attempts have been made for SLOCC classification [11–16]. Despite these efforts, a SLOCC classification for general n-qubit states which results in a finite number of families with Greenberger-Horne-Zeilinger (GHZ) and W states belonging to different families is still beyond reach.

This paper is organized as follows. We first construct a matrix for an n -qubit state and we show that the rank of the matrix is preserved under SLOCC. The rank provides a simple way of classifying n-qubit states into a number of SLOCC-inequivalent families. We then exemplify the use of the rank in distinguishing n-qubit GHZ and W states as well as some four-, five-, and six-qubit states. The determinant of the matrix gives rise to a polynomial invariant of degree 2^k ($k \leqslant n/2+1$) for n qubits and this construction allows one to derive the expressions for these polynomial invariants explicitly. Intriguingly, the even n-qubit concurrence, the even n-tangle, the odd n-tangle, and polynomial invariants of degree $2^{n/2}$ for even n qubits all turn out to be special cases of polynomial invariants of degree 2^k. We also discuss the properties of entanglement measures built upon the polynomial invariants of degree 2^k for product states.

II. THE INVARIANCE OF THE RANK

We follow the notation of [14]. Let $|\psi'\rangle = \sum_{k=0}^{2^n-1} a_k|k\rangle$ be any pure state of any n qubits. We let $C^{(n)}_{1,2,\dots,i}(|\psi'\rangle)$ be the $2^i \times 2^{n-i}$ coefficient matrix of the state $|\psi'\rangle$, whose entries are the coefficients $a_0,a_1,\dots,a_{2^n-1}$ of the state $|\psi'\rangle$ arranged in ascending lexicographical order. Here the bits 1 to i and $i+1$ to n are referred to as the row bits and column bits, respectively. In particular, when $i=0$, $C^{(n)}_{\emptyset}(|\psi'\rangle)$ reduces to the row vector $(a_0,\dots,a_{2^n-1})$ and, when $i=n$, $C^{(n)}_{1,\dots,n}(|\psi'\rangle)$ reduces to the column vector $(a_0,\dots,a_{2^n-1})^T$. Note that qubits $q_1,q_2,\dots,q_i$ can also be chosen as row bits.

Recall that two n-qubit states $|\psi\rangle$ and $|\psi'\rangle$ are SLOCC equivalent if and only if there are local invertible operators $\mathcal{A}_1$, $\mathcal{A}_2$, …, and $\mathcal{A}_n$ such that [2]

$$|\psi'\rangle = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_n|\psi\rangle. \tag{1}$$

When qubits $q_1,q_2,\dots,q_i$ are chosen as row bits, the coefficient $2^i \times 2^{n-i}$ matrix $C^{(n)}_{q_1,q_2,\dots,q_i}(|\psi'\rangle)$ satisfies the following matrix equation [13,14]:

$$C^{(n)}_{q_1,q_2,\dots,q_i}(|\psi'\rangle) = (\mathcal{A}_{q_1}\otimes\cdots\otimes\mathcal{A}_{q_i})C^{(n)}_{q_1,q_2,\dots,q_i}(|\psi\rangle)(\mathcal{A}_{q_{i+1}}\otimes\cdots\otimes\mathcal{A}_{q_n})^T. \tag{2}$$

1

Taking the transpose of both sides of Eq. (2) and after some algebra, we obtain

$$C^{(n)}_{q_1,q_2,\dots,q_i}(|\psi'\rangle)\upsilon^{\otimes(n-i)}\big[C^{(n)}_{q_1,q_2,\dots,q_i}(|\psi'\rangle)\big]^T = (\mathcal{A}_{q_1}\otimes\cdots\otimes\mathcal{A}_{q_i})C^{(n)}_{q_1,q_2,\dots,q_i}(|\psi\rangle)$$

$$\times\left(\mathcal{A}_{q_{i+1}}^{T}\upsilon\mathcal{A}_{q_{i+1}}\otimes\cdots\otimes\mathcal{A}_{q_n}^{T}\upsilon\mathcal{A}_{q_n}\right)$$
$$\times\left[C_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi\rangle)\right]^{T}(\mathcal{A}_{q_1}\otimes\cdots\otimes\mathcal{A}_{q_i})^{T}, \quad (3)$$

where $\upsilon=\sqrt{-1}\sigma_y$ and $\sigma_y$is the Pauli operator.

