ON THE SIZE OF GRAPHS WITHOUT REPEATED CYCLE LENGTHS

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Abstract

In 1975, P. Erdös proposed the problem of determining the maximum number $f(n)$ of edges in a graph of n vertices in which any two cycles are of different lengths. In this paper, it is proved that

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $t = 1260r + 169 \ \ (r \geq 1)$ and $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$. Consequently, $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2 + \frac{7654}{19071}}$, which is better than the previous bounds $\sqrt{2}$ [Y. Shi, Discrete Math. 71(1988), 57-71], $\sqrt{2.4}$ [C. Lai, Australas. J. Combin. 27(2003), 101-105]. The conjecture $\lim_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} = \sqrt{2.4}$ is not true.

1 Introduction

Let $f(n)$ be the maximum number of edges in a graph on n vertices in which no two cycles have the same length. In 1975, Erdös raised the problem of determining $f(n)$ (see Bondy and Murty [1], p.247, Problem 11). Shi[14] proved a lower bound.

Theorem 1 (Shi[14])

$$f(n) \geq n + [(\sqrt{8n-23} + 1)/2]$$

for $n \geq 3$.

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Chen, Lehel, Jacobson, and Shreve [3], Jia[5], Lai[6,7,8], Shi[15,17,18,19] obtained some results.

Boros, Caro, Füredi and Yuster[2] proved an upper bound.

Theorem 2 (Boros, Caro, Füredi and Yuster[2]) For n sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [9] improved the lower bound.

Theorem 3 (Lai [9])

$$f(n) \geq n + \sqrt{2.4}\sqrt{n}(1 - o(1))$$

and proposed the following conjecture:

Conjecture 4 (Lai [9])

$$\lim_{n\to\infty} \frac{f(n) - n}{\sqrt{n}} = \sqrt{2.4}.$$

Lai [6] raised the following Conjecture:

Conjecture 5 (Lai [6])

$$\liminf_{n\to\infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$

Markström [12] raised the following problem:

Problem 6 (Markström [12]) Determine the maximum number of edges in a hamiltonian graph on n vertices with no repeated cycle lengths.

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on n vertices in which no two cycles have the same length. The result can be found in [2,3,14].

The survey article on this problem can be found in Tian[20], Zhang[23], Lai and Liu [10].

The progress of all 50 problems in [1] can be find in Stephen C. Locke, Unsolved problems: http://math.fau.edu/locke/Unsolved.htm

A related topic concerns Entringer problem. Determine which simple graph G have exactly one cycle of each length l, $3 \leq l \leq v$ (see problem 10 in [1]), this problem was posed in 1973 by R. C. Entringer. For the developments on this topic, see[4,11,12,13,16,21,22].

In this paper, we construct a graph G having no two cycles with the same length which leads the following result.

Theorem 7 Let $t = 1260r + 169$ $(r \geq 1)$, then

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$. The Conjecture 4 is not true.

2 Proof of the theorem 7

Proof. Let $t = 1260r + 169, r \geq 1$, $n_t = \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$, $n \geq n_t$. We shall show that there exists a graph G on n vertices with $n + \frac{107}{3}t + \frac{7}{3}$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq 20t, 27t \leq i \leq 28t + 64, 29t - 734 \leq i \leq 29t + 267, 30t - 531 \leq i \leq 30t + 57, 31t - 741 \leq i \leq 31t + 58, 32t - 740 \leq i \leq 32t + 57, 33t - 741 \leq i \leq 33t + 57, 34t - 741 \leq i \leq 34t + 52, 35t - 746 \leq i \leq 35t + 60, 36t - 738 \leq i \leq 36t + 60, 37t - 738 \leq i \leq 37t + 799, i = 20t + j (1 \leq j \leq \frac{t-7}{6}), i = 20t + \frac{t-1}{6} + j (1 \leq j \leq \frac{t-7}{6}), i = 21t + 2j + 1 (0 \leq j \leq t - 1), i = 21t + 2j (0 \leq j \leq \frac{t-1}{2}), i = 23t + 2j + 1 (0 \leq j \leq \frac{t-1}{2})$, and $i = 20t + \frac{t-1}{6}$, $i = 20t + \frac{t-1}{3} + \frac{t-1}{6} - 1$, $i = 20t + \frac{t-1}{3} + \frac{t-1}{6}$, $i = 20t + \frac{2t-2}{3}$, $i = 21t - 2$, $i = 21t - 1$, $i = 26t)$.

