On the number of edges in some graphs ⋆

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph with n vertices in which any two cycles are of different lengths. The sequence $(c_1, c_2, \cdots, c_n)$ is the cycle length distribution of a graph G with n vertices, where c_i is the number of cycles of length i in G. Let $f(a_1, a_2, \cdots, a_n)$ denote the maximum possible number of edges in a graph which satisfies $c_i \leq a_i$, where a_i is a nonnegative integer. In 1991, Shi posed the problem of determining $f(a_1, a_2, \cdots, a_n)$ which extended the problem due to Erdős. It is clear that $f(n) = f(1, 1, \cdots, 1)$. Let $g(n, m) = f(a_1, a_2, \cdots, a_n)$, where $a_i = 1$ if i/m is an integer, and $a_i = 0$ otherwise. It is clear that $f(n) = g(n, 1)$. We prove that $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2 + \frac{40}{99}}$, which is better than the previous bounds $\sqrt{2}$ (Shi, 1988), and $\sqrt{2 + \frac{7654}{19071}}$ (Lai, 2017). We show that $\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} > \sqrt{2.444}$, for all even integers m. We make the following conjecture: $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444}$.

Key words: Graph, cycle, number of edges.
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1 Introduction

Let $f(n)$ be the maximum number of edges in a graph with n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see Bondy and Murty [1], p.247, Problem 11). Shi [11] proved a lower bound.

Theorem 1 (Shi [11])

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$.

Chen, Lehel, Jacobson and Shreve [3], Jia [4], Lai [5–7], Shi [13,14] obtained some additional related results.

Boros, Caro, Füredi and Yuster [2] proved an upper bound as follows.

Theorem 2 (Boros, Caro, Füredi and Yuster [2]) For n sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [8] improved the lower bound by Shi as follows.

Theorem 3 (Lai [8]) Let $t = 1260r + 169 \ \ (r \geq 1)$, then

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$.

Lai [5] proposed the following conjecture:

Conjecture 4 (Lai [5])

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3}.$$

It would be nice to prove that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3+\frac{3}{5}}.$$

Survey papers on this problem can be found in Tian [15], Zhang [16], Lai and Liu [9].

The progress of all 50 problems in [1] can be found in Locke [10].

The sequence $(c_1, c_2, \cdots, c_n)$ is the cycle length distribution of a graph G with n vertices, where c_i is the number of cycles of length i in G. Let $f(a_1, a_2, \cdots, a_n)$ denote the maximum possible number of edges in a graph which satisfies $c_i \leq a_i$, where a_i is a nonnegative integer. Shi [12] posed the problem of determining $f(a_1, a_2, \cdots, a_n)$ which extended the problem due to Erdős. It is clear that $f(n) = f(1, 1, \cdots, 1)$. Let $g(n, m) = f(a_1, a_2, \cdots, a_n)$, where $a_i = 1$ if i/m is an integer, and $a_i = 0$ otherwise. It is clear that $f(n) = g(n, 1)$.

In this paper, we obtain the following results.

Theorem 5 Let m be even, $s_1 > s_2, s_1 + 3s_2 > k$, then

$$g(n, m) \geq n + (k + s_1 + 2s_2 + 1)t - 1$$

for $n \geq (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1$.

Theorem 6 Let $t = 1260r + 169 \ (r \geq 1)$, then

$$f(n) \geq n + \frac{119}{3}t - \frac{26399}{3}$$

for $n \geq \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$.

2 Proof of Theorem 5

Proof. Let $n_t = (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1$, m be even, $s_1 > s_2, s_1 + 3s_2 > k$, $n \geq n_t$. It suffice to show that there exists a graph G on n vertices with $n + (k + s_1 + 2s_2 + 1)t - 1$ edges such that all cycles in G have distinct lengths and all the lengths of cycles are the multiple of m.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq s_1t, i = s_1t + j \ (1 \leq j \leq s_2t), i = s_1t + s_2t + j \ (1 \leq j \leq t))$.