We let

$$\Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi'\rangle = C_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi'\rangle)\upsilon^{\otimes(n-i)}\left[C_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi'\rangle)\right]^{T}. \quad (4)$$

It is clear that $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi'\rangle$ is a square matrix of order 2^i for n qubits. Invoking the fact that $\mathcal{A}_k^T\upsilon\mathcal{A}_k=(\det\mathcal{A}_k)\upsilon$, we may rewrite Eq. (3) as

$$\Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi'\rangle) = \left(\Pi_{k=i+1}^{n}\det\mathcal{A}_{q_k}\right)(\mathcal{A}_{q_1}\otimes\cdots\otimes\mathcal{A}_{q_i}) \times\Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi\rangle)(\mathcal{A}_{q_1}\otimes\cdots\otimes\mathcal{A}_{q_i})^{T}. \quad (5)$$

In the following, we will omit the state $|\psi'\rangle$ and simply write $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ whenever the state is clear from the context. It immediately follows from Eq. (5) that the rank of the square matrix $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ of an n-qubit state is invariant under SLOCC. This leads to the following theorem.

Theorem. If two n-qubit states are SLOCC equivalent then their square matrices $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ given above have the same rank.

Restated in the contrapositive the theorem reads: If two square matrices $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ associated with two n-qubit states differ in their ranks, then the two states belong necessarily to different SLOCC classes.

III. APPLICATIONS TO SLOCC CLASSIFICATION

A. Classification of four-qubit states

Verstraete *et al.* partitioned four-qubit states into nine SLOCC-inequivalent families, two of which are L_{abc_2} and L_{ab_3} [3]. Later, it was pointed out that L_{ab_3} is equivalent to the subfamily $L_{abc_2}(a=c)$ of L_{abc_2} obtained by setting $a=c$ [6]. In [13], we showed that $L_{ab_3}(a=b=0)$ is inequivalent to $L_{abc_2}(a=c)$ using the rank of coefficient matrices. Alternatively, Sharma *et al.* proved that $L_{abc_2}(a=c)$ and L_{ab_3} are not SLOCC equivalent using negativity fonts [17]. Here by elaborating further on the relationship between L_{ab_3} and $L_{abc_2}(a=c)$, we show by using the rank of the square matrix that, when $a\neq 0$, L_{ab_3} is contained in L_{abc_2}. In Table I we list the rank of $\Omega_{1,2}^{(4)}$ for L_{ab_3} and $L_{abc_2}(a=c)$.

It follows from Table I that, when $a=0$, L_{ab_3} and $L_{abc_2}(a=c)$ are inequivalent to each other. Furthermore, $L_{ab_3}(a=0)$ and $L_{ab_3}(a\neq 0)$ are inequivalent to each other. Likewise, $L_{abc_2}(a=c=0)$ and $L_{abc_2}(a=c\neq 0)$ are inequivalent to each other. It turns out that, when $a\neq 0$, L_{ab_3} and $L_{abc_2}(a=c)$ are equivalent to each other. This can be seen as follows. A tedious calculation shows that, when $a\neq 0$, $L_{abc_2}(a=c)$ and L_{ab_3} satisfy the following equation:

$$L_{abc_2}(a=c)=\mathcal{A}_1\otimes\mathcal{A}_2\otimes\mathcal{A}_3\otimes\mathcal{A}_4\, L_{ab_3}, \quad (6)$$

where $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ are invertible local operators given by

$$\mathcal{A}_1=\begin{pmatrix}\frac{1}{2a^{3/2}} & 0\\ 0 & \frac{1}{2\sqrt{2}a^2}\end{pmatrix},\quad \mathcal{A}_2=\begin{pmatrix}0 & 1\\ -\sqrt{2a} & 0\end{pmatrix},$$
$$\mathcal{A}_3=\begin{pmatrix}-\sqrt{2a} & 0\\ -i\sqrt{2} & 2a\end{pmatrix},\quad \mathcal{A}_4=\begin{pmatrix}-i & \sqrt{2}a\\ -\sqrt{a} & 0\end{pmatrix}.$$

Therefore, when $a=0$, L_{ab_3} and $L_{abc_2}(a=c)$ are inequivalent to each other, whereas when $a\neq 0$, L_{ab_3} and $L_{abc_2}(a=c)$ are equivalent to each other. In particular, when $a\neq 0$, we have SLOCC equivalence between L_{ab_3} and $L_{abc_2}(a=c)$ for the following cases: (i) $a=b$; (ii) $a=-b$; (iii) $b=3a$; (iv) $b=-3a$; (v) $b=0$; (vi) $ab\neq 0$ and $a\neq\pm b$ and $b\neq\pm-3a$.

