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RESEARCH ARTICLE

Largest adjacency, signless Laplacian, and Laplacian H-eigenvalues of loose paths

Junjie YUE[1,2], **Liping ZHANG**[1], **Mei LU**[1]

1 Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China
2 State Key Laboratory of Space Weather, Chinese Academy of Sciences, Beijing 100910, China

Abstract We investigate k-uniform loose paths. We show that the largest H-eigenvalues of their adjacency tensors, Laplacian tensors, and signless Laplacian tensors are computable. For a k-uniform loose path with length $\ell \geqslant 3$, we show that the largest H-eigenvalue of its adjacency tensor is $((1+\sqrt{5})/2)^{2/k}$ when $\ell = 3$ and $\lambda(\mathscr{A}) = 3^{1/k}$ when $\ell = 4$, respectively. For the case of $\ell \geqslant 5$, we tighten the existing upper bound 2. We also show that the largest H-eigenvalue of its signless Laplacian tensor lies in the interval $(2,3)$ when $\ell \geqslant 5$. Finally, we investigate the largest H-eigenvalue of its Laplacian tensor when k is even and we tighten the upper bound 4.

Keywords H-eigenvalue, hypergraph, adjacency tensor, signless Laplacian tensor, Laplacian tensor, loose path
MSC 74B99, 15A18, 15A69

1 Introduction

In recent years, the study of spectral hypergraph theory via tensors [4,7,8,10–12,14,18,20,21] has attracted extensive attention and interest since the work of [4,11,16,18]. As was in [16], a real tensor $\mathscr{T} = (t_{i_1\cdots i_k})$ of order k and dimension n refers to a multidimensional array (also called hypermatrix) with entries $t_{i_1\cdots i_k}$ such that $t_{i_1\cdots i_k} \in \mathbb{R}$ for all $i_j \in [n] := [1,\ldots,n]$ and $j \in [k]$. Given a vector $x \in \mathbb{R}^n$, $\mathscr{T}x^{k-1}$ is defined as an n-dimensional vector such that its ith element being

$$\sum_{i_2,\ldots,i_k\in[n]} t_{ii_2\cdots i_k} x_{i_2}\cdots x_{i_k}, \quad i \in [n].$$

Let $\mathscr{I}$ be the identity tensor of appropriate dimension, e.g., $i_{i_1\cdots i_k} = 1$ if and

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Corresponding author: Liping ZHANG, E-mail: lzhang@math.tsinghua.edu.cn

only if $i_1 = \cdots = i_k \in [n]$, and zero otherwise, when the dimension is n. The following definition was introduced by Qi [16].

Definition 1 Let $\mathscr{T}$ be a k-th order n-dimensional real tensor. For some $\lambda \in \mathbb{R}$, if polynomial system $(\lambda \mathscr{I} - \mathscr{T})x^{k-1} = 0$ has a solution $x \in \mathbb{R}^n \backslash \{0\}$, then λ is called an H-eigenvalue and x an H-eigenvector.

Obviously, H-eigenvalues are real number. By [6,16], we have the number of H-eigenvalues of a real tensor is finite. By [18], we have all the tensors considered in this paper have at least one H-eigenvalue. Hence, we can denote by $\lambda(\mathscr{T})$ as the largest H-eigenvalue of a real tensor $\mathscr{T}$.

As was in [18], a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V, which is labeled as $[n]$, and edge set E. By k-uniformity, we mean that for every edge $e \in E$, the cardinality $|e|$ of e is equal to k. Throughout this paper, $k \geqslant 3$ and $n \geqslant k$. Moreover, since the trivial hypergraph (i.e., $E = \emptyset$) is of less interest, we consider only hypergraphs having at least one edge (i.e., nontrivial) in this paper. The following definition was introduced by Qi [18].

Definition 2 Let $G = (V, E)$ be a k-uniform hypergraph. The adjacency tensor of G is defined as the k-th order n dimensional tensor $\mathscr{A}$ whose $(i_1 \cdots i_k)$-entry is

$$a_{i_1 \cdots i_k} := \begin{cases} \dfrac{1}{(k-1)!}, & \{i_1, \ldots, i_k\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathscr{D}$ be a k-th order n-dimensional diagonal tensor with its diagonal element $d_{i \cdots i}$ being d_i, the degree of vertex i, for all $i \in [n]$. Then $\mathscr{L} := \mathscr{D} - \mathscr{A}$ is the Laplacian tensor of the hypergraph G, and $\mathscr{Q} := \mathscr{D} + \mathscr{A}$ is the signless Laplacian tensor of the k-uniform hypergraph G.

By [18], zero is always the smallest H-eigenvalue of $\mathscr{L}$ and $\mathscr{Q}$, and we have

$$d \leqslant \lambda(\mathscr{L}) \leqslant \lambda(\mathscr{Q}) \leqslant 2d,$$

where d is the maximum degree of G. By [4, Theorem 3.8], we have

$$\overline{d} \leqslant \lambda(A) \leqslant d,$$

where $\overline{d}$ be the average degree of G.

Recently, Hu et al. [10] introduced the class of cored hypergraphs and power hypergraphs, and investigated the properties of their Laplacian H-eigenvalues. Power hypergraphs are cored hypergraphs, but not vice versa. Loose paths are power hypergraphs. They showed that when k is even, the largest Laplacian H-eigenvalue and the largest signless Laplacian H-eigenvalue of a cored hypergraph are the same [10, Proposition 3.2]. Especially, they showed that when k is odd, the largest Laplacian H-eigenvalue of a k-uniform loose path is equal to the maximum degree, i.e., 2 [10, Proposition 4.2]. They also computed the Laplacian H-spectra of the loose path of length 3. Actually, there are still changing open problems for the loose path which are worthy of being investigated. First, it is known that [10,18] the largest Laplacian H-eigenvalue is between 2 and 4 when k is even. Can we compute its largest Laplacian H-eigenvalue when k is even or can we tighten its upper bound? Second, can we describe the properties of its largest adjacency and signless Laplacian H-eigenvalues? It is known that its largest adjacency H-eigenvalue is less than or equal to 2 [4, Theorem 3.8]. Can we tighten the upper bound? Motivated by these questions, we study the adjacency and signless Laplacian H-eigenvalues of the class of loose paths in this paper.

We first investigate the properties of H-eigenvectors of adjacency tensor and signless Laplacian tensors for power hypergraphs. We next establish some facts on the class of loose paths. We investigate the largest H-eigenvalues of its adjacency tensor and signless Laplacian tensor. For a k-uniform loose path with length $\ell \geqslant 3$, we show that $\lambda(\mathscr{A})$ equals $((1+\sqrt{5}\,)/2)^{2/k}$ when $\ell = 3$ and $3^{1/k}$ when $\ell = 4$, respectively. We establish an upper bound for $\lambda(\mathscr{A})$ that is better than 2 when $\ell \geqslant 5$. We also give a good upper bound for $\lambda(\mathscr{Q})$ which is better than 4 and we show $2 \leqslant \lambda(\mathscr{Q}) \leqslant 3$. By [10, Proposition 3.2], this conclusion also holds for $\lambda(\mathscr{L})$ when k is even. Very recently, Qi et al. [19] studied many properties of regular uniform hypergraphs, s-cycles, s-paths, and their largest Laplacian H-eigenvalues. When $s = 1$, s-paths [19] are just the loose paths in our paper. However, our results are not different. They mainly established that a k-uniform s-path is odd-bipartite ($k \geqslant 4$, $1 \leqslant s \leqslant k-1$).

The rest of this paper is organized as follows. We recall some notations and establish some facts on H-eigenvectors of cored hypergraphs and power hypergraphs in the next section. We investigate the largest adjacency H-eigenvalue of loose paths in Section 3. We investigate the largest signless Laplacian and Laplacian H-eigenvalues of loose paths in Section 4. We also give some numerical experiments to compute the largest signless Laplacian and Laplacian H-eigenvalues in Section 5. Some final remarks are given in Section 6.

2 Preliminaries

In this section, we list some essential notions of uniform hypergraphs which will be used in the sequel. Please refer to [1–3,5,9,18] for comprehensive references. In this paper, unless stated otherwise, a hypergraph means an undirected simple k-uniform hypergraph G with vertex set V and edge set E. For a subset $S \subset [n]$, we denote by E_S the set of edges $\{e \in E \mid S \cap e \neq \emptyset\}$. For a vertex $i \in V$, we simplify $E_{\{i\}}$ as E_i. It is the set of edges containing the vertex i, i.e.,

$$E_i := \{e \in E \mid i \in e\}.$$

The cardinality $|E_i|$ of the set E_i is defined as the degree of the vertex i, which is denoted by d_i. Two different vertices i and j are connected to each other (or the pair i and j is connected), if there is a sequence of edges $(e_1, \ldots, e_m)$ such that $i \in e_1$, $j \in e_m$, and $e_r \cap e_{r+1} \neq \emptyset$ for all $r \in [m-1]$. A hypergraph is called connected, if every pair of different vertices of G is connected. In the sequel, unless stated otherwise, all the notations introduced above are reserved for the specific meanings. For the sake of simplicity, we mainly consider connected hypergraphs in the subsequent analysis. By the techniques in [9,18], the conclusion on connected hypergraphs can be easily generalized to general hypergraphs.

