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Note

On the size of graphs with all cycle having distinct length

Chunhui Lai
Department of Mathematics, Zhangzhou Teachers College, Zhangzhou, Fujian 363000, China

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Abstract

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the question of determining $f(n)$ (see Bondy and Murty (1976)). In this note, we prove that for $n \geqslant 36 \cdot 5t^2 - 4 \cdot 5t + 1$ one has $f(n) \geqslant n + 9t - 1$. We conjecture that $\underline{\lim}(f(n)-n)/\sqrt{n} \leqslant \sqrt{3}$.

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdős raised the question of determining $f(n)$ (see 1, p. 247, Problem 11]). In 1988, Shi [2] proved that

$$f(n) \geqslant n + [(\sqrt{8n-23}+1)/2] \quad \text{for all } n \geqslant 3.$$

He conjectured that

$$f(n) = n + [(\sqrt{8n-23}+1)/2] \quad \text{for all } n \geqslant 3,$$

and he proved that this conjecture for $3 \leqslant n \leqslant 17$ is true. In 1989 and 1990, *I* [3,4] proved that

$$n + (4/15)\sqrt{30n} - C \leqslant f(n) < n - 2 + \sqrt{n \cdot \ln(4n/(2m+3)) + 2n} + \log_2(n+6)$$

for all $n \geqslant e^{2m}(2m+3)/4$ ($m = 1, 2, 3, \dots$), and the conjecture of Yongbing Shi is not true. In this note n, t, i are all integers.

Correspondence to: Chunhui Lai, Department of Mathematics, Zhangzhou Teachers College, Zhangzhou, Fujian 363000, China.

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Theorem. *Let* $t \geqslant 1$, *then* $f(n) \geqslant n+9t-1$ *for* $n \geqslant 36 \cdot 5t^2 - 4 \cdot 5t + 1$.

Proof. We give an example G consisting of 2-connected blocks, $B_i, (1 \leqslant i \leqslant 7t)$. These blocks all have a common vertex x, otherwise their vertex sets are pairwise disjoint. For $i \leqslant 6t$ B_i is simply the cycle of length i. The block B_{6t+i} (for $1 \leqslant i \leqslant t$) is obtained from a cycle

$$C_{13t+3i} = xx_{1,i}x_{2,i} \cdots x_{13t+3i-1,i}x$$

such that the vertex x is connected to $x_{4t+i,i}$ by a path

$$xx_{13t+3i,i}x_{13t+3i+1,i} \cdots x_{15t+3i-2,i}x_{4t+i,i},$$

and by another path

$$xx_{15t+3i-1,i}x_{15t+3i,i} \cdots x_{17t+3i-3,i}x_{7t+2i,i}$$

to the vertex $x_{7t+2i,i}$. Here B_{6t+i} contains cycles of lengths $6t+i, 7t+i, 8t+i, 9t+2i, 11t+2i$, and $13t+3i$. Then the theorem easily follows from the inequality $f(n+1) \geqslant f(n)+1$. $\square$

From the theorem of this paper we have

$$\underline{\lim}(f(n)-n)/\sqrt{n} \geqslant \sqrt{2+16/73}.$$

We may make the following conjecture.

Conjecture. $\underline{\lim}(f(n)-n)/\sqrt{n} \leqslant \sqrt{3}$.

References

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