TABLE I. The rank of $\Omega_{1,2}^{(4)}$ for L_{ab_3} and $L_{abc_2}(a=c)$.

	$a=0$, $b=0$	$a=0$, $b\neq 0$	$a\neq 0$, $b=0$	$a\neq 0$, $b\neq 0$
L_{ab_3}	1	2	3	4
$L_{abc_2}(a=c)$	0	1	3	4

B. Classification of some five-qubit and six-qubit states

We list in Table II the rank of $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ for the five-qubit states and six-qubit states introduced in [18]. Consulting the table, the five-qubit states $|\Psi_2\rangle$, $|\Psi_4\rangle$, $|\Psi_5\rangle$, and $|\Psi_6\rangle$ are inequivalent to one another under SLOCC and they can be distinguished via the rank of $\Omega_1^{(5)}$ and $\Omega_3^{(5)}$. Similarly, the six-qubit states $|\Xi_2\rangle$, $|\Xi_4\rangle$, $|\Xi_5\rangle$, $|\Xi_6\rangle$, and $|\Xi_7\rangle$ are inequivalent to one another under SLOCC and they can be distinguished via the rank of $\Omega_{1,2}^{(6)}$ and $\Omega_{1,2,3,4}^{(6)}$.

TABLE II. SLOCC classification of some five-qubit and six-qubit states.

Five-qubit state	Rank of $\Omega_1^{(5)}$	Rank of $\Omega_3^{(5)}$	Six-qubit state	Rank of $\Omega_{1,2}^{(6)}$	Rank of $\Omega_{1,2,3,4}^{(6)}$
$\vert\Psi_2\rangle$	2	2	$\vert\Xi_2\rangle$	2	2
$\vert\Psi_4\rangle$	0	0	$\vert\Xi_4\rangle$	0	2
$\vert\Psi_5\rangle$	0	1	$\vert\Xi_5\rangle$	0	3
$\vert\Psi_6\rangle$	1	1	$\vert\Xi_6\rangle$	1	3
			$\vert\Xi_7\rangle$	2	3

C. Classification of n-qubit states

We exemplify the classification with n-qubit GHZ and W states. We find that $\Omega_1^{(n)}$ has rank 2 for n-qubit GHZ states, rank 1 for 3-qubit W states, and rank 0 for n-qubit W states for $n\geqslant 4$. Hence n-qubit GHZ states can be distinguished from n-qubit W states under SLOCC via the rank of $\Omega_1^{(n)}$. In addition, one may also distinguish cluster states from GHZ (W) states using the ranks of $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}$. For example, the cluster state of four qubits can be readily distinguished from a four-qubit GHZ (four-qubit W) state using the ranks of $\Omega_1^{(4)}$ and $\Omega_{1,2}^{(4)}$.

More generally, let σ denote the sequence $q_1,q_2,\ldots,q_i$ and F_r^{σ} be the set of n-qubit states with the rank of $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ being equal to r. Thus, n-qubit states are partitioned into 2^i+1

TABLE III. The polynomial invariant $|\det \Omega_{\emptyset}^{(n)}|$ of degree 2.

Qubits	Expressions	Polynomial invariants
$n = 2$	$\vert \det \Omega_{\emptyset}^{(2)}\vert$	Concurrence [29]
even n	$\vert \det \Omega_{\emptyset}^{(n)}\vert$	n-qubit concurrence [28,30]
odd n	$\vert \det \Omega_{\emptyset}^{(n)}\vert = 0$	[a]

[a]Note that for odd n, $|\det \Omega_{\emptyset}^{(n)}| = 0$. This reveals that no such nontrivial polynomial invariant of degree 2 exists for odd-n qubits.

SLOCC-inequivalent families by the theorem above, i.e., F_0^σ, $F_1^\sigma, \ldots$, and $F_{2^i}^\sigma$.

IV. POLYNOMIAL INVARIANTS OF DEGREE 2^k

Several approaches have been proposed to construct polynomial invariants [19–23]. However, the computational complexity grows very rapidly as the number of qubits increases (e.g., some methods are not readily generalized to more complicated Hilbert spaces). Accordingly, the expressions of polynomial invariants have thus far been given only up to five qubits. Recently, a few attempts have been made to construct polynomial invariants with explicit expressions [24–28].

