

# Some open problems on cycles \*

*Chunhui Lai*<sup>1,2</sup>, *Mingjing Liu*<sup>1</sup>

1. Department of Mathematics and Information Science,  
Minnan Normal University,  
Zhangzhou, Fujian 363000, CHINA.
  2. Center for Discrete Mathematics and Theoretical Computer Science,  
Fuzhou University,  
Fuzhou, Fujian 350003, CHINA.
- e-mail: laich@winmail.cn; laichunhui@mnnu.edu.cn  
(Chunhui Lai, Corresponding author)  
liumingjing1983@163.com (Mingjing Liu)

## Abstract

Let  $f(n)$  be the maximum number of edges in a graph on  $n$  vertices in which no two cycles have the same length. Erdős raised the problem of determining  $f(n)$ . Erdős conjectured that there exists a positive constant  $c$  such that  $ex(n, C_{2k}) \geq cn^{1+1/k}$ . Hajós conjecture that every simple even graph on  $n$  vertices can be decomposed into at most  $n/2$  cycles. We present the problems, conjectures related to these problems and we summarize the know results. We do not think Hajós conjecture is true.

**Key words:** Hajós conjecture; even graph; Turan number; cycle; the maximum number of edges

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## Erdős Problem 1

Let  $f(n)$  be the maximum number of edges in a graph on  $n$  vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining  $f(n)$  (see Bondy and Murty [1], p.247, Problem 11). Shi[41] proved a lower bound.

**Theorem 1.1 (Shi[41])**

$$f(n) \geq n + [(\sqrt{8n - 23} + 1)/2]$$

for  $n \geq 3$ .

Chen, Lehel, Jacobson, and Shreve [4], Jia[25], Lai[28,29,30,31,32,33,34], Shi[42,43,44,45,46,47], Y. Shi, Y. Tang, H. Tang, L. Gong and L. Xu[48] obtained some results.

Boros, Caro, Füredi and Yuster[3] proved an upper bound.

**Theorem 1.2 ( Boros, Caro, Füredi and Yuster[3])** For  $n$  sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [35] improved the lower bound.

**Theorem 1.3 (Lai [35])**

$$f(n) \geq n + \sqrt{2.4}\sqrt{n}(1 - o(1))$$

and proposed the following conjecture:

**Conjecture 1.4 (Lai [35])**

$$\lim_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} = \sqrt{2.4}.$$

It seems difficult to prove this conjecture. It would be nice to prove one of the following weaker conjectures:

**Conjecture 1.5 (Lai[30])**

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$

**Conjecture 1.6 (Lai[31])**

$$\liminf_{n \rightarrow \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{2.4}.$$

Let  $f_2(n)$  be the maximum number of edges in a 2-connected graph on  $n$  vertices in which no two cycles have the same length.

Shi[44] proved that

**Theorem 1.7 (Shi[44])** For every integer  $n \geq 3$ ,  $f_2(n) \leq n + [\frac{1}{2}(\sqrt{8n - 15} - 3)]$ .

Chen, Lehel, Jacobson, and Shreve [4] proved that

**Theorem 1.8 (Chen, Lehel, Jacobson, and Shreve [4])**  $f_2(n) \geq n + \sqrt{n/2} - o(\sqrt{n})$

Boros, Caro, Füredi and Yuster [3] improved this lower bound significantly:

**Theorem 1.9 (Boros, Caro, Füredi and Yuster [3])**  $f_2(n) \geq n + \sqrt{n} - O(n^{\frac{9}{20}})$ .

**Corollary 1.10 (Boros, Caro, Füredi and Yuster [3])**

$$\sqrt{2} \geq \limsup \frac{f_2(n) - n}{\sqrt{n}} \geq \liminf \frac{f_2(n) - n}{\sqrt{n}} \geq 1.$$

Boros, Caro, Füredi and Yuster [3] made the following conjecture:

**Conjecture 1.11 ( Boros, Caro, Füredi and Yuster [3])**

$$\lim \frac{f_2(n) - n}{\sqrt{n}} = 1.$$

It is easy to see that Conjecture 1.11 implies the (difficult) upper bound in the Erdős-Turan Theorem [9,12](see Boros, Caro, Füredi and Yuster [3]).