Now we define these B_is. These subgraphs all only have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq s_2t$, let the subgraph B_{s_1t+i} consists of a cycle

$$xa_i^1a_i^2...a_i^{ms_1t+2ms_2t+mi-1}x$$

and a path:

$$xa_{i,1}^1 a_{i,1}^2 ... a_{i,1}^{\frac{ms_1t-ms_2t+mi}{2}-1} a_i^{\frac{ms_1t+ms_2t+mi}{2}}.$$

Based on the construction, B_{s_1t+i} contains exactly three cycles of lengths:

$$ms_1t+mi, ms_1t+ms_2t+mi, ms_1t+2ms_2t+mi.$$

For $1 \leq i \leq t$, let the subgraph $B_{s_1t+s_2t+i}$ consists of a cycle

$$C_{s_1t+s_2t+i} = xy_i^1 y_i^2 ... y_i^{ms_1t+3ms_2t+mk(k+1)t+mi-1} x$$

and k paths sharing a common vertex x, the other end vertices are on the cycle $C_{s_1t+s_2t+i}$:

$$xy_{i,p}^1 y_{i,p}^2 ... y_{i,p}^{\frac{ms_1t+3ms_2t-mkt+m(p-1)t+mi}{2}-1} y_i^{\frac{ms_1t+3ms_2t+mk(2p-1)t+m(p-1)t+mi}{2}} (p=1,2,...,k).$$

As a cycle with k chords contains $\binom{k+2}{2}$ distinct cycles, $B_{s_1t+s_2t+i}$ contains exactly $\frac{(k+2)(k+1)}{2}$ cycles of lengths:

$$ms_1t+3ms_2t+mkht+(h+j-1)mt+mi(j \geq 1, h \geq 0, k+1 \geq j+h).$$

B_0 is a path with an end vertex x and length $n-n_t$. The other B_i is simply a cycle of length mi.

Then $g(n,m) \geq n+(k+s_1+2s_2+1)t-1$, for $n \geq n_t$.

This completes the proof.

From Theorem 5, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} \geq$$

$$\sqrt{\frac{(k+s_1+2s_2+1)^2}{(\frac{3}{4}k^2+\frac{1}{2}ks_1+\frac{3}{2}ks_2+\frac{1}{2}s_1^2+\frac{3}{2}s_1s_2+\frac{9}{4}s_2^2+k+s_1+3s_2+\frac{1}{2})}},$$

for all even integers m.

Let $s_1 = 28499066, s_2 = 4749839, k = 14249542$, then

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} > \sqrt{2.444},$$

for all even integers m.

3 Proof of Theorem 6

Proof. Let $n_t = \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$, $t = 1260r + 169, r \geq 1$, $n \geq n_t$. It suffice to show that there exists a graph G on n vertices with $n + \frac{119}{3}t - \frac{26399}{3}$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq 22t, i = 22t + j$ $(1 \leq j \leq \frac{5t-8}{3})$, $i = 23t + \frac{2t-2}{3} + j$ $(1 \leq j \leq \frac{5t-8}{3})$, $i = 32t + j - 60$ $(58 \leq j \leq t - 742))$.

Now we define these $B_i s$. These subgraphs all only have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph B_{22t+i} consists of a cycle

$$xa_i^1 a_i^2 ... a_i^{28t+\frac{2t-2}{3}+2i-3} x$$

and a path:

$$xa_{i,1}^1 a_{i,1}^2 ... a_{i,1}^{\frac{56t-2}{6}} a_i^{\frac{76t-4}{6}+i}.$$

Based on the construction, B_{22t+i} contains exactly three cycles of lengths:

$$22t + i, 25t + \frac{t-1}{3} + i - 1, 28t + \frac{2t-2}{3} + 2i - 2.$$

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph $B_{23t+\frac{2t-2}{3}+i}$ consists of a cycle

$$xb_i^1 b_i^2 ... b_i^{28t+\frac{2t-2}{3}+2i-2} x$$

and a path:

$$xb_{i,1}^1 b_{i,1}^2 ... b_{i,1}^{11t-1} b_i^{\frac{76t-4}{6}+i}.$$

Based on the construction, $B_{23t+\frac{2t-2}{3}+i}$ contains exactly three cycles of lengths:

$$23t+\frac{2t-2}{3}+i, 27t+i-1, 28t+\frac{2t-2}{3}+2i-1.$$

For $58 \leq i \leq t-742$, let the subgraph $B_{32t+i-60}$ consists of a cycle

$$C_{32t+i-60} = xy_i^1 y_i^2 ... y_i^{137t+11i+890} x$$

and ten paths sharing a common vertex x, the other end vertices are on the cycle $C_{32t+i-60}$:

$$xy_{i,1}^1 y_{i,1}^2 ... y_{i,1}^{11t-2} y_i^{21t-59+i}$$

$$xy_{i,2}^1 y_{i,2}^2 ... y_{i,2}^{12t-2} y_i^{31t-53+2i}$$

$$xy_{i,3}^1 y_{i,3}^2 ... y_{i,3}^{12t-2} y_i^{41t+156+3i}$$

$$xy_{i,4}^1 y_{i,4}^2 ... y_{i,4}^{13t-2} y_i^{51t+155+4i}$$

$$xy_{i,5}^1 y_{i,5}^2 ... y_{i,5}^{13t-2} y_i^{61t+155+5i}$$

$$xy_{i,6}^1 y_{i,6}^2 ... y_{i,6}^{14t-2} y_i^{71t+154+6i}$$

$$xy_{i,7}^1 y_{i,7}^2 ... y_{i,7}^{14t-2} y_i^{81t+153+7i}$$

$$xy_{i,8}^1 y_{i,8}^2 ... y_{i,8}^{15t-2} y_i^{91t+147+8i}$$

$$xy_{i,9}^1 y_{i,9}^2 ... y_{i,9}^{15t-2} y_i^{101t+149+9i}$$

$$xy_{i,10}^1 y_{i,10}^2 ... y_{i,10}^{16t-2} y_i^{111t+151+10i}.$$

As a cycle with d chords contains $\binom{d+2}{2}$ distinct cycles, $B_{32t+i-60}$ contains exactly 66 cycles of lengths:

$32t+i-60$,	$33t+i+4$,	$34t+i+207$,	$35t+i-3$,
$36t+i-2$,	$37t+i-3$,	$38t+i-3$,	$39t+i-8$,
$40t+i$,	$41t+i$,	$42t+i+739$,	$43t+2i-54$,
$43t+2i+213$,	$45t+2i+206$,	$45t+2i-3$,	$47t+2i-3$,
$47t+2i-4$,	$49t+2i-9$,	$49t+2i-6$,	$51t+2i+2$,
$51t+2i+741$,	$53t+3i+155$,	$54t+3i+212$,	$55t+3i+206$,
$56t+3i-4$,	$57t+3i-4$,	$58t+3i-10$,	$59t+3i-7$,
$60t+3i-4$,	$61t+3i+743$,	$64t+4i+154$,	$64t+4i+212$,
$66t+4i+205$,	$66t+4i-5$,	$68t+4i-10$,	$68t+4i-8$,
$70t+4i-5$,	$70t+4i+737$,	$74t+5i+154$,	$75t+5i+211$,
$76t+5i+204$,	$77t+5i-11$,	$78t+5i-8$,	$79t+5i-6$,
$80t+5i+736$,	$85t+6i+153$,	$85t+6i+210$,	$87t+6i+198$,
$87t+6i-9$,	$89t+6i-6$,	$89t+6i+735$,	$95t+7i+152$,
$96t+7i+204$,	$97t+7i+200$,	$98t+7i-7$,	$99t+7i+735$,
$106t+8i+146$,	$106t+8i+206$,	$108t+8i+202$,	$108t+8i+734$,
$116t+9i+148$,	$117t+9i+208$,	$118t+9i+943$,	$127t+10i+150$,
$127t+10i+949$,	$137t+11i+891$.		

B_0 is a path with an end vertex x and length $n-n_t$. The other B_i is simply a cycle of length i.

Then $f(n) \geq n + \frac{119}{3}t - \frac{26399}{3}$, for $n \geq n_t$.

This completes the proof.

From Theorem 6, we have

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}},$$

which is better than the previous bounds $\sqrt{2}$ (see [11]), and $\sqrt{2+\frac{7654}{19071}}$ (see [8]).

Combining this with Boros, Caro, Füredi and Yuster's upper bound, namely Theorem 2, we get

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

From the proof of Theorem 6, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} \geq \sqrt{2+\frac{40}{99}},$$

for all integers m.

If $m = 1$, $1 \leq i \leq t$, there exists the subgraph similar to $B_{s_1t+s_2t+i}$ consists of a cycle $C_{s_1t+s_2t+i}$ and k paths sharing a common vertex x, the other end vertices are on the cycle $C_{s_1t+s_2t+i}$ such that all cycles in $B_{s_1t+s_2t+i}$ have distinct lengths, then we could obtain

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444} > \sqrt{2+\frac{40}{99}}.$$

But we only for $m = 1$, $58 \leq i \leq t - 742$, construct a subgraph similar to $B_{s_1t+s_2t+i}$ consists of a cycle $C_{s_1t+s_2t+i}$ and ten paths sharing a common vertex x, the other end vertices are on the cycle $C_{s_1t+s_2t+i}$ such that all cycles in $B_{s_1t+s_2t+i}$ have distinct lengths and obtain

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

Since the liminf for $\frac{g(n,m)-n}{\sqrt{\frac{n}{m}}}$ for even m is $\sqrt{2.444}$, it is reasonable to suspect that such a lower bound also holds for $\frac{f(n)-n}{\sqrt{n}}$.

We make the following conjecture:

Conjecture 7

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444}.$$

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