Corollary 1. Let $|\psi\rangle$ and $|\psi'\rangle$ be two SLOCC-equivalent states of n qubits, then the following equation holds for $0 \leqslant i \leqslant n$:

$$\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi'\rangle) = \det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi\rangle)\left(\Pi_{k=1}^{n} \det \mathcal{A}_k\right)^{2^i}. \tag{7}$$

Proof. Taking the determinant of both sides of Eq. (5) yields the desired result. ■

As an immediate consequence of Corollary 1, $\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ is a polynomial invariant of degree 2^{i+1}, where $0 \leqslant i \leqslant n/2$. Otherwise, $\Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ is not full rank.

Polynomial invariants constructed above include as special cases several well-known polynomial invariants in the literature. Below are some examples.

Example 1. Set $i = 0$. Then $|\det \Omega_{\emptyset}^{(n)}|$ is a polynomial invariant of degree 2. See Table III.

Example 2. We set $i = 1$. Then $\det \Omega_1^{(n)}$ is a polynomial invariant of degree 4. In particular, $4|\det \Omega_1^{(3)}|$ is equal to the 3-tangle [25,31]. Furthermore, $4|\det \Omega_1^{(n)}|$ is a natural extension of the 3-tangle to general n qubits. See Table IV.

We remark that for n even, $\det \Omega_1^{(n)}$ (i.e., even n-tangle) is invariant under permutations [33]. For n odd, we may choose qubit j, $j = 1, \ldots, n$, as the row bit for the coefficient matrix $C_j^{(n)}$. This yields n polynomial invariants $\det \Omega_j^{(n)}$ of degree 4 for odd n ($\geqslant$5) qubits (for five qubits, see [23]) [25,33,34].

TABLE IV. The polynomial invariant $\det \Omega_1^{(n)}$ of degree 4.

Qubits	Expressions	Polynomial invariants
$n = 3$	$4\vert \det \Omega_1^{(3)}\vert$	3-tangle [25,31]
even n	$4\vert \det \Omega_1^{(n)}\vert$	even n-tangle [32]
odd n	$4\vert \det \Omega_1^{(n)}\vert$	odd n-tangle [25]

We emphasize that these n polynomial invariants $\det \Omega_j^{(n)}$ of degree 4 for any odd $n \geqslant 5$ qubits are linearly independent (this is particularly true for five qubits [23]). This can be proved by resorting to the following properties of $\det \Omega_j^{(n)}$ for n-odd qubits [34]:

(1) $(i,j) \det \Omega_j^{(n)} = \det \Omega_i^{(n)}$, where (i,j) is the transposition of qubits i and j.

(2) $\det \Omega_j^{(n)}$ is invariant under any permutation of qubits not involving qubit j.

Example 3. Let n be even and $i = n/2$. Then $C_{1\cdots(n/2)}^{(n)}$ is a square matrix. In view of Eqs. (4) and (7), we have

$$\det C_{1\cdots(n/2)}^{(n)}(|\psi'\rangle) = \det C_{1\cdots(n/2)}^{(n)}(|\psi\rangle)\left(\Pi_{k=1}^{n} \det \mathcal{A}_k\right)^{2^{(n-2)/2}}. \tag{8}$$

As an immediate consequence, $\det C_{1\cdots(n/2)}^{(n)}$ is a determinant invariant of degree $2^{n/2}$ and we recover the result in [26] (in particular we recover the polynomial invariants of degree 4 for four qubits given in [19]).

In light of Eq. (7), we may determine whether two n-qubit states are inequivalent to each other under SLOCC via the vanishing or not of their associated polynomial invariants $\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}$. More precisely, we have the following result.

Corollary 2. For any two SLOCC-equivalent pure states $|\psi\rangle$ and $|\psi'\rangle$ of n qubits, either both $\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi'\rangle)$ and $\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi\rangle)$ vanish or neither vanishes. In other words, if one of $\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi'\rangle)$ and $\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}(|\psi\rangle)$ vanishes while the other does not, then the two states $|\psi\rangle$ and $|\psi'\rangle$ are SLOCC inequivalent.

For example, the n-qubit GHZ and W states can also be distinguished under SLOCC as $\det \Omega_{\emptyset}^{(n)} = 0$ for the W state and $\det \Omega_{\emptyset}^{(n)} \neq 0$ for the GHZ state.