In the following, we recall the definitions of cored hypergraphs and power hypergraphs introduced in [10]. We also list the definition of loose paths introduced in [10,13,15].

Definition 3 Let $G = (V, E)$ be a k-uniform hypergraph. If for every edge $e \in E$, there is a vertex $i_e \in e$ such that the degree of the vertex i_e is one, then G is a cored hypergraph. A vertex with degree one is a cored vertex, and a vertex with degree larger than one is an intersectional vertex.

Definition 4 Let $G = (V, E)$ be a 2-uniform graph. For any $k \geqslant 3$, the kth power of G, $G^k := (V^k, E^k)$ is defined as the k-uniform hypergraph with the set of edges

$$E^k := \{e \cup \{i_{e,1}, \ldots, i_{e,k-2}\} \mid e \in E\},$$

and the set of vertices

$$V^k := V \cup \{i_{e,1}, \ldots, i_{e,k-2}, e \in E\}.$$

Definition 5 Let $G = (V, E)$ be a k-uniform hypergraph. If we can number the vertex set V as

$$V := \{i_{1,1}, \ldots, i_{1,k}, i_{2,2}, \ldots, i_{2,k}, \ldots, i_{\ell-1,2}, \ldots, i_{\ell-1,k}, i_{\ell,2}, \ldots, i_{\ell,k}\}$$

for some positive integer ℓ such that

$$E = \{\{i_{1,1}, \ldots, i_{1,k}\}, \{i_{1,k}, i_{2,2}, \ldots, i_{2,k}\}, \ldots, \{i_{\ell-1,k}, i_{\ell,2}, \ldots, i_{\ell,k}\}\},$$

then G is a loose path. ℓ is the length of the loose path.

It is easy to see that the class of power hypergraphs is a subclass of cored hypergraphs and not all cored hypergraphs are power hypergraphs. Power hypergraphs contain loose paths [10,13,15]. Recently, the Laplacian tensor of a k-uniform loose path was investigated by Hu et al. [10]. It is shown that its largest Laplacian H-eigenvalue is 2 when k is odd. They also investigated the properties of Laplacian H-eigenvalues for a loose path of length $\ell = 3$ when k is odd. In this paper, one of our purposes is to compute the largest H-eigenvalues of adjacency tensor and signless Laplacian tensor of a k-uniform loose path with length $\ell \geqslant 3$. The other is to tighten the upper bound of its largest Laplacian H-eigenvalue when k is even.

In the following, we establish some facts on H-eigenvectors of cored hypergraphs and power hypergraphs, which will be used in the sequel.

By Definition 1 and the notation of core vertices, we immediately get the following results via the similar proof of [10, Lemma 3.1].

Lemma 1 *Let $G=(V,E)$ be a k-uniform cored hypergraph, and let $x\in\mathbb{R}^n$ be an H-eigenvector of its adjacency tensor $\mathscr{A}$ corresponding to an H-eigenvalue $\lambda\neq 0$. If there are two core vertices i and j in an edge $e\in E$, then $|x_i|=|x_j|$. Moreover, $x_i=x_j$ when k is an odd number.*

Lemma 2 *Let $G=(V,E)$ be a k-uniform cored hypergraph, and let $x\in\mathbb{R}^n$ be an H-eigenvector of its signless Laplacian tensor $\mathscr{Q}$ corresponding to an H-eigenvalue $\lambda\neq 1$. If there are two cored vertices i and j in an edge $e\in E$, then $|x_i|=|x_j|$. Moreover, $x_i=x_j$ when k is an odd number.*

By Definition 1, Lemmas 1 and 2, we get the following lemmas on odd-uniform power hypergraphs in the similar way of [10, Lemma 4.1].

Lemma 3 *Let k be odd, and let $G=(V,E)$ be a k-uniform power hypergraph. Let $x\in\mathbb{R}^n$ be an H-eigenvector of its adjacency tensor $\mathscr{A}$ corresponding to an H-eigenvalue λ. Let $e\in E$ be an arbitrary but fixed edge.*

(i) *If e has only one intersectional vertex i, and $x_s\neq 0$ for some cored vertex $s\in e$, then $\lambda x_s=x_i$.*

(ii) *If e has two intersectional vertices i and j, and $x_s\neq 0$ for some cored vertex $s\in e$, then $\lambda x_s^2=x_ix_j$.*

Lemma 4 *Let k be odd, and let $G=(V,E)$ be a k-uniform power hypergraph. Let $x\in\mathbb{R}^n$ be an H-eigenvector of its signless Laplacian tensor $\mathscr{Q}$ corresponding to an H-eigenvalue λ. Let $e\in E$ be an arbitrary but fixed edge.*

(i) *If e has only one intersectional vertex i, and $x_s\neq 0$ for some cored vertex $s\in e$, then $(\lambda-1)x_s=x_i$.*

(ii) *If e has two intersectional vertices i and j, and $x_s\neq 0$ for some cored vertex $s\in e$, then $(\lambda-1)x_s^2=x_ix_j$.*

The following results are given for even-uniform power hypergraphs.

Lemma 5 *Let k be even, let $G=(V,E)$ be a k-uniform power hypergraph, and let $x\in\mathbb{R}^n$ be an H-eigenvector of its adjacency tensor $\mathscr{A}$ corresponding to an H-eigenvalue $\lambda\neq 0$. Let $e\in E$ be an arbitrary but fixed edge, and let e' be the set of its intersectional vertices. Let α be the cardinality of the set $\{i\in e\backslash e'\mid x_i<0\}$.*

(i) *If e has only one intersectional vertex i, and $x_s\neq 0$ for some cored vertex $s\in e$, then $\lambda x_i>0$ when α is even and $\lambda x_i<0$ when α is odd. Here, $x_s=x_i/\lambda$ or $-x_i/\lambda$.*

(ii) *If e has two intersectional vertices i and j, and $x_s\neq 0$ for some cored vertex $s\in e$, then $\lambda x_ix_j>0$ when α is even and $\lambda x_ix_j<0$ when α is odd. Here, $x_s=\pm\sqrt{x_ix_j/\lambda}$ or $\pm\sqrt{-x_ix_j/\lambda}$.*

Proof Let $x_+=|x_s|$. By Definition 1, we have

$$\lambda x_s^k=\prod_{t\in e}x_t. \tag{1}$$

(i) If α is even, then we have

$$(1) \Longleftrightarrow \lambda x_+^k = x_+^{k-1} x_i \Longleftrightarrow \lambda x_+ = x_i.$$

If α is odd, then we have

$$(1) \Longleftrightarrow \lambda x_+^k = -x_+^{k-1} x_i \Longleftrightarrow \lambda x_+ = -x_i.$$

(ii) If α is even, then

$$(1) \Longleftrightarrow \lambda x_+^k = x_+^{k-2} x_i x_j \Longleftrightarrow \lambda x_+^2 = x_i x_j.$$

If α is odd, then we have

$$(1) \Longleftrightarrow \lambda x_+^k = -x_+^{k-2} x_i x_j \Longleftrightarrow \lambda x_+^2 = -x_i x_j. \qquad \square$$

Lemma 6 *Let k be even, let $G = (V, E)$ be a k-uniform power hypergraph, and let $x \in \mathbb{R}^n$ be an H-eigenvector of its Laplacian tensor $\mathscr{L}$ corresponding to an H-eigenvalue $\lambda \neq 1$. Let $e \in E$ be an arbitrary but fixed edge, and let e' be the set of its intersectional vertices. Let α be the cardinality of the set $\{i \in e \backslash e' \mid x_i < 0\}$.*

(i) *If e has only one intersectional vertex i, and $x_s \neq 0$ for some cored vertex $s \in e$, then $(1-\lambda)x_i > 0$ when α is even and $(1-\lambda)x_i < 0$ when α is odd. Here, $x_s = x_i/(1-\lambda)$ or $x_i/(\lambda-1)$.*

(ii) *If e has two intersectional vertices i and j, and $x_s \neq 0$ for some cored vertex $s \in e$, then $(1-\lambda)x_i x_j > 0$ when α is even and $(1-\lambda)x_i x_j < 0$ when α is odd. Here, $x_s = \pm\sqrt{x_i x_j/(1-\lambda)}$ or $\pm\sqrt{x_i x_j/(\lambda-1)}$.*

Proof Let $x_+ = |x_s|$. By Definition 1, we have

$$(\lambda - 1)x_s^k = -\prod_{t \in e} x_t. \tag{2}$$

(i) If α is even, then we have

$$(2) \Longleftrightarrow (1-\lambda) x_+^k = x_+^{k-1} x_i \Longleftrightarrow (1-\lambda) x_+ = x_i.$$

If α is odd, then we have

$$(2) \Longleftrightarrow \lambda x_+^k = -x_+^{k-1} x_i \Longleftrightarrow (1-\lambda) x_+ = -x_i.$$

(ii) If α is even, then

$$(2) \Longleftrightarrow (1-\lambda) x_+^k = x_+^{k-2} x_i x_j \Longleftrightarrow (1-\lambda) x_+^2 = x_i x_j.$$

If α is odd, then we have

$$(2) \Longleftrightarrow (1-\lambda) x_+^k = -x_+^{k-2} x_i x_j \Longleftrightarrow (1-\lambda) x_+^2 = -x_i x_j. \qquad \square$$

The proof for the following lemma is similar to that of Lemma 6.

