# VERTEX ALGEBRAS AND QUANTUM MASTER EQUATION

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ABSTRACT. We study the effective Batalin-Vilkovisky quantization theory for chiral deformation of two dimensional conformal field theories. We establish an exact correspondence between renormalized quantum master equations for effective functionals and Maurer-Cartan equations for chiral vertex operators. The generating functions are proven to have modular property with mild holomorphic anomaly. As an application, we construct an exact solution of quantum B-model (BCOV theory) in complex one dimension that solves the higher genus mirror symmetry conjecture on elliptic curves.

## CONTENTS

## 1. INTRODUCTION

Quantum field theory provides a rich source of mathematical thoughts. One important feature of quantum field theory that lies secretly behind many of its surprising mathematical predictions is about its nature of infinite dimensionality. A famous example is the mysterious mirror symmetry conjecture between symplectic and complex geometries, which can be viewed as a version of infinite dimensional Fourier transform. Typically, many quantum problems are formulated in terms of "path integrals", which require measures that are mostly not yet known to mathematicians. Nevertheless, asymptotic analysis can always be performed with the help of the celebrated idea of renormalization.

Despite the great success of renormalization theory in physics applications, its use in mathematics is relatively limited but extremely powerful when it does apply. One such example is Kontsevich's solution [21] to the deformation quantization problem on arbitrary Poisson manifolds. Kontsevich's explicit formula of star product is obtained via graph integrals on a compactification of configuration space on the disk, which can be viewed as a geometric renormalization of the perturbative expansion of Poisson sigma model (see also [5]). Another recent example is Costello's homotopic theory [7] of effective renormalizations in the Batalin-Vilkovisky formalism. This leads to a systematic construction of factorization algebras via quantum field theories [9]. For example, a natural geometric interpretation of the Witten genus is obtained in such a way [8].

To facilitate geometric applications of effective renormalization methods, it would be key to connect renormalized quantities to geometric objects. We will be mainly interested in quantum field theory with gauge symmetries. The most general framework of quantizing gauge theories is the Batalin-Vilkovisky formalism [4], where the quantum consistency of gauge transformations is described by the so-called *quantum master equation*. There have developed several mathematical approaches to incorporate Batalin-Vilkovisky formalism with renormalizations since their birth. The central quantity of all approaches lies in the renormalized quantum master equation. In this paper, we will mainly discuss the formalism in [7], which has developed a convenient framework that is also rooted in the homotopic culture of derived algebraic geometry. A brief introduction to the philosophy of this approach is discussed in Section 2.

The simplest nontrivial example is given by quantum mechanical models, which can be viewed as quantum field theories in one dimension. The renormalized Batalin-Vilkovisky quantization in the above fashion is analyzed in [18, 23] for topological quantum mechanics. In particular, it is shown in [23] that the renormalized quantum master equation can be identified with the geometric equation of Fedosov's abelian connection [14] on Weyl bundles over symplectic manifolds. The algebraic nature of this correspondence is reviewed in Section 2.4. Such a correspondence leads to a simple geometric approach to algebraic index theorem [15, 31], where the index formula follows from the homotopic renormalization group flow together with an equivariant localization of BV integration [23].

In this paper, we study systematically the renormalized quantum master equation in two dimensions. We will focus on quantum theories obtained by chiral deformations of free CFT's (see Section 3.2 for our precise set-up). One important feature of such two dimensional chiral theories is that they are free of ultra-violet divergence (see Theorem 3.9). This greatly simplifies the analysis of quantization since singular counter-terms are not required. However, the renormalized quantum master equation requires quantum corrections by chiral local functionals. Such quantum corrections could in principle be very complicated.

One of our main results in this paper (Theorem 3.11) is an exact description of the quantum corrections in terms of vertex algebras. Briefly speaking, Theorem 3.11 states that the renormalized quantum master equations (QME) is equivalent to quantum corrected chiral vertex operators that satisfies Maurer-Cartan (MC) equations. In other words, we have an exact description of the quantization of chiral deformation of two dimensional conformal field theories

$$\boxed{\text{renormalized QME}} \Longleftrightarrow \boxed{\text{MC equations for chiral vertex operators}}$$

The Maurer-Cartan equation serves as an integrability condition for chiral vertex operators, which is often related to integrable hierarchies in concrete cases. We discuss such an example in Section 4. Furthermore, we prove a general result on the modularity property of the generating functions and their holomorphic anomaly (Theorem 3.21). This work is also motivated from understanding Dijkgraaf's description [13] of chiral deformation of conformal field theories.

The above correspondence can be viewed as the two dimensional vertex algebra analogue of the one dimensional result in [23]. In fact, one main motivation of the current work is to explore the analogue of index theorem for chiral vertex operators in terms of the method of equivariant localization in BV integration as proceeded in [23]. It allows us to solve many quantization problems in terms of powerful techniques in vertex algebras.

As an application in Section 4, we construct an exact solution of quantum B-model on elliptic curves, which leads to the solution of the corresponding higher genus mirror symmetry conjecture. Mirror symmetry is a famous duality between symplectic (A-model) and complex (B-model) geometries that arises from superconformal field theories. It has been a long-standing challenge for mathematicians to construct quantum B-model on compact Calabi-Yau manifolds. In [10], we construct a gauge theory of polyvector fields on Calabi-Yau manifolds (called BCOV theory) as a generalization of the Kodaira-Spencer gauge theory [3]. It is proposed in [10] (as a generalization of [3]) that the Batalin-Vilkovisky quantization of BCOV theory leads to quantum B-model that is mirror to the A-model Gromov-Witten theory of counting higher genus curves. Our construction in Section 4 gives a concrete realization of this program. This leads to the first mathematically fully established example of quantum B-model on compact Calabi-Yau manifolds.

Our result in Section 4 also leads to an interesting result in physics. Quantum BCOV theory can be viewed as a complete description of topological B-twisted closed string field theory in the sense of Zwiebach [33]. Zwiebach's closed string field theory describes the dynamics of closed strings in term of the so-called string vertices. Despite the beauty of this construction, string vertices are very difficult to compute and few concrete examples are known. Our exact solution in Section 4 can be viewed as giving an explicit realization of Zwiebach's string vertices for B-twisted topological string on elliptic curves.

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**Conventions**

- Let $V$ be a $\mathbb{Z}$-graded $k$-vector space. We use $V_m$ to denote its degree $m$ component. Given $a \in V_m$, we let $\bar{a} = m$ be its degree.
  - $V[n]$ denotes the degree shifting of $V$ such that $V[n]_m = V_{n+m}$.
  - $V^*$ denotes its dual such that $V^*_m = \mathrm{Hom}_k(V_{-m}, k)$. Our base field $k$ will mainly be $\mathbb{R}$ or $\mathbb{C}$.
  - $\mathrm{Sym}^m(V)$ and $\wedge^m(V)$ denote the graded symmetric product and graded skew-symmetric product respectively. We also denote

    $$\mathrm{Sym}(V) := \bigoplus_{m\geq 0} \mathrm{Sym}^m(V), \quad \widehat{\mathrm{Sym}}(V) := \prod_{m\geq 0} \mathrm{Sym}^m(V).$$

    Given $I = \sum\limits_{m\geq 0} I_m \in \widehat{\mathrm{Sym}}(V^*)$, and $a_1, \cdots, a_m \in V$, we denote its $m$-order Taylor coefficient

    $$\frac{\partial}{\partial a_1} \cdots \frac{\partial}{\partial a_m} I(0) := I_m(a_1, \cdots, a_m)$$

    where we have viewed $I_m$ as a multi-linear map $V^{\otimes m} \to k$.
  - Given $P \in \mathrm{Sym}^2(V)$, it defines a "second order operator" $\partial_P$ on $\mathrm{Sym}(V^*)$ or $\widehat{\mathrm{Sym}}(V^*)$ by

    $$\partial_P : \mathrm{Sym}^m(V^*) \to \mathrm{Sym}^{m-2}(V^*), \quad I \to \partial_P I,$$

    where for any $a_1, \cdots, a_{m-2} \in V$,

    $$\partial_P I(a_1, \cdots a_{m-2}) := I(P, a_1, \cdots a_{m-2}).$$
  - $V[z]$, $V[[z]]$ and $V((z))$ denote polynomial series, formal power series and Laurent series respectively in a variable $z$ valued in $V$.
- Let $\mathcal{A}$ be a graded commutative algebra. $[-,-]$ always means the graded commutator, i.e., for elements $a, b$ with specific degrees,

  $$[a, b] := a \cdot b - (-1)^{\bar{a}\bar{b}} b \cdot a.$$

  We always assume Koszul sign rule in dealing with graded objects.
- $\otimes$ without subscript means tensoring over the real numbers $\mathbb{R}$.
- Given a manifold $X$, we denote the space of real smooth forms by

  $$\Omega^\bullet(X) = \bigoplus_k \Omega^k(X)$$

where $\Omega^k(X)$ is the subspace of $k$-forms. If furthermore X is a complex manifold, we denote the space of complex smooth forms by

$$\Omega^{\bullet,\bullet}(X) = \bigoplus_{p,q} \Omega^{p,q}(X) = \Omega^\bullet(X) \otimes \mathbb{C}$$

where $\Omega^{p,q}(X)$ is the subspace of $(p,q)$-forms.

- $\mathrm{Dens}(X)$ denotes the density bundle on a manifold $X$. When $X$ is oriented, we naturally identify $\mathrm{Dens}(X)$ with top differential forms on $X$.
- Let E be a vector bundle on a manifold $X$. $\mathcal{E} = \Gamma(X, E)$ denotes the space of smooth sections, and $\mathcal{E}' = D'(X, E)$ denotes the distributional sections. If $E^*$ is the dual bundle of $E$, then we have a natural pairing

$$\mathcal{E}' \otimes \Gamma(X, E^* \otimes \mathrm{Dens}(X)) \to \mathbb{R}.$$

- **H** denotes the upper half plane.

## 2. Batalin-Vilkovisky formalism and effective renormalization

In this section, we collect basics and fix notations on the quantization of gauge theories in the Batalin-Vilkovisky (BV) formalism. We explain Costello's homotopic renormalization theory of Batalin-Vilkovisky quantization and present a one-dimensional example to motivate our discussions in two dimensions.

### 2.1. **Batalin-Vilkovisky algebras and the master equation.**

**Definition 2.1.** A differential Batalin-Vilkovisky (BV) algebra is a triple $(\mathcal{A}, Q, \Delta)$

- $\mathcal{A}$ is a $\mathbb{Z}$-graded commutative associative unital algebra.
- $Q : \mathcal{A} \to \mathcal{A}$ is a derivation of degree 1 such that $Q^2 = 0$.
- $\Delta : \mathcal{A} \to \mathcal{A}$ is a second-order operator of degree 1 such that $\Delta^2 = 0$.
- $Q$ and $\Delta$ are compatible: $[Q, \Delta] = Q\Delta + \Delta Q = 0$.

Here $\Delta$ is called the BV operator. $\Delta$ being "second-order" means the following: define the *BV bracket* $\{-,-\}$ as measuring the failure of $\Delta$ being a derivation

$$\{a, b\} := \Delta(ab) - (\Delta a)b - (-1)^{\bar{a}} a \Delta b.$$

Then $\{-,-\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ defines a Poisson bracket of degree 1 satisfying

- $\{a, b\} = (-1)^{\bar{a}\bar{b}} \{b, a\}$.
- $\{a, bc\} = \{a, b\} c + (-1)^{(\bar{a}+1)\bar{b}} b \{a, c\}$.
- $\Delta \{a, b\} = - \{\Delta a, b\} - (-1)^{\bar{a}} \{a, \Delta b\}$.

The $(Q, \Delta)$-compatibility condition implies the following Leibniz rule

$$Q\{a,b\} = -\{Qa,b\} - (-1)^{\bar{a}}\{a,Qb\}.$$

**Definition 2.2.** Let $(\mathcal{A}, Q, \Delta)$ be a differential BV algebra. A degree 0 element $I \in \mathcal{A}_0$ is said to satisfy *classical master equation* (CME) if

$$QI + \frac{1}{2}\{I,I\} = 0.$$

If $I$ solves CME, then it is easy to see that $Q + \{I, -\}$ defines a differential on $\mathcal{A}$, which can be viewed as a Poisson deformation of $Q$. However, it may not be compatible with $\Delta$. A sufficient condition for the compatibility is the "divergence freeness" $\Delta I = 0$. A slight generalization of this is the following.

**Definition 2.3.** Let $(\mathcal{A}, Q, \Delta)$ be a differential BV algebra. A degree 0 element $I \in \mathcal{A}[[\hbar]]$ is said to satisfy *quantum master equation* (QME) if

$$QI + \hbar\Delta I + \frac{1}{2}\{I,I\} = 0.$$

Here $\hbar$ is a formal variable representing the quantum parameter.

The "second-order" property of $\Delta$ implies that QME is equivalent to

$$(Q + \hbar\Delta)e^{I/\hbar} = 0.$$

If we decompose $I = \sum_{g\geq 0} I_g \hbar^g$, then the $\hbar \to 0$ limit of QME is precisely CME

$$QI_0 + \frac{1}{2}\{I_0, I_0\} = 0.$$

We can rephrase the $(Q, \Delta)$-compatibility as the nilpotency of $Q + \hbar\Delta$. It is direct to check that QME implies the nilpotency of $Q + \hbar\Delta + \{I, -\}$, which can be viewed as a compatible deformation.

2.2. **Odd symplectic space and the toy model.** We discuss a toy model of differential BV algebra via $(-1)$-shifted symplectic space. This serves as the main motivating resources of our quantum field theory examples.

Let $(V, Q)$ be a finite dimensional dg vector space. The differential $Q : V \to V$ induces a differential on various tensors of $V, V^*$, still denoted by $Q$. Let

$$\omega \in \wedge^2 V^*, \quad Q(\omega) = 0,$$

be a $Q$-compatible symplectic structure such that $\deg(\omega) = -1$. It identifies

$$V^* \simeq V[1].$$

Let $K = \omega^{-1} \in \mathrm{Sym}^2(V)$ be the Poisson kernel of degree 1 under

$$\begin{array}{cc} \wedge^2 V^* & \simeq \mathrm{Sym}^2(V)[2] \\ \omega & K \end{array}$$

where we have used the canonical identification $\wedge^2(V[1]) \simeq \mathrm{Sym}^2(V)[2]$. Let

$$\mathcal{O}(V) := \widehat{\mathrm{Sym}}(V^*) = \prod_n \mathrm{Sym}^n(V^*).$$

Then $(\mathcal{O}(V), Q)$ is a graded-commutative dga.

The degree 1 Poisson kernel $K$ defines the following BV operator

$$\Delta_K : \mathcal{O}(V) \to \mathcal{O}(V) \quad \text{by}$$

$$\Delta_K(\varphi_1 \cdots \varphi_n) = \sum_{i,j} \pm (K, \varphi_i \otimes \varphi_j) \varphi_1 \cdots \hat{\varphi}_i \cdots \hat{\varphi}_j \cdots \varphi_n, \quad \varphi_i \in V^*.$$

Here $(K, \varphi_i \otimes \varphi_j)$ denotes the natural paring between $V \otimes V$ and $V^* \otimes V^*$. $\pm$ is the Koszul sign by permuting $\varphi_i$'s. The following lemma is well-known.

**Lemma 2.4.** *$(\mathcal{O}(V), Q, \Delta_K)$ defines a differential BV algebra.*

The above construction can be summarized as

$$(-1)\text{-shifted dg symplectic} \Longrightarrow \text{differential BV}.$$

*Remark* 2.5. Since we only use $K$ to define BV operator, the above process is well-defined for $(-1)$-shifted dg Poisson structure where $K$ may be degenerate. We will see such an example in Section 4.

2.3. **UV problem and homotopic renormalization.** Let us now move on to discuss examples of quantum field theory that we will be mainly interested in.

2.3.1. *The ultra-violet problem.* One important feature of quantum field theory is about its infinite dimensionality. It leads to the main challenge in mathematics to construct measures on infinite dimensional space (called the path integrals). It is also the source of the difficulty of ultra-violet divergence and the motivation for the celebrated idea of renormalization in physics. Let us address some of these issues via the Batalin-Vilkovisky formalism.

In the previous section, we discuss the $(-1)$-shifted dg symplectic space $(V, Q, \omega)$. There $V$ is assumed to be finite dimensional. This is why we call it "toy model". Typically in quantum field theory, $V$ will be modified to be the space of smooth sections of certain vector bundles on a smooth manifold, while the differential $Q$ and the pairing $\omega$ come from something "local". Such $V$ will be called the *space* *of fields*, which is evidently a very large space with delicate topology. Nevertheless, let us naively perform similar constructions as that in the toy model.

