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Quasi-range-preserving Operator *)

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Abstract This paper introduces a new concept, called quasi-range-preserving operator, and gives necessary and sufficient conditions for a linear operator to be quasi-range-preserving. As a special case of its corollary Glicksberg's problem for special case is affirmatively answered.

Key Words and Phrases Quasi-range-preserving Operator; Convex Closed Hull; Uniform Algebra; Glicksberg's Problem

Glicksberg[1] asked whether any proper uniform algebra A on a compact Hausdorff space X (i. e., $C(X)\backslash A$ is nonempty) is uncomplemented, that is, whether there is no bounded projection from $C(X)$ onto A. Glicksberg, Rosenthal, Pelczyński, Kisljakov, Etcheberry and other mathematicians' work on this problem has been summarized in Pelczyński's monograph (see [2]). Sidney[3], Kislyakov[4] and Lai[5, 6] have obtained some results too.

For studying this problem, I introduce a new concept of quasi-range-preserving operator. Let X, Y be topological spaces, and Z a normed linear space with $Z\neq\{0\}$. By $M(X\to Z)$ we denote the normed linear space of all bounded mappings from X to Z with the norm $\|f\|=\sup\limits_{x\in X}\|f(x)\|$. The constant mapping $f(x)\equiv z$ is identified with z. Let A be a linear subspace of $M(X\to Z)$ which contains the constant mappings, and T a linear operator from A to $M(Y\to Z)$. T is called a quasi-range-preserving operator if $R(Tf)\subset$the convex closed hull of $R(f)$ (here, $R(f)=\{f(x)\,|\,x\in X\}$, $R(Tf)=\{Tf(y)\,|\,y\in Y\}$), for all $f\in A$. We prove that (1) if Z is an inner product space, then a necessary and sufficient condition for T to be a quasi-range-preserving operator is: $\|T\|=1$, $Tz=z$, for all $z\in Z$; (2) if Z is a normed linear space with the norm $\|\cdot\|$, then a necessary and sufficient condition for T to be a quasi-range-preserving operator is: $\|T\|'=1$, $Tz=z$ for all norms $\|\cdot\|'$ which are equivalent to $\|\cdot\|$ and for all $z\in Z$. As a special case of its corollary Glicksberg's problem for special case is affirmatively answered.

Theorem 1 If T is a quasi-range-preserving operator, then $\|T\|=1$, $Tz=z$ for all $z\in Z$.

Proof Suppose $\|f\|\leqslant 1$, $f\in A$. Since $\{z\,|\,\|z\|\leqslant 1,\ z\in Z\}$ is a convex closed set and T is a quasi-range-preserving operator, we have $R(Tf)\subset$the convex closed hull of $R(f)\subset\{z\,|\,\|z\|\leqslant 1,\ z\in Z\}$. Hence $\|Tf\|\leqslant 1$. Since $\{z\}$ is a convex closed set, we have $Tz=z$ for all $z\in Z$. Hence

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$\|T\|=1$.

Theorem 2 Let H be a real (complex) Hilbert space with $H\neq\{0\}$, A be a linear subspace of $M(X\to H)$ which contains the constant mappings, and T be a linear operator from A to $M(Y\to H)$. Then a necessary and sufficient condition for T to be a quasi-range-preserving operator is

$$\|T\|=1, \quad Th=h \qquad \text{for all } h\in H.$$

Proof Since a complex Hilbert space can be regarded as a real Hilbert space, without loss of generality we may assume that H is a real Hilbert space.

(1) By Theorem 1, the condition is necessary.

(2) Sufficiency. Suppose T is not a quasi-range-preserving operator. Then there is an $f\in A$ such that $R(Tf)\not\subset$ the convex closed hull of $R(f)\overset{\text{d.f.}}{=\!=}S$, that is, there is $y_0\in Y$, such that $(Tf)(y_0)\notin S$. It is easy to prove that there is a bounded linear functional g and a constant b such that

$$g[(Tf)(y_0)]>b\geqslant g(s) \qquad \text{for all } s\in S.$$

Then there is $h_1\in H$ such that

$$(h,h_1)=g(h) \qquad \text{for all } h\in H.$$

Hence

$$((Tf)(y_0),h_1)>b\geqslant(s,h_1) \qquad \text{for all } s\in S.$$

Since $\{h_1/\|h_1\|\}$ is a normal orthogonal system of H, there is a complete normal orthogonal system E of H such that $E\supset\{h_1/\|h_1\|\}$.