Lemma 7 *Let k be even, let $G = (V, E)$ be a k-uniform power hypergraph, and let $x \in \mathbb{R}^n$ be an H-eigenvector of its signless Laplacian tensor $\mathscr{Q}$ corresponding to an H-eigenvalue $\lambda \neq 1$. Let $e \in E$ be an arbitrary but fixed edge, and let e' be the set of its intersectional vertices. Let α be the cardinality of the set $\{i \in e \backslash e' \,|\, x_i < 0\}$.*

(i) *If e has only one intersectional vertex i, and $x_s \neq 0$ for some cored vertex $s \in e$, then $(\lambda - 1)x_i > 0$ when α is even and $(\lambda - 1)x_i < 0$ when α is odd. Here, $x_s = x_i/(\lambda - 1)$ or $x_i/(1 - \lambda)$.*

(ii) *If e has two intersectional vertices i and j, and $x_s \neq 0$ for some cored vertex $s \in e$, then $(\lambda - 1)x_i x_j > 0$ when α is even and $(\lambda - 1)x_i x_j < 0$ when α is odd. Here, $x_s = \pm\sqrt{x_i x_j/(\lambda - 1)}$ or $\pm\sqrt{x_i x_j/(1 - \lambda)}$.*

3 Largest adjacency H-eigenvalue of loose paths

For a k-uniform loose path G with length $\ell \geqslant 3$, there are few results on the spectral radius $\lambda(\mathscr{A})$ of its adjacency tensor$\mathscr{A}$ in the literature. It is known [10] that its maximum degree is 2 and $\lambda(\mathscr{A}) \leqslant 2$ [4, Theorem 3.8]. In this section, based on the lemmas given in Section 2, we show that $\lambda(\mathscr{A}) = ((1+\sqrt{5}\,)/2)^{2/k}$ when $\ell = 3$ and $\lambda(\mathscr{A}) = 3^{1/k}$ when $\ell = 4$. For the case of $\ell \geqslant 5$, we tighten the existing upper bound 2.

Theorem 1 *Let $G = (V, E)$ be a k-uniform loose path with length 3, and let $\mathscr{A}$ be its adjacency tensor. Then the spectral radius of $\mathscr{A}$, i.e., the largest H-eigenvalue*

$$\lambda(\mathscr{A}) = \Big(\frac{1+\sqrt{5}}{2}\Big)^{2/k}.$$

Proof It is not restrictive to assume that

$$E = \{\{1, \ldots, k\}, \{k, \ldots, 2k-1\}, \{2k-1, \ldots, 3k-2\}\}.$$

By [22, Theorem 3.20], [17, Theorem 4], and [14, Lemma 3.1], if we can find a positive H-eigenvector $x \in \mathbb{R}^n$ of $\mathscr{A}$ corresponding to an H-eigenvalue λ, then $\lambda = \lambda(\mathscr{A})$. By Lemmas 1, 3, and 5, for such an eigenpair (x, λ), we have

$$x_1 = \cdots = x_{k-1} = \frac{x_k}{\lambda}$$

if $x_s \neq 0$ for some $s \in \{1, \ldots, k-1\}$,

$$x_{k+1} = \cdots = x_{2k-2} = \sqrt{\frac{x_k x_{2k-1}}{\lambda}}$$

if $x_s \neq 0$ for some $s \in \{k+1, \ldots, 2k-2\}$, and

$$x_{2k} = \cdots = x_{3k-2} = \frac{x_{2k-1}}{\lambda}$$

if $x_s \neq 0$ for some $s \in \{2k, \ldots, 3k-2\}$. Hence, the equations of the H-eigenvalue λ become

$$\lambda x_k^{k-1} = x_{2k-1}\Big(\sqrt{\frac{x_k x_{2k-1}}{\lambda}}\Big)^{k-2} + \Big(\frac{x_k}{\lambda}\Big)^{k-1},$$
$$\lambda x_{2k-1}^{k-1} = x_k\Big(\sqrt{\frac{x_k x_{2k-1}}{\lambda}}\Big)^{k-2} + \Big(\frac{x_{2k-1}}{\lambda}\Big)^{k-1}.$$

That is,

$$(\lambda^k - 1)x_k^{k/2} = \lambda^{k/2} x_{2k-1}^{k/2}, \quad (\lambda^k - 1)x_{2k-1}^{k/2} = \lambda^{k/2} x_k^{k/2}. \tag{3}$$

Define $a := \lambda^k - 1$ and $b := \lambda^{k/2}$. Then the first equality in (3) implies $a > 0$. Note that the determinant

$$\begin{vmatrix} a & -b \\ -b & a \end{vmatrix} = (a+b)(a-b).$$

Hence, if

$$p(\lambda) := \lambda^{k/2}(\lambda^{k/2} - 1) - 1 = 0,$$

then system (3) has a nonzero solution. It is easy to see that $p(\lambda)$ is monotone increasing when $\lambda > 1$ and $\lambda(\mathscr{A})$ is a root of $p(\lambda)$. Since

$$p(1) = -1, \quad p(2) = 2^k - 2^{k/2} - 1 > 0,$$

the equation $p(\lambda) = 0$ has a unique root in the interval $(1, 2)$. By direct computation, it is $((1+\sqrt{5})/2)^{2/k}$. So, we have

$$\lambda(\mathscr{A}) = \Big(\frac{1+\sqrt{5}}{2}\Big)^{2/k}.$$ □

Theorem 2 *Let $G = (V, E)$ be a k-uniform loose path with length $\ell \geqslant 4$. Let $\mathscr{A}$ be its adjacency tensor. Then we have*

(i) $\lambda(\mathscr{A}) = 3^{1/k}$ *when $\ell = 4$.*

(ii) $\lambda(\mathscr{A}) \leqslant \lambda^*$ *when $\ell = 5$, where λ^* is the unique root of*

$$(\lambda^k - 1)(\lambda^{k/2} - 1) - \lambda^{k/2} = 0$$

in the interval $(1, 2)$.

(iii) $\lambda(\mathscr{A}) \leqslant \lambda^*$ *when $\ell \geqslant 6$, where λ^* is the unique root of*

$$a[(b+\sqrt{b^2-4})^{(\ell-2)/2} - (b-\sqrt{b^2-4})^{(\ell-2)/2}]$$
$$-2b[(b+\sqrt{b^2-4})^{(\ell-4)/2} + (b-\sqrt{b^2-4})^{(\ell-4)/2}] = 0$$

in the interval $(4^{1/k}, 2)$. Here, a and b is defined as $a = \lambda^k - 1$ and $b = \lambda^{k/2}$, respectively.