More precisely, let $X$ be a smooth oriented manifold without boundary. Let $E^\bullet$ be a complex of vector bundles on $X$

$$\cdots E^{-1} \xrightarrow{Q} E^0 \xrightarrow{Q} E^1 \cdots$$

where $Q$ is the differential. We assume that $(E^\bullet, Q)$ is an elliptic complex. Our space of fields $\mathcal{E}$ replacing $V$ will be the space of smooth global sections

$$\mathcal{E} = \Gamma(X, E^\bullet),$$

with the induced differential, still denoted by $Q$. The symplectic pairing will be

$$\omega(s_1, s_2) := \int_X (s_1, s_2), \quad s_1, s_2 \in \mathcal{E},$$

where

$$(-,-) : E^\bullet \otimes E^\bullet \to \mathrm{Dens}(X)$$

is a non-degenerate graded skew-symmetric pairing of degree $-1$. To perform the toy model construction, we need the following steps

(1) The dual vector space $\mathcal{E}^*$ (analogue of $V^*$). This can be defined via the space of distributions on $\mathcal{E}$

$$\mathcal{E}^* := \mathrm{Hom}(\mathcal{E}, \mathbb{R})$$

where Hom is the space of continuous maps.

(2) The tensor space $(\mathcal{E}^*)^{\otimes k}$ (analogue of $(V^*)^{\otimes k}$). This can be defined via the completed tensor product for distributions

$$(\mathcal{E}^*)^{\otimes k} = \mathcal{E}^* \hat{\otimes} \cdots \hat{\otimes} \mathcal{E}^*$$

where $(\mathcal{E}^*)^{\otimes k}$ is the distributions on the bundle $E^\bullet \boxtimes \cdots \boxtimes E^\bullet$ over $X \times \cdots \times X$. $\mathrm{Sym}^k(\mathcal{E}^*)$ is defined similarly by taking care of the graded permutation. Then we have a well-defined notion (via distributions)

$$\mathcal{O}(\mathcal{E}) := \prod_{k \geq 0} \mathrm{Sym}^k(\mathcal{E}^*)$$

as the analogue of $\mathcal{O}(V)$.

(3) The Poisson kernel $K_0 = \omega^{-1}$ (the analogue of $K$). The pairing $\omega$ does not induce an identification between $\mathcal{E}[1]$ and its dual $\mathcal{E}^*$ in this case. Since $\omega$ is defined via integration, the Poisson kernel $K_0$ is the $\delta$-function

representing integral kernel of the identity operator. Therefore $K_0$ is a distributional section of $\mathcal{E}\hat{\otimes}\mathcal{E}$ supported on the diagonal $X \hookrightarrow X \times X$

$$K_0 \in \mathrm{Sym}^2(\mathcal{E}').$$

See Conventions for $\mathcal{E}'$. It is at Step (3) where we get trouble. In fact, if we naively define the BV operator

$$\Delta_{K_0} : \mathcal{O}(\mathcal{E}) \stackrel{?}{\to} \mathcal{O}(\mathcal{E}),$$

then $\Delta_{K_0}$ is *ill-defined*, since we can not pair a distribution $K_0$ with another distribution from $\mathcal{O}(\mathcal{E})$. This difficulty originates from the infinite dimensional nature of the problem.

2.3.2. *Homotopic renormalization.* The solution to the above problem requires the method of renormalization in quantum field theory. There are several different approaches to renormalizations, and we will adopt Costello's homotopic theory [7] in this paper that will be convenient for our applications.

The key observations are

(1) $K_0$ is a $Q$-closed distribution: $Q(K_0) = (Q \otimes 1 + 1 \otimes Q)K_0 = 0$.
(2) elliptic regularity: there is a canonical isomorphism of cohomologies

$$H^*(\text{smooth}, Q) \cong H^*(\text{distribution}, Q).$$

It follows that we can find a distribution $P_r \in \mathrm{Sym}^2(\mathcal{E}')$ and a smooth element $K_r \in \mathrm{Sym}^2(\mathcal{E})$ such that

$$K_0 = K_r + Q(P_r).$$

$P_r$ is the familiar notion of a parametrix.

**Definition 2.6.** $K_r$ will be called the renormalized BV kernel with respect to the parametrix $P_r$.

Since $K_r$ is smooth, there is no problem to pair $K_r$ with distributions. The same formula as in the toy model leads to

**Lemma/Definition 2.7.** *We define the renormalized BV operator* $\Delta_{K_r}$

$$\Delta_{K_r} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$$

*via the smooth renormalized BV kernel* $K_r$*. The triple* $(\mathcal{O}(\mathcal{E}), Q, \Delta_{K_r})$ *defines a differential BV algebra, called the renormalized differential BV algebra (with respect to* $P_r$*).*

Therefore $(\mathcal{O}(\mathcal{E}), Q, \Delta_{K_r})$ can be viewed as a homotopic replacement of the original naive problematic differential BV algebra. As the formalism suggests, we need to understand relations between difference choices of the parametrices.

Let $P_{r_1}$ and $P_{r_2}$ be two parametrices, $K_{r_1}$ and $K_{r_2}$ be the corresponding renormalized BV kernels. Let us denote

$$P_{r_1}^{r_2} := P_{r_2} - P_{r_1}.$$

Since $Q(P_{r_1}^{r_2}) = K_{r_1} - K_{r_2}$ is smooth, $P_{r_1}^{r_2}$ is smooth itself by elliptic regularity.

**Definition 2.8.** $P_{r_1}^{r_2}$ will be called the regularized propagator.

**Example 2.9.** Typically, suppose we have an adjoint operator $Q^\dagger$ such that $[Q, Q^\dagger]$ is a generalized Laplacian. Then given $t > 0$, the integral kernel $K_t$ for the heat operator $e^{-t[Q,Q^\dagger]}$ can be viewed as a renormalized BV kernel. In this case, the regularized propagator is given by

$$P_{t_1}^{t_2} = \int_{t_1}^{t_2} (Q^\dagger \otimes 1) K_t dt.$$

$P_{r_1}^{r_2}$ can be viewed as a homotopy linking two different renormalized differential BV algebras. In fact, similar to the definition of renormalized BV operator,

**Definition 2.10.** We define $\partial_{P_{r_1}^{r_2}} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$ as the second-order operator of contracting with the smooth kernel $P_{r_1}^{r_2} \in \mathrm{Sym}^2(\mathcal{E})$ (see also Conventions).

**Lemma 2.11.** *The following equation holds formally as operators on $\mathcal{O}(\mathcal{E})[[\hbar]]$*

$$\left(Q + \hbar\Delta_{K_{r_2}}\right) e^{\hbar\partial_{P_{r_1}^{r_2}}} = e^{\hbar\partial_{P_{r_1}^{r_2}}} \left(Q + \hbar\Delta_{K_{r_1}}\right),$$

*i.e., the following diagram commutes*

$$\begin{array}{ccc} \mathcal{O}(\mathcal{E})[[\hbar]] & \xrightarrow{Q+\hbar\Delta_{K_{r_1}}} & \mathcal{O}(\mathcal{E})[[\hbar]] \\ {\scriptstyle \exp\left(\hbar\partial_{P_{r_1}^{r_2}}\right)} \downarrow & & \downarrow {\scriptstyle \exp\left(\hbar\partial_{P_{r_1}^{r_2}}\right)} \\ \mathcal{O}(\mathcal{E})[[\hbar]] & \xrightarrow[Q+\hbar\Delta_{K_{r_2}}]{} & \mathcal{O}(\mathcal{E})[[\hbar]] \end{array}$$

*Sketch.* This follows from the observation that

$$\left[Q, \partial_{P_{r_1}^{r_2}}\right] = \Delta_{K_{r_1}} - \Delta_{K_{r_2}}.$$

See for example [7] for further discussions. □

**Definition 2.12.** Let

$$\mathcal{O}^+(\mathcal{E})[[\hbar]] := \mathrm{Sym}^{\geq 3}(\mathcal{E}^*) + \hbar\mathcal{O}(\mathcal{E})[[\hbar]] \subset \mathcal{O}(\mathcal{E})[[\hbar]]$$

be the subspace of those functionals which are at least cubic modulo $\hbar$. Given any two parametrices $P_{r_1}, P_{r_2}$, we define the homotopic renormalization group (HRG) operator

$$W(P_{r_1}^{r_2}, -) : \mathcal{O}^+(\mathcal{E})[[\hbar]] \to \mathcal{O}^+(\mathcal{E})[[\hbar]]$$
$$W(P_{r_1}^{r_2}, I) := \hbar \log \left( e^{\hbar \partial_{P_{r_1}^{r_2}}} e^{I/\hbar} \right).$$

The real content of the above formula is

$$W(P_{r_1}^{r_2}, I) = \sum_{\Gamma \text{ connected}} W_\Gamma(P_{r_1}^{r_2}, I)$$

where the summation is over all connected Feynman graphs with $P_{r_1}^{r_2}$ being the propagator and $I$ being the vertex (see for example [2] for Feynman graph techniques). Wick's Theoreom identifies the above two formula. In particular, the graph expansion formula implies that HRG operator $W(P_{r_1}^{r_2}, -)$ is well-defined on $\mathcal{O}^+(\mathcal{E})[[\hbar]]$. We refer to [7] for a thorough discussion in the current context.

Lemma 2.11 motivates the following definition.

**Definition 2.13** ([7])**.** A solution of effective quantum master equation is an assignement $I[r] \in \mathcal{O}(\mathcal{E})[[\hbar]]$ for each parametrix $P_r$ satisfying

- Renormalized quantum master equation (RQME)
$$(Q + \hbar\Delta_{K_r})e^{I[r]/\hbar} = 0.$$
- Homotopic renormalization group flow equation (HRG): for any two parametrices $P_{r_1}, P_{r_2}$,
$$I[r_2] = W(P_{r_1}^{r_2}, I[r_1]).$$

RQME and HRG are compatible by Lemma 2.11. A solution of effective quantum master equation is completely determined by its value at a fixed parametrix. Functionals at other paramatrices are obtained via HRG.

*Remark* 2.14. Here we adopt the name "homotopic RG flow" as opposed to the name "RG flow" in [7]. If the manifold preserves a rescaling symmetry, it will induce a rescaling action on the solution space of effective quantum master equations. Such flow equation will be called RG flow to be consistent with the physics terminology.

2.3.3. *Locality and counter-term technique.* In practice, we obtain solutions of renormalized quantum master equations via local functionals with the help of the method of counter-terms. We explain this construction in this subsection.

**Definition 2.15.** A functional $I \in \mathcal{O}(\mathcal{E})$ is called *local* if it can be expressed in terms of an integration of a Lagrangian density $\mathcal{L}$

$$I(\phi) = \int_X \mathcal{L}(\phi), \quad \phi \in \mathcal{E}.$$

$\mathcal{L}$ can be viewed as a $\mathrm{Dens}(X)$-valued function on the jet bundle of $\mathcal{E}$. The subspace of local functionals of $\mathcal{O}(\mathcal{E})$ is denoted by $\mathcal{O}_{loc}(\mathcal{E})$. We also denote

$$\mathcal{O}^+_{loc}(\mathcal{E})[[\hbar]] = \mathcal{O}^+(\mathcal{E})[[\hbar]] \cap \mathcal{O}_{loc}(\mathcal{E})[[\hbar]].$$

Let us assume we are in the situation of Example 2.9. Given $L > 0$, we have a smoothly regularized BV kernel $K_L$ in terms of the heat kernel. Let $P^L_\epsilon$ be the regularized propagator for $0 < \epsilon < L < \infty$.

Given any local functional $I \in \mathcal{O}^+_{loc}(\mathcal{E})$, we can find an $\epsilon$-dependent local functional $I^{CT}(\epsilon) \in \hbar\mathcal{O}_{loc}(\mathcal{E})[[\hbar]]$, such that the following limit exists

$$I[L] = \lim_{\epsilon \to 0} W(P^L_\epsilon, I + I^{CT}(\epsilon)) \in \mathcal{O}^+(\mathcal{E})[[\hbar]].$$

Here $I^{CT}(\epsilon)$ has a singular dependence on $\epsilon$ when $\epsilon \to 0$, and this singularity exactly cancels those that come from the naive graph integral $W(P^L_\epsilon, I)$. $I^{CT}(\epsilon)$ is called the *counter-term*, which plays an important role in the renormalization theory of quantum fields. For a proof of the existence of counter-terms in the current context, see for example [7, Appendix 1].

By construction, $I[L]$ satisfies HRG for all heat kernel regularizations:

$$I[L_2] = W(P^{L_2}_{L_1}, I[L_1]), \quad \forall 0 < L_1, L_2 < \infty.$$

It remains to analyze the quantum master equation.

Firstly, although the BV operator $\Delta_L$ becomes singular as $L \to 0$, its associated BV bracket is in fact well-defined on local functionals.

**Definition 2.16.** We define the classical BV bracket on $\mathcal{O}_{loc}(\mathcal{E})$ by

$$\{I_1, I_2\} := \lim_{L \to 0} \{I_1, I_2\}_L, \quad I_1, I_2 \in \mathcal{O}_{loc}(\mathcal{E}).$$

The reason that the above limit exists lies in the observation that the delta-function can be naturally paired with integration.

**Definition 2.17.** A local functional $I \in \mathcal{O}_{loc}(\mathcal{E})$ is said to satisfy classical master equation (CME) if

$$QI + \frac{1}{2}\{I, I\} = 0.$$

Therefore classical master equation does not require renormalization. In practice, any local functional with a gauge symmetry can be completed into a local functional that satisfies the classical master equation. This fact lies in the heart of the Batalin-Vilkovisky formalism.

*Remark* 2.18. Given $I_0 \in \mathcal{O}_{loc}(\mathcal{E})$ satisfying the classical master equation, $Q + \{I_0, -\}$ defines a nilpotent vector field on $\mathcal{E}$. The physical meaning is that it generates the infinitesimal gauge transformation. In mathematical terminology, it defines a (local) $L_\infty$ structure on $\mathcal{E}$.

To proceed to construct solutions of effective quantum master equations, let us naively use counter-terms to construct a family $I[L]$ as above from $I_0$. It can be shown that $I[L]$ satisfies renormalized quantum master equation modulo $\hbar$

$$QI[L] + \hbar\Delta_L I[L] + \frac{1}{2}\{I[L], I[L]\}_L = O(\hbar).$$

Then we need to find quantum corrections

$$I_0 \to I_0 + \hbar I_1 + \hbar^2 I_2 + \cdots \in \mathcal{O}^+_{loc}(\mathcal{E})[[\hbar]]$$

such that the renormalized quantum master equation holds true at all orders of $\hbar$. This can be formulated as a deformation problem. The quantum corrections may become very complicated in general, and intrinsic obstructions for solving the quantum master equation could exist at certain $\hbar$-order (gauge anomalies). Nevertheless, one of our main purposes here is to understand the geometric meaning of such quantum corrections and find their solutions.

2.4. **Example: Topological quantum mechanics.** The simplest nontrivial example is when $X$ is one-dimensional. This corresponds to quantum mechanical models. We explain the one dimensional Chern-Simons theory that is studied in detail in [23]. In such a geometric situation, the renormalized BV master equation is related to Fedosov's abelian connection on Weyl bundles [14]. In the next section, we generalize this analysis to two dimensional models.

Let $X = S^1$. Let $V$ be a graded vector space with a degree 0 symplectic pairing

$$(-,-) : \wedge^2 V \to \mathbb{R}.$$

The space of fields will be

$$\mathcal{E} = \Omega^\bullet(S^1) \otimes V.$$

Let us denote $X_{dR}$ by the super-manifold whose underlying topological space is $X$ and whose structure sheaf is the de Rham complex on $X$. We can identify $\varphi \in \mathcal{E}$ with a map $\hat{\varphi}$ between super-manifolds

$$\hat{\varphi} : X_{dR} \to V$$

whose underlying map on topological spaces is the constant map to $0 \in V$.

The differential $Q = d_{S^1}$ on $\mathcal{E}$ is the de Rham differential on $\Omega^\bullet(S^1)$. The $(-1)$-shifted symplectic pairing is the pairing

$$\omega(\varphi_1, \varphi_2) := \int_{S^1} (\varphi_1, \varphi_2).$$

The induced differential BV structure is just the AKSZ-formalism [1] applied to the one-dimensional $\sigma$-model. Given $I \in \mathcal{O}(V)$ of degree $k$, it induces an element $\hat{I} \in \mathcal{O}(\mathcal{E})$ of degree $k-1$ via

$$\hat{I}(\varphi) := \int_{S^1} \hat{\varphi}^*(I), \quad \forall \varphi \in \mathcal{E}.$$

We choose the standard flat metric on $S^1$ and use the heat kernel regularization as in Example 2.9. The following Theorem is a consequence of [23].

**Theorem 2.19** ([23])**.** *Given $I \in \mathrm{Sym}^{\geq 3}(V^*) + \hbar\mathcal{O}(V)[[\hbar]]$ of degree 1, the limit*

$$\hat{I}[L] = \lim_{\epsilon \to 0} W(P^L_\epsilon, \hat{I})$$

*exists as an degree 0 element of $\mathcal{O}^+(\mathcal{E})[[\hbar]]$. The family $\{\hat{I}[L]\}_{L>0}$ solves the effective quantum master equation if and only if*

$$[I, I]_\star = 0.$$

*Here $\mathcal{O}(V)[[\hbar]]$ inherits a natural Moyal product $\star$ from the linear symplectic form $(-,-)$. $[-,-]_\star$ is the commutator with respect to the Moyal product.*

In [23], the above theorem is formulated as a family version of $V$ parametrized by a symplectic manifold. Then the effective quantum master equation is equivalent to a flat connection gluing the Weyl bundle (i.e. the bundle of associative algebra with fiberwise Moyal product). A further analysis of the partition function leads to a simple formulation of the algebraic index theorem.

## 3. Vertex algebra and BV master equation

In this section we establish our main theorem on the correspondence between solutions of the renormalized quantum master equation for two dimensional chiral theories and Maurer-Cartan elements for chiral vertex operators (Theorem 3.11). We prove the modularity property of generating functions of chiral theories on elliptic curves and establish the polynomial nature of their antiholomorphic dependence on the moduli (Theorem 3.21).

3.1. **Vertex algebra.** In this section we collect some basics on vertex algebras that will be used in this paper. We refer to [16,19] for details.

3.1.1. *Definition of vertex algebras.* In this section $\mathcal{V}$ will always denote a $\mathbb{Z}/2\mathbb{Z}$-graded superspace over $\mathbb{C}$. It has two components with different parities

$$\mathcal{V} = \mathcal{V}_{\bar{0}} \oplus \mathcal{V}_{\bar{1}}, \quad \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}.$$

We say $v \in \mathcal{V}$ has parity $p(a) \in \mathbb{Z}/2\mathbb{Z}$ if $v \in \mathcal{V}_{p(v)}$.

**Definition 3.1.** A *field* on $\mathcal{V}$ is a power series

$$A(z) = \sum_{k\in\mathbb{Z}} A_{(k)} z^{-k-1} \in \mathrm{End}(\mathcal{V})[[z, z^{-1}]]$$

such that for any $v \in \mathcal{V}$, $A(z)v \in \mathcal{V}((z))$. We will denote

$$A(z)_+ = \sum_{k<0} A_{(k)} z^{-k-1}, \quad A(z)_- = \sum_{k\geq 0} A_{(k)} z^{-k-1}.$$

- Given two fields $A(z), B(z)$, we define their *normal ordered product*

  $$: A(z)B(w) := A(z)_+ B(w) + (-1)^{p(A)p(B)} B(w) A(z)_-.$$

  Note that $\mathrm{End}(\mathcal{V})$ is naturally a superspace. $p(A), p(B)$ are the parities.
- Two fields $A(z), B(z)$ are called *mutually local* if

  $$A(z)B(w) = \sum_{k=0}^{N-1} \frac{C_k(w)}{(z-w)^{k+1}} + : A(z)B(w) : .$$

  for some $N \in \mathbb{Z}^{\geq 0}$ and fields $C_k$. The first term on the right is called the *singular part*, and we shall write

  $$A(z)B(w) \sim \sum_{k=0}^{N-1} \frac{C_k(w)}{(z-w)^{k+1}}.$$

  The above two formulae are called the *operator product expansion* (OPE).

**Definition 3.2.** A *vertex algebra* is a collection of data

- (space of states) a superspace $\mathcal{V}$;
- (vacuum) a vector $|0\rangle \in \mathcal{V}_{\bar{0}}$;
- (translation operator) an even linear operator $T : \mathcal{V} \to \mathcal{V}$;
- (state-field correspondence) an even linear operation (vertex operators)

$$Y(\cdot, z) : \mathcal{V} \to \operatorname{End} \mathcal{V}[[z, z^{-1}]]$$

  taking each $A \in \mathcal{V}$ to a field $Y(A, z) = \sum_{n \in Z} A_{(n)} z^{-n-1}$;

satisfying the following axioms:

- (vaccum axiom) $Y(|0\rangle, z) = \mathrm{Id}_{\mathcal{V}}$. Furthermore, for any $A \in \mathcal{V}$ we have

$$Y(A, z)|0\rangle \in \mathcal{V}[[z]], \quad \text{and} \quad \lim_{z \to 0} Y(A, z)|0\rangle = A.$$

- (translation axiom) $T|0\rangle = 0$. For any $A \in \mathcal{V}$, $[T, Y(A, z)] = \partial_z Y(A, z)$;
- (locality axiom) All fields $Y(A, z)$, $A \in \mathcal{V}$, are mutually local.

Let $A, B \in \mathcal{V}$. Their OPE can be expanded as

$$Y(A, z) Y(B, w) = \sum_{n \in \mathbb{Z}} \frac{Y(A_{(n)} \cdot B, w)}{(z - w)^{n+1}},$$

where $\left\{ A_{(n)} \cdot B \right\}_{n \in \mathbb{Z}}$ can be viewed as defining an infinite tower of products.

**Definition 3.3.** A vertex algebra $\mathcal{V}$ is called *conformal*, of *central charge* $c \in \mathbb{C}$, if there is a vector $\omega_{vir} \in \mathcal{V}$ (called a conformal vector) such that under the state-field correspondence $Y(\omega_{vir}, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$:

- $L_{-1} = T$, $L_0$ is diagonalizable on $V$;
- $\{L_n\}_{n \in \mathbb{Z}}$ span the Virasoro algebra with central charge $c$.

Let $\mathcal{V}$ be a conformal vertex algebra. The field $Y(\omega, z)$ will often be denoted by $T(z)$, called the *energy momentum tensor*. $\mathcal{V}$ can be decomposed as

$$\mathcal{V} = \bigoplus_{\alpha} \mathcal{V}^{\alpha}, \quad L_0|_{\mathcal{V}_{\alpha}} = \alpha.$$

$A \in \mathcal{V}^{\alpha}$ is said to have conformal weight $\alpha$. Its field will often be expressed by

$$Y(A, z) = \sum_{k} A_k z^{-k-\alpha}, \quad A_k : \mathcal{V}^{\bullet} \to \mathcal{V}^{\bullet - k},$$

where $A_k$ has conformal weight $-k$. In previous notations, $A_k = A_{(k+\alpha-1)}$.

3.1.2. *Modes Lie algebra.* Following the presentation in [16], we associate a canonical Lie algebra to a vertex algebra via Fourier modes of vertex operators.

**Definition 3.4.** Given a vertex algebra $\mathcal{V}$, we define a Lie algebra, denoted by $\oint \mathcal{V}$, as follows. As a vector space, the Lie algebra $\oint \mathcal{V}$ has a basis given by $A_{(k)}$'s

$$\oint \mathcal{V} := \mathrm{Span}_{\mathbb{C}} \left\{ \oint dz z^k Y(A,z) := A_{(k)} \right\}_{A\in\mathcal{V}, k\in\mathbb{Z}}.$$

The Lie bracket is determined by the OPE (Borcherds commutator formula)

$$\left[ A_{(m)}, B_{(n)} \right] = \sum_{j\geq 0} \binom{m}{j} \left( A_{(j)} B \right)_{m+n-j}.$$

Here we use a different notation $\oint$ than $U(\cdot)$ in [16, Section 4.1] to emphasize its nature of mode expansion. The commutator relations are better illustrated formally in terms of residues

$$\left[ \oint dz z^m Y(A,z), \oint dw w^n Y(B,w) \right] = \oint dw w^n \oint_w dz z^m \sum_{j\in\mathbb{Z}} \frac{Y(A_{(j)}\cdot B, w)}{(z-w)^{j+1}},$$

where $\oint_w dz := \int_{C_w} \frac{dz}{2\pi i}$ is the integration over a small loop $C_w$ around $w$.

Equivalently, the vector space $\oint \mathcal{V}$ can be described as

$$\oint \mathcal{V} = \mathcal{V}[z, z^{-1}] / \operatorname{im} \partial$$

where $\partial(A \otimes z^k) := T(A) \otimes z^k + kA \otimes z^{k-1}, A \in V$. Then $A \otimes z^k$ represents the Fourier mode $\oint dz z^k Y(A,z)$ and $\operatorname{im}\partial$ represents the space of total derivatives.

3.1.3. *Examples.* Let $\mathbf{h} = \oplus_{\alpha\in\mathbb{Q}} \mathbf{h}^\alpha$, where $\mathbf{h}^\alpha = \mathbf{h}^\alpha_{\bar{0}} \oplus \mathbf{h}^\alpha_{\bar{1}}$, be a $\mathbb{Q}$-graded superspace. Here the $\mathbb{Q}$-grading is the conformal weight and we assume for simplicity only finitely many weights appear in $\mathbf{h}$. Let $\mathbf{h}$ be equipped with an even symplectic pairing

$$\langle -, - \rangle : \wedge^2 \mathbf{h} \to \mathbb{C}$$

which is of conformal weight $-1$, that is, the only nontrivial pairing is

$$\langle -, - \rangle : \mathbf{h}^\alpha \otimes \mathbf{h}^{1-\alpha} \to \mathbb{C}.$$

For each $a \in \mathbf{h}^\alpha$, we associate a field

$$a(z) = \sum_{r\in\mathbb{Z}-\alpha} a_r z^{-r-\alpha}.$$

We define their OPE by

$$a(z)b(w) \sim \left( \frac{i\hbar}{\pi} \right) \frac{\langle a, b \rangle}{(z-w)}, \quad \forall a, b \in \mathbf{h}.$$

This is equivalent to the commutator relations

$$[a_r, b_s] = \frac{i\hbar}{\pi} \langle a, b \rangle \, \delta_{r+s,0}, \quad \forall a, b \in \mathbf{h}, r \in \mathbb{Z} - \alpha, s \in \mathbb{Z} + \alpha.$$

The vertex algebra $\mathcal{V}[\mathbf{h}]$ that realizes the above OPE relations is given by the Fock representation space. The vaccum vector satisfies

$$a_r|0\rangle = 0, \quad \forall a \in \mathbf{h}^\alpha, r + \alpha > 0,$$

and $\mathcal{V}[\mathbf{h}]$ is freely generated from the vaccum by the operators $\{a_r\}_{r+\alpha\leq 0}, a \in \mathbf{h}^\alpha$. For any $a \in \mathbf{h}$, $a(z)$ becomes a field acting naturally on the Fock space $\mathcal{V}[\mathbf{h}]$. $\mathcal{V}[\mathbf{h}]$ is a conformal vertex algebra structure, with energy momentum tensor

$$T(z) = \frac{1}{2} \sum_{i,j} \omega_{ij} : \partial a^i(z) a^j(z) :$$

where $\{a^i\}$ is a basis of $\mathbf{h}$, $\omega_{ij}$ is the inverse matrix of $\langle a^i, b^j \rangle$: $\sum_k \omega_{ik} \langle a^k, a^j \rangle = \delta_i^j$. The central charge is $\dim \mathbf{h}_{\bar{0}} - \dim \mathbf{h}_{\bar{1}}$.

*Remark* 3.5. When $\mathbf{h} = \mathbf{h}_{\bar{0}}$ is purely bosonic, the associated vertex algebra is the $\beta - \gamma$ system (or the chiral differential operators [17,29]). When $\mathbf{h} = \mathbf{h}_{\bar{1}}$ is purely fermionic, the associated vertex algebra is the $b - c$ system.

Under the state-field correspondence, the vertex algebra $\mathcal{V}[\mathbf{h}]$ can be identified with the polynomial algebra

$$\mathcal{V}[\mathbf{h}] \cong \mathbb{C}\left[\partial^k a^i\right], \quad a^i \text{ is a basis of } \mathbf{h}, k \geq 0.$$

We also denote its formal completion by

$$\mathcal{V}[[\mathbf{h}]] := \mathbb{C}[[\partial^k a^i]].$$

$\mathcal{V}[\mathbf{h}], \mathcal{V}[[\mathbf{h}]]$ can be viewed as the chiral analogue of the Weyl algebra.

3.2. **Two dimensional chiral QFT.** Let $\Sigma$ be a complex curve, $K_\Sigma$ be its canonical line bundle. We will be interested in the following data as a BV set-up:

- $(E^\bullet, \delta)$ is a differential complex of holomorphic bundles on $\Sigma$;
- A degree 0 symplectic pairing of complexes

$$(-,-) : E^\bullet \otimes E^\bullet \to K_\Sigma.$$

Here $K_\Sigma$ is viewed as a complex concentrated at degree 0.

The space of quantum fields associated to the above data will be

$$\mathcal{E} = \Omega^{0,\bullet}(\Sigma, E^\bullet).$$

The $(-1)$-symplectic pairing $\omega$ on $\mathcal{E}$ is obtained via

$$\begin{array}{ccc} \mathcal{E} \otimes \mathcal{E} & \xrightarrow{(-,-)} & \Omega^{0,\bullet}(\Sigma, K_\Sigma) \\ & {\scriptstyle \omega} \searrow \quad \swarrow {\scriptstyle \int_\Sigma} & \\ & \mathbb{C} & \end{array}$$

Then the triple $(\mathcal{E}, Q = \bar{\partial} + \delta, \omega)$ defines an infinite dimensional $(-1)$-symplectic structure in the sense of section 2. Our main goal in this section is to study the associated renormalized quantum master equation.

We will be mainly concerning the flat situation in this paper when $\Sigma = \mathbb{C}, \mathbb{C}^*$ or the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ $(\tau \in \mathbf{H})$. We assume this from now on.

**Definition 3.6.** We will fix a coordinate $z$ on $\Sigma$: this is the linear coordinate on $\mathbb{C}$, $z \sim z + 1 \sim z + \tau$ on $E_\tau$, and $e^{2\pi i z}$ parametrizes $\mathbb{C}^*$. Our convention of the volume form is

$$d^2 z := \frac{i}{2} dz \wedge d\bar{z}.$$

We also use the following notation: $f(\overrightarrow{z})$ means a smooth function on $z$ while $f(z)$ means a holomorphic function.

Since $\Sigma$ is flat, we will focus on the situation in this paper when $(E^\bullet, (-,-))$ comes from linear data $(\mathbf{h}, \langle -,- \rangle)$ as follows:

- $\mathbf{h} = \bigoplus_{m \in \mathbb{Z}} \mathbf{h}_m$ is a $\mathbb{Z}$-graded vector space (the grading is the cohomology degree);
- $\langle -,- \rangle : \wedge^2 \mathbf{h} \to \mathbb{C}$ is a degree 0 symplectic pairing;
- $E^\bullet = \mathcal{O}_\Sigma \otimes \mathbf{h}$, and $\mathcal{E} = \Omega^{0,\bullet}(\Sigma) \otimes \mathbf{h}$;
- $\delta \in \mathbb{C}\left[\frac{\partial}{\partial z}\right] \otimes \bigoplus_m \mathrm{Hom}(\mathbf{h}^m, \mathbf{h}^{m+1})$. Here $\mathbb{C}\left[\frac{\partial}{\partial z}\right]$ represents the space of translation invariant holomorphic differential operators on $\Sigma$;
- $(-,-)$ is induced from fiberwise $\langle -,- \rangle$ via the identification $K_\Sigma \cong \mathcal{O}_\Sigma dz$

$$(\varphi_1, \varphi_2) = \int dz \wedge \langle \varphi_1, \varphi_2 \rangle .$$

We consider the dual $\mathbb{Z}$-graded vector space $\mathbf{h}^*$ with the induced symplectic pairing still denoted by $\langle -,- \rangle$. The $\mathbb{Z}/2\mathbb{Z}$-grading associated to the $\mathbb{Z}$-grading defines the parity on $\mathbf{h}^*$. The extra $\mathbb{Z}$-grading of conformal weight will not be explicit at this stage. At this stage, we assume for simplicity that the pairing

$\langle -, - \rangle$ on $\mathbf{h}^*$ has conformal weight $-1$ and obtain the vertex algebra $\mathcal{V}[[\mathbf{h}^*]]$ as in Section 3.1.3. The same discussion can be easily generalized when the conformal weight $\langle -, - \rangle$ is different from $-1$. We describe such an example in Section 4.

