QUANTIZABLE FUNCTIONS ON KÄHLER MANIFOLDS AND NON-FORMAL QUANTIZATION

KWOKWAI CHAN, NAICHUNG CONAN LEUNG, AND QIN LI

ABSTRACT. Applying the Fedosov connections constructed in [6], we find a (dense) subsheaf of smooth functions on a Kähler manifold X which admits a non-formal deformation quantization. When X is prequantizable and the Fedosov connection satisfies an integrality condition, we prove that this subsheaf of functions can be quantized to a sheaf of twisted differential operators (TDO), which is isomorphic to that associated to the prequantum line bundle. We also show that examples of such quantizable functions are given by images of quantum moment maps.

CONTENTS

1. INTRODUCTION

The quantization of the phase space (X, ω) of a classical mechanical system is the procedure of associating functions in a dense subspace $\mathcal{A} \subset C^\infty(X)$ to operators on a Hilbert

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space $\mathcal{H}$ such that the composition gives a deformation of the classical pointwise multiplication. The most important two schemes of quantization in mathematics are deformation quantization and geometric quantization, which focus on different aspects of the quantization picture. Deformation quantization is by definition a formal deformation $(C^\infty(X)[[\hbar]], \star)$ of the commutative algebra $C^\infty(X)$, where $\hbar$ is a formal variable, such that the first term of the commutator is precisely the Poisson bracket; while geometric quantization focuses on the Hilbert space $\mathcal{H}$ and its operators.

In general, it is too optimistic to expect that the Hilbert space in geometric quantization is really a module over the deformation quantization algebra. We take the Berezin-Toeplitz quantization as an example ([2, 13–16, 18, 19]): a smooth function f acts on the Hilbert spaces $\mathcal{H}_k := H^0(X, L^{\otimes k})$ of holomorphic sections of tensor powers of the prequantum line bundle via Toeplitz operators. However, the composition of Toeplitz operators is only *asymptotic* to a sum of Toeplitz operators as $k \to \infty$. In particular, each Hilbert space $\mathcal{H}_k$ for a fixed k does not form a module over the deformation quantization algebra. This is also one of the reasons for using the formal variable $\hbar$ in Berezin-Toeplitz quantization (instead of complex values).

These drawbacks led to some dissatisfaction among physicists. They even claimed that "deformation quantization is not quantization" (see [11, Section 1.4] for a more detailed explanation of this comment). To solve this problem, Gukov and Witten [11] proposed a new scheme of quantization by considering the A-model of a suitable complexification of a symplectic manifold X. In this *brane quantization* picture, the Hilbert space and the algebra of operators acting on it are both morphism spaces between certain branes, whose definitions are still mysterious to both physicists and mathematicians. We also do not know in general which symplectic manifolds admit such a quantization (see, however, [1, 11]).

In this paper, we give a mathematical construction of non-formal quantizations of Kähler manifolds. The naive idea is to take the evaluation of $\hbar$ in deformation quantization to some complex numbers to get rid of the formal variable. But there would be convergence issues in general. To overcome this, we exploit the Fedosov connections constructed in [6] which have nice finiteness properties. We also restrict to a subalgebra of smooth functions, since the star product $f \star g$ of two functions will be a formal power series in $\hbar$ and is in general divergent after evaluating $\hbar$ at complex values. We will be able to show that this subspace of functions can be quantized to holomorphic differential operators acting on the Hilbert space $H^0(X, L^{\otimes k})$ in geometric quantization. This yields a dense subspace of *quantizable functions* as $k \to \infty$.

To have a glimpse of the idea, consider the flat space $\mathbb{C}^n$ equipped with the Wick product on $C^\infty(\mathbb{C}^n)[[\hbar]]$, a natural choice of the subalgebra of $C^\infty(\mathbb{C}^n)$ is the space of polynomials where the formal variable can be evaluated at any complex number. This is because the Wick product of any two polynomials is still a polynomial in $\hbar$ (see equation (2.2) for an explicit formula of Wick product). On a general Kähler manifold (X, ω, J), we use Fedosov's flat connections [8] to globalize the local computations and find a subspace of functions whose noncommutativity under the star product is polynomially controlled.