Proof Without loss of generality, we assume

$$E=\{\{1,\dots,k\},\{k,\dots,2k-1\},\{2k-1,\dots,3k-2\},\dots,\\ \{(\ell-1)k-\ell+2,\dots,\ell k-\ell,\ell k-\ell+1\}\}.$$

By [22, Theorem 3.20], [17, Theorem 4], and [14, Lemma 3.1], if we can find a positive H-eigenvector $x\in\mathbb{R}^n$ of $\mathscr{A}$ corresponding to an H-eigenvalue λ, then $\lambda=\lambda(\mathscr{A})$. By Lemmas 2, 3, and 5, for such an eigenpair (x,λ), we have

$$x_1=\cdots=x_{k-1}=\frac{x_k}{\lambda}$$

if $x_s\neq 0$ for some $s\in\{1,\dots,k-1\}$, and

$$x_{k+1}=\cdots=x_{2k-2}=\sqrt{\frac{x_kx_{2k-1}}{\lambda}}$$

if $x_s\neq 0$ for some $s\in\{k+1,\dots,2k-2\}$. The rest may be deduced by analogy,

$$x_{(\ell-2)k-\ell+4}=\cdots=x_{(\ell-1)k-\ell+1}=\sqrt{\frac{x_{(\ell-2)k-\ell+3}x_{(\ell-1)k-\ell+2}}{\lambda}}$$

if $x_s\neq 0$ for some $s\in\{(\ell-2)k-\ell+4,\dots,(\ell-1)k-\ell+1\}$, and

$$x_{(\ell-1)k-\ell+3}=\cdots=x_{\ell k-\ell+1}=\frac{x_{(\ell-1)k-\ell+2}}{\lambda}$$

if $x_s\neq 0$ for some $s\in\{(\ell-1)k-\ell+3,\dots,\ell k-\ell+1\}$. Hence, by straightforward computation, the equations of the H-eigenvalue λ become

$$\begin{cases}(\lambda^k-1)x_k^{k/2}=\lambda^{k/2}x_{2k-1}^{k/2},\\ \lambda^{k/2}x_{2k-1}^{k/2}=x_k^{k/2}+x_{3k-2}^{k/2},\\ \dots,\\ \lambda^{k/2}x_{(\ell-2)k-\ell+3}^{k/2}=x_{(\ell-3)k-\ell+4}^{k/2}+x_{(\ell-1)k-\ell+2}^{k/2},\\ (\lambda^k-1)x_{(\ell-1)k-\ell+2}^{k/2}=\lambda^{k/2}x_{(\ell-2)k-\ell+3}^{k/2}.\end{cases}\tag{4}$$

Define $a:=\lambda^k-1$ and $b:=\lambda^{k/2}$. Then the first equality in (4) implies $a>0$. Let

$$AX=\begin{pmatrix}a & -b & & & \\ -1 & b & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & b & -1\\ & & & -b & a\end{pmatrix}_{(\ell-1)\times(\ell-1)}\begin{pmatrix}x_k^{k/2}\\ x_{2k-1}^{k/2}\\ x_{3k-2}^{k/2}\\ \vdots\\ x_{(\ell-3)k-\ell+4}^{k/2}\\ x_{(\ell-2)k-\ell+3}^{k/2}\\ x_{(\ell-1)k-\ell+2}^{k/2}\end{pmatrix}_{(\ell-1)\times 1}.$$

In order to compute $\lambda(\mathscr{A})$, we divide into three cases to discuss under what conditions $AX = 0$ has nonzero solutions.

Case 1 $\ell = 4$.

If

$$\begin{vmatrix} a & -b & \\ -1 & b & -1 \\ & -b & a \end{vmatrix} = ab(a-2) = 0,$$

i.e., $a - 2 = 0$, then $AX = 0$ has nonzero solutions. This induces $\lambda^k - 3 = 0$. Hence, we have $\lambda(\mathscr{A}) = 3^{1/k}$.

Case 2 $\ell = 5$.

If

$$\begin{vmatrix} a & -b & & \\ -1 & b & -1 & \\ & -1 & b & -1 \\ & & -b & a \end{vmatrix} = (b-(b+1)a)(b-(b-1)a) = 0,$$

i.e.,

$$p_1(\lambda) := b - (b+1)a = 0$$

or

$$p_2(\lambda) := b - (b-1)a = 0,$$

then $AX = 0$ has nonzero solutions. Since $a > 0$, we have $p_2(\lambda) > p_1(\lambda)$. Hence, $\lambda(\mathscr{A})$ must be the largest real root of $p_2(\lambda) = 0$. It is easy to see that

$$p_2(1) = 1, \quad p_2(2) = -2^{3k/2} + 2^k + 2^{\frac{k}{2}+1} - 1 = (2-2^k)(2^{k/2}-1) - 1 < 0.$$

By straightforward computation, we have

$$p_2'(\lambda) = \frac{k}{2}\lambda^{\frac{k}{2}-1}(-3\lambda^k + 2\lambda^{k/2} + 2).$$

Since $a > 0$, $p_2'(\lambda) < 0$ for $\lambda \in (1,2)$. Hence, $p_2(\lambda)$ is monotone decreasing in the interval $(1,2)$. So, $p_2(\lambda) = 0$ has the unique root $\lambda^* \in (1,2)$. Thus, we have $\lambda(\mathscr{A}) \leqslant \lambda^*$.

Case 3 $\ell \geqslant 6$.

We consider the nonzero solutions of $AX = 0$ in $(4^{1/k}, 2)$. It is sufficient to discuss under what conditions $|A|$, the determinant of A, equals zero. Clearly, we have $b^2 > 4$ in this interval. Define

$$D_n = \begin{vmatrix} b & -1 & & & \\ -1 & b & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & b & -1 \\ & & & -1 & b \end{vmatrix}_{n\times n}.$$

Hence, we have

$$D_n = \frac{(b+\sqrt{b^2-4}\,)^{n+1} - (b-\sqrt{b^2-4}\,)^{n+1}}{2^{n+1}\sqrt{b^2-4}}. \tag{5}$$

By straightforward computation and (5), we have

$$\begin{aligned}
|A| = 0 &\Longleftrightarrow a^2 D_{\ell-3} - 2abD_{\ell-4} + b^2 D_{\ell-5} = 0\\
&\Longleftrightarrow a^2(w^{\ell-2} - u^{\ell-2}) - 4ab(w^{\ell-3} - u^{\ell-3}) + 4b^2(w^{\ell-4} - u^{\ell-4}) = 0\\
&\Longleftrightarrow [a(w^{(\ell-2)/2} + u^{(\ell-2)/2}) - 2b(w^{(\ell-4)/2} - u^{(\ell-4)/2})]\\
&\qquad \times [a(w^{(\ell-2)/2} - u^{(\ell-2)/2}) - 2b(w^{(\ell-4)/2} + u^{(\ell-4)/2})] = 0,
\end{aligned}$$

where

$$w = b + \sqrt{b^2-4}\,, \quad u = b - \sqrt{b^2-4}\,.$$

Obviously, $w > 0$ and $u > 0$. Let

$$f_1(\lambda) = a(w^{(\ell-2)/2} + u^{(\ell-2)/2}) - 2b(w^{(\ell-4)/2} - u^{(\ell-4)/2})$$

and

$$f_2(\lambda) = a(w^{(\ell-2)/2} - u^{(\ell-2)/2}) - 2b(w^{(\ell-4)/2} + u^{(\ell-4)/2}).$$

Since $b > 2$ and $a > 3$, we have

$$\begin{aligned}
f_1(\lambda) &= (aw - 2b)w^{(\ell-4)/2} + (au + 2b)u^{(\ell-4)/2}\\
&> (aw - 2b)w^{(\ell-4)/2}\\
&> (a-2)bw^{(\ell-4)/2}\\
&> 0.
\end{aligned}$$

Hence, it is sufficient to discuss the roots of $f_2(\lambda) = 0$ in the interval $(4^{1/k}, 2)$. Obviously, $f_2(4^{1/k}) < 0$ and

$$\begin{aligned}
f_2(2) &> 2(2^{k/2} + \sqrt{2^k-4}\,)^{(\ell-4)/2}(\sqrt{2^k-4}\,(2^k-1) - 2^{(K+2)/2})\\
&> 2(2^{k/2} + \sqrt{2^k-4}\,)^{(\ell-4)/2}(7\sqrt{2^k-4} - 2^{(K+2)/2})\\
&> 0,
\end{aligned}$$

where the second inequality holds due to $k \geqslant 3$. Hence, $f_2(\lambda) = 0$ has a root in $(4^{1/k}, 2)$. Let

$$a' = k\lambda^{k-1}, \quad b' = \frac{k}{2}\,\lambda^{(k-2)/2}.$$

By straightforward computation, we obtain

$$\begin{aligned}
f_2'(\lambda) = \frac{1}{\sqrt{b^2-4}}\Big\{&\Big[\Big(a'\sqrt{b^2-4} + \frac{\ell-2}{2}\,ab'\Big)w - 2b'\sqrt{b^2-4} - b(\ell-4)b'\Big]w^{(\ell-4)/2}\\
&+ \Big[\Big(\frac{\ell-2}{2}\,ab' - a'\sqrt{b^2-4}\Big)u - 2b'\sqrt{b^2-4} + b(\ell-4)b'\Big]u^{(\ell-4)/2}\Big\}.
\end{aligned}$$

It follows from the facts $b > 2$ and $a > 3$ that

$$wa' - 2b' > 2a' - 2b' > 0,$$

$$\frac{\ell-2}{2}aw - b(\ell-4) > \frac{\ell-2}{2}ab - b(\ell-4) = b\Big(\frac{\ell-2}{2}a - \ell + 4\Big) > b\frac{\ell+2}{2} > 0,$$

and

$$b(\ell-4)b' - 2b'\sqrt{b^2-4} = b'(b(\ell-4) - 2\sqrt{b^2-4}\,) > b'(b(\ell-4) - 2b) \geqslant 0.$$

Due to

$$(\lambda^k - 1)^2 - \lambda^k(\lambda^k - 4) = 1 + 2\lambda^k > 0,$$

we have

$$\frac{\ell-2}{2}ab' - a'\sqrt{b^2-4} = \frac{\ell-2}{4}k\lambda^{(k-2)/2}[(\lambda^k-1) - \lambda^{k/2}\sqrt{\lambda^k-4}\,] > 0.$$

Consequently, we obtain $f_2'(\lambda) > 0$ for all $\lambda \in (4^{1/k}, 2)$. Hence, $f_2(\lambda) = 0$ has the unique root $\lambda^* \in (4^{1/k}, 2)$ and we also have $\lambda(\mathscr{A}) \leqslant \lambda^*$. □

4 Largest signless Laplacian H-eigenvalue of loose paths

For a k-uniform loose path G with length $\ell \geqslant 3$, there are some known results on the largest H-eigenvalues $\lambda(\mathscr{Q})$ and $\lambda(\mathscr{L})$ of its signless Laplacian tensor $\mathscr{Q}$ and Laplacian tensor $\mathscr{L}$ in the literature. It is known [10,18] that

$$2 \leqslant \lambda(\mathscr{Q}) = \lambda(\mathscr{L}) \leqslant 4$$

when k is even. By [10, Proposition 4.2], $\lambda(\mathscr{L}) = 2$ when k is odd. In this section, based on the lemmas given in Section 2, we mainly show that

$$2 \leqslant \lambda(\mathscr{Q}) < 3$$

when $k \geqslant 5$. By [10, Proposition 3.2], we tighten the existing upper bound 4 of $\lambda(\mathscr{Q})$ and $\lambda(\mathscr{L})$ when k is even and we also investigate the largest signless Laplacian H-eigenvalue $\lambda(\mathscr{Q})$ when k is odd.

The following theorem shows that $2 < \lambda(\mathscr{Q}) < 3$ when the length $\ell = 3$.

Theorem 3 *Let $G = (V, E)$ be a k-uniform loose path with length 3, and let $\mathscr{Q}$ be its signless Laplacian tensor. Then its largest H-eigenvalue $\lambda(\mathscr{Q}) \in (2, 3)$.*

Proof It is not restrictive to assume that

$$E = \{\{1, \ldots, k\}, \{k, \ldots, 2k-1\}, \{2k-1, \ldots, 3k-2\}\}.$$

By [22, Theorem 3.20], [17, Theorem 4], and [14, Lemma 3.1], if we can find a positive H-eigenvector $x \in \mathbb{R}^n$ of $\mathscr{Q}$ corresponding to an H-eigenvalue λ, then $\lambda = \lambda(\mathscr{Q}) \geqslant 2$. By Lemmas 1, 4, and 7, for such an eigenpair (x, λ), we have

$$x_1 = \cdots = x_{k-1} = \frac{x_k}{\lambda - 1}$$

if $x_s \neq 0$ for some $s \in \{1, \ldots, k-1\}$,

$$x_{k+1} = \cdots = x_{2k-2} = \sqrt{\frac{x_k x_{2k-1}}{\lambda - 1}}$$

if $x_s \neq 0$ for some $s \in \{k+1, \ldots, 2k-2\}$, and

$$x_{2k} = \cdots = x_{3k-2} = \frac{x_{2k-1}}{\lambda - 1}$$

if $x_s \neq 0$ for some $s \in \{2k, \ldots, 3k-2\}$. Hence, the equations of the H-eigenvalue λ become

$$\begin{cases} (\lambda - 2)x_k^{k-1} = x_{2k-1}\left(\sqrt{\frac{x_k x_{2k-1}}{\lambda - 1}}\right)^{k-2} + \left(\frac{x_k}{\lambda - 1}\right)^{k-1}, \\ (\lambda - 2)x_{2k-1}^{k-1} = x_k\left(\sqrt{\frac{x_k x_{2k-1}}{\lambda - 1}}\right)^{k-2} + \left(\frac{x_{2k-1}}{\lambda - 1}\right)^{k-1}. \end{cases}$$

By straightforward computation, we have

$$\begin{cases} ((\lambda - 2)(\lambda - 1)^{k-1} - 1)x_k^{k/2} = (\lambda - 1)^{k/2} x_{2k-1}^{k/2}, \\ ((\lambda - 2)(\lambda - 1)^{k-1} - 1)x_{2k-1}^{k/2} = (\lambda - 1)^{k/2} x_k^{k/2}. \end{cases} \tag{6}$$

Define

$$a := (\lambda - 2)(\lambda - 1)^{k-1} - 1, \quad b = (\lambda - 1)^{k/2}.$$

Then the first equality in (6), together with $b > 0$, $x_k > 0$, and $x_{2k-1} > 0$, implies $a > 0$. Note that the determinant

$$\begin{vmatrix} a & -b \\ -b & a \end{vmatrix} = (a + b)(a - b).$$

Hence, if $f(\lambda) = 0$, then system (6) has a nonzero solution, where

$$f(\lambda) = (\lambda - 2)(\lambda - 1)^{k-1} - 1 - (\lambda - 1)^{k/2}.$$

It is easy to see that $\lambda(\mathcal{Q})$ is a root of $f(\lambda) = 0$. Since $f(2) = -2 < 0$ and

$$f(3) = 2^{k-1} - 2^{k/2} - 1 = \frac{1}{2}\left[(2^{k/2} - 1)^2 - 1\right] > \frac{1}{2}\left[(\sqrt{8} - 1)^2 - 3\right] > 0,$$

where the first inequality holds due to $k \geqslant 3$, the equation $f(\lambda) = 0$ must has a root in the interval $(2, 3)$. Let

$$t = \lambda - 1, \quad p(t) = t^{\frac{k}{2}-1}(t - 1), \quad q(t) = 1 + \frac{1}{t^{k/2}}.$$

Then $t > 1$ due to $\lambda > 2$ and

$$f(\lambda) = t^{k/2}(p(t) - q(t)).$$

Clearly, $p(t)$ is monotone increasing and $q(t)$ is monotone decreasing when $t > 1$. Hence, $f(\lambda)$ is monotone increasing when $\lambda > 2$. Thus, $f(\lambda) = 0$ has the unique root in the interval $(2, 3)$. That is, $\lambda(\mathcal{Q})$. Hence, $2 < \lambda(\mathcal{Q}) < 3$. □

Subsequently, we investigate the largest signless Laplacian H-eigenvalue for the case of $\ell \geqslant 4$.

Theorem 4 *Let $G = (V, E)$ be a k-uniform loose path with length $\ell \geqslant 4$, and let $\mathcal{Q}$ be its signless Laplacian tensor. Then we have*

(i) *if $\ell = 4$, then $3 < \lambda(\mathcal{Q}) < 4$ for $k = 3$ and $2 < \lambda(\mathcal{Q}) < 3$ for $k \geqslant 4$;*

(ii) *if $\ell = 5$, then $3 < \lambda(\mathcal{Q}) < 4$ for $k = 3$ and $2 < \lambda(\mathcal{Q}) < 3$ for $k \geqslant 4$;*

(iii) *if $\ell \geqslant 6$, then $2 \leqslant \lambda(\mathcal{Q}) \leqslant 4$ for $k = 3$ or $k = 4$ and $2 \leqslant \lambda(\mathcal{Q}) < 3$ for $k \geqslant 5$.*

Proof Without loss of generality, we assume that

$$E = \{\{1, \ldots, k\}, \{k, \ldots, 2k-1\}, \{2k-1, \ldots, 3k-2\}, \ldots,\\ \{(\ell-1)k-\ell+2, \ldots, \ell k-\ell, \ell k-\ell+1\}\}.$$

By [22, Theorem 3.20], [17, Theorem 4], and [14, Lemma 3.1], if we can find a positive H-eigenvector $x \in \mathbb{R}^n$ of $\mathcal{Q}$ corresponding to an H-eigenvalue λ, then

$$\lambda = \lambda(\mathcal{Q}) \geqslant 2.$$

By Lemmas 2, 4, and 7, for such an eigenpair (x, λ), we have

$$x_1 = \cdots = x_{k-1} = \frac{x_k}{\lambda - 1}$$

if $x_s \neq 0$ for some $s \in \{1, \ldots, k-1\}$, and

$$x_{k+1} = \cdots = x_{2k-2} = \sqrt{\frac{x_k x_{2k-1}}{\lambda - 1}}$$

if $x_s \neq 0$ for some $s \in \{k+1, \ldots, 2k-2\}$. The rest may be deduced by analogy:

$$x_{(\ell-2)k-\ell+4} = \cdots = x_{(\ell-1)k-\ell+1} = \sqrt{\frac{x_{(\ell-2)k-\ell+3}\, x_{(\ell-1)k-\ell+2}}{\lambda - 1}}$$

if $x_s \neq 0$ for some $s \in \{(\ell-2)k-\ell+4, \ldots, (\ell-1)k-\ell+1\}$, and

$$x_{(\ell-1)k-\ell+3} = \cdots = x_{\ell k-\ell+1} = \frac{x_{(\ell-1)k-\ell+2}}{\lambda - 1}$$

if $x_s \neq 0$ for some $s \in \{(\ell-1)k-\ell+3, \ldots, \ell k-\ell+1\}$. Hence, the equations of the H-eigenvalue λ become

$$\begin{cases} (\lambda-2)x_k^{k-1} = x_{2k-1}\left(\sqrt{\dfrac{x_k x_{2k-1}}{\lambda-1}}\right)^{k-2} + \left(\dfrac{x_k}{\lambda-1}\right)^{k-1}, \\ (\lambda-2)x_{2k-1}^{k-1} = x_k\left(\sqrt{\dfrac{x_k x_{2k-1}}{\lambda-1}}\right)^{k-2} + x_{3k-2}\left(\sqrt{\dfrac{x_{2k-1}x_{3k-2}}{\lambda-1}}\right)^{k-2}, \\ \dots, \\ (\lambda-2)x_{(\ell-2)k-\ell+3}^{k-1} = x_{(\ell-3)k-\ell+4}\left(\sqrt{\dfrac{x_{(\ell-3)k-\ell+4}x_{(\ell-2)k-\ell+3}}{\lambda-1}}\right)^{k-2} \\ \qquad\qquad + x_{(\ell-1)k-\ell+2}\left(\sqrt{\dfrac{x_{(\ell-2)k-\ell+3}x_{(\ell-1)k-\ell+2}}{\lambda-1}}\right)^{k-2}, \\ (\lambda-2)x_{(\ell-1)k-\ell+2}^{k-1} = x_{(\ell-2)k-\ell+3}\left(\sqrt{\dfrac{x_{(\ell-2)k-\ell+3}x_{(\ell-1)k-\ell+2}}{\lambda-1}}\right)^{k-2} \\ \qquad\qquad + \left(\dfrac{x_{(\ell-1)k-\ell+2}}{\lambda-1}\right)^{k-1}. \end{cases}$$

By straightforward computation, we have

$$\begin{cases} ((\lambda-2)(\lambda-1)^{k-1}-1)x_k^{k/2} = (\lambda-1)^{k/2}x_{2k-1}^{k/2}, \\ (\lambda-2)(\lambda-1)^{(k-2)/2}x_{2k-1}^{k/2} = x_k^{k/2} + x_{3k-2}^{k/2}, \\ \dots, \\ (\lambda-2)(\lambda-1)^{(k-2)/2}x_{(\ell-2)k-\ell+3}^{k/2} = x_{(\ell-3)k-\ell+4}^{k/2} + x_{(\ell-1)k-\ell+2}^{k/2}, \\ ((\lambda-2)(\lambda-1)^{k-1}-1)x_{(\ell-1)k-\ell+2}^{k/2} = (\lambda-1)^{k/2}x_{(\ell-2)k-\ell+3}^{k/2}. \end{cases} \tag{7}$$

Define

$$a = (\lambda-2)(\lambda-1)^{k-1}-1, \quad b = (\lambda-1)^{k/2}, \quad c = (\lambda-2)(\lambda-1)^{(k-2)/2}.$$

Then the first inequality in (7) implies $a > 0$. Let

$$AX = \begin{pmatrix} a & -b & & & \\ -1 & c & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c & -1 \\ & & & -b & a \end{pmatrix}_{(d-1)\times(d-1)} \begin{pmatrix} x_k^{k/2} \\ x_{2k-1}^{k/2} \\ x_{3k-2}^{k/2} \\ \vdots \\ x_{(d-3)k-d+4}^{k/2} \\ x_{(d-2)k-d+3}^{k/2} \\ x_{(d-1)k-d+2}^{k/2} \end{pmatrix}_{(d-1)\times 1}.$$

In order to compute $\lambda(\mathscr{Q})$, we divide into three cases to discuss under what conditions $AX = 0$ has nonzero solutions.

Case 1 $\ell = 4$.

If

$$\begin{vmatrix} a & -b & \\ -1 & c & -1 \\ & -b & a \end{vmatrix} = a(ac-2b) = 0,$$

i.e., $ac - 2b = 0$, then $AX = 0$ has nonzero solutions. This shows that $\lambda(\mathscr{Q})$ must be a solution of

$$(\lambda-2)^2(\lambda-1)^{k-1} - 3\lambda + 4 = 0.$$

Let

$$f(\lambda) = (\lambda-2)^2(\lambda-1)^{k-1} - 3\lambda + 4.$$

By straightforward computation, we have

$$f'(\lambda) = (\lambda-2)(\lambda-1)^{k-2}(\lambda k + \lambda - 2k) - 3.$$

Clearly, $f'(\lambda)$ is monotone increasing when $\lambda > 2$. Moreover, we have

$$f'(2) = -3 < 0, \quad \lim_{\lambda\to+\infty} f'(\lambda) = +\infty.$$

Hence, $f(\lambda)$ is first monotone decreasing and then monotone increasing. On the other hand,

$$f(2) = -2 < 0, \quad \lim_{\lambda\to+\infty} f(\lambda) = +\infty.$$

Hence, $f(\lambda) = 0$ has only one real root when $\lambda > 2$.

For $k = 3$, we have

$$f(3) = -1, \quad f(4) = 28.$$

So the real root of $f(\lambda) = 0$ is just $\lambda(\mathscr{Q}) \in (3,4)$. For $k \geqslant 4$, we have

$$f(3) = 2^{k-1} - 5 > 0.$$

So the real root $\lambda(\mathscr{Q})$ is in the interval $(2,3)$.

Case 2 $\ell = 5$.

If

$$\begin{vmatrix} a & -b & & \\ -1 & c & -1 & \\ & -1 & c & -1 \\ & & -b & a \end{vmatrix} = (b-(c+1)a)(b-(c-1)a) = 0,$$

i.e.,

$$p_1(\lambda) := b - (c+1)a = 0$$

or

$$p_2(\lambda) := b - (c-1)a = 0,$$

then $AX = 0$ has nonzero solutions. Since $a > 0$, $c > 1$, we have $p_2(\lambda) > p_1(\lambda)$. Hence, $\lambda(\mathscr{Q})$ must be the largest real root of $p_2(\lambda) = 0$. Clearly,

$$p_2(2) = 0, \quad p_2(3) > 0, \quad p_2(4) < 0, \quad k = 3.$$

When $k \geqslant 4$, we have

$$\begin{aligned} p_2(3) &= 2^{k/2} - (2^{(k-2)/2} - 1)(2^{k-1} - 1) \\ &\leqslant 2^{k/2} - 2^{k-1} + 1 \\ &= 2^{k/2}(1 - 2^{(k-2)/2}) + 1 \\ &\leqslant -2^{k/2} + 1 \\ &< 0. \end{aligned}$$

By direct computation, we obtain

$$\begin{aligned} p_2'(\lambda) &= (\lambda - 1)^{(k-4)/2}\Big[\frac{k}{2}(\lambda - 1) - \frac{1}{2}(\lambda k - 2k + 2)((\lambda - 2)(\lambda - 1)^{k-1} - 1) \\ &\quad - (\lambda k - 2k + 1)(\lambda - 1)^{k/2}((\lambda - 2)(\lambda - 1)^{\frac{k}{2}-1} - 1)\Big] \\ &< (\lambda - 1)^{(k-4)/2}\Big[\frac{k}{2}(\lambda - 1) - \frac{1}{2}(\lambda k - 2k + 2)((\lambda - 2)(\lambda - 1)^{k-1} - 1)\Big] \\ &\leqslant (\lambda - 1)^{(k-4)/2}\Big[\frac{k}{2}(\lambda - 1) - \frac{3}{2}(\lambda k - 2k + 2)\Big] \\ &= (\lambda - 1)^{(k-4)/2}\Big(-k\lambda + \frac{5}{2}k - 3\Big) \\ &< 0 \end{aligned}$$

for $\lambda, k \geqslant 3$. This shows that $p_2(\lambda)$ is monotone decreasing when $\lambda \geqslant 3$. Hence, it is easy to see that $3 < \lambda(\mathscr{Q}) < 4$ when $k = 3$ and $2 \leqslant \lambda(\mathscr{Q}) < 3$ when $k \geqslant 4$.

Case 3 $\ell \geqslant 6$.

We consider the nonzero solutions of $AX = 0$ in $(2, 4)$. It is sufficient to discuss under what conditions $|A|$, the determinant of A, equals zero. Clearly, we have $c > 0$ in this interval. Define

$$D_n = \begin{vmatrix} c & -1 & & & \\ -1 & c & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & c & -1 \\ & & & -1 & c \end{vmatrix}_{n\times n}.$$

By straightforward computation, we obtain

$$|A| = a^2 D_{\ell-3} - 2abD_{\ell-4} + b^2 D_{\ell-5}.$$

By [18], we known that $2 \leqslant \lambda(\mathscr{Q}) \leqslant 4$. So, we discuss the values of λ such that $|A| = 0$ when $\lambda > 3$. If there is no λ such that $|A| = 0$, then $2 \leqslant \lambda(\mathscr{Q}) \leqslant 3$

must hold. Clearly, we have $c > 2$ and $b > 1$ when $\lambda > 3$ and $k \geqslant 5$. In this situation, we compute

$$D_n = \frac{(c+\sqrt{c^2-4}\,)^{n+1} - (c-\sqrt{c^2-4}\,)^{n+1}}{2^{n+1}\sqrt{c^2-4}}.$$

Let

$$w = c+\sqrt{c^2-4}, \quad u = c-\sqrt{c^2-4}.$$

Then, $w > c > 2$ and $u > 0$. We have

$$\begin{aligned} |A| = 0 &\iff a^2(w^{\ell-2} - u^{\ell-2}) - 4ab(w^{\ell-3} - u^{\ell-3}) + 4b^2(w^{\ell-4} - u^{\ell-4}) = 0 \\ &\iff [a(w^{(\ell-2)/2} + u^{(\ell-2)/2}) - 2b(w^{(\ell-4)/2} - u^{(\ell-4)/2})] \\ &\quad\quad \times [a(w^{(\ell-2)/2} - u^{(\ell-2)/2}) - 2b(w^{(\ell-4)/2} + u^{(\ell-4)/2})] = 0. \end{aligned}$$

Let

$$f_1(\lambda) = a(w^{(\ell-2)/2} + u^{(\ell-2)/2}) - 2b(w^{(\ell-4)/2} - u^{(\ell-4)/2})$$

and

$$f_2(\lambda) = a(w^{(\ell-2)/2} - u^{(\ell-2)/2}) - 2b(w^{(\ell-4)/2} + u^{(\ell-4)/2}). \tag{8}$$

Due to $a = bc - 1$ and $c > 2$, $b > 1$, we have

$$\begin{aligned} f_1(\lambda) &= (wa-2b)w^{(\ell-4)/2} + (ua+2b)u^{(\ell-4)/2} \\ &\geqslant w^{(\ell-4)/2}(wa-2b) \\ &= w^{(\ell-4)/2}[b(cw-2) - w] \\ &> w^{(\ell-4)/2}(w-2) \\ &> w^{(\ell-4)/2}\sqrt{c^2-4} \\ &> 0. \end{aligned}$$

Hence, it is sufficient to discuss the root of $f_2(\lambda) = 0$ in the interval $(3, 4)$. We obtain via computation that

$$\begin{aligned} f_2(3) &> (2^{(k-2)/2} - \sqrt{2^{k-2}-4}\,)^{(\ell-4)/2}(2\sqrt{2^{k-2}-4}\,(2^{k-1}-1) - 2^{(k+4)/2}) \\ &\geqslant 2^{\frac{k}{2}+2}(2^{\frac{k}{2}-1}-1) - 4 \\ &> 2^{\frac{k}{2}+2} - 4 \\ &> 0. \end{aligned} \tag{9}$$

Define

$$a' = (\lambda-1)^{k-2}(\lambda k - 2k + 1), \quad b' = \frac{k}{2}\,(\lambda-1)^{(k-2)/2},$$
$$c' = \frac{1}{2}\,(\lambda-1)^{(k-4)/2}(\lambda k - 2k + 2).$$

Then, the derivative of $f_2(\lambda)$ is

$$f_2'(\lambda) = \frac{1}{\sqrt{c^2-4}}\Big\{\Big[\Big(a'\sqrt{c^2-4} + \frac{\ell-2}{2}\,ac'\Big)w - 2b'\sqrt{c^2-4} - b(\ell-4)c'\Big]w^{(\ell-4)/2} + \Big[\Big(-a'\sqrt{c^2-4} + \frac{\ell-2}{2}\,ac'\Big)u - 2b'\sqrt{c^2-4} + b(\ell-4)c'\Big]u^{(\ell-4)/2}\Big\}. \tag{10}$$

First, due to $\lambda > 3$ and $k \geqslant 5$, we have

$$a'w - 2b' = (c + \sqrt{c^2-4})(bc' + cb') - 2b' > 2(bc' + cb') - 2b' > 0 \tag{11}$$

and

$$\begin{aligned}\frac{\ell-2}{2}\,aw - b(\ell-4) &= \frac{\ell-2}{2}\,a(c+\sqrt{c^2-4}\,) - b(\ell-4)\\ &> (\ell-2)a - b(\ell-4)\\ &= (\ell-2)(bc-1) - b(\ell-4)\\ &= [(\ell-2)c - (\ell-4)]b - (\ell-2)\\ &\geqslant (\ell-2)c - (\ell-4) - \ell + 2\\ &> 2(\ell-2) - (\ell-4) - \ell + 2\\ &= 2\\ &> 0.\end{aligned} \tag{12}$$

Second, we have

$$\begin{aligned}&-2b'\sqrt{c^2-4} + b(\ell-4)c'\\ &= \frac{\ell-4}{2}\,(\lambda-1)^{k/2}(\lambda-1)^{(k-4)/2}(\lambda k - 2k + 2) - k(\lambda-1)^{(k-2)/2}\sqrt{c^2-4}\\ &\geqslant (\lambda-1)^{k/2}(\lambda-1)^{(k-4)/2}(\lambda k - 2k + 2) - k(\lambda-1)^{(k-2)/2}\sqrt{c^2-4}\\ &= (\lambda-1)^{(k-2)/2}[(\lambda-1)^{(k-2)/2}(\lambda k - 2k + 2) - k\sqrt{c^2-4}\,]\\ &\geqslant (\lambda-1)^{(k-2)/2}(\lambda k - 2k + 2) - k\sqrt{c^2-4}.\end{aligned}$$

Since

$$\begin{aligned}&(\lambda-1)^{k-2}(\lambda k - 2k + 2)^2 - k^2(c^2-4)\\ &= 4(\lambda-1)^{k-2} + 4k(\lambda-2)(\lambda-1)^{k-2} + 4k^2\\ &> 0,\end{aligned}$$

we have

$$(\lambda-1)^{(k-2)/2}(\lambda k - 2k + 2) > k\sqrt{c^2-4}.$$

This yields

$$-2b'\sqrt{c^2-4} + b(\ell-4)c' > 0. \tag{13}$$

Similarly, from the fact $\ell \geqslant 6$, we have

$$\begin{aligned}
&-a'\sqrt{c^2-4}+\frac{\ell-2}{2}ac'\\
&=\frac{\ell-2}{4}a(\lambda-1)^{(k-4)/2}(\lambda k-2k+2)-(\lambda-1)^{k-2}(\lambda k-2k+1)\sqrt{c^2-4}\\
&>a(\lambda-1)^{(k-4)/2}(\lambda k-2k+2)-(\lambda-1)^{k-2}(\lambda k-2k+1)\sqrt{c^2-4}\\
&=(\lambda-1)^{(k-4)/2}[a(\lambda k-2k+2)-(\lambda-1)^{k/2}(\lambda k-2k+1)\sqrt{c^2-4}].
\end{aligned}$$

Since

$$\begin{aligned}
&a^2(\lambda k-2k+2)^2-(\lambda-1)^k(\lambda k-2k+1)^2(c^2-4)\\
&\quad=(\lambda k-2k+1)^2+2\lambda(\lambda-1)^{k-1}(\lambda k-2k+1)^2+1+2(\lambda k-2k+1)\\
&\qquad+(\lambda-2)^2(\lambda-1)^{2k-2}-2(\lambda-2)(\lambda-1)^{k-1}\\
&\qquad+2(\lambda-2)^2(\lambda-1)^{2k-2}(\lambda k-2k+1)-4(\lambda-2)(\lambda-1)^{k-1}(\lambda k-2k+1)\\
&\quad\geqslant(\lambda-2)^2(\lambda-1)^{2k-2}-2(\lambda-2)(\lambda-1)^{k-1}\\
&\qquad+2(\lambda-2)^2(\lambda-1)^{2k-2}(\lambda k-2k+1)-4(\lambda-2)(\lambda-1)^{k-1}(\lambda k-2k+1)\\
&\quad=(\lambda-2)(\lambda-1)^{k-1}[(\lambda-2)(\lambda-1)^{(k-2)/2}(\lambda-1)^{k/2}-2]\\
&\qquad+(\lambda-2)(\lambda-1)^{k-1}(\lambda k-2k+1)[2(\lambda-2)(\lambda-1)^{k-1}-4]\\
&\quad>(\lambda-2)(\lambda-1)^{k-1}(c-2)+(\lambda-2)(\lambda-1)^{k-1}(\lambda k-2k+1)\\
&\quad\geqslant 2(\lambda-2)(\lambda-1)^{(k-2)/2}(\lambda-1)^{k/2}-4\\
&\quad\geqslant 2c(\lambda-1)^{k/2}-4\\
&\quad\geqslant 2c-4>0,
\end{aligned}$$

we have

$$-a'\sqrt{c^2-4}+\frac{\ell-2}{2}ac'>0. \tag{14}$$

Combining (10)–(14), we have $f_2'(\lambda)>0$ when $\lambda>3$ and $k\geqslant 5$. This indicates that $f_2(\lambda)$ is monotone increasing when $\lambda>3$, which together with (9) implies $2<\lambda(\mathscr{Q})<3$ for $k\geqslant 5$ and $\ell\geqslant 6$. Thus, we have the desired results. That is, when the length $\ell\geqslant 6$, we have $2\leqslant\lambda(\mathscr{Q})\leqslant 4$ for $k=3,4$ and $2\leqslant\lambda(\mathscr{Q})<3$ for $k\geqslant 5$. □

By [10, Proposition 3.2] and Theorems 3 and 4, we immediately have the following result.