The relevant chiral local functionals will be described by $\oint \mathcal{V}[[\mathbf{h}^*]]$. Precisely,

**Definition 3.7.** Given $I \in \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$, we extend it $\Omega^{0,\bullet}[[\hbar]]$-linearly to a map

$$I : \mathcal{E} \to \Omega^{0,\bullet}[[\hbar]].$$

Explicitly, if $I = \sum \partial^{k_1} a_1 \cdots \partial^{k_m} a_m \in \mathcal{V}[[\mathbf{h}^*]]$, where $a_i \in \mathbf{h}^*$, $\varphi \in \mathcal{E}$, then

$$I(\varphi) = \sum \pm \partial_z^{k_1} a_1(\varphi) \cdots \partial_z^{k_m} a_m(\varphi),$$

Here $a_i(\varphi) \in \Omega^{0,\bullet}$ comes from the natural pairing between $\mathbf{h}^*$ and $\mathbf{h}$. $\partial_z$ is the holomorphic derivative with respect to our prescribed linear coordinate $z$ on $\Sigma$. $\pm$ is the Koszul sign. We associate a local functional $\hat{I} \in \mathcal{O}_{loc}(\mathcal{E})[[\hbar]]$ on $\mathcal{E}$ by

$$\hat{I}(\varphi) := i \int_\Sigma dz\, I(\varphi), \quad \varphi \in \mathcal{E}.$$

Note that $\deg(\hat{I}) = \deg(I) - 1$.

We will fix the standard flat metric on $\Sigma$. Let $\bar{\partial}^*$ denote the adjoint of $\bar{\partial}$, and $h_t \in C^\infty(\Sigma \times \Sigma), t > 0$, be the heat kernel function of the Laplacian operator $H = \left[\bar{\partial}, \bar{\partial}^*\right]$. It is normalized by

$$(e^{-tH} f)(\overrightarrow{z}_1) = \int_\Sigma d^2 z_2\, h_t(\overrightarrow{z}_1, \overrightarrow{z}_2) f(\overrightarrow{z}_2), \quad t > 0.$$

For $\Sigma = \mathbb{C}$, we have $\bar{\partial}^* = -2\partial_z \iota_{\bar{\partial}_{\bar{z}}}$ and $H = -2\partial_z \bar{\partial}_{\bar{z}} = \frac{1}{2}\left(\partial_x^2 + \partial_y^2\right)$ where $z = x + iy$, and $\iota_{\bar{\partial}_{\bar{z}}}$ is the contraction with the vector field $\frac{\partial}{\partial \bar{z}}$. Then explicitly

$$h_t(\overrightarrow{z}_1, \overrightarrow{z}_2) = \frac{1}{2\pi t} e^{-|z_1 - z_2|^2/2t}.$$

When $\Sigma = \mathbb{C}^*$ or $E_\tau$, $h_t$ is obtained from the above heat kernel on $\mathbb{C}$ by a further summation over the relevant lattices.

The regularized BV kernel $K_L \in \mathrm{Sym}^2(\mathcal{E})$ is given by

$$K_L(\overrightarrow{z}_1, \overrightarrow{z}_2) = i\, h_L(\overrightarrow{z}_1, \overrightarrow{z}_2)(d\bar{z}_1 \otimes 1 - 1 \otimes d\bar{z}_2) C_{\mathbf{h}}.$$

Here $C_{\mathbf{h}} = \sum_{i,j} \omega_{ij}(a^i \otimes a^j)$ is the Casimir element where $\{a^i\}$ is a basis of $\mathbf{h}$, $\omega_{ij}$ is the inverse matrix of $\langle a^i, b^j \rangle$. The normalization constant is chosen such that

$$(e^{-LH}\varphi)(\overrightarrow{z}_1) = -\frac{1}{2} \int_\Sigma dz_2 \wedge \left\langle K_L(\overrightarrow{z}_1, \overrightarrow{z}_2), \varphi(\overrightarrow{z}_2) \right\rangle, \quad \forall \varphi \in \mathcal{E}$$

where $\langle -, - \rangle$ inside the integral is the pairing in the $z_2$-component. The factor $\frac{1}{2}$ is the symmetry factor, while the extra minus sign comes froming passing $K_L$ through $dz_2$. Then $K_0$ is precisely the desired singular Poisson tensor. The regularized propagator is

$$P_\epsilon^L = \int_\epsilon^L (\bar{\partial}^* \otimes 1) K_u du, \quad 0 < \epsilon < L < \infty.$$

**Definition 3.8.** We define the subspace $\mathcal{V}^+[[\mathbf{h}^*]][[\hbar]] \subset \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$ by saying that $I \in \mathcal{V}^+[[\mathbf{h}^*]][[\hbar]]$ if and only if $\lim_{\hbar \to 0} I$ is at least cubic when $\mathcal{V}[[\mathbf{h}^*]]$ is viewed as a (formal) polynomial ring.

**Theorem 3.9.** *Given $I \in \mathcal{V}^+[[\mathbf{h}^*]][[\hbar]]$ of degree 1 and $L > 0$, the limit*

$$\hat{I}[L] := \lim_{\epsilon \to 0} W(P_\epsilon^L, \hat{I})$$

*exists as an element of $\mathcal{O}^+(\mathcal{E})[[\hbar]]$. $\{\hat{I}[L]\}_{L>0}$ defines a family of $\mathcal{O}^+(\mathcal{E})[[\hbar]]$ satisfying the homotopic RG flow equation, and $\lim_{L \to 0} \hat{I}[L] = \hat{I}$. Here $\hat{I}$ is defined in Definition 3.7.*

*Proof.* The limit behavior of $\epsilon$ and L is a consequence of [25, Lemma 3.1 and Proposition B.1]. The homotopic RG flow equation is similar to the discussion in Section 2.3.3. □

We remark that the order of limit is important in the proof of Theorem 3.9. The chiral nature of the problem implies that all potential singularities in $W(P_\epsilon^L, I)$ in fact vanish upon integration by parts before taking the limit $\epsilon \to 0$ [25].

It remains to analyze the renormalized quantum master equation for $\hat{I}[L]$.

3.3. **Quantum master equation and OPE.** The differential $\delta$ induces naturally a differential on the formal polynomial ring $\mathcal{V}[[\mathbf{h}^*]]$ (denoted by the same symbol)

$$\delta : \mathcal{V}[[\mathbf{h}^*]] \to \mathcal{V}[[\mathbf{h}^*]].$$

Let us write $\delta = D \otimes \phi$, where $D \in \mathbb{C}\left[\frac{\partial}{\partial z}\right]$, $\phi \in \mathrm{Hom}(\mathbf{h}, \mathbf{h})$. Let $\phi^* \in \mathrm{Hom}(\mathbf{h}^*, \mathbf{h}^*)$ be the dual of $\phi$. Then in terms of generators,

$$\delta(\partial_z^k a) := \partial_z^k D(\phi^*(a)), \quad a \in \mathbf{h}^*.$$

This further induces a differential

$$\delta : \oint \mathcal{V}[[\mathbf{h}^*]] \to \oint \mathcal{V}[[\mathbf{h}^*]].$$

Recall we have a natural Lie bracket defined on $\oint \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$ as in Section 3.1.2.

**Lemma 3.10.** *The differential $\delta$ and the Lie bracket $[,]$ define a structure of differential graded Lie algebra on $\oint \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$.*

*Proof.* We leave this formal check to the interested reader. ☐

The main theorem in this paper is the following, which can be viewed as the two dimensional chiral analogue of Theorem 2.19.

**Theorem 3.11.** *Let $\hat{I}[L]$ be defined in Theorem 3.9. Then the family $\hat{I}[L]$ satisfies renormalized quantum master equation if and only if*

$$\delta \oint dzI + \frac{1}{2}\left(\frac{i\hbar}{\pi}\right)^{-1}\left[\oint dzI, \oint dzI\right] = 0,$$

*i.e.. $\oint dzI$ is a Maurer-Cartan element of $\oint \mathcal{V}[[\mathbf{h}]][[\hbar]]$.*

*Proof.* It is enough to work with $\Sigma = \mathbb{C}$, since $\mathbb{C}^*, \Sigma_\tau$ are quotients by translations and our $\delta$ and $I$ are translation invariant. Moreover, the renormalized quantum master equations

$$(\dagger) \qquad Q\hat{I}[L] + \hbar\Delta_L\hat{I}[L] + \frac{1}{2}\left\{\hat{I}[L], \hat{I}[L]\right\}_L = 0, \quad \text{or } (Q+\Delta_L)\, e^{\hat{I}[L]/\hbar} = 0,$$

are equivalent for each $L$, therefore it is enough to analyze the limit $L \to 0$. By Theorem 3.9,

$$\begin{aligned}(Q+\hbar\Delta_L)\, e^{\hat{I}[L]/\hbar} &= \lim_{\epsilon\to 0} (Q+\hbar\Delta_L)\left(e^{\hbar\partial_{P_\epsilon^L}} e^{\hat{I}/\hbar}\right)\\ &= \lim_{\epsilon\to 0} e^{\hbar\partial_{P_\epsilon^L}}\left((Q+\hbar\Delta_\epsilon)\, e^{\hat{I}/\hbar}\right)\\ &= \frac{1}{\hbar}\lim_{\epsilon\to 0} e^{\hbar\partial_{P_\epsilon^L}}\left(Q\hat{I} + \hbar\Delta_\epsilon\hat{I} + \frac{1}{2}\left\{\hat{I}, \hat{I}\right\}_\epsilon\right) e^{\hat{I}/\hbar}\\ &= \frac{1}{\hbar}\lim_{\epsilon\to 0} e^{\hbar\partial_{P_\epsilon^L}}\left(\delta\hat{I} + \frac{1}{2}\left\{\hat{I}, \hat{I}\right\}_\epsilon\right) e^{\hat{I}/\hbar}.\end{aligned}$$

Here we have observed

- $\Delta_\epsilon\hat{I} = 0$, since $\hat{I}$ is local and $K_\epsilon$ becomes zero when restricted to the diagonal (the factor $d\bar{z}_1 - d\bar{z}_2$ vanishes when $z_1 = z_2 = z$).
- $\bar{\partial}\hat{I} = 0$, since it contributes to a total derivative.

Therefore formally

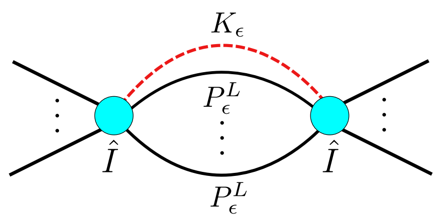
$$\hbar e^{-\hat{I}/\hbar}\lim_{L\to 0}(Q+\hbar\Delta_L)\, e^{\hat{I}[L]/\hbar} = \delta\hat{I} + \frac{1}{2}\lim_{L\to 0}\lim_{\epsilon\to 0} e^{-\hat{I}/\hbar} e^{\hbar\partial_{P_\epsilon^L}}\left(\left\{\hat{I}, \hat{I}\right\}_\epsilon e^{\hat{I}/\hbar}\right).$$

The first term matches with that in the theorem up to a factor of $i$ (recall Definition 3.7). It remains to analyze the second term involving effective BV bracket. We proceed in two steps.

*Step 1*: Reduction to two-vertex diagram. We claim

$$\lim_{L\to 0}\lim_{\epsilon\to 0} e^{-\hat{I}/\hbar} e^{\hbar\partial_{P_\epsilon^L}}\left(\left\{\hat{I},\hat{I}\right\}_\epsilon e^{\hat{I}/\hbar}\right) = \lim_{L\to 0}\lim_{\epsilon\to 0}\sum_{m\geq 0} W_{\Gamma_m}(\hat{I}, P_\epsilon^L, K_\epsilon).$$

Here $\Gamma_m$ is the diagram involving only two vertices by $\hat{I}$, $m$ propagators by $P_\epsilon^L$(draw by solid lines), and one extra propagator by $K_\epsilon$ (draw by dashed line):




This says that only diagrams with two vertices contribute to the limit $L \to 0$. This claim follows from the $L \to 0$ limit of [25, Proposition B.2].

*Step 2*: It remains to show

$$\lim_{L\to 0}\lim_{\epsilon\to 0}\sum_{m\geq 0} W_{\Gamma_m}(\hat{I}, P(\epsilon, L), K_\epsilon) = \frac{\pi}{i\hbar}\left[\widehat{\oint dzI, \oint dzI}\right].$$

$[\oint dzI, \oint dzI]$ is computed by Wick contractions and OPE's (see [19]). Comparing with the OPE formula, taking care of factor $i$ in Definition 3.7 and symmetric factors of the Feynman graphs, it is enough to establish the following identity

$$\begin{aligned}\lim_{\epsilon\to 0}\sum_{\sigma\in S_{k+1}}\frac{1}{m!}\int_{\mathbb{C}} d^2z_1\int_{\mathbb{C}} d^2z_2 A(\overrightarrow{z}_1)B(\overrightarrow{z}_2)\partial_{z_1}^{k_{\sigma(0)}} K_\epsilon(\overrightarrow{z}_1,\overrightarrow{z}_2)\prod_{i=1}^m \partial_{z_1}^{k_{\sigma(i)}} P_\epsilon^L(\overrightarrow{z}_1,\overrightarrow{z}_2)\\ = \frac{\pi}{(m+1)!}\int_{\mathbb{C}} d^2z_2 B(\overrightarrow{z}_2)\oint_{z_2} dz_1 A(\overrightarrow{z}_1)\prod_{i=0}^m\left(\partial_{z_1}^{k_i}\frac{i}{\pi(z_1-z_2)}\right).\end{aligned}$$

for $k_0, \cdots, k_m \in \mathbb{Z}^{\geq 0}$, and smooth functions $A, B$ with compact support. Here

$$\begin{aligned}K_\epsilon(\overrightarrow{z}_1,\overrightarrow{z}_2) &= \frac{i}{2\pi\epsilon}e^{-|z_1-z_2|^2/2\epsilon}\\ P_\epsilon^L(\overrightarrow{z}_1,\overrightarrow{z}_2) &= (-2i)\int_\epsilon^L dt\partial_{z_1}h_t(\overrightarrow{z}_1,\overrightarrow{z}_2) = (-2i)\int_\epsilon^L\frac{dt}{2\pi t}\frac{\bar{z}_2-\bar{z}_1}{2t}e^{-|z_1-z_2|^2/2t}\end{aligned}$$

where we have used the same symbols but only keep factors of the BV kernel and the regularized propagator that are relevant in this computation. $A, B$ are test functions viewed as collecting all inputs on external edges of the two vertices. $\oint_{z_2} dz_1$ means a loop integral of $z_1$ around $z_2$ normalized by

$$\oint_{z_2} \frac{dz_1}{z_1 - z_2} = 1.$$

Our notation $\oint_{z_2} dz_1 A(\overrightarrow{z}_1) \prod_{i=0}^{m} \left(\partial_{z_1}^{k_i} \frac{1}{z_1 - z_2}\right)$ means only picking up the holomorphic derivative of $A$ according to the pole condition, i.e.,

$$\oint_{z_2} dz_1 A(\overrightarrow{z}_1) \frac{1}{(z_1 - z_2)^{m+1}} := \frac{1}{m!} \partial_{z_2}^m A(\overrightarrow{z}_2).$$

$\sigma$ is the summation over permutations of $\{0, 1, \cdots, m\}$. The holomorphic derivatives are distributed symmetrically since the local functional $\hat{I}$ is symmetric and contains only holomorphic derivatives.

The above identity essentially follows from [25, Proposition B.2]. We give a different but more direct computation below for reader's convenience.

Let us change coordinates by

$$(z_1, z_2) \to (z = z_1 - z_2, z_2), \quad z = re^{i\theta}.$$

Let us focus on the term when $\sigma$ is the trivial permutation.

$$\begin{aligned}
&\frac{1}{m!} \int_{\mathbb{C}} d^2 z_1 \int_{\mathbb{C}} d^2 z_2 A(\overrightarrow{z}_1) B(\overrightarrow{z}_2) \partial_{z_1}^{k_0} K_\epsilon(\overrightarrow{z}_1, \overrightarrow{z}_2) \prod_{i=1}^{m} \partial_{z_1}^{k_i} P_\epsilon^L(\overrightarrow{z}_1, \overrightarrow{z}_2) \\
=& \frac{i^{m+1}(-2)^m}{m!} \int_{\mathbb{C}} d^2 z_2 B(\overrightarrow{z}_2) \int_{\mathbb{C}} d^2 z \frac{A(\overrightarrow{z}_2 + \overrightarrow{z})}{(-z)^{k_0 + \cdots + k_m + m}} \left( \int_\epsilon^L \prod_{i=1}^{m} \frac{dt_i}{2\pi t_i} \right) e^{-\frac{r^2}{2\epsilon} - \sum_{i=1}^{m} \frac{r^2}{2t_i}} \frac{1}{2\pi\epsilon} \left(\frac{r^2}{2\epsilon}\right)^{k_0} \prod_{i=1}^{k} \left(\frac{r^2}{2t_i}\right)^{k_i+1} \\
=& \frac{i^{m+1}(-2)^m}{m!} \int_{\mathbb{C}} d^2 z_2 B(\overrightarrow{z}_2) \\
& \int_0^\infty r dr \left( \int_\epsilon^L \prod_{i=1}^{m} \frac{dt_i}{2\pi t_i} \right) e^{-\frac{r^2}{2\epsilon} - \sum_{i=1}^{m} \frac{r^2}{2t_i}} \frac{1}{2\pi\epsilon} \left(\frac{r^2}{2\epsilon}\right)^{k_0} \prod_{i=1}^{k} \left(\frac{r^2}{2t_i}\right)^{k_i+1} \int_{|z|=r} d\theta \frac{A(\overrightarrow{z}_2 + \overrightarrow{z})}{(-z)^{k_0 + \cdots + k_m + m}}.
\end{aligned}$$

To compute $\int_{|z|=r} d\theta \frac{A(\overrightarrow{z}_2 + \overrightarrow{z})}{z^{k_0 + \cdots + k_m + m}}$, we need to do Taylor expansion of $A(\overrightarrow{z}_2 + \overrightarrow{z})$ around $z = 0$ to get

$$A(\overrightarrow{z}_2 + \overrightarrow{z}) \sim A(\overrightarrow{z}_2) + \partial_{z_2} A(\overrightarrow{z}_2) z + \bar{\partial}_{z_2} A(\overrightarrow{z}_2) \bar{z} + \cdots.$$

However, as shown by [25, Proposition B.2], only terms involving holomorphic derivatives will survive in the $\epsilon \to 0$ limit. Therefore the $\theta$ integral can be replaced by

$$\int_{|z|=r} d\theta \frac{A(\overrightarrow{z}_2 + \overrightarrow{z})}{(-z)^{k_0 + \cdots + k_m + m}} \to 2\pi \oint_{z_2} \frac{dz_1}{z_1 - z_2} \frac{A(\overrightarrow{z}_1)}{(z_2 - z_1)^{k_0 + \cdots + k_m + m}},$$

where the meaning of $\oint_{z_2} dz_1$ is explained above. In particular, its value does not depend on $r$. Therefore the $r$-integral can be evaluated as

$$\lim_{\epsilon\to 0}\int_0^\infty rdr\left(\int_\epsilon^L \prod_{i=1}^m \frac{dt_i}{2\pi t_i}\right)e^{-\frac{r^2}{2\epsilon}-\sum_{i=1}^m \frac{r^2}{2t_i}}\frac{1}{2\pi\epsilon}\left(\frac{r^2}{2\epsilon}\right)^{k_0}\prod_{i=1}^k\left(\frac{r^2}{2t_i}\right)^{k_i+1}$$

$$\overset{r^2=2\epsilon u_0}{\underset{t_i=\epsilon u_0/u_i}{=\!=\!=}}\frac{1}{(2\pi)^{m+1}}\int_0^\infty du_0\int_0^{u_0}\prod_{i=1}^k du_i e^{-\sum_{i=0}^m u_i}\prod_{i=0}^m u_i^{k_i}$$

$$=\frac{1}{(2\pi)^{m+1}}\int_{0\le u_i\le u_0, 1\le i\le m}\prod_{i=0}^m du_i\prod_{i=0}^m\left(u_i^{k_i}e^{-u_i}\right).$$

By summation over the permutations of $\{0,1,\cdots,m\}$, this integral leads to

$$\frac{1}{(2\pi)^{m+1}}\frac{1}{(m+1)}\int_0^\infty\prod_{i=0}^m du_i\prod_{i=0}^m\left(u_i^{k_i}e^{-u_i}\right)=\frac{1}{(2\pi)^{m+1}}\frac{\prod_{i=0}^m k_i!}{(m+1)}.$$

It follows by combining the above computations that

$$\lim_{\epsilon\to 0}\sum_{\sigma\in S_{k+1}}\frac{1}{m!}\int_{\mathbb{C}}d^2z_1\int_{\mathbb{C}}d^2z_2 A(\overrightarrow{z}_1)B(\overrightarrow{z}_2)\partial_{z_1}^{k_{\sigma(0)}}h_\epsilon(\overrightarrow{z}_1,\overrightarrow{z}_2)\prod_{i=1}^m\partial_{z_1}^{k_{\sigma(i)}}P_\epsilon^L(\overrightarrow{z}_1,\overrightarrow{z}_2)$$

$$=\frac{i^{m+1}(-2)^m\prod_{i=0}^m k_i!}{(2\pi)^m(m+1)!}\int d^2z_2 B(\overrightarrow{z}_2)\oint_{z_2}dz_1\frac{A(\overrightarrow{z}_1)}{\prod_{i=0}^m(z_2-z_1)^{k_i+1}}$$

$$=\frac{\pi}{(m+1)!}\int d^2z_2 B(\overrightarrow{z}_2)\oint_{z_2}dz_1 A(\overrightarrow{z}_2)\prod_{i=0}^m\partial_{z_1}^{k_i}\frac{i}{\pi(z_1-z_2)}.$$

□

### 3.4. **Generating function and modularity.**

3.4.1. *Generating function.* We describe the generating function when $\Sigma = E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ is an elliptic curve.

Let $I \in \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$ satisfy the Maurer-Cartan equation in Theorem 3.11. $\hat{I}[L]$ be the associated family solving the renormalized quantum master equation

$$(Q+\hbar\Delta_L)\,e^{\hat{I}[L]/\hbar}=0.$$

**Definition 3.12.** We define the generating function $\hat{I}\,[E_\tau] \in \mathcal{O}(\mathbf{h}[\epsilon])[[\hbar]]$ as a formal function on $\mathbf{h}[\epsilon]$ where

- $\epsilon$ is an odd element of degree 1, representing the generator $\frac{d\bar{z}}{\operatorname{Im}\tau}$ of harmonics $\mathbb{H}^{0,1}(E_\tau)$.

- Under the identification $H^\bullet(\mathcal{E},\bar\partial) \cong \mathbf{h}\otimes\mathbb{H}^{0,\bullet}(E_\tau) \cong \mathbf{h}[\epsilon]$, $\hat I[E_\tau]$ is the restriction of $\hat I[\infty]$ to harmonic elements $\mathbf{h}\otimes\mathbb{H}^{0,\bullet}(E_\tau)$.
- Given $a_1,\cdots,a_m\in\mathbf{h}[\epsilon]$, we denote its $m$-th Taylor coeffeicient by

$$\langle a_1,\cdots,a_m\rangle_g = \frac{\partial}{\partial a_1}\cdots\frac{\partial}{\partial a_m}\hat I_g[E_\tau](0).$$

*Remark* 3.13. Elements of $\mathbf{h}\otimes\mathbb{H}^{0,\bullet}(E_\tau)$ are ofter called *zero modes*. Our definition of $\hat I[E_\tau]$ is just the effective theory on zero modes following physics terminology.

The space $\mathbf{h}[\epsilon]$ carries naturally a $(-1)$-symplectic structure. The nontrivial pairing is between $\mathbf{h}$ and $\mathbf{h}\epsilon$ where

$$\omega(a,b\epsilon)=\langle a,b\rangle\,,\quad a,b\in\mathbf{h}.$$

Let $\Delta$ denote the associated BV operator on $\mathcal{O}(\mathbf{h}[\epsilon])$. It is not hard to see that $\Delta$ can be identified with $\Delta_\infty$. The differential $\delta$ also induces a differential on $\mathbf{h}[\epsilon]$. Here we only need to keep the constant part of the differential operator in $\delta$, since $\frac{\partial}{\partial z}$ will annihilate the harmonics $\mathbb{H}^{0,\bullet}(E_\tau)$.

**Proposition 3.14.** *The triple $(\mathcal{O}(\mathbf{h}[\epsilon]),\delta,\Delta)$ is a differential BV algebra. The generating function $\hat I[\Sigma_\tau]$ satisfies the BV master equation*

$$(\delta+\hbar\Delta)\,e^{\hat I[E_\tau]/\hbar}=0.$$

*Proof.* We observe that the BV kernel $K_L$ lies in $\mathrm{Sym}^2(\mathbf{h}[\epsilon])$ when $L\to\infty$, which defines the BV operator $\Delta$ as that on $\mathbf{h}[\epsilon]$. The proposition is just the quantum master equation

$$(Q+\hbar\Delta_\infty)e^{\hat I[\infty]/\hbar}=0$$

restricted to harmonic subspaces $\mathbf{h}[\epsilon]$.

□

3.4.2. *Modularity.* Now we analyze the dependence of $\hat I[E_\tau]$ on the complex structure $\tau$. We consider the modular group $SL(2,\mathbb{Z})$, which acts on the upper half plane $\mathbf{H}$ by

$$\tau\to\gamma\tau:=\frac{A\tau+B}{C\tau+D},\quad \text{for }\gamma\in\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in SL(2,\mathbb{Z}),\quad \tau\in\mathbf{H}.$$

Recall that a function $f:\mathbf{H}\to\mathbb{C}$ is said to have modular weight $k$ under the modular transformation $SL(2,\mathbb{Z})$ if

$$f(\gamma\overrightarrow{\tau})=(C\tau+D)^k f(\overrightarrow{\tau}),\quad \text{for }\gamma\in\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in SL(2,\mathbb{Z}).$$

**Definition 3.15.** We extend the $SL(2,\mathbb{Z})$ action to $\mathbb{C}^n \times \mathbf{H}$ by

$$\gamma : (z_1, \cdots, z_n, \tau) \to (\gamma z_1, \cdots, \gamma z_n, \gamma\tau) := \left(\frac{z_1}{C\tau + D}, \cdots, \frac{z_n}{C\tau+D}, \frac{A\tau + B}{C\tau + D}\right),$$

for $\gamma \in \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{Z})$.

It is easy to see that this defines a group action, in other words,

$$\gamma_1(\gamma_2(z_1, \cdots, z_n, \tau)) = (\gamma_1\gamma_2)(z_1, \cdots, z_n, \tau), \quad \gamma_1, \gamma_2 \in SL(2,\mathbb{Z}).$$

**Definition 3.16.** A differential form $\Omega$ on $\mathbb{C}^n \times \mathbf{H}$ is said to have modular weight $k$ under the above $SL(2,\mathbb{Z})$ action if

$$\gamma^*\Omega = (C\tau + D)^k \Omega.$$

When $n = 0$ and $\Omega$ being a 0-form on $\mathbf{H}$, this reduces to the above modular function of weight $k$.

Let $h_L$ be the heat kernel function on $E_\tau$ as before. Pulled back to the universal cover $\mathbb{C}$ of $E_\tau$, $h_L$ gives rise to a function $\tilde{h}_L$ on $\mathbb{C} \times \mathbb{C} \times \mathbf{H}$ by

$$\tilde{h}_L(\vec{z}_1, \vec{z}_2; \vec{\tau}) = \frac{1}{2\pi L} \sum_{\lambda \in \Lambda_\tau} e^{-|z_1 - z_2 + \lambda|^2/2L}, \quad \Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau.$$

$SL(2,\mathbb{Z})$ transforms the lattice $\Lambda$ by

$$\Lambda_{\gamma\tau} = \frac{1}{C\tau + D}\Lambda_\tau, \quad \gamma \in \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{Z}).$$

It follows that the heat kernel $\tilde{h}_L$ transforms under $SL(2,\mathbb{Z})$ as

$$\tilde{h}_L(\gamma\vec{z}_1, \gamma\vec{z}_2; \gamma\tau) = |C\tau + D|^2 \, h_{|C\tau+D|^2 L}(\vec{z}_1, \vec{z}_2; \vec{\tau}), \quad \forall \gamma \in SL(2,\mathbb{Z}).$$

The regularized BV kernel $K_L$ and propagator $P_\epsilon^L$ are

$$K_L(\vec{z}_1, \vec{z}_2; \vec{\tau}) = i \, h_L(\vec{z}_1, \vec{z}_2; \vec{\tau}) \, (d\bar{z}_1 \otimes 1 - 1 \otimes d\bar{z}_2) \, C_{\mathbf{h}}$$

and

$$P_\epsilon^L(\vec{z}_1, \vec{z}_2; \vec{\tau}) = -2i \int_\epsilon^L du \partial_{z_1} h_u(\vec{z}_1, \vec{z}_2; \vec{\tau}) C_{\mathbf{h}}.$$

We define similarly $\tilde{K}_L, \tilde{P}_\epsilon^L$ as $\tilde{h}_L$ on the universal cover $\mathbb{C}$ of $E_\tau$. The following lemma is straight-forward.

**Lemma 3.17.** *We have the following modular transformation properties of the regularized propagators and effective BV kernel*

$$\gamma^* \tilde{P}_{\epsilon}^{L} = (C\tau + D)\tilde{P}_{|C\tau+D|^2\epsilon}^{|C\tau+D|^2 L}, \quad \gamma^* \tilde{K}_L = (C\tau + D)\tilde{K}_{|C\tau+D|^2 L}.$$

*In particular, $\tilde{P}_0^{\infty}, \tilde{K}_{\infty}$ have weight 1 in the sense of Definition 3.16. More generally, the k-th holomorphic derivative $\partial_{z_1}^k \tilde{P}_0^{\infty}(\overrightarrow{z}_1, \overrightarrow{z}_2; \overrightarrow{\tau})$ has modular weight $k+1$.*

*Remark* 3.18. $K_{\infty}(\overrightarrow{z}_1, \overrightarrow{z}_2; \overrightarrow{\tau}) = \frac{d\bar{z}_1 \otimes 1 - 1 \otimes d\bar{z}_2}{2 \operatorname{Im} \tau} C_{\mathbf{h}}$.

**Definition 3.19.** A quantization $\hat{I}[L]$ defined by $I = \sum_{g \geq 0} I_g \hbar^g \in \oint \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$ in Theorem 3.9 is called *modular invariant* if each $I_g$ contains exactly $g$ holomorphic derivatives.

*Remark* 3.20. This definition may vary according to the conformal weight of the pairing $\langle -, - \rangle$ on $\mathbf{h}^*$. See Remark 4.5 for a specific example.

**Theorem 3.21.** *Let $\hat{I}[E_\tau] = \sum_{g \geq 0} \hat{I}_g[E_\tau]\hbar^g$ be the generating function of a modular invariant quantization, $\hat{I}_g[E_\tau] \in \mathcal{O}(\mathbf{h}[\epsilon])$. Then for any $a_1, \cdots, a_k \in \mathbf{h}, b_1, \cdots, b_m \in \mathbf{h}\epsilon$, the Taylor coefficient of $\hat{I}_g[E_\tau]$*

$$\langle a_1, \cdots, a_k, b_1, \cdots, b_m \rangle_g$$

*is modular of weight $m + g - 1$ as a function on $\mathbf{H}$. Moreover, It has the following expansion*

$$\langle a_1, \cdots, a_k, b_1, \cdots, b_m \rangle_g = \sum_{i=0}^{N} \frac{f_i(\tau)}{(\operatorname{Im} \tau)^i}$$

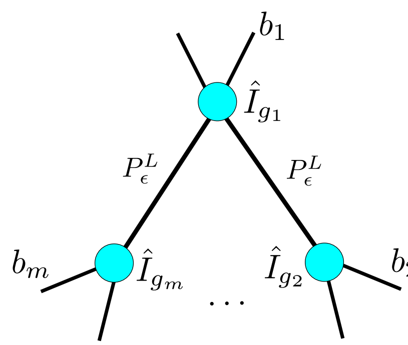
*where $f_i(\tau)$'s are holomorphic functions in $\tau$ and $N < \infty$ is an integer.*

*Proof.* Let us write

$$\hat{I}[\infty] = \sum_{\Gamma \text{ connected}} W(\Gamma, I).$$

Let $\Gamma$ be a Feynman graph which contributes to $\langle a_1, \cdots, a_k, b_1, \cdots, b_m \rangle_g$. By the type reason (the propagator only contains 0-forms on $E_\tau$), $\Gamma$ contains $m$ vertices

$\{v_j\}_{j=1}^m$ each of which has an external input given by $b_j$, $j = 1, \cdots m$.




Assume the vertex $v_j$ has genus $g_j$, i.e., given by the local functional $I_{g_j}$. Let $E$ be the set of propagators in $\Gamma$. Then

$$m + g = \sum_{j=1}^m g_j + |E| + 1.$$

The graph integral $W(\Gamma, I)$ can be written as

$$W(\Gamma, I) = \lim_{\substack{\epsilon\to 0\\ L\to\infty}} A \prod_{i=1}^m \int_{E_\tau} \frac{d^2 z_i}{\operatorname{Im}\tau} \prod_{e\in E} \partial_{z_{h(e)}}^{n_e} P_\epsilon^L(\vec{z}_{h(e)}, \vec{z}_{t(e)}; \vec{\tau})$$

where $A$ is a combinatorial coefficient not depending on $\tau$.

$$h, t : E \to \{1, \cdots, m\}$$

denote the head and the tale of the edge, where we have chosen an arbitrary orientation on edges. $n_e$ denotes the number of holomorphic derivatives applied to propagator at the edge $e$. Since all the external inputs $a_i, b_j$'s are harmonics, all holomorphic derivatives in the vertex $I_{g_j}$ will go to the propagators. By the modular invariance of the quantization, $I_{g_j}$ contains exactly $g_j$ holomorphic derivatives, hence

$$\sum_{e\in E} n_e = \sum_{j=1}^m g_j.$$

It follows from the modular property of the measure $\frac{d^2 z}{\operatorname{Im}\tau}$ and the propagator $\hat{P}_0^\infty$ that $W(\Gamma, I)$ is a modular function on $\mathbf{H}$ of weight

$$\sum_{e\in E} (n_e + 1) = |E| + \sum_{j=1}^m g_j = m + g - 1.$$

The polynomial dependence on $\frac{1}{\operatorname{Im}\tau}$ follows from [25, Proposition 5.1]. □

Given a function $f$ on $\mathbf{H}$ of the form $f = \sum_{i=0}^{N} \frac{f_i(\tau)}{(\operatorname{Im}\tau)^i}$, we denote

$$\lim_{\bar{\tau}\to\infty} f := f_0(\tau).$$

It is shown in [20] that $f$ is determined by the leading term $f_0$ and modular property. In particular, the operation $\lim_{\bar{\tau}\to\infty}$ identifies the space of almost holomorphic modular forms with the space of quasi-modular forms [20]. In general, the $\bar{\tau} \to \infty$ limit of the generating function $\hat{I}[E_\tau]$ will be reduced to certain characters on vertex algebras. This is argued in [13] by the method of contact terms, and realized in [26] by a method of cohomological localization via the quantum master equation. This pheonomenon will be systematically studied in [27].

3.5. **Example: Poisson $\sigma$-model.** We illustrate the application of Theorem 3.11 by the example of the AKSZ formalism of Poisson $\sigma$-model as described in [6].

Let $V = \mathbb{R}^n$ and $P$ be a Poisson bi-vector field on $V$. Let $\Sigma$ be a flat surface as before. We consider the BV formalism of Poisson sigma model in the formal neighborhood of constant maps from $\Sigma$ to the origin of $V$. The space of fields is

$$\mathcal{E} = \Omega^\bullet(\Sigma) \otimes (V \oplus V^*[1]).$$

The differential on $\mathcal{E}$ is the de Rham differential $d_\Sigma$ on $\Omega^\bullet(\Sigma)$. The $(-1)$-shifted symplectic pairing is

$$\omega(\varphi, \eta) := \int_\Sigma (\varphi, \eta), \quad \text{where } \varphi \in \Omega^\bullet(\Sigma) \otimes V, \eta \in \Omega^\bullet(\Sigma) \otimes V^*[1].$$

Let us choose linear coordinates $x^i$ on $V$ and $P = \sum_{i,j} P^{ij}(x)\partial_{x^i} \wedge \partial_{x^j}$. The above fields $\varphi, \eta$ in coordinate components are

$$\varphi = \{\varphi^i\}_{i=1}^n, \quad \eta = \{\eta_i\}_{i=1}^n, \quad \text{where } \varphi^i \in \Omega^\bullet(\Sigma), \eta_i \in \Omega^\bullet(\Sigma)[1].$$

The action functional is given by [6]

$$S = \sum_i \int_\Sigma \eta_i d_\Sigma \varphi^i + \sum_{i,j} \int_\Sigma P^{ij}(\varphi)\eta_i\eta_j.$$

Let us split the differential $d_\Sigma = Q + \delta$, where $Q = \bar{\partial}$ and $\delta = \partial$ are the $(0,1)$-differential and $(1,0)$-differential on $\Sigma$. Then the above theory falls into the setting of Section 3.2, and we can apply Theorem 3.11 to study its quantization via chiral deformations. In terms of notations in Section 3.2, we have

$$\mathbf{h} = V[dz] \oplus V^*[dz][1],$$

where $V[dz] = V \otimes \mathbb{C}[dz], V^*[dz] = V^* \otimes \mathbb{C}[dz]$. The relevant vertex algebra is generated by $\varphi^i, \eta_i$: $\varphi^i$ represent components in $V[dz]$ and $\eta_i$ represent components in $V^*[dz][1]$. The nontrivial OPEs are given by

$$\varphi^i(z)\eta_j(w) \sim \delta_{i,j}\left(\frac{i\hbar}{\pi}\right)\frac{dz - dw}{z - w}.$$

Here it is understood that we have to match the corresponding components in $dz, dw$ in the above formula. For example, if we write $\varphi^i(z) = \varphi^i_0(z) + \varphi^i_1(z)dz$ and $\eta_i(z) = \eta_{i0}(z) + \eta_{i1}(z)dz$, then matching the $dw$ component we find

$$\varphi^i_0(z)\eta_{j1}(w) \sim \delta_{i,j}\left(\frac{i\hbar}{\pi}\right)\frac{-1}{z - w}.$$

The classical interaction is represented by the vertex operator

$$\oint I, \quad \text{where } I = \sum_{i,j} P^{ij}(\varphi)\eta_i\eta_j.$$

Here it is understood by the type reason that only the $dz$-component of $I$ contributes to $\oint I$ when we expand the fields $\varphi^i, \eta_i$ into forms in $z$.

The classical master equation is satisfied by two independent equations

$$\delta \oint I = 0, \quad \left\{\oint I, \oint I\right\} = 0.$$

The first term vanishes since it produces a total derivative while the vanishing of the second term follows from Jacobi identity for $P$ [6].

**Proposition 3.22.** *The classical interaction $\oint I$ satisfies the quantum master equation of Theorem 3.11.*

*Sketch of proof.* We only need to prove $[\oint I, \oint I] = 0$ since $\delta \oint I = 0$. By definition, $[\oint I, \oint I]$ is computed by Wick contractions and OPE's (see [19]). A single contraction gives rise to the classical bracket $\{\oint I, \oint I\}$. If we have two or more contractions between the fields, the form of $I$ implies that each contraction contributes a factor of $dz - dw$. However, the product of two copies of $dz - dw$ is vanishing by the type reason. This implies $[\oint I, \oint I] = \{\oint I, \oint I\} = 0$. □

This proposition says that no quantum correction is needed at all! This remarkable fact lies in cancellations between bosons and fermions, which is just a incidence of supersymmetry. This result gives a natural interpretation of Kontsevich's graph formula of star product [21], which is argued in [5] by the requirement of BV quantum master equation. We remark that the tadpole diagrams (i.e. with edges that start and end at the same vertex) that appeared in [5] vanish by our renormalization scheme: the regularized BV kernel $K_\epsilon$ and propagator $P_\epsilon^L$ on the tadpole are zero before we take the limit $\epsilon \to 0$ (see the proof of Theorem 3.11). In particular, our construction here should lead to a rigorous formulation of [5] by gluing the above linear case to Poisson manifolds.

In general, when the surface $\Sigma$ is a compact Riemann surface which is no longer flat, further obstructions may exist for quantization. One such example is the topological B-model from a genus $g$ surface $\Sigma$ to a complex manifold $X$. The perturbative BV quantization in the current sense is analyzed in [22]. It is found that the tadpole diagram gives rise to the obstruction class (anomaly) by $(2g-2)c_1(X)$, requiring for a Calabi-Yau geometry to be quantizable. It would be very interesting to extend the results in this paper systematically to arbitrary surfaces and to the string-theoretical formulation of coupling with 2d gravity.

## 4. Application: quantum B-model on elliptic curves

In this section, we apply our theory to solve the higher genus $B$-model on elliptic curves. Part of the results are presented in [26] based mainly on symmetry argument. Using the technique we have developed in this paper, we give stronger results on the exact solution of the full system. Combining with the A-model results in [32] and the B-model computations in [26], it leads to the establishment of higher genus mirror symmetry on elliptic curves.

### 4.1. **BCOV theory on elliptic curves.**

We consider topological B-model on Calabi-Yau geometry, which concerns with the geometry of complex structures. In [3], Bershadsky, Cecotti, Ooguri, and Vafa proposed *Kodaira-Spencer gauge theory* on Calabi-Yau 3-folds as the leading approximation of B-twisted closed string field theory. This is fully generalized in [10] to arbitrary Calabi-Yau manifolds, giving rise to a complete description of B-twisted closed string field theory in the sense of Zwiebach [33]. We shall call this BCOV theory. An earlier related work on the finite dimensional toy model of BCOV theory appeared in [28] in the absence of the issue of renormalization.

In this section, we describe BCOV theory on elliptic curves [26] and study its quantum geometry in terms of the tools we have developed.

Let $\mathrm{E}_\tau = \mathbb{C}/(\mathbb{Z}\oplus Z\tau)$ as before. $z$ denotes the linear holomorphic coordinate. The space of fields of BCOV theory on $\mathrm{E}_\tau$ is given by [10]

$$\mathcal{E} = \Omega^{0,\bullet}(\mathrm{E}_\tau, \mathcal{O}_{\mathrm{E}_\tau})[[t]] \oplus \Omega^{0,\bullet}(\mathrm{E}_\tau, T_{\mathrm{E}_\tau}[1])[[t]].$$

Here $T_{E_\tau}[1]$ is the holomorphic tangent bundle sitting at degree $-1$. $t$ is a formal variable of cohomology degree 0 that represents "gravitational descendants". Note that we have used a different grading from [10]. The differential is given by

$$Q = \bar{\partial} + \delta, \quad \delta = t\partial,$$

where $\partial : T_{E_\tau} \to \mathcal{O}_{E_\tau}$ is the divergence operator with respect to the holormophic volume form $dz$. In terms of the set-up in section 3.2, we have

- $\mathbf{h} = \mathbb{C}[[t,\theta]], \deg(t) = 0, \deg(\theta) = -1$. Here $\theta$ represents the global vector field $\partial_z$. Then $\mathcal{E} \cong \Omega^{0,\bullet}(E_\tau) \otimes \mathbf{h}$.
- $\delta = \frac{\partial}{\partial z} \otimes t\frac{\partial}{\partial \theta}$.
- However, $\mathcal{E}$ is $(-1)$-shifted Poisson instead of symplectic. Let

  $$h_L(\overrightarrow{z}_1, \overrightarrow{z}_2) = \sum_{\lambda \in \Lambda} \frac{1}{2\pi L} e^{-|z_1 - z_2 + \lambda|^2/2L}, \quad \Lambda = \mathbb{Z} + \mathbb{Z}\tau,$$

  be the heat kernel function on $E_\tau$. The regularized BV kernel is given by

  $$K_L = i\partial_{z_1} h_L(\overrightarrow{z}_1, \overrightarrow{z}_2)(d\bar{z}_1 \otimes 1 - 1 \otimes d\bar{z}_2)C_{\mathbf{h}}$$

  where $C_{\mathbf{h}} := t^0 \otimes t^0 \in \mathbf{h} \otimes \mathbf{h}$. The regularized propagator is

  $$P_\epsilon^L = -2i \int_\epsilon^L du\, \partial_{z_1}^2 h_u(\overrightarrow{z}_1, \overrightarrow{z}_2)C_{\mathbf{h}}.$$

The situation differs a bit from our set-up in Section 3.2. There is one more holomorphic derivative for our BV kernel and the factor $C_{\mathbf{h}}$ is highly degenerate. Nevertheless, techniques in section 3 can be applied to the Poisson case without much change (see also Remark 2.5). In particular,

- $K_L$ defines a regularized BV operator $\Delta_L$ such that $(\mathcal{O}(\mathcal{E}), Q, \Delta_L)$ is a differential BV algebra.
- $K_0$ well-defines a BV-bracket on local functionals as in Definition 2.16

  $$\{-,-\} : \mathcal{O}_{loc}(\mathcal{E}) \otimes \mathcal{O}_{loc}(\mathcal{E}) \to \mathcal{O}_{loc}(\mathcal{E}).$$

Introduce

$$\langle - \rangle_0 : \mathrm{Sym}^\bullet(\mathbb{C}[[t]]) \to \mathbb{C}, \quad \left\langle t^{k_1} \otimes \cdots \otimes t^{k_n} \right\rangle_0 = \binom{n-3}{k_1, \cdots, k_n}.$$

$\langle - \rangle_0$ represents intersection numbers of $\psi$-classes on moduli space of stable rational curves. We extend it $\Omega^{0,\bullet}[\theta]$-linearly to

$$\langle - \rangle_0 : \mathrm{Sym}^\bullet(\mathcal{E}) \to \Omega^{0,\bullet}[\theta].$$

**Definition 4.1** ([10])**.** We define the classical BCOV interaction $I^{BCOV} \in \mathcal{O}_{loc}(\mathcal{E})$ by the local functional

$$I^{BCOV}(\varphi) = i \int_{E_\tau} dz \int d\theta \, \langle e^{\varphi} \rangle_0 \,, \quad \varphi \in \mathcal{E}.$$

Here $\langle e^{\varphi} \rangle_0$ is understood as

$$\langle e^{\varphi} \rangle_0 = \sum_{k \geq 3} \left\langle \frac{\varphi^{\otimes k}}{k!} \right\rangle_0 .$$

The fermionic integral $\int d\theta$ means taking the coefficient of the term with $\theta$

$$\int d\theta (a + \theta b) = b.$$

$I^{BCOV}$ satisfies the classical master equation [10]

$$Q I^{BCOV} + \frac{1}{2} \left\{ I^{BCOV}, I^{BCOV} \right\} = 0.$$

*Remark* 4.2. By Remark 2.18, $Q + \{I^{BCOV}, -\}$ defines a $L_\infty$-structure on $\mathcal{E}$. It is shown in [10] that this $L_\infty$-structure is quasi-isomorphic to the standard dg Lie algebra structure with differential $Q$ and Schouten-Nijenhuis bracket.

Since the Poisson kernel is degenerate and contains one more holomorphic derivative, the application of Theorem 3.11 to our situation requires slight modification. Let us parametrize

$$\varphi = \sum_{k \geq 0} b_k t^k + \eta_k \theta t^k, \quad \varphi \in \mathbf{h}.$$

We introduce mutually local fields $b_k(z), \eta_k(z)$ with OPE relations

$$b_0(z) b_0(w) \sim \frac{i\hbar}{\pi} \frac{1}{(z-w)^2}, \quad b_k(z) b_m(w) \sim 0, k+m > 0.$$

$$b_\bullet(z) \eta_\bullet(w) \sim 0, \quad \eta_\bullet(z) \eta_\bullet(w) \sim 0.$$

The OPEs exactly respect the structure of the our Poisson kernel. The associated vertex algebra $\mathcal{V}[[\mathbf{h}^*]]$ is the tensor product of a Heisenberg vertex algebra (generated by the field $b_0(z)$) with several copies of commutative vertex algebra (generated by the fields $(b_{>0}(z), \eta_\bullet(z))$). Equivalently, if we collect the fields with parameter $t$

$$b(z,t) = \sum_{k \geq 0} b_k(z) t^k, \quad \eta(z,t) = \sum_{k \geq 0} \eta_k(z) t^k,$$

then the OPE's can be simply written as

$$b(z_1, t_1) b(z_2, t_2) \sim \frac{i\hbar}{\pi} \frac{1}{(z_1 - z_2)^2}, \quad b(z_1, t_1) \eta(z_2, t_2) \sim 0, \quad \eta(z_1, t_1) \eta(z_2, t_2) \sim 0.$$

The differential $\delta$ is then dually expressed as

$$\delta b(z,t) = t\partial_z \eta(z,t), \quad \delta\eta(z,t) = 0.$$

In terms of components,

$$\delta b_{k+1} = \partial_z \eta_k, \quad \delta \eta_k = 0.$$

We adapt notations in section 3.2. The classical BCOV interaction can be expressed as $I^{BCOV} = \hat{I}_0$, where $I_0 \in \mathcal{V}[[\mathbf{h}^*]]$ is defined similarly to Definition 4.1

$$I_0(b,\eta) = \left\langle e^b \otimes \eta \right\rangle_0.$$

It satisfies the following Maurer-Cartan equation modulo $\hbar$

$$\delta \oint dz I_0 + \frac{1}{2}\frac{\pi}{i\hbar}\left[\oint dz I_0, \oint dz I_0\right] = O(\hbar).$$

Our goal is to find $I = \sum\limits_{g\geq 0} \hbar^g I_g \in \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$ as a quantum correction of the classical BCOV interaction $I_0$ satisfying the exact Maurer-Cartan equation

$$\delta \oint dz I + \frac{1}{2}\frac{\pi}{i\hbar}\left[\oint dz I, \oint dz I\right] = 0.$$

We leave it to the reader to check that a slight modification of Theorem 3.11 implies that the above Maurer-Cartan equation is equivalent to the renormalized quantum master equation of our BCOV theory.