Markström [27] raised the following problem:

**Problem 1.12 (Markström [27])** Determine the maximum number of edges in a hamiltonian graph on  $n$  vertices with no repeated cycle lengths.

Let  $g(n)$  denote the least number of edges of a graph which contains a cycle of length  $k$  for every  $1 \leq k \leq n$ . Jia[25] proved the following results:

**Theorem 1.13 (Jia[25])**

When  $n$  is sufficiently large,

$$n + \log_2 n - 1 \leq g(n) \leq n + \frac{3}{2} \log_2 n + 1.$$

**Theorem 1.14 (Jia[25])**

For a sufficiently large positive integer  $n$ ,  $g(n) \leq n + \log_2 n + \frac{3}{2} \log_2 \log_2 n + O(1)$

**Corollary 1.15 (Jia[25])**

For  $n$  sufficiently large,  $g(n) = n + \log_2 n + O(\log_2 \log_2 n)$ .

Jia[25] made the following conjecture:

**Conjecture 1.16 (Jia[25])**

$$g(n) = n + \log_2 n + O(1),$$

as  $n \rightarrow \infty$ .

The sequence  $(c_1, c_2, \dots, c_n)$  is the cycle length distribution of a graph  $G$  of order  $n$  where  $c_i$  is the number of cycles of length  $i$  in  $G$ . Let  $f(a_1, a_2, \dots, a_n)$  denote the maximum possible number of edges in a graph which satisfies  $c_i \leq a_i$  where  $a_i$  is a nonnegative integer. Shi posed the problem of determining  $f(a_1, a_2, \dots, a_n)$  which extended the problem due to Erdős, it is clearly that  $f(n) = f(1, 1, \dots, 1)$  (see Xu and Shi[52]).

The lower bound  $f(0, 0, 2, \dots, 2)$  is given by Xu and Shi[52].

**Theorem 1.17 (Xu and Shi[52])** For  $n \geq 3$ ,

$$f(0, 0, 2, \dots, 2) \geq n - 1 + [(\sqrt{11n - 20})/2],$$

and the equality holds when  $3 \leq n \leq 10$ .

Given a graph  $H$ , what is the maximum number of edges of a graph with  $n$  vertices not containing  $H$  as a subgraph? This number is denoted  $ex(n, H)$ , and is known as the Turan number.

We denote by  $m_i(n)$  the numbers of cycles of length  $i$  in the complete graph  $K_n$  on  $n$  vertices. Obviously,

$$\begin{aligned} & ex(n, C_k) \\ &= f(0, 0, m_3(n), \dots, \\ & m_{k-1}(n), 0, m_{k+1}(n), \dots, m_n(n)) \\ &= f(0, 0, 2^{\frac{n(n-1)}{2}}, \dots, \\ & 2^{\frac{n(n-1)}{2}}, 0, 2^{\frac{n(n-1)}{2}}, \dots, 2^{\frac{n(n-1)}{2}}). \end{aligned}$$

Therefore, finding  $ex(n, C_k)$  is a special case of determining  $f(a_1, a_2, \dots, a_n)$ .

## Erdős conjecture 2

P. Erdős conjectured that there exists a positive constant  $c$  such that  $ex(n, C_{2k}) \geq cn^{1+1/k}$  (see Erdős[11]). Erdős [8] posed the problem of determining  $ex(n, C_4)$ .

Erdős [10] (published without proof) and Bondy and Simonovits [2] obtained that

**Theorem 2.1 (Erdős [10] and Bondy and Simonovits [2])**

$$ex(n, C_{2k}) \leq ckn^{1+1/k}$$

Wenger [49] proved the following:

**Theorem 2.2 (Wenger [49])**

$$ex(n, C_4) \geq \left(\frac{n}{2}\right)^{3/2},$$

$$ex(n, C_6) \geq \left(\frac{n}{2}\right)^{4/3},$$

$$ex(n, C_{10}) \geq \left(\frac{n}{2}\right)^{6/5}.$$

Füredi[18] proved that

**Theorem 2.3 (Füredi[18])** If  $q$  is a power of 2, then

$$ex(q^2 + q + 1, C_4) = q(q + 1)^2/2.$$

Füredi[19] also showed the following:

**Theorem 2.4 (Füredi[19])** Let  $G$  be a quadrilateral-free graph with  $e$  edges on  $q^2 + q + 1$  vertices, and suppose that  $q \geq 15$ . Then  $e \leq q(q + 1)^2/2$ .

