On the Erdős problem

Chunhui Lai

Minnan Normal University
joint work with Shaoqiang Liu

Qingdao, September 22, 2024

Outline

Outline

Let G be a graph with m edges on n vertices, then G contains at least $m - n + 1$ distinct cycles. Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see J.A. Bondy and U.S.R. Murty [Graph Theory with Applications (Macmillan, New York, 1976).], p.247, Problem 11). Let $f^*(n)$ be the maximum number of edges in a simple graph on n vertices in which any two cycles are of different lengths.

Let M_n be the set of simple graphs on n vertices and $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. Let $sc(n)$ be the second-largest cycle length for all $G \in M_n$. Let $tc(n)$ be the third-largest cycle length for all $G \in M_n$. A natural question is what is the numbers of $mc(n)$, $sc(n)$, $tc(n)$. Let $mcn(n)$ be the maximum cycle numbers for all $G \in M_n$. A natural question is what is the numbers of $mcn(n)$. Let $b(n)$ be the maximum 2-connected block numbers for all $G \in M_n$. A natural question is what is the numbers of $b(n)$.

Y. Shi[On maximum cycle-distributed graphs, Discrete Math. 71(1988), 57-71. MR0954686 (89i:05169)] proved that
Theorem 1.1 (Shi 1988)

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$ and

$$f(n) = f^*(n-1) + 3$$

for $n \geq 3$.

C. Lai[Upper bound and lower bound of $f(n)$, J. Zhangzhou Teachers College(Natural Science Edition) 4(1)(1990) 29,30-34. MR1078010] proved that
Theorem 1.2 (Lai 1990) for $n \geq e^{2m}(2m+3)/4$,

$$f(n) < n - 2 + \sqrt{nln(4n/(2m+3)) + 2n} + log_2(n+6).$$

Chen, Lehel, Jacobson and Shreve[Note on graphs without repeated cycle lengths, J. Graph Theory 29(1998),11-15. MR1633908] gave a quick proof of this result.

Y. Shi [The number of edges in a maximum graphs with uni-length cycles, Kexue Tongbao 33(10)(1988), 795 - 796.], Y. Shi [Some theorems on simple MCD-graphs, J. Math. Res. Exposition 11(4)(1991), 551-554. MR1144956 (92i:05182)], Y. Shi [The number of edges in a maximum cycle distributed graph, Discrete Math. 104(1992), 205-209. MR1172850 (93d:05083)], Y. Shi [On simple MCD-graphs, Chinese Quarterly J. Math. 7(3)(1992),41-47.], Y. Shi [On simple MCD graphs containing a subgraph homemorphic to K_4, Discrete Math. 126(1994), 325-338. MR1264498 (95f:05072)],

Y. Shi [An upper bound on the size of simple MCD-graphs, Pure Appl. Math. 10(1994), Special Issue, 210-216. MR1369978 (96j:05063)], Y. Shi [A class of almost uniquely pancyclic graphs, J. Systems Sci. Math. Sci. 26(4) (2006), 433-439. MR2270870], Y. Shi, Y. Tang, H. Tang, L. Gong, L. Xu [Two classes of simple MCD graphs, Discrete geometry, combinatorics and graph theory, 177-188, Lecture Notes in Comput. Sci., 4381, Springer, Berlin, 2007. MR2364761], Y. Shi, L. Xu, X. Chen, M. Wang [Almost uniquely pancyclic graphs, Adv. Math. (China) 35(5) (2006), 563-569. MR2410416],

X. Jia[Some extremal problems on cycle distributed graphs, Congr. Number., 121(1996), 216-222.], G. Chen, J. Lehel, M. S.Jacobson, and W. E.Shreve [Note on graphs without repeated cycle lengths, J. Graph Theory, 29(1998),11-15.],

C. Lai [On the Erdős problem, J. Zhangzhou Teachers College(Natural Science Edition) 3(1)(1989) 55-59. MR1025502], C. Lai [On the size of graphs with all cycle having distinct length, Discrete Math. 122(1993) 363-364. MR1246693], C. Lai[On the maximum number of edges in a graph in which no two cycles have the same length, Combinatorics, graph theory, algorithms and applications (Beijing, 1993), 447–450, World Sci. Publ., River Edge, NJ, 1994. MR1313999 (96b:05080)],

C. Lai [The edges in a graph in which no two cycles have the same length, J. Zhangzhou Teachers College (Natural Science) 8(4)(1994), 30-34. MR1336586], C. Lai [The numbers of edge of one class graphs, J. of Zhangzhou Teachers College (Natural Science) 12(2)(1999), 7-9, 20. MR1723698], C. Lai [A lower bound for the number of edges in a graph containing no two cycles of the same length, Electron. J. Combin. 8(2001), Note 9, 1 - 6. MR1877662],

C. Lai [The number of edges of some graphs in which no two cycles have the same length, J. of Zhangzhou Teachers College (Natural Science) 15(1) (2002), 10-14. MR1895073], C. Lai [Graphs without repeated cycle lengths, Australas. J. Combin. 27(2003), 101-105. MR1955391], C. Lai [On the size of graphs without repeated cycle lengths, Discrete Appl. Math. 232 (2017), 226-229. MR3711962] obtained some results.

E. Boros, Y. Caro, Z. Füredi and R. Yuster[Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284.] proved that

Theorem 1.3 (Boros, Caro, Füredi and Yuster 2001)

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

C. Lai [On the size of graphs without repeated cycle lengths, Discrete Appl. Math. 232 (2017), 226-229.] improved the lower bound.

Theorem 1.4 (Lai 2017) Let $t = 1260r + 169 \ (r \geq 1)$, then

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$.

C. Lai[On the size of graphs with all cycle having distinct length, Discrete Math. 122(1993) 363-364.] proposed the following conjecture:

Conjecture 1.5 (Lai 1993)

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3}.$$

It would be nice to prove that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3+\frac{3}{5}}.$$

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3+\frac{1}{3}}.$$

C. Lai [On the number of edges in some graphs, Discrete Applied Mathematics 283 (2020), 751-755] construct a graph G having no two cycles with the same length which leads to the following result.

Theorem 1.6 (Lai 2020) Let $t = 1260r + 169 \ \ (r \geq 1)$, then

$$f(n) \geq n + \frac{119}{3}t - \frac{26399}{3}$$

for $n \geq \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$.

From Theorem 1.6, we have

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}},$$

which is better than the previous bounds $\sqrt{2}$ (see Shi 1988), $\sqrt{2+\frac{7654}{19071}}$ (see Lai 2017).

Combining this with Boros, Caro, Füredi and Yuster's upper bound, we get

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

The sequence $(c_1, c_2, \cdots, c_n)$ is the cycle length distribution of a graph G of order n, where c_i is the number of cycles of length i in G. Let $f(a_1, a_2, \cdots, a_n)$ denote the maximum possible number of edges which satisfies $c_i \leq a_i$ where a_i is a nonnegative integer. Y. Shi posed the problem of determining $f(a_1, a_2, \cdots, a_n)$, which extended the problem due to Erdős, it is clearly that $f(n) = f(1, 1, \cdots, 1)$ (see [Y. Shi, Some problems of cycle length distribution, J. Nanjing Univ. (Natural Sciences), Special Issue On Graph Theory, 27(1991), 233-234]).

Let $g(n,m) = f(a_1, a_2, \cdots, a_n)$, $a_i = 1$ for all i/m be integer, $a_i = 0$ for all i/m be not integer. It is clearly that $f(n) = g(n,1)$.

From the Theorem 1.6, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} \geq \sqrt{2+\frac{40}{99}},$$

for all integer m.

We obtain the following result.

Theorem 1.7 (Lai 2020) Let m be even, $s_1 > s_2, s_1 + 3s_2 > k$, then

$$g(n, m) \geq n + (k + s_1 + 2s_2 + 1)t - 1$$

for
$n \geq (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1.$

From Theorem 1.7, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} \geq$$

$$\sqrt{\frac{(k+s_1+2s_2+1)^2}{(\frac{3}{4}k^2+\frac{1}{2}ks_1+\frac{3}{2}ks_2+\frac{1}{2}s_1^2+\frac{3}{2}s_1s_2+\frac{9}{4}s_2^2+k+s_1+3s_2+\frac{1}{2})}},$$

for all even integer m.