V. INVARIANT-BASED ENTANGLEMENT MEASURES

The explicit expressions of these polynomial invariants $\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}$ make it possible for us to investigate the properties of $\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}$. We next explore the properties of $|\det \Omega_{q_1,q_2,\ldots,q_i}^{(n)}|$ by use of the product state $|\psi\rangle_{1\cdots n} = |\phi\rangle_{j_1\cdots j_\ell} \otimes |\varphi\rangle_{j_{\ell+1}\cdots j_n}$, where $|\phi\rangle_{j_1\cdots j_\ell}$ is a state of ℓ qubits, $j_1, \ldots, j_\ell$, and $|\varphi\rangle_{j_{\ell+1}\cdots j_n}$ is a state of the remaining $(n-\ell)$ qubits, $j_{\ell+1}, \ldots, j_n$. We let $C_{q_1,\ldots,q_i}^{(n)}(|\psi\rangle_{1\cdots n})$ be the coefficient matrix associated with the state $|\psi\rangle_{1\cdots n}$, where $q_1, \ldots$ and q_i are chosen as row bits. We let $C_{q_1^*,\ldots,q_s^*}^{(\ell)}(|\phi\rangle_{j_1\cdots j_\ell})$ be the $2^s \times 2^{\ell-s}$ coefficient matrix associated with the ℓ-qubit state $|\phi\rangle_{j_1\cdots j_\ell}$. Here $\{q_1^*, \ldots, q_s^*\} = \{q_1, \ldots, q_i\} \cap \{j_1, \ldots, j_\ell\}$ are the row bits. We let $C_{q_1',\ldots,q_t'}^{(n-\ell)}(|\varphi\rangle_{j_{\ell+1}\cdots j_n})$ be the $2^t \times 2^{n-\ell-t}$ coefficient matrix associated with the $(n-\ell)$-qubit state $|\varphi\rangle_{j_{\ell+1}\cdots j_n}$. Here $\{q_1', \ldots, q_t'\} = \{q_1, \ldots, q_i\} \cap \{j_{\ell+1}, \ldots, j_n\}$ are the row bits. Note that $s + t = i$. From [14], we have

$$C_{q_1,\ldots,q_i}^{(n)}(|\phi\rangle_{j_1\cdots j_\ell} \otimes |\varphi\rangle_{j_{\ell+1}\cdots j_n}) = C_{q_1^*,\ldots,q_s^*}^{(\ell)}(|\phi\rangle_{j_1\cdots j_\ell}) \otimes C_{q_1',\ldots,q_t'}^{(n-\ell)}(|\varphi\rangle_{j_{\ell+1}\cdots j_n}). \tag{9}$$

Using the notation of Eq. (4), a simple calculation yields

$$\left| \det \Omega^{(n)}_{q_1,\ldots,q_i}(|\phi\rangle_{j_1\cdots j_\ell} \otimes |\varphi\rangle_{j_{\ell+1}\cdots j_n})\right|$$
$$= \left| \det \Omega^{(\ell)}_{q_1^*,\ldots,q_s^*}(|\phi\rangle_{j_1\cdots j_\ell})\right|^{2^t} \left| \det \Omega^{(n-\ell)}_{q_1',\ldots,q_t'}(|\varphi\rangle_{j_{\ell+1}\cdots j_n})\right|^{2^s}. \tag{10}$$

Clearly, the absolute value of the SLOCC polynomial invariant $\det \Omega^{(n)}_{q_1,q_2,\ldots,q_i}$ is not additive for product states. Note that it vanishes for product states with $s > \ell/2$ or $t > (n-\ell)/2$. Consider, for example, a product state of four qubits $|\psi\rangle_{1234} = |\phi\rangle_{13} \otimes |\varphi\rangle_{24}$. Then a straightforward calculation yields that $\det \Omega^{(4)}_{12}(|\psi\rangle_{1234}) = [\det \Omega^{(2)}_1(|\phi\rangle_{13})]^2 [\det \Omega^{(2)}_2(|\varphi\rangle_{24})]^2$.

Recently, it was shown that a positive homogeneous SLOCC polynomial invariant defines an n-qubit entanglement monotone if and only if the homogeneous degree is less than or equal to 4 [35]. Accordingly, the absolute value of the polynomial invariant $\det \Omega^{(n)}_{q_1,q_2,\ldots,q_i}$ with degree $\leqslant 4$ is an entanglement monotone and it can be considered as an entanglement measure for n qubits.

VI. CONCLUSION

We have constructed a matrix whose rank is preserved under SLOCC and given examples of classifying n-qubit states via the rank for n up to 6. Polynomial invariants in the form of determinants of the square matrix not only have explicit expressions but also, as special cases, recover several existing polynomial invariants in the literature. We have also studied the properties of the entanglement measures built from the absolute values of polynomial invariants on product states. We expect that the proposed approach for classifying n-qubit states and constructing polynomial invariants may find further applications.

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