**Corollary 2.5 (Füredi[19])** If  $q$  is a prime power greater than 13,  $n = q^2 + q + 1$ . Then

$$ex(n, C_4) = q(q + 1)^2/2.$$

Füredi, Naor and Verstraete[20] proved that

**Theorem 2.6 (Füredi, Naor and Verstraete[20])**

$$ex(n, C_6) > 0.5338n^{4/3}$$

for infinitely many  $n$  and

$$ex(n, C_6) < 0.6272n^{4/3}$$

if  $n$  is sufficiently large.

This refutes the Erdős-Simonovits conjecture in 1982 for hexagons (see [20]).

The survey article on this Erdős conjecture can be found in Chung [5].

There are not good sufficient and necessary conditions of when a graph on  $n$  vertices contains  $k$  cycle. For  $k = n$ , it is Hamiltonian problem, the survey article on Hamiltonian problem can be found in Gould [21,22].

### Hajós conjecture 3

An *eulerian graph* is a graph (not necessarily connected) in which each vertex has even degree. Let  $G$  be an eulerian graph. A *circuit decomposition* of  $G$  is a set of edge-disjoint circuits  $C_1, C_2, \dots, C_t$  such that  $E(G) = C_1 \cup C_2 \cup \dots \cup C_t$ . It is well known that every eulerian graph has a circuit decomposition. A natural question is to find the smallest number  $t$  such that  $G$  has a circuit decomposition of  $t$  circuits? Such smallest number  $t$  is called the *circuit decomposition number* of  $G$ , denoted by  $cd(G)$ . For each edge  $xy \in E(G)$ , let  $m(xy)$  be the number of edges between  $x$  and  $y$ . The *multiple number* of  $G$  is defined by  $m(G) = \sum_{uv \in E(G)} (m(uv) - 1)$ . A

reduction of a graph  $G$  is a graph obtained from  $G$  by recursively applying the following operations:

1. Remove the edges of circuit.
2. Delete an isolated vertex (0-vertex).
3. Delete a 2-vertex with two distinct neighbors and add a new edge joining its two neighbors.
4. If  $u$  is a 4-vertex with 4 distinct neighbors  $\{x, y, z, w\}$  such that  $xy \in E(G)$  and  $zw \notin E(G)$ , then delete  $u$  and join  $x$  and  $y$  with a new parallel edge, and add a new edge between  $z$  and  $w$ .

A reduction is proper if it is not the original graph (see [16]). The following conjecture is due to Hajós (see [36]).

**Hajós conjecture:**

$$cd(G) \leq \frac{|V(G)|}{2}$$

for every simple eulerian graph  $G$ .

Lovasz [36] proved the following:

**Theorem 3.1 (Lovasz [36])** A graph of  $n$  vertices can be covered by  $\leq \lfloor n/2 \rfloor$  disjoint paths and circuits.

Jiang [26] and Seyffarth [40] considered planar eulerian graphs.

**Theorem 3.2 (Jiang [26] and Seyffarth [40])**  $cd(G) \leq \frac{|V(G)|-1}{2}$  for every simple planar eulerian graph  $G$ .

Granville and Moisiadis [23] and Favaron and Kouider [17] extended to multigraphs.

**Theorem 3.3 (Granville and Moisiadis [23] and Favaron and Kouider [17])** If  $G$  is an even multigraph of order  $n$ , of size  $m$ , with  $\Delta(G) \leq 4$ , then  $cd(G) \leq \frac{n+M-1}{2}$  where  $M = m - m^*$  and  $m^*$  is the size of the simple graph induced by  $G$ .

Fan and Xu [16] proved that:

**Theorem 3.4 (Fan and Xu [16])** If  $G$  is an eulerian graph with

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2}$$

then  $G$  has a reduction  $H$  such that

$$cd(H) > \frac{|V(H)| + m(H) - 1}{2}$$

and the number of vertices of degree less than six in  $H$  plus  $m(H)$  is at most one.

**Corollary 3.5 (Fan and Xu [16])** Hajós conjecture is valid for projective graphs.

**Corollary 3.6 (Fan and Xu [16])** Hajós conjecture is valid for  $K_6^-$  minor free graphs.

Xu[50] also proved the following two results:

**Theorem 3.7 (Xu[50])** If  $G$  is an eulerian graph with

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2}$$

such that

$$cd(H) \leq \frac{|V(H)| + m(H) - 1}{2}$$

for each proper reduction of  $G$ , then  $G$  is 3-connected. Moreover, if  $S = \{x, y, z\}$  is a 3-cut of  $G$ , letting  $G_1$  and  $G_2$  be the two induced subgraph of  $G$  such that  $V(G_1) \cap V(G_2) = S$  and  $E(G_1) \cup E(G_2) = E(G)$ , then either  $S$  is not an independent set, or  $G_1$  and  $G_2$  are both eulerian graphs.

**Corollary 3.8 (Xu[50])** *To prove Hajós' conjecture, it suffices to prove*

$$cd(G) \leq \frac{|V(G)| + m(G) - 1}{2}$$

for every 3-connected eulerian graph  $G$ .

Fan [14] proved that

**Theorem 3.9 (Fan [14])** Every eulerian graph on  $n$  vertices can be covered by at most  $\lfloor \frac{n-1}{2} \rfloor$  circuits such that each edge is covered an odd number of times.

This settles a conjecture made by Chung in 1980(see[14]).

Xu and Wang[51] gave the following result:

**Theorem 3.10 (Xu and Wang[51])** *The edge set of each even toroidal graph can be decomposed into at most  $(n + 3)/2$  circuits in  $O(mn)$  time, where a toroidal graph is a graph embedable on the torus.*

**Theorem 3.11 (Xu and Wang[51])** *The edge set of each toroidal graph can be decomposed into at most  $3(n - 1)/2$  circuits and edges in  $O(mn)$  time.*

**We do not think Hajós conjecture is true.**

By the proof of Lemma 3.3 in N. Dean [What is the smallest number of dicycles in a dicycle decomposition of an Eulerian digraph? J. Graph Theory 10 (1986), no. 3, 299–308.], if exists  $k$  vertices counterexample, then will exist  $ik - i + 1 (i = 2, 3, 4, \dots)$  vertices counterexamples.

A related problem was conjectured by Gallai (see [36]):

**Conjecture 3.12 (Gallai's conjecture)** Every simple connected graph on  $n$  vertices can be decomposed into at most  $(n + 1)/2$  paths.

Lovasz [36] proved that

**Theorem 3.13 (Lovasz [36])** If a graph has  $u$  odd vertices and  $g$  even vertices ( $g \geq 1$ ), then it can be covered by  $u/2 + g - 1$  disjoint paths.

**Theorem 3.14 (Lovasz [36])** Let a locally finite graph have only odd vertices. Then it can be covered by disjoint finite paths so that every vertex is the endpoint of just one covering path.

The path number of a graph  $G$ , denoted  $p(G)$ , is the minimum number of edge-disjoint paths covering the edges of  $G$ . Donald [7] proved the following:

**Theorem 3.15 (Donald [7])** If a graph with  $u$  vertices of odd degree and  $g$  nonisolated vertices of even degree. Then

$$p(G) \leq u/2 + \lceil \frac{3}{4}g \rceil \leq \lceil \frac{3}{4}n \rceil.$$

Pyber [37] proved that

**Theorem 3.16 (Pyber [37])** A graph  $G$  of  $n$  vertices can be covered by  $n - 1$  circuits and edges.

**Theorem 3.17 (Pyber [37])** Let  $G$  be a graph of  $n$  vertices and  $\{C_1, \dots, C_k\}$  be a set of circuits and edges such that  $\cup_{i=1}^k E(C_i) = E(G)$  and  $k$  is minimal. Then we can choose  $k$  different edges,  $e_i \in E(C_i)$ , such that these edges form a forest in  $G$ .

**Theorem 3.18 (Pyber [37])** Let  $G$  be a graph of  $n$  vertices not containing  $C_4$ . Then  $G$  can be covered by  $\lceil (n - 1)/2 \rceil$  circuits and  $n - 1$  edges.

Pyber [38] proved that

**Theorem 3.19 (Pyber [38])** Every connected graph  $G$  on  $n$  vertices can be covered by  $n/2 + O(n^{3/4})$  paths.

**Theorem 3.20 (Pyber [38])** Every connected graph on  $n$  vertices with  $e$  edges can be covered by  $n/2 + 4(e/n)$  paths.

Reed [39] proved that

**Theorem 3.21 (Reed [39])** Any connected cubic graph  $G$  of order  $n$  can be covered by  $\lceil n/9 \rceil$  vertex disjoint paths.

Dean, Kouider [6] and Yan [53] proved that

**Theorem 3.22 (Dean, Kouider [6] and Yan [53])** If a graph(possibly disconnected) with  $u$  vertices of odd degree and  $g$  nonisolated vertices of even degree. Then

$$p(G) \leq u/2 + \lceil \frac{2}{3}g \rceil.$$

Fan [13] proved that

**Theorem 3.23 (Fan [13])** Every connected graph on  $n$  vertices can be covered by at most  $\lceil n/2 \rceil$  paths.

This settles a conjecture made by Chung in 1980(see[13]).

**Theorem 3.24 (Fan [13])** Every 2-connected graph on  $n$  vertices can be covered by at most  $\lfloor \frac{2n-1}{3} \rfloor$  circuits.

This settles a conjecture made by Bondy in 1990(see[13]).

**Corollary 3.25 (Fan [13])** Let  $G$  be a 2-edge-connected graph on  $n$  vertices. Then  $G$  can be covered by at most  $\lfloor \frac{3(n-1)}{4} \rfloor$  circuits.

Fan [15] define a graph operation, called  $\alpha$ -operation and proved that

**Definition 3.26(Fan [15])** Let  $H$  be a graph. A pair  $(S, y)$ , consisting of an independent set  $S$  and a vertex  $y \in S$ , is called an  $\alpha$ -pair if the following holds: for every vertex  $v \in S \setminus \{y\}$ , if  $d_H(v) \geq 2$ , then (a)  $d_H(u) \leq 3$  for all  $u \in N_H(v)$  and (b)  $d_H(u) = 3$  for at most two vertices  $u \in N_H(v)$ . (That is, all the neighbors of  $v$  have degree at most 3, at most two of which have degree exactly 3.) An  $\alpha$ -operation on  $H$  is either (i) add an isolated vertex or (ii) pick an  $\alpha$ -pair  $(S, y)$  and add a vertex  $x$  joined to each vertex of  $S$ , in which case the ordered triple  $(x, S, y)$  is called the  $\alpha$ -triple of the  $\alpha$ -operation.

**Definition 3.27(Fan [15])** An  $\alpha$ -graph is a graph that can be obtained from the empty set via a sequence of  $\alpha$ -operations.

**Theorem 3.28 (Fan [15])** Let  $G$  be a graph on  $n$  vertices ( not necessarily connected). The E-subgraph of  $G$  is the subgraph induced by the vertices of even degree in  $G$ . If the E-subgraph of  $G$  is an  $\alpha$ -graph, then  $G$  can be decomposed into  $\lfloor n/2 \rfloor$  paths.

**Corollary 3.29 (Fan [15])** Let  $G$  be a graph on  $n$  vertices ( not necessarily connected). If each block of the E-subgraph of  $G$  is a triangle-free graph with maximum degree at most 3, then  $G$  can be decomposed into  $\lfloor n/2 \rfloor$  paths.

Harding and McGuinness [24] proved that

**Theorem 3.30 (Harding and McGuinness [24])** For every simple graph  $G$  have girth  $g \geq 4$ , with  $u$  vertices of odd degree and  $w$  nonisolated vertices of even degree, there is a path-decomposition having at most  $u/2 + \lfloor \frac{g+1}{2g} w \rfloor$  paths.

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