Let

$$k>\max\left\{\frac{\frac{b^2}{\|h_1\|^2}+\|f\|^2-\|(Tf)(y_0)\|^2}{\frac{2}{\|h_1\|}[((Tf)(y_0),h_1)-b]},\ \sup_{s\in S}\left|\left(s,\frac{h_1}{\|h_1\|}\right)\right|\right\}.$$

Then

$$0<\left(f(x),\frac{h_1}{\|h_1\|}\right)+k=\left(f(x)+k\frac{h_1}{\|h_1\|},\frac{h_1}{\|h_1\|}\right)\leqslant\frac{b}{\|h_1\|}+k.$$

Hence

$$\left[\left(f(x)+k\frac{h_1}{\|h_1\|},\frac{h_1}{\|h_1\|}\right)\right]^2\leqslant\left(\frac{b}{\|h_1\|}+k\right)^2.$$

Since

$$\left\|T\left(f+k\frac{h_1}{\|h_1\|}\right)\right\|\geqslant\left\|(Tf)(y_0)+k\frac{h_1}{\|h_1\|}\right\|$$
$$=\left[\|(Tf)(y_0)\|^2+k^2+\frac{2k}{\|h_1\|}((Tf)(y_0),h_1)\right]^{\frac{1}{2}},$$
$$\left\|f+k\frac{h_1}{\|h_1\|}\right\|=\sup_{x\in X}\left\{\left[\left(f(x)+k\frac{h_1}{\|h_1\|},\frac{h_1}{\|h_1\|}\right)\right]^2+\sum_{\substack{e\in E\\ e\neq h_1/\|h_1\|}}(f(x),e)^2\right\}^{\frac{1}{2}}$$

$$\leqslant\left[\left(\frac{b}{\|h_1\|}+k\right)^2+\|f\|^2\right]^{\frac{1}{2}}<\left\|T\left(f+k\frac{h_1}{\|h_1\|}\right)\right\|,$$

one gets

$$\|T\|>1.$$

This contradicts $\|T\|=1$.

Theorem 3 Let Z be a real (complex) inner product space with $Z\neq\{0\}$, A be a linear subspace of $M(X\to Z)$ which contains the constant mappings, and T be a linear operator from A to $M(Y\to Z)$. Then a necessary and sufficient condition for T to be a quasi-range-preserving operator is

$$\|T\|=1,\quad Tz=z\qquad \text{for all } z\in Z.$$

Proof (1) By Theorem 1, we obtain the necessity.

(2) Sufficiency. Without loss of generality we may assume that Z is a real inner product space. We denote the completion of Z by H. Then $Z\subset H$. Hence A can be regarded as a linear subspace of $M(X\to H)$.

Let A' denote the linear subspace spaned by $A\cup H$. For any $\varphi=af+bh$, where a, $b\in R$ (all real numbers), $f\in A$, and $h\in H$, in A' we define $T'(\varphi)=aTf+bh$. It is easy to prove that $T'(\varphi)$ has nothing to do with the choice of a, b, f, h. Obviously, $T'h=h$ for all $h\in H$. For any a, $b\in R$, $f\in A$, and $h\in H$, there are $z_n\in Z$ such that $\lim\limits_{n\to\infty} z_n=h$. Thus

$$\|T'(af+bh)\|=\|aTf+bh\|=\|aTf+b\lim_{n\to\infty}z_n\|$$

$$=\|\lim_{n\to\infty}T(af+bz_n)\|\leqslant\overline{\lim_{n\to\infty}}\|T\|\|af+bz_n\|=\|af+bh\|.$$

Hence

$$\|T'\|=1.$$

By Theorem 2, T' is a quasi-range-preserving operator. Hence for any f in A

$$R(Tf)=R(T'f)\cap Z\subset\{\text{the convex closed hull of } R(f) \text{ in } H\}\cap Z$$

$$=\text{the convex closed hull of } R(f) \text{ in } Z,$$

that is, T is a quasi-range-preserving operator.

Theorem 4 Let Z be a real (complex) inner product space with $Z\neq\{0\}$, a be a real (complex) constant, A be a linear subspace of $M(X\to Z)$ which contains the constant mappings, and T be a linear operator from A to $M(Y\to Z)$. Then a necessary and sufficient condition for $R(Tf)\subset a\,\{\text{the convex closed hull of } R(f)\}\overset{\text{d. f.}}{=\!=}\{az\,|\,z\in \text{the convex closed hull of } R(f)\}\ \forall\, f\in A$ is

$$\|T\|=|a|,\quad Tz=az\qquad \text{for all } z\in Z.$$

Proof I. If $a=0$, Theorem 4 is trivial.

II. For $a\neq0$, since T is a linear operator, so is $\frac{1}{a}T$. Applying Theorem 3 to $\frac{1}{a}T$ we can obtain Theorem 4.

Theorem 5 Let Z be a real (complex) normed linear space with the norm $\|\cdot\|$ and $Z\neq\{0\}$, A be a linear subspace of $M(X\to Z)$ which contains the constant mappings, and T be a linear operator from A to $M(Y\to Z)$. Then a necessary and sufficient condition for T to be a quasi-range-preserving operator is

$$\|T\|'=1, \qquad Tz=z$$

for all norms $\|\cdot\|'$ which are equivalent to $\|\cdot\|$ and for all $z\in Z$.

Proof (1) The necessity is obvious.

(2) Sufficiency. Without loss of generality we may assume that Z is a real normed linear space.

Suppose T is not a quasi-range-preserving operator. Then there is $f\in A$ such that $R(Tf)\not\subset$ the convex closed hull of $R(f)\overset{\text{d. f.}}{=\!=}S$, that is, there is $y_0\in Y$, such that $(Tf)(y_0)\notin S$. Hence there is a bounded linear functional g and a constant b such that

$$g[(Tf)(y_0)]>b\geqslant g(s) \qquad \text{for all } s\in S.$$

Thus there is a positive integer n such that $g[(Tf)(y_0)]>b+\frac{1}{n}$. It is easy to prove that there is $z_0\in Z$ such that $g(z_0)=\inf_{s\in S} g(s)$. Then

$$g(s)\geqslant g(z_0) \qquad \text{for all } s\in S,$$

that is,

$$g(s)-g(z_0)\geqslant 0.$$

Therefore

$$\begin{aligned} g[(Tf)(y_0)-z_0] &> b+\frac{1}{n}-g(z_0)\geqslant g(s)-g(z_0)+\frac{1}{n}\\ &=\frac{1}{n}+|g(s)-g(z_0)|=|g(s-z_0)|+\frac{1}{n}\\ &=|g(-(s-z_0))|+\frac{1}{n}. \end{aligned}$$

For $t_1, t_2, \cdots, t_m\left(\sum_{i=1}^{m}t_i=1\right)$ in the closed interval $[0,1]$ and $p_1, p_2, \cdots, p_m$ in the set $(S-z_0)\cup[-(S-z_0)]$

$$\begin{aligned} \left|g\left(\sum_{i=1}^{m}t_ip_i\right)\right|+\frac{1}{n} &\leqslant \sum_{i=1}^{m}t_i\left(|g(p_i)|+\frac{1}{n}\right)\\ &\leqslant \sum_{i=1}^{m}t_ig((Tf)(y_0)-z_0)=g((Tf)(y_0)-z_0). \end{aligned}$$

Hence, for any d in the convex closed hull of $(S-z_0)\cup[-(S-z_0)]\overset{\text{d. f.}}{=\!=}D$

$$g((Tf)(y_0)-z_0)\geqslant |g(d)|+\frac{1}{n}>g(d).$$

Therefore, $(Tf)(y_0)-z_0\notin D$. It is clear that if $z\in D$ then $-z\in D$. Hence $0=\frac{1}{2}z+\frac{1}{2}(-z)\in D$. Since D is closed, we have $d((Tf)(y_0)-z_0, D)>0$ ($d((Tf)(y_0)-z_0, D)$ is the distance of D and $(Tf)(y_0)-z_0$). Let W be all the points whose distance from D is less than or equal to $\frac{1}{2}d((Tf)(y_0)-z_0, D)$. It is clear that the points whose distance from 0 is less than or equal to

$\frac{1}{2}d((Tf)(y_0)-z_0, D)$ are all in W, and $(Tf)(y_0)-z_0 \notin W$. It is easy to prove that W is a balanced absorbing bounded convex closed set which contains D. Hence the Minkowski functional of W

$$P(z) = \inf\{a \mid a > 0,\ a^{-1}z \in W\}$$

is a norm of Z, denoted by $\|\cdot\|'$. It is clear that $\|\cdot\|'$ is equivalent to $\|\cdot\|$, and $w \in W$ if and only if $\|w\|' \leqslant 1$. Since

$$f(x) - z_0 \in D \subset W \qquad \text{for all } x \in X,$$

we have

$$\|f(x) - z_0\|' \leqslant 1.$$

Since $\|T\|' = 1$, we have

$$\|T(f - z_0)\|' \leqslant \|T\|' \|f - z_0\|' \leqslant 1.$$

Thus

$$\|(Tf)(y_0) - z_0\|' \leqslant \|Tf - z_0\|' = \|T(f - z_0)\|' \leqslant 1.$$

Then $(Tf)(y_0)-z_0 \in W$. This contradicts $(Tf)(y_0)-z_0 \notin W$.

Theorem 6 Let Z be a real (complex) normed linear space with the norm $\|\cdot\|$ and $Z \neq \{0\}$, a be a real (complex) constant, A be a linear subspace of $M(X \to Z)$ which contains the constant mappings, and T be a linear operator from A to $M(Y \to Z)$. Then a necessary and sufficient condition for $R(Tf) \subset a\{$the convex closed hull of $R(f)\}$ $\forall f \in A$ is

$$\|T\|' = |a|, \qquad Tz = az$$

for all norms $\|\cdot\|'$ which are equivalent to $\|\cdot\|$ and for all $z \in Z$.

Proof I. If $a=0$, Theorem 6 is trivial.

II. For $a \neq 0$, since T is a linear operator, so is $\frac{1}{a}T$. Applying Theorem 5 to $\frac{1}{a}T$ we can obtain Theorem 6.

Corollary Let B be a selfadjoint (that is, if $f \in B$, then the complex conjugate function (of f) $\bar{f} \in B$) linear subspace of $M(X \to C)$ (C denotes all complex numbers) which contains the constant functions, A be a linear subspace of $M(Y \to C)$ which contains the constant functions, and T be a linear operator from B onto A. If there is a non-zero constant a such that

$$\|T\| = |a|, \qquad T1 = a,$$

then A is selfadjoint.

Proof It is clear that $\frac{1}{a}T$ is a linear operator from B onto A and

$$\left\|\frac{1}{a}T\right\| = 1, \qquad \left(\frac{1}{a}T\right)(1) = 1.$$

By Theorem 2, $\frac{1}{a}T$ is a quasi-range-preserving operator. Since a straight line segment is a convex closed set, $\frac{1}{a}T$ maps a real valued function into a real valued function. Hence, for any $f \in B$, $\frac{1}{a}T$ maps the complex conjugate function (of f) $\bar{f}$ into $\overline{\left(\frac{1}{a}T\right)(f)}$. Thus A is selfadjoint.

As a special case of the corollary we assume that $Y=X$ and X is a compact Hausdorff space, $a=1$, $B=C(X)$, and A is a proper uniform algebra of $C(X)$. We know that A is not selfadjoint. Hence there is no projection from $C(X)$ onto A with norm 1.

References

[1] Glicksberg, I., Some uncomplemented function algebras, *Trans. Amer. Math. Soc.*, **111**(1964), 121—137.

[2] Pelczyński, A., Banach space of analytic functions and absolutely summing operators, *CBMS Regional Conf. Ser. Math.*, No. 30, Amer. Math. Soc., Providence, R. I., 1977, pp. 28—35.

[3] Sidney, S. J., Uniform algebras and projections, *Proc. Amer. Math. Soc.*, **91**(1984), 381—382.

[4] Kislyakov, S. V., Proper uniform algebras are uncomplemented, *Dokl. Akad. Nauk SSSR*, **309**(4)(1989), 795—798.

[5] Lai Chunhui, On a problem of Glicksberg concerning uniform algebras, *J. Zhangzhou Teachers College* (Nat. Sci.), (1)(1987), 101—103, 96.

[6] Lai Chunhui, Interpolation set of weak function algebra and function algebra, *J. Xiamen Univ.* (Nat. Sci.), **28**(1989), 138—141.