Let $s_1 = 28499066, s_2 = 4749839, k = 14249542$, then

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} > \sqrt{2.444},$$

for all even integer m.

C. Lai [On the number of edges in some graphs, Discrete Applied Mathematics 283 (2020), 751-755] make the following conjecture:

Conjecture 1.8 (Lai 2020)

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444}.$$

We make the following conjecture:

Conjecture 1.9

$$\lim_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2+\frac{4}{9}}.$$

C. Lai [On the Erdős problem, J. Zhangzhou Teachers College(Natural Science Edition) 3(1)(1989) 55-59. MR1025502] proved that

Theorem 1.10 (Lai 1989)

$$mc(n) \leq n - 1$$

for $n \geq \sum_{i=1}^{71} i - 8 \times 18$.

C. Lai [An old problem of Erdős: a graph without two cycles of the same length,Discrete Applied Mathematics 337 (2023) , 42-45. arXiv:2110.04696] proved that

Theorem 1.11 (Lai 2023) For n sufficiently large,

$$mc(n) \leq \frac{15}{16}n.$$

and make the following conjecture:

Conjecture 1.12 (Lai 2023)

$$\lim_{n\to\infty} \frac{mc(n)}{n} = 0.$$

It is clear that $mcn(n) \le mc(n) - 2$.
By theorem 1.6, it is clear that

$$mcn(n) \ge \sqrt{2 + \frac{40}{99}}\sqrt{n}(1 - o(1)).$$

It would be nice to determin $f(n)$ for infinitely many n.
It would be nice to determin $g(n, m)$ for infinitely many (n, m).

It would be nice to determin $mc(n)$ for infinitely many n.
It would be nice to determin $sc(n)$ for infinitely many n.
It would be nice to determin $tc(n)$ for infinitely many n.
It would be nice to determin $mcn(n)$ for infinitely many n.

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length.

Y. Shi[On maximum cycle-distributed graphs, Discrete Math. 71(1988) 57-71.] proved that

Theorem 1.13 (Shi 1988) For every integer $n \geq 3$, $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n-15} - 3)]$.

G. Chen, J. Lehel, M. S.Jacobson, and W. E.Shreve [Note on graphs without repeated cycle lengths, J. Graph Theory, 29(1998),11-15.] proved that

Theorem 1.14 (Chen, Lehel, Jacobson, and Shreve 1998) $f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

E. Boros, Y. Caro, Z. Füredi and R. Yuster [Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284.] improved this lower bound significantly:

Theorem 1.15 (Boros, Caro, Füredi and Yuster 2001)
$f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$.

Combining this with Shi's results we know:

Corollary 1.16 (Boros, Caro, Füredi and Yuster 2011)

$$\sqrt{2} \geq \limsup \frac{f_2(n) - n}{\sqrt{n}} \geq \liminf \frac{f_2(n) - n}{\sqrt{n}} \geq 1.$$

Boros, Caro, Füredi and Yuster [E. Boros, Y. Caro, Z. Füredi and R. Yuster, Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284.] made the following conjecture:

Conjecture 1.17 (Boros, Caro, Füredi and Yuster 2001) $\lim \frac{f_2(n)-n}{\sqrt{n}} = 1.$

It is easy to see that Conjecture 1.17 implies the (difficult) upper bound in the Erdős Turan Theorem [P. Erdős , On a problem of Sidon in additive number theory and on some related problems. Addendum, J. Lond. Math. Soc. 19 (1944), 208.][P. Erdős and P. Turan, On a problem of Sidon in additive number theory, and on some related problems, J. Lond. Math. Soc. 16 (1941), 212–215.](see [E. Boros, Y. Caro, Z. Füredi and R. Yuster, Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284.]).

Jie Ma, Tianchi Yang [Non-repeated cycle lengths and Sidon sequences, Israel J. Math. 245(2) (2021), 639-674. MR4358259] prove a conjecture of Boros, Caro, Füredi and Yuster on the maximum number of edges in a 2-connected graph without repeated cycle lengths, which is a restricted version of a longstanding problem of Erdős. Their proof together with the matched lower bound construction of Boros, Caro, Füredi and Yuster show that this problem can be conceptually reduced to the seminal problem of finding the maximum Sidon sequences in number theory.