Now we define these B_i's. These subgraphs all have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq \frac{t-7}{6}$, let the subgraph B_{20t+i} consist of a cycle

$$xa_i^1 a_i^2 ... a_i^{\frac{62t-8}{3}+2i} x$$

and a path:

$$xa_{i,1}^1 a_{i,1}^2 ... a_{i,1}^{\frac{59t-5}{6}} a_i^{\frac{61t-1}{6}+i}$$

Based the construction, B_{20t+i} contains exactly three cycles of lengths:

$$20t + i, 20t + \frac{t-1}{3} + i - 1, 20t + \frac{2t-2}{3} + 2i - 1.$$

For $1 \leq i \leq \frac{t-7}{6}$, let the subgraph $B_{20t+\frac{t-1}{6}+i}$ consist of a cycle

$$xb_i^1 b_i^2 ... b_i^{\frac{62t-5}{3}+2i} x$$

and a path:

$$xb_{i,1}^1 b_{i,1}^2 ... b_{i,1}^{10t-1} b_i^{\frac{61t-1}{6}+i}$$

Based the construction, $B_{20t+\frac{t-1}{6}+i}$ contains exactly three cycles of lengths:

$$20t + \frac{t-1}{6} + i, 20t + \frac{t-1}{3} + \frac{t-1}{6} + i, 20t + \frac{2t-2}{3} + 2i.$$

For $0 \leq i \leq t - 1$, let the subgraph $B_{21t+2i+1}$ consist of a cycle

$$xu_i^1 u_i^2 ... u_i^{25t+2i-1} x$$

and a path:

$$xu_{i,1}^1 u_{i,1}^2 ... u_{i,1}^{(19t+2i-1)/2} u_i^{(23t+2i+1)/2}$$

Based the construction, $B_{21t+2i+1}$ contains exactly three cycles of lengths:

$$21t+2i+1, 23t+2i, 25t+2i.$$

For $0 \leq i \leq \frac{t-3}{2}$, let the subgraph B_{21t+2i} consist of a cycle

$$xv_i^1v_i^2...v_i^{25t+2i}x$$

and a path:

$$xv_{i,1}^1v_{i,1}^2...v_{i,1}^{9t+i-1}v_i^{12t+i}$$

Based the construction, B_{21t+2i} contains exactly three cycles of lengths:

$$21t+2i, 22t+2i+1, 25t+2i+1.$$

For $0 \leq i \leq \frac{t-3}{2}$, let the subgraph $B_{23t+2i+1}$ consist of a cycle

$$xw_i^1w_i^2...w_i^{26t+2i+1}x$$

and a path:

$$xw_{i,1}^1w_{i,1}^2...w_{i,1}^{(21t+2i-1)/2}w_i^{(25t+2i+1)/2}$$

Based the construction, $B_{23t+2i+1}$ contains exactly three cycles of lengths:

$$23t+2i+1, 24t+2i+2, 26t+2i+2.$$

For $58 \leq i \leq t-742$, let the subgraph $B_{27t+i-57}$ consist of a cycle

$$C_{27t+i-57} = xy_i^1y_i^2...y_i^{132t+11i+893}x$$

and ten paths sharing a common vertex x, the other end vertices are on the cycle $C_{27t+i-57}$:

$$xy_{i,1}^1y_{i,1}^2...y_{i,1}^{(17t-1)/2}y_i^{(37t-115)/2+i}$$
$$xy_{i,2}^1y_{i,2}^2...y_{i,2}^{(19t-1)/2}y_i^{(57t-103)/2+2i}$$
$$xy_{i,3}^1y_{i,3}^2...y_{i,3}^{(19t-1)/2}y_i^{(77t+315)/2+3i}$$
$$xy_{i,4}^1y_{i,4}^2...y_{i,4}^{(21t-1)/2}y_i^{(97t+313)/2+4i}$$
$$xy_{i,5}^1y_{i,5}^2...y_{i,5}^{(21t-1)/2}y_i^{(117t+313)/2+5i}$$
$$xy_{i,6}^1y_{i,6}^2...y_{i,6}^{(23t-1)/2}y_i^{(137t+311)/2+6i}$$
$$xy_{i,7}^1y_{i,7}^2...y_{i,7}^{(23t-1)/2}y_i^{(157t+309)/2+7i}$$
$$xy_{i,8}^1y_{i,8}^2...y_{i,8}^{(25t-1)/2}y_i^{(177t+297)/2+8i}$$
$$xy_{i,9}^1y_{i,9}^2...y_{i,9}^{(25t-1)/2}y_i^{(197t+301)/2+9i}$$

$$xy^1_{i,10}y^2_{i,10}\cdots y^{(27t-1)/2}_{i,10}y_i^{(217t+305)/2+10i}.$$

As a cycle with d chords contains $\binom{d+2}{2}$ distinct cycles, $B_{27t+i-57}$ contains exactly 66 cycles of lengths:

$27t+i-57$,	$28t+i+7$,	$29t+i+210$,	$30t+i$,
$31t+i+1$,	$32t+i$,	$33t+i$,	$34t+i-5$,
$35t+i+3$,	$36t+i+3$,	$37t+i+742$,	$38t+2i-51$,
$38t+2i+216$,	$40t+2i+209$,	$40t+2i$,	$42t+2i$,
$42t+2i-1$,	$44t+2i-6$,	$44t+2i-3$,	$46t+2i+5$,
$46t+2i+744$,	$48t+3i+158$,	$49t+3i+215$,	$50t+3i+209$,
$51t+3i-1$,	$52t+3i-1$,	$53t+3i-7$,	$54t+3i-4$,
$55t+3i-1$,	$56t+3i+746$,	$59t+4i+157$,	$59t+4i+215$,
$61t+4i+208$,	$61t+4i-2$,	$63t+4i-7$,	$63t+4i-5$,
$65t+4i-2$,	$65t+4i+740$,	$69t+5i+157$,	$70t+5i+214$,
$71t+5i+207$,	$72t+5i-8$,	$73t+5i-5$,	$74t+5i-3$,
$75t+5i+739$,	$80t+6i+156$,	$80t+6i+213$,	$82t+6i+201$,
$82t+6i-6$,	$84t+6i-3$,	$84t+6i+738$,	$90t+7i+155$,
$91t+7i+207$,	$92t+7i+203$,	$93t+7i-4$,	$94t+7i+738$,
$101t+8i+149$,	$101t+8i+209$,	$103t+8i+205$,	$103t+8i+737$,
$111t+9i+151$,	$112t+9i+211$,	$113t+9i+946$,	$122t+10i+153$,
$122t+10i+952$,	$132t+11i+894$.		

B_0 is a path with an end vertex x and length $n-n_t$. Other B_i is simply a cycle of length i.

Then $f(n)\geq n+\frac{107}{3}t+\frac{7}{3}$, for $n\geq n_t$.

From the above result, we have

$$\liminf_{n\to\infty}\frac{f(n)-n}{\sqrt{n}}\geq\sqrt{2+\frac{7654}{19071}},$$

which is better than the previous bounds $\sqrt{2}$ (see [14]), $\sqrt{2+\frac{2}{5}}$ (see [9]).

The Conjecture 4 is not true. This completes the proof.

Combining this with Boros, Caro, Füredi and Yuster's upper bound, we get

$$1.98\geq\limsup_{n\to\infty}\frac{f(n)-n}{\sqrt{n}}\geq\liminf_{n\to\infty}\frac{f(n)-n}{\sqrt{n}}\geq\sqrt{2+\frac{7654}{19071}}.$$

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976). MR3138588

[2] E. Boros, Y. Caro, Z. Füredi and R. Yuster, Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284. MR1842115 (2002g:05130)

[3] G. Chen, J. Lehel, M. S. Jacobson and W. E. Shreve, Note on graphs without repeated cycle lengths, J. Graph Theory 29(1998),11-15. MR1633908 (2000d:05059)

[4] J. C. George, A. Khodkar, W. D. Wallis, Pancyclic and bipancyclic graphs, Springer Briefs in Mathematics, Springer, Cham, 2016. MR3495006

[5] X. Jia, Some extremal problems on cycle distributed graphs, Congr. Numer. 121(1996), 216-222. MR1431994 (97i:05068)

[6] C. Lai, On the size of graphs with all cycle having distinct length, Discrete Math. 122(1993) 363-364. MR1246693(94i:05048)

[7] C. Lai, The edges in a graph in which no two cycles have the same length, J. Zhangzhou Teachers College (Natural Science) 8(4)(1994), 30-34. MR1336586 (96h:05121)

[8] C. Lai, A lower bound for the number of edges in a graph containing no two cycles of the same length, Electron. J. Combin. 8(2001), #N9, 1 - 6. MR1877662 (2002k:05124)

[9] C. Lai, Graphs without repeated cycle lengths, Australas. J. Combin. 27(2003), 101-105. MR1955391 (2003m:05102)

[10] C. Lai, M. Liu, Some open problems on cycles, J. Combin. Math. Combin. Comput., 91(2014), 51-64. MR3287706

[11] S. Liu, On r-(k)-pancyclic graphs, Ars Combin., accepted.

[12] K. Markström, A note on uniquely pancyclic graphs, Australas. J. Combin. 44 (2009), 105-110. MR2527003 (2010i:05180)

[13] Y. Shi, Some theorems of uniquely pancyclic graphs, Discrete Math. 59 (1986), no. 1-2, 167–180. MR0837964

[14] Y. Shi, On maximum cycle-distributed graphs, Discrete Math. 71(1988), 57-71. MR0954686 (89i:05169)

[15] Y. Shi, The number of edges in a maximum graphs with uni-length cycles, Kexue Tongbao 33(10)(1988), 795 - 796.

[16] Y. Shi, H. P. Yap, S. K. Teo, On uniquely r-pancyclic graphs, Graph theory and its applications: East and West (Jinan, 1986), 487-499, Ann. New York Acad. Sci., 576, New York Acad. Sci., New York, 1989. MR1110849 (93d:05088)

[17] Y. Shi, Some theorems on simple MCD-graphs, J. Math. Res. Exposition 11(4)(1991), 551-554. MR1144956 (92i:05182)

[18] Y. Shi, The number of edges in a maximum cycle distributed graph, Discrete Math. 104(1992), 205-209. MR1172850 (93d:05083)

[19] Y. Shi, On simple MCD graphs containing a subgraph homemorphic to K_4, Discrete Math. 126(1994), 325-338. MR1264498 (95f:05072)

[20] F. Tian, The progress on some problems in graph theory, Qufu Shifan Daxue Xuebao Ziran Kexue Ban 1986, no. 2, 30-36. MR0865617

[21] H. P. Yap, S. K. Teo, On uniquely r-pancyclic graphs, Lecture Notes in Math.,1073(1984), 334-335.

[22] C. T. Zamfirescu, (2)-pancyclic graphs, Discrete Appl. Math., 161(2013), 1128-1136. MR3030597

[23] K. Zhang, Progress of some problems in graph theory, J. Math. Res. Exposition 27(3) (2007), 563-576. MR2349503