Corollary 1 *Let k be even, and let $G=(V,E)$ be a k-uniform loose path with length $\ell\geqslant 3$. Let $\mathscr{L}$ be its Laplacian tensor. Then, the following statements hold*:

(i) $\lambda(\mathscr{L})\in(2,3)$ *when* $\ell=3,4,5$;

(ii) *when* $\ell\geqslant 6$ *and* $k=4$, $2\leqslant\lambda(\mathscr{L})\leqslant 4$;

(iii) *when* $\ell\geqslant 6$ *and* $k>5$, $2\leqslant\lambda(\mathscr{L})<3$.

5 Numerical experiments

In this section, according the proof process of Theorems 3 and 4, we design a procedure to compute the largest signless Laplacian H-eigenvalue of loose paths for some numerical experiments. The discussion for $\lambda(\mathscr{A})$ is similar, we omit them here.

Procedure Step 1 For fixed k and ℓ, we find out all the real roots of the polynomial equation in a certain interval.

(i) When $\ell = 3$ or $\ell = 4$, we can find out the unique real root of

$$(\lambda - 2)(\lambda - 1)^{k-1} - (\lambda - 1)^{k/2} - 1 = 0$$

or

$$(\lambda - 2)^2(\lambda - 1)^{(3k-4)/2} - (\lambda - 2)(\lambda - 1)^{(k-2)/2} - 2(\lambda - 1)^{k/2} = 0$$

by dichotomy.

(ii) When $\ell = 5$ and $k \geqslant 4$, we can find out all the real roots of

$$(\lambda-2)(\lambda-1)^{k-1}+(\lambda-1)^{k/2}-(\lambda-2)^2(\lambda-1)^{(3k-4)/2}-(\lambda-2)(\lambda-1)^{(k-2)/2}-1=0$$

by judging all discrete points in $(2, 3)$.

(iii) First, we find out the unique real root t of the equations

$$(\lambda - 2)(\lambda - 1)^{(k-2)/2} = 0, \quad \lambda > 2.$$

When $k \geqslant 5$, we compute the unique real root of $f_2(\lambda) = 0$, where $f_2(\lambda)$ is defined as in (8), in $(t, 3)$ by dichotomy and find out another real roots by judging all discrete points in $(2, t)$. When $k = 3$ or $k = 4$, we compute the unique real root in $(t, 4)$ by dichotomy and find out another real roots by judging all discrete points in $(2, t)$.

Step 2 Check whether all the real roots are the H-eigenvalues of $\mathscr{Q}$ or not.

Substituting the real roots into equations

$$AX = A(x_1, \ldots, x_n)^{\mathrm{T}} = 0,$$

determine whether there is $X > 0$ or not. Let A be an $n \times n$ matrix. If $|A| = 0$, then we have $\operatorname{rank}(A) = n - 1$. So $x_1, \ldots, x_{n-1}$ can be expressed as $c_i x_1$, $i \in \{1, \ldots, n-1\}$. If c_i, $i \in \{1, \ldots, n-1\}$, are greater than zero, we have $X > 0$. The computational complexity is $O(n)$.

Step 3 Choose the largest real roots which satisfy Steps 1 and 2.

From the above procedure, the computational complexity of this algorithm is

$$O\Big(\frac{k\ell(\ell-2)}{2} + \frac{1}{\varepsilon} + \log_2 \frac{1}{\varepsilon}\Big),$$

where ε is a given tolerance. We also can find out the largest adjacency H-eigenvalue of loose paths by the similar procedure as above. The results of the largest signless Laplacian H-eigenvalues and the largest adjacency H-eigenvalues are presented in Figs. 1 and 2. Here, we take $3 \leqslant k, \ell \leqslant 50$. The curve in Fig. 1 (a) is the function with respect to the variable k when ℓ is fixed, while the one in Fig. 1 (b) is the function with respect to the variable ℓ when k is fixed. As can be seen in Fig. 1 that $\lambda(\mathscr{Q})$ with respect to k is a strictly decreasing sequence when ℓ is fixed, the numerical experiments show that [10, Conjecture 4.1] is true for loose paths. It also can be discerned from Fig. 1 that the function $\lambda(\mathscr{Q})$ with respect to k is convergent when ℓ trends to $+\infty$.

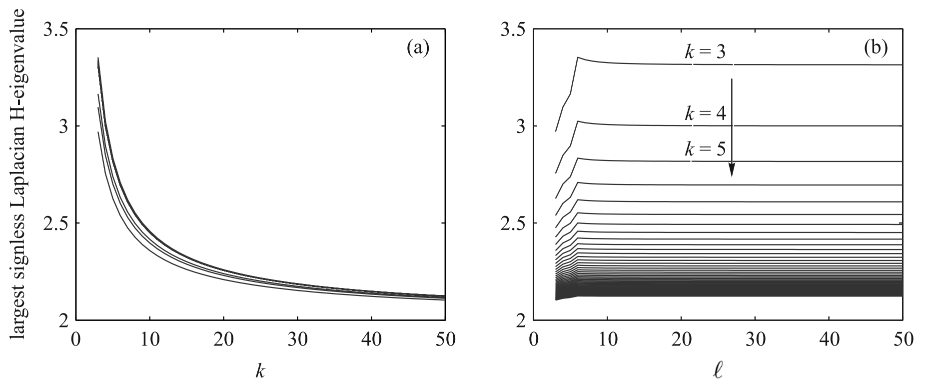

Fig. 1 Largest signless Laplacian H-eigenvalue $\lambda(\mathscr{Q})$ of loose paths. Function $\lambda(\mathscr{Q})$ with respect to (a) variable k when ℓ is fixed and (b) variable ℓ when k is fixed.

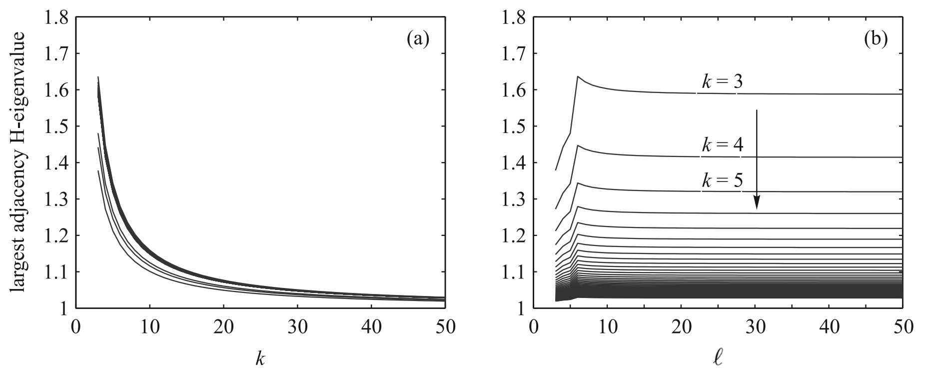

Fig. 2 Largest adjacency H-eigenvalue $\lambda(\mathscr{A})$ of loose paths. Function $\lambda(\mathscr{A})$ with respect to (a) variable k when ℓ is fixed and (b) variable ℓ when k is fixed.

6 Concluding remarks

We investigate the spectral theory of loose paths. We compute the largest H-eigenvalue $\lambda(\mathscr{A})$ of its adjacency tensor and the largest H-eigenvalue $\lambda(\mathscr{Q})$ of its signless Laplacian tensor. We tighten their bounds. The numerical results shows that $\lambda(\mathscr{Q})$ with respect to k is a strictly decreasing sequence for a k-uniform loose path with fixed length ℓ. This is a conjecture to be presented here for future research.

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