Theorem 1.18 (Ma, Yang 2021)

Any n-vertex 2-connected graph with no two cycles of the same length contains at most $n + \sqrt{n} + o(\sqrt{n})$ edges.

It would be nice to determin $f_2(n)$ for infinitely many n.

Let $f_2(n,k)$ be the maximum number of edges in a graph G on n vertices in which no two cycles have the same length and G which consists of k 2-connected blocks. A natural question is what is the maximum number of edges $f_2(n,k)$. It is clearly that $f_2(n,1) = f_2(n)$. Let $g_2(n,k)$ be the maximum number of edges in a graph G on n vertices in which no two cycles have the same length and G which consists of less than or equal to k 2-connected blocks. It is clearly that $g_2(n,1) = f_2(n)$, $g_2(n,k) \leq g_2(n,k+1)$.

By Theorem 1.3, it is clear that

$$f_2(n,k) \leq f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

$$g_2(n,k) \leq f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

K. Markström [A note on uniquely pancyclic graphs, Australas. J. Combin. 44 (2009), 105-110.] raised the following problem:

Problem 1.19 (Markström 2009) Determine the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.

Let $g(n)$ be the maximum number edges in an n-vertex, Hamiltonian graph with no repeated cycle length. J. Lee, C. Timmons [A note on the number of edges in a Hamiltonian graph with no repeated cycle length, Australas. J. Combin. 69(2)(2017), 286-291.] prove the following.

Theorem 1.20 (Lee and Timmons 2017) If q is a power of a prime and $n = q^2 + q + 1$, then

$$g(n) \geq n + \sqrt{n - 3/4} - 3/2$$

A simple counting argument shows that $g(n) < n + \sqrt{2n} + 1$.

It would be nice to determining $g(n)$ for infinitely many n.

Let MH_n be the set of Hamiltonian graphs on n vertices and $g(n)$ edges in which any two cycles are of different lengths. Let $mcn_H(n)$ be the maximum cycle numbers for all $G \in MH_n$.

It would be nice to determin $mcn_H(n)$ for infinitely many n.

The lower bound $f(0,0,2,\cdots,2)$ is given by J. Xu and Y. Shi [The maximum possible number of edges in a simple graph with at most two cycles having the same lengty, J.Shanghai Normal Univ.(Natural Sciences), 32(3)(2003), 26-32.].

Theorem 1.21 (Xu and Shi 2003)

For $n \geq 3$,

$f(0,0,2,...,2) \geq n-1+[(\sqrt{11n-20})/2]$,

and the equality holds when $3 \leq n \leq 10$.

Huiqiu Lin, Mingqing Zhai, Yanhua Zhao[Spectral Radius, Edge-Disjoint Cycles and Cycles of the Same Length,The electronic journal of combinatorics 29(2) (2022), P2.1] proved that

Theorem 1.22 (Lin, Zhai, Zhao 2022)

Let G be a graph of order $n \geq 26$. If $\rho(G) \geq \rho(K_{1,n-1}^{+})$, then G contains two cycles of the same length unless $G \cong K_{1,n-1}^{+}$.

and asked the following problem.

Problem 1.23 (Lin, Zhai, Zhao 2022)

What is the maximum spectral radius among all 2-connected n-vertex graphs without two cycles of the same length?

Y. Shi [An upper bound on the size of simple MCD-graphs, Pure Appl. Math. 10(1994), Special Issue, 210-216. MR1369978 (96j:05063)]proved that

Theorem 1.24 (Shi 1994)

$$b(n) \leq [(\sqrt{8n+1}-5)/2]+1$$

It would be nice to prove

$$\liminf_{n\to\infty} \frac{b(n)}{\sqrt{n}} \le \sqrt{29}/4.$$

It would be nice to determin $b(n)$ for infinitely many n.

It would be nice to determin the maximum number of edges in some special graph on n vertices in which no two cycles have the same length. Such as outerplanar graph, k-partite graph, subgraph of k-Cube, etc.

From Theorem 1.7, we have

$$\liminf_{n\to\infty} \frac{g(n,2)-n}{\sqrt{\frac{n}{2}}} > \sqrt{2.444},$$

for 2-partite graph(the cycle length of 2-partite graph is even).

Xu Yue Can, Shi Yong Bing [A class of graphs determined by their cycle length distributions. (Chinese) Shanghai Shifan Daxue Xuebao Ziran Kexue Ban 23 (1994), no. 2, 110 – 116. MR1330667], Wang Min, Wang Ming Lei, Shi Yong Bing [Bipartite graphs determined by their cycle length distributions. (Chinese) Adv. Math. (China) 34 (2005), no. 2, 167 – 172. MR2228996], Wang Min, Shi Yong Bing[Uniqueness of cycle length distribution of the bipartite graph $K_{n,r} \backslash A(|A| \leq 3)$. (Chinese) J. Math. Res. Exposition 26 (2006), no. 1, 149 – 155. MR2208593], S. Liu [On $r-(k)$-pancyclic graphs, Ars Combin. 140(2018), 277-291. MR3822005] obtained some results.

Given a graph H, what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turan number.

We denote by $m_i(n)$ the numbers of cycles of length i in the complete graph K_n on n vertices. Obviously,

$$ex(n, C_k)$$

$$= f(0, 0, m_3(n), \cdots,$$

$$m_{k-1}(n), 0, m_{k+1}(n), \cdots, m_n(n))$$

$$= f(0, 0, 2^{\frac{n(n-1)}{2}}, \cdots,$$

$$2^{\frac{n(n-1)}{2}}, 0, 2^{\frac{n(n-1)}{2}}, \cdots, 2^{\frac{n(n-1)}{2}}).$$

Therefore, finding $ex(n, C_k)$ is a special case of determining $f(a_1, a_2, \cdots, a_n)$.

There are not good sufficient and necessary condition of when a graph on n vertices in which no two cycles have the same length. There are also not good sufficient and necessary condition of when a 2-connected graph on n vertices in which no two cycles have the same length. It would be nice to give good sufficient and necessary condition of when a graph on n vertices in which no two cycles have the same length. It would be nice to give good sufficient and necessary condition of when a 2-connected graph on n vertices in which no two cycles have the same length.

The survey article on this problem can be found in

Tian, Feng[The progress on some problems in graph theory. (Chinese) Qufu Shifan Daxue Xuebao Ziran Kexue Ban 1986, no. 2, 30–36. MR0865617 (87m:05160)],

Zhang, Ke Min[Progress of some problems in graph theory. (Chinese) J. Math. Res. Exposition 27 (2007), no. 3, 563–576. MR2349503] and

Chunhui Lai, Mingjing Liu[Some open problems on cycles, Journal of Combinatorial Mathematics and Combinatorial Computing 91 (2014), 51-64. MR3287706]

Chunhui Lai, Shaoqiang Liu, Erdos Problem In book: Some Problems on Paths and Cycles
https://www.researchgate.net/profile/Chunhui-Lai/research

Also see Douglas B. West [Introduction to Graph Theory (Second edition), Prentice Hall, 2001] ,p77, Exercises 2.1.41 , Douglas B. West [Combinatorial Mathematics, Cambridge University Press,Cambridge, 2020] ,p249, Exercise 5.4.27 , RMM 2019 - Romanian Master of Mathematics 2019 http://rmms.lbi.ro/rmm2019/index.php?id=home, The 11th Romanian Master of Mathematics Competition Problem 3.

The progress of all 50 problems in [J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976)] can be find in Stephen C. Locke, Unsolved problems: http://math.fau.edu/locke/Unsolved.htm

Acknowledgement:

Project supported by the National Science Foundation of China (No.61379021; No. 11401290), Fujian Provincial Natural Science Foundation of China (2015J01018; 2020J01795, 2021J02048), Fujian Provincial Training Foundation for "Bai-Quan-Wan Talents Engineering", Project of Fujian Education Department (JZ160455), Fujian Key Laboratory of Granular Computing and Applications(Minnan Normal University),the Institute of Meteorological Big Data-Digital Fujian and Fujian Key Laboratory of Data Science and Statistics.

The authors would like to thank Professor Y. Caro, R. Yuster for their advice and sending some papers to us. The authors would like to thank Professor E. Boros, G. Fan, R. Gould, G.O.H. Katona,Z. Song, C. Zhao for their advice. The authors would like to thank Professor Y. Chen, T. Yang for their sending some papers to us.

The End

Thanks for your attention!