4.2. **Hodge weight and dilaton equation.** The first simplification we will make is to use rescaling symmetries of quantum master equation.

We assigne the following gradings in $\mathcal{V}[[\mathbf{h}^*]][[\hbar]]$ and $\oint \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$.

| | $\partial_z^m b_k$ | $\partial_z^m \eta_k$ | $\hbar$ | $\delta$ | $z$ | $\oint dz$ | $[-,-]$ |
|---|---|---|---|---|---|---|---|
| cohomology degree (deg) | 0 | 1 | 0 | 1 | 0 | 0 | 0 |
| conformal weight (cw) | -k+m+1 | -k+m | 0 | 1 | -1 | -1 | 0 |
| dilaton dimension (dim) | m | m | -2 | 1 | 0 | 0 | -1 |
| Hodge weight (hw=cw-dim) | -k+1 | -k | 2 | 0 | 0 | -1 | 1 |

They are all compatible with quantum master equation. The classical BCOV interaction has

$$\deg(I_0) = 1, \quad \mathrm{cw}(I_0) = 2, \quad \dim(I_0) = 0.$$

**Definition 4.3.** A solution $I \in \mathcal{V}[[\mathbf{h}^*]][[\hbar]]$ of quantum master equation for which

$$\deg(I) = 1, \quad \mathrm{cw}(I) = 2, \quad \dim(I) = 0,$$

will be called an *equivariant quantization*.

*Remark* 4.4. For an equivariant quantization,

$$hw(I_g) = \mathrm{cw}(I_g) - \dim(I_g) = 2 - 2g.$$

This is exactly the Hodge weight condition for elliptic curves described in [10]. The condition $\dim(I) = 0$ is essentially equivalent to the dilaton equation [26]. Therefore the equivariance condition is naturally viewed as imposing the Hodge weight condition and the dilation equation.

*Remark* 4.5. The dilaton dimension condition implies that the genus $g$ correction $I_g$ contains exactly $2g$ holomorphic derivatives. This is the modification of modular invariant quantization (Definition 3.19). It changes the number of holomorphic derivatives from $g$ to $2g$ exactly because there exists an extra holomorphic derivative on our propagator. In particular, an analgue of Theorem 3.21 holds in this situation. See also [26].

**Definition 4.6.** Let us denote the homogeneous component

$$\mathcal{V}[[\mathbf{h}^*]]^d_w := \{a \in \mathcal{V}[[\mathbf{h}^*]] | \deg(a) = d, \mathrm{cw}(a) = w\}.$$

We will focus on equivariant quantizations, which are given by elements $I \in \mathcal{V}[[\mathbf{h}^*]]^1_2$ satisfying the Maurer-Cartan equation. In this case, we can use the rescaling symmetry to set $\hbar = \pi/i$. The power of $\hbar$ can be recovered from counting the number of derivatives.

From now on, we will work with the normalized OPE

$$b_0(z)b_0(w) \sim \frac{1}{(z-w)^2}, \quad b_k(z)b_m(w) \sim 0, k+m > 0.$$
$$b_\bullet(z)\eta_\bullet(w) \sim 0, \quad \eta_\bullet(z)\eta_\bullet(w) \sim 0.$$

4.3. **Reduction to Fedosov's equation.** In this section, we use boson-fermion correspondence to further reduce quantum master equation to a deformation quantization problem of Fedosov's equation [14].

**Definition 4.7.** Let us package the fields $\{b_{>0}, \eta_\bullet\}$ into new series denoted by

$$\tilde{b}(z) = \sum_{k\geq 1} \frac{t^k}{k!} b_k(z), \quad \tilde{\eta}(z) = \sum_{k\geq 1} \frac{t^k}{k!} \eta_{k-1}(z).$$

These fields do not appear in the propagator, and will be called the *background fields*. $b_0$ does appear in the propagator, and will be called the *dynamical field*.

The deformation quantization problem arises from viewing the linear coordinate $z$ and the descendant variable $t$ as a Darboux system of a holomorphic symplectic structure on $\mathbb{C}^2$:

$$\omega = dz \wedge dt.$$

We consider the following differential ring freely generated by two generators $\tilde{b}, \tilde{\eta}$ and two derivatives $\partial_z, \partial_t$:

$$\mathcal{B} = \mathbb{C}[[\partial_z^\bullet \partial_t^\bullet \tilde{b}, \partial_z^\bullet \partial_t^\bullet \tilde{\eta}]].$$

Here $\tilde{b}$ has even parity of cohomology degree 0, $\tilde{\eta}$ has odd parity of cohomology degree 1. Here we have confused ourselves to use the same symbols $\tilde{b}, \tilde{\eta}$ for our generators in $\mathcal{B}$. Later on, we will plug into expressions in terms of the background fields in Definition 4.7 when the meaning is clear from the context.

The symplectic form induces a Poisson structure on $\mathcal{B}$ by

$$\{F, G\} = \partial_t F \partial_z G - \partial_z F \partial_t G, \quad F, G \in \mathcal{B}.$$

We also introduce a differential $\delta$ as an analogue of that in our BCOV theory

$$\delta : \mathcal{B} \to \mathcal{B}, \quad \tilde{b} \to \partial_z \tilde{\eta}.$$

It is easy to check that $\delta$ is compatible with the Poisson bracket, hence $\mathcal{B}$ becomes a dg Poisson algebra. It has a natural deformation quantization in terms of the Moyal product $\star$

$$\mathcal{B} \star \mathcal{B} \to \mathcal{B}, \quad F \star G = \sum_{k_1, k_2 \geq 0} \frac{(-1)^{k_2}}{2^{k_1+k_2} k_1! k_2!} \left(\partial_t^{k_1} \partial_z^{k_2} F\right) \left(\partial_t^{k_2} \partial_z^{k_1} G\right).$$

The following lemma is straight-forward.

**Lemma 4.8.** *The triple $(\mathcal{B}, \star, \delta)$ defines an associative differential graded algebra. In particular, $(\mathcal{B}, \delta, [-,-]_\star)$ is a DGLA, where $[-,-]_\star$ is the commutator with respect to the Moyal product.*

We also introduce the analogue grading of conformal weight by

$$\mathrm{cw}(\partial_z^m \partial_t^k \tilde{b}) = m - k + 1, \quad \mathrm{cw}(\partial_z^m \partial_t^k \tilde{\eta}) = m - k + 1.$$

Note that $\tilde{\eta}$ has now $\mathrm{cw} = 1$ by the shift in our Definition 4.7. Then

$$\mathrm{cw}(\star) = 0, \quad \mathrm{cw}(\delta) = 1.$$

We denote the homogeneous component

$$\mathcal{B}_w^d = \{u \in \mathcal{B} | \deg(u) = d, \quad \mathrm{cw}(u) = w\}.$$

Our goal in this section is to construct a morphism of DGLA preserving the conformal weight

$$\Phi : (\mathcal{B}, \delta, [-,-]_\star) \to \left( \oint (\mathcal{V}[[\mathbf{h}^*]]), \delta, [-,-] \right)$$

and construct a canonical solution of Maurer-Cartan equation in $\mathcal{B}$. This leads to a solution of quantum master equation for our BCOV theory.

4.3.1. *boson-fermion correspondence.* We will construct $\Phi$ in terms of boson-fermion correspondence. Let us first fix our notations here and refer details to [19, 30]. We introduce a pair of fermions $\psi, \psi^\dagger$ with OPE

$$\psi(z)\psi^\dagger(w) \sim \frac{1}{z-w}.$$

We introduce a free boson with OPE

$$\phi(z)\phi(w) \sim \log(z-w), \quad \phi(z) = \sum_{k \neq 0} \frac{\alpha_n z^{-n}}{-n} + \alpha_0 \log z + p.$$

$p$ is the momemtum creation operator as a conjugate of $\alpha_0$. The boson-fermion correspondence says that a free boson is equivalent to a pair of fermions, under the following correspondence rule

$$\psi =: e^{\phi} :_B, \quad \psi^\dagger =: e^{-\phi} :_B, \quad \text{and} \quad \partial\phi =: \psi\psi^\dagger :_F.$$

Here $: - :_B, : - :_F$ denote the normal ordering for bosonic fields and fermionic fields respectively. The following fundamental relation holds

$$: \psi(z)\psi^\dagger(w) :_F = \frac{1}{z-w} \left( : e^{\phi(z)-\phi(w)} :_B - 1 \right).$$

Let us expand by

$$: e^{\phi(z)-\phi(w)} :_B = 1 + \sum_{k \geq 1} \frac{(z-w)^k}{k!} W^{(k)}(w), \quad W^{(k)}(z) = \sum_{n \in \mathbb{Z}} z^{-n-k} W_n^{(k)},$$

then $W_n^{(k)}$ generate the so-called $W_{1+\infty}$ algebra. In terms of bosons,

$$W^{(k)}(\partial_z \phi) = \sum_{\sum_{i \geq 1} i k_i = k} \frac{k!}{\prod_i k_i!} : \prod_i \left( \frac{1}{i!} \partial^i \phi \right)^{k_i} :_B .$$

Note that $W^{(k)}$ only depends on $\partial_z \phi$. We can also express it in terms of fermions

$$W^{(k)}(\psi, \psi^\dagger) = k : (\partial^{k-1}\psi)\psi^\dagger :_F.$$

It follows from the fermionic expression that the Fourier modes $\oint dzz^m W^{(k)}$ generates a central extension of the Lie algebra of differential operators on the circle

$$kz^m\partial_z^{k-1} \rightsquigarrow \oint dzz^m W^{(k)}, \quad k \geq 1, m \in \mathbb{Z}.$$

If we only look at non-negative modes, then we have a Lie algebra isomorphism

$$\begin{aligned} \mathbf{W} : \mathbb{C}\left[z, \partial_z\right] &\stackrel{\cong}{\to} \mathrm{Span}_{\mathbb{C}}\left\{\oint dzz^m W^{(k)}\right\}_{k\geq 1, m\geq 0} \\ kz^m\partial_z^{k-1} &\to \oint dzz^m W^{(k)}. \end{aligned}$$

Our field $b_0$ of Heisenberg vertex algebra can be identified via free boson by

$$b_0(z) = \partial_z\phi(z).$$

The other fields $b_{>0}, \eta_\bullet$ generate holomorphic(commutative) vertex algebra. Under the boson-fermion correspondence,

$$\mathcal{V}[[\mathbf{h}^*]][[e^{\pm p}]] \cong \mathbb{C}[[\partial_z^m\psi, \partial_z^m\psi^\dagger, \partial_z^m b_{>0}, \partial_z^m \eta_\bullet]]_{m\geq 0}.$$

If we introduce the charge grading,

$$\mathrm{charge}(\psi) = 1, \quad \mathrm{charge}(\psi^\dagger) = -1, \quad \mathrm{charge}(\mathrm{others}) = 0,$$

then $\mathcal{V}[[\mathbf{h}^*]e^{kp}$ corresponds the homogenous component of charge $k$ on the fermionic side. We will be mainly interested in the charge 0 component. The operator $\delta$ does not involve $b_0$. The only nontrivial part of $\delta$ is

$$\delta(\partial_z^m b_{k+1}) = \partial_z^{m+1}\eta_k, \quad \forall k, m \geq 0.$$

4.3.2. *Reduction to Fedosov's equation.* Now we construct the map $\Phi$.

**Definition 4.9.** We define $\Phi : \mathcal{B} \to \oint(\mathcal{V}[[\mathbf{h}^*]])$ by

$$\Phi(J) = \sum_{k\geq 0} \frac{1}{k+1} \oint dzW^{(k+1)}(b_0) \oint dtt^{-k-1} e^{\frac{1}{2}\frac{\partial}{\partial z}\frac{\partial}{\partial t}} J(\tilde{b}, \tilde{\eta}), \quad J \in \mathcal{B}.$$

In this expression, we need to substitute the generators $\tilde{b}, \tilde{\eta}$ in terms of the background fields as in Definition 4.7. $\oint dt$ is the same as taking the residue at $t = 0$. $W^{(k+1)}(b_0)$ is defined in the previous subsection under $b_0 = \partial_z\phi$.

It is easy to see that $\Phi$ preserves the conformal weight. Moreover,

**Proposition 4.10.** $\Phi$ *is a morphism of DGLA's*

$$\Phi : (\mathcal{B}, \delta, [-,-]_\star) \to \left(\oint(\mathcal{V}[[\mathbf{h}^*]]), \delta, [-,-]\right).$$

*Proof.* In terms of notations in the previous subsection, $\Phi$ can be written as

$$\Phi(J) = \mathbf{W}\left(\sum_{k\geq 0}\left(\oint dt t^{-k-1} e^{\frac{1}{2}\partial_z\partial_t} J\right)\partial_z^k\right).$$

To clarify the meaning of this formula, let

$$\rho : \mathbb{C}[z,t] \to \mathbb{C}[z,\partial_z], \quad \rho(z^m t^k) = z^m \partial_z^k.$$

An equivalent formal description is

$$\rho(f) = \sum_{k\geq 0}\left(\oint dt t^{-k-1} f(t,z)\right)\partial_z^k, \quad f \in \mathbb{C}[z,t].$$

Let $\star$ denote the Moyal product on $\mathbb{C}[z,t]$

$$f \star g = e^{\frac{1}{2}(\partial_{t_1}\partial_{z_2} - \partial_{z_1}\partial_{t_2})}(f(z_1,t_1)g(z_2,t_2))\Big|_{z_i=z,t_i=t}, \quad f,g \in \mathbb{C}[z,t].$$

Consider

$$\begin{aligned} e^{\frac{1}{2}\partial_z\partial_t}(f\star g) &= e^{\frac{1}{2}\partial_z\partial_t}\left(e^{\frac{1}{2}(\partial_{t_1}\partial_{z_2} - \partial_{z_1}\partial_{t_2})}(f(z_1,t_1)g(z_2,t_2))\Big|_{z_i=z,t_i=t}\right) \\ &= e^{\frac{1}{2}(\partial_{z_1}+\partial_{z_2})(\partial_{t_1}+\partial_{t_2})} e^{\frac{1}{2}(\partial_{t_1}\partial_{z_2} - \partial_{z_1}\partial_{t_2})}(f(z_1,t_1)g(z_2,t_2))\Big|_{z_i=z,t_i=t} \\ &= e^{\partial_{t_1}\partial_{z_2}}(e^{\frac{1}{2}\partial_{z_1}\partial_{t_1}} f(z_1,t_1) e^{\frac{1}{2}\partial_{z_2}\partial_{t_2}} g(z_2,t_2))\Big|_{z_i=z,t_i=t}. \end{aligned}$$

Comparing with the associative composition $\circ$ of differential operators, we find

$$\rho(e^{\frac{1}{2}\partial_z\partial_t}(f\star g)) = \rho(e^{\frac{1}{2}\partial_z\partial_t} f)\circ\rho(e^{\frac{1}{2}\partial_z\partial_t} g).$$

In particular,

$$\rho(e^{\frac{1}{2}\partial_z\partial_t}[f,g]_\star) = \left[\rho(e^{\frac{1}{2}\partial_z\partial_t} f), \rho(e^{\frac{1}{2}\partial_z\partial_t} g)\right].$$

It follows from this algebraic fact and $\mathbf{W}$ being a Lie algebra morphism that

$$\Phi([J_1,J_2]_\star) = [\Phi(J_1),\Phi(J_2)], \quad \forall J_1,J_2 \in \mathcal{B}.$$

The compatibility of $\Phi$ with $\delta$ is easy to verify. □

To construct an equivariant solution of quantum master equation, we only need to find $J \in \mathcal{B}_1^1$ satisfying

$$\delta J + \frac{1}{2}[J,J]_\star = 0.$$

This can be viewed as a version of Fedosov's abelian connection [14].

**Lemma 4.11.**

$$H^\bullet(\mathcal{B},\delta) = \mathbb{C}[[\partial_t^k\tilde{\eta}]].$$

*Proof.* This follows from the observation that the complex $(\mathcal{B},\delta)$ can be identified with the de Rham complex of $\mathbb{C}[[\partial_z^\bullet\partial_t^\bullet\tilde{b}]]$ valued in the vector space $\mathbb{C}[[\partial_t^k\tilde{\eta}]]$. ☐

**Corollary 4.12.**

$$H^1(\mathcal{B},\delta)_1=\mathbb{C}\tilde{\eta},\quad H^2(\mathcal{B},\delta)_2=0.$$

*Here $H^k(\mathcal{B},\delta)_w$ denotes the component of $H^k(\mathcal{B},\delta)$ of conformal weight $w$.*

*Proof.* $H^1(\mathcal{B},\delta)_1=\mathbb{C}\tilde{\eta}$ is obvious. The only possible term in $\mathbb{C}[[\partial_t^k\tilde{\eta}]]$ with deg $=$ 2 and cw $=2$ is $\tilde{\eta}^2$, which vanishes by the odd parity of $\tilde{\eta}$. ☐

To solve the above equation, let us introduce an auxiliary grading by

$$T(\partial_z^m\partial_t^k\tilde{b})=k,\quad T(\partial_z^m\partial_t^k\tilde{\eta})=k,$$

i.e., $T$ counts the number of $\partial_t$'s. Let us introduce the operator

$$\delta^*:\mathcal{B}\to\mathcal{B},\quad \partial_z^{m+1}\partial_t^k\tilde{\eta}\to\partial_z^m\partial_t^k\tilde{b},(m\geq 0),\quad \partial_t^k\tilde{\eta}\to 0.$$

Let $N=\delta\delta^*+\delta^*\delta$. We define

$$\delta^{-1}:\alpha\to\begin{cases}\frac{1}{m}\delta^*\alpha & \text{if } N\alpha=m\alpha\\ 0 & \text{if } N\alpha=0\end{cases}$$

Then $\delta^{-1}$ can be viewed as a homotopic inverse of $\delta$, where $1-[\delta,\delta^{-1}]$ is the projection to $\mathbb{C}[[\partial_t^k\tilde{\eta}]]\cong H^\bullet(\mathcal{B},\delta)$.

**Lemma 4.13.** *There exists a unique $J^B\in\mathcal{B}_1^1$ satisfying*

(1) $\delta J^B+\frac{1}{2}\left[J^B,J^B\right]=0$;
(2) $\lim_{\lambda\to 0}\lambda^T(J^B)=\tilde{\eta}$;
(3) $\delta^{-1}J^B=0$.

*Proof.* Let us decompose $J^B=\sum\limits_{k\geq 0}J_{(k)}$, where $T(J_{(k)})=kJ_{(k)}$. The initial condition (2) is $J_{(0)}=\tilde{\eta}$. We show that other $J_{(k)}$'s can be uniquely solved satisfying conditions (1) and (3). Let us denote $J_{(<k)}=\sum\limits_{0\leq i<k}J_{(i)}$. Suppose we have solved $J_{(<k)}$. The $J_{(k)}$ we are looking for satisfies

$$\delta J_{(k)}=-\frac{1}{2}\left[J_{(<k)},J_{(<k)}\right]\Big|_{(k)}.$$

Here the subscript $|_{(k)}$ on the right hand side means the component containing $k$ $\partial_t$'s (i.e. $T$-eigenvalue $k$). By the standard deformation theory argument,

$-\frac{1}{2}\left[J_{(<k)}, J_{(<k)}\right]\Big|_{(k)}$ is annihilated by $\delta$ and is of conformal weight 2. Then the existence of $J_{(k)} \in \mathcal{B}_1^1$ follows from $H^2(\mathcal{B}, \delta)_2 = 0$. In particular,

$$J_{(k)} = \delta^{-1}\left(-\frac{1}{2}\left[J_{(<k)}, J_{(<k)}\right]\Big|_{(k)}\right).$$

solves equation (1) and (3) up to $J_{<(k+1)}$.

Assume $J_{(k)} + U$ is another solution, then $U$ satisfies

$$\deg(U) = 1, \mathrm{cw}(U) = 1, T(U) = k > 0, \quad \text{and } \delta U = \delta^{-1}U = 0.$$

Since $H^1(\mathcal{B}, \delta)_1$ is spanned by $\tilde{\eta}$, while $T(\tilde{\eta}) = 0$. It follows that $U = 0$. ☐

4.4. **Exact solution of quantum BCOV theory.**

**Definition 4.14.** Let $J^B$ be in Lemma 4.13. We denote $\Phi(J^B) = \oint dz I^B$.

By Proposition 4.10 and Lemma 4.13, $\oint dz I^B$ defines a Maurer-Cartan element of $\mathcal{V}[[\mathbf{h}^*]]$. To justify that $\oint dz I^B$ indeed defines a quantization of our BCOV theory, we are left to check the following two properties:

(1) [Integrality]: only terms with even number of derivatives contributes to $\oint dz I^B$.
(2) [Classical limit]: the term in $\oint dz I^B$ containing no derivatives coincide with our classical BCOV interaction.

Property (1) on integrality comes from our discussion in Section 4.2 on dilaton dimension. A term with $m$ holomorphic derivatives contributes to $\hbar^{m/2}$, while we only allow integer powers of $\hbar$ to appear in our quantization.

Property (2) is just about the classical limit.

We will explicitly check (1) and (2) below.

*Remark* 4.15. Half integer powers of $\hbar$ appear naturally when open strings are included. In [11], we have also developed an open-closed BCOV theory. The terms in $J^B$ with odd number of holomorphic derivatives will be total derivatives. They vanish upon integration, but may couple nontrivially with open string sectors. It would be extremely interesting to see how open string would play into a role here.

4.4.1. *Integrality.* Let us consider the following transformation

$$R:\mathcal{B}\to\mathcal{B},\quad \partial_z^k\partial_t^m\tilde{b}\to(-\partial_z)^k\partial_t^m\tilde{b},\ \ \partial_z^k\partial_t^m\tilde{\eta}\to(-\partial_z)^k\partial_t^m\tilde{\eta},$$

i.e., $R$ is the reflection $\partial_z\to-\partial_z$. It is easy to check that $R(J^B)$ also satisfies (1)(2)(3) in Lemma 4.13. It follows from the uniqueness that

$$R(J^B)=J^B.$$

Let us identify $b_0=\partial_z\phi$ as in Section 4.3.1. Then

$$\begin{aligned}\Phi(J^B)&=\sum_{k\geq 0}\frac{1}{k+1}\oint dzW^{(k+1)}(b_0)\oint dtt^{-k-1}e^{\frac12\partial_z\partial_t}J^B\\&=\oint dz\sum_{k\geq 0}\frac{W^{(k+1)}(b_0)}{(k+1)!}\oint\frac{dt}{t}\partial_t^k e^{\frac12\partial_z\partial_t}J^B\\&=\oint dz\oint\frac{dt}{t}\frac{:e^{\phi(z+\partial_t)-\phi(z)}:_B-1}{\partial_t}e^{\frac12\partial_z\partial_t}J^B\\&=\oint dz\oint\frac{dt}{t}e^{\frac12\partial_z\partial_t}\left(\frac{:e^{\phi(z+\frac12\partial_t)-\phi(z-\frac12\partial_t)}:_B-1}{\partial_t}J^B\right)\\&=\oint dz\oint\frac{dt}{t}\frac{:e^{\phi(z+\frac12\partial_t)-\phi(z-\frac12\partial_t)}:_B-1}{\partial_t}J^B.\end{aligned}$$

Here we have formally identified $\phi(z+\partial_t)=\sum\limits_{k\geq 0}\partial_z^k\phi\frac{\partial_t^k}{k!}$ in the above manipulation and used the fact that the operator $e^{\frac12\partial_t\partial_z}$ amounts to shifting $z\to z+\frac12\partial_t$. In the last line, we have thrown away terms which are total derivatives in $z$. Now $\phi(z+\frac12\partial_t)-\phi(z-\frac12\partial_t)$ contains only even number of $\partial_z$'s in terms of the field $b_0=\partial_z\phi$. From $R(J^B)=J^B$, we know that $J^B$ also contains only even number of $\partial_z$'s. Therefore $\oint dzI^B=\Phi(J^B)$ satisfies Property (1) on integrality.

4.4.2. *classical limit.* We check that the classical limit $\oint dzI_0^B$ of $\oint dzI^B$ coincides with our classical BCOV interaction. By dilaton dimension, the classical limit is related to the component $J_0^B$ of $J^B$ which does not involve any $\partial_z$ and satisfies

$$\delta J_0^B+\frac12\{J_0^B,J_0^B\}=0,$$

where $\{-,-\}$ is the Poisson bracket $\{A,B\}=(\partial_tA\partial_zB-\partial_zA\partial_tB)$. Smilarly, $J^B$ is uniquely determined by further imposing conditions (2)(3) in Lemma 4.13.

**Lemma 4.16.**

$$J_0^B=\tilde{\eta}+\sum_{k\geq 1}\frac{\partial_t^{k-1}}{k!}\left(\tilde{b}^k\partial_t\tilde{\eta}\right).$$

*Proof.* Let $\hat{J}_0^B = \tilde{\eta} + \sum_{k\geq 1} \frac{\partial_t^{k-1}}{k!}\left(\tilde{b}^k \partial_t \tilde{\eta}\right)$. Let us first rewrite the above formula as

$$\partial_t \hat{J}_0^B = \sum_{k\geq 0} \frac{\partial_t^k}{k!}(\tilde{b}^k \partial_t \tilde{\eta}) = \oint_0 \frac{d\lambda}{\lambda} e^{\lambda \partial_t} \frac{1}{1-\lambda^{-1}\tilde{b}} \partial_t \tilde{\eta}$$
$$= \oint_0 d\lambda \frac{\partial_t \tilde{\eta}(z, t+\lambda)}{\lambda - \tilde{b}(z, t+\lambda)} = \left.\frac{\partial_t \tilde{\eta}(z, t+\lambda)}{1-\partial_t \tilde{b}(z, t+\lambda)}\right|_{\lambda=\tilde{b}(z,t+\lambda)}.$$

Here the substitution $\lambda = \tilde{b}(z, t+\lambda)$ means solving $\lambda = \sum_{k\geq 0} \frac{\lambda^k}{k!}\partial_t^k \tilde{b}$ for $\lambda$ as a power series in $\partial_t^k \tilde{b}$, then plugging $\lambda$ into $\partial_t \tilde{\eta}(z, t+\lambda) = \sum_{k\geq 0} \frac{\lambda^k}{k!}\partial_t^{k+1}\tilde{\eta}$ and $\partial_t \tilde{b}(z, t+\lambda) = \sum_{k\geq 0} \frac{\lambda^k}{k!}\partial_t^{k+1}\tilde{b}$. This implies via a simple chain rule computation

$$\hat{J}_0^B = \tilde{\eta}(z, t+\lambda)|_{\lambda=\tilde{b}(z,t+\lambda)}.$$

Similar computations lead to

$$\delta \hat{J}_0^B = \left.\frac{\partial_z \tilde{\eta}(t+\lambda)\partial_t \tilde{\eta}(t+\lambda)}{1-\partial_t \tilde{b}(t+\lambda)}\right|_{\lambda=\tilde{b}(t+\lambda)}$$
$$\partial_z \hat{J}_0^B = \left.\left(\partial_z \tilde{\eta}(t+\lambda) + \frac{\partial_z \tilde{b}(t+\lambda)}{1-\partial_t \tilde{b}(t+\lambda)}\partial_t \tilde{\eta}(t+\lambda)\right)\right|_{\lambda=\tilde{b}(t+\lambda)}.$$

It follows that

$$\delta \hat{J}_0^B + \frac{1}{2}\left\{\hat{J}_0^B, \hat{J}_0^B\right\} = \delta \hat{J}_0^B + \partial_t \hat{J}_0^B \partial_z \hat{J}_0^B = \left.\frac{\partial_z \tilde{b}(t+\lambda)}{\left(1-\partial_t \tilde{b}(t+\lambda)\right)^2}\left(\partial_t \tilde{\eta}(t+\lambda)\right)^2\right|_{\lambda=\tilde{b}(t+\lambda)} = 0,$$

where we have used the odd parity of $\tilde{\eta}$. Also, $\hat{J}_0^B = 0$ satisfies condition (2)(3) in Lemma 4.13. It follows from the uniqueness that $\hat{J}_0^B = J_0^B$. □

**Corollary 4.17.** *$\oint dz I_0^B$ coincides with the classical BCOV interaction. In particular, $\oint dz I^B$ is a quantization of the classical BCOV theory satisfying the Hodge weight condition and dilaton equation.*

*Proof.* The above lemma allows us to compute $\oint dz I_0^B$ explicitly

$$\oint dz I_0^B = \sum_{k>0} \frac{1}{k} \oint dz b_0^k \oint dt t^{-k} J_0^B$$
$$= \oint dz \sum_{k,l\geq 0, k_i>0} \frac{(k-1)!}{k_1! \cdots k_m! l!} \frac{b_0^k b_{k_1} \cdots b_{k_m} \eta_l}{k!} \oint dt t^{-k} \partial_t^{m-1}\left(t^{k_1+\cdots+k_m+l}\right)$$
$$= \oint dz \sum_{\substack{k,l\geq 0, k_i>0 \\ k_1+\cdots+k_m+l=k+m-2}} \frac{(k_1+\cdots+k_m+l)!}{k_1! \cdots k_m! l!} \frac{b_0^k b_{k_1} \cdots b_{k_m} \eta_l}{k!}$$

$$= \oint dz \left\langle e^b \otimes \eta \right\rangle_0 .$$

□

4.5. **Higher genus mirror symmetry.** We briefly explain the work [26] and discuss how our exact solution of quantum BCOV theory on elliptic curves is related to the Gromov-Witten theory on mirror elliptic curves. We refer to [26] for further details.

First of all, it is proved in [10, 24] that up to gauge equivalence, there exists at most one solution of quantum master equation of BCOV theory on elliptic curves that satisfies the Hodge weight condition and the dilaton equation. Such solution will also satisfy a set of Virasoro equations [26] that is mirror to that in [32]. It follows that the solution $\oint dzI^B$ we explicitly find here is the canonical quantization of BCOV theory satisfying Virasoro equations.

The Virasoro equations on elliptic curves reduce the computation of Gromov-Witten invariants to the so-called *stationary sector* [32]. There is a mirror story of this described in [26]. If we think about the Maurer-Cartan equation

$$\delta \oint dzI^B + \frac{1}{2}\left[\oint dzI^B, \oint dzI^B\right] = 0$$

as an evolution equation on the vertex algebra, then the stationary sector of our BCOV theory can be viewed as the initial condition. More precisely, the stationary sector of our fields is defined in the B-model by

$$b_{>0} = 0, \quad \eta_k = \text{constant}.$$

This represents the local $\delta$-cohomology. From our construction, the restriction of $\oint dzI^B$ to our stationary sector, denoted by $\oint dzI^S$, is given by

$$\oint dzI^S = \Phi(\tilde{\eta}) = \sum_{k\geq 0} \oint dz \frac{W^{(k+2)}}{k+2} \eta_k.$$

The quantum master equation in the stationary sector becomes

$$\left[\oint dzI^S, \oint dzI^S\right] = 0.$$

Expanding the coefficients $\eta_k$'s, this is equivalent to

$$\left[\oint dz \frac{W^{(k+2)}}{k+2}, \oint dz \frac{W^{(m+2)}}{m+2}\right] = 0, \quad k, m \geq 0,$$

which represents infinite number of commuting Hamiltonians.

Finally, since our quantization is modular invariant (see Remark 4.5), the generating functions in the stationary sector will be given by almost holomorphic modular forms, whose $\bar{\tau} \to \infty$ limit can be computed by the character [26]

$$\mathrm{Tr}_{\mathcal{H}}\, q^{L_0 - \frac{1}{24}} e^{\frac{1}{\hbar} \sum_{k \geq 0} \oint dz \eta_k \frac{W^{(k+2)}}{k+2}}, \quad q = e^{2\pi i \tau},$$

where $\mathcal{H}$ is the Heisenberg vertex algebra generated by $b_0$. This coincides with the A-model computation [32] under the boson-fermion correspondence. This can be viewed as a full generalization of [12] on the mirror interpretation of the cubic interaction $W^{(3)}$.

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