The key lies in the fact that the Fedosov connections on Kähler manifolds constructed in [6] are quantizations of Kapranov's L_∞ structure [12]. The special form of these Fedosov connections allows us to take the evaluation $\hbar = 1/k$ for any $k \in \mathbb{C} \setminus \{0\}$, yielding a flat connection $D_{\alpha,k}$; here α (called the Karabegov form) is a $(1,1)$-form representing a class in $\hbar H^2_{dR}(X)[\hbar]$. We call k the *level* and define *quantizable functions of level* k as those functions whose corresponding flat sections under the connection $D_{\alpha,k}$ have only finite degree anti-holomorphic parts (see Definition 2.16). In particular, the star product of these functions is still quantizable and only has finite $\hbar$ power expansion, so there are no convergence issues. These quantizable functions form a sheaf $\mathcal{C}^\infty_{\alpha,k}$ of algebras on the Kähler manifold X under the star product. In Section 3, we show that this gives an example of so-called *sheaves of twisted differential operators* (TDO for short) on X, which appeared in the theory of D-modules [9]:

Theorem 1.1 (= Theorems 3.6 + 3.7)**.** *Let X be a Kähler manifold. For any closed $(1,1)$-form $\alpha \in \mathcal{A}^{1,1}(X)[[\hbar]]$ and level k, the sheaf $\mathcal{C}^\infty_{\alpha,k}$ of quantizable functions (under the Fedosov connection $D_{\alpha,k}$) is a TDO on X with characteristic class $[\omega - \alpha]$.*

Note that the Karabegov form of the Fedosov connection is precisely given by $\frac{1}{\hbar}(\omega - \alpha)$. When it satisfies an integrality condition (see equation (4.4)), we can prove that the sheaf $\mathcal{C}^\infty_{\alpha,k}$ is isomorphic to the sheaf of holomorphic differential operators on some holomorphic line bundles. A particularly important case is when the Karabegov form is the same as that in the *Berezin-Toeplitz quantization*. In this case, the line bundles are tensor powers $L^{\otimes k}$'s of the prequantum line bundle L. An extension of Fedosov's method allows us to construct a sheaf $\mathcal{F}_{L^{\otimes k}}$ of modules over the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}$ equipped with a compatible Fedosov flat connection, and we have the following result analogous to the ono-to-one correspondence between smooth functions and flat sections of the Weyl bundle in Fedosov quantization:

Theorem 1.2 (= Theorem 4.4)**.** *On a Kähler manifold X with prequantum line bundle L, the symbol map gives a canonical isomorphism from flat sections of $\mathcal{F}_{L^{\otimes k}}$ on any open set $U \subset X$ under the Fedosov connection to holomorphic sections of $L^{\otimes k}$ on U.*

Now the compatibility between the Fedosov connections on the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}$ and $\mathcal{F}_{L^{\otimes k}}$ implies that quantizable functions of level k act on the (local) holomorphic sections of $L^{\otimes k}$. Since locality is obvious, this gives the desired non-formal quantization described by holomorphic differential operators on the Hilbert space $H^0(X, L^{\otimes k})$:

Theorem 1.3 (= Theorem 4.6)**.** *Suppose that X is a Kähler manifold equipped with a prequantum line bundle L. Choose α so that the Karabegov form coincides with that of the Berezin-Toeplitz quantization of X. Then for any positive integer k, there is a natural isomorphism (of TDOs)*

$$\varphi : \mathcal{C}^\infty_{\alpha,k} \to \mathcal{D}(L^{\otimes k})$$

from the sheaf of algebras of level k quantizable functions to the sheaf of holomorphic differential operators on $L^{\otimes k}$.

This sheaf-theoretic description of our quantization procedure provides an example of gluing of quantizations over open sets to global ones in the Kähler setting. Theorem 1.3 can be generalized beyond the case of Berezin-Toeplitz quantization by changing the Karabegov form of the Fedosov quantization, so that holomorphic differential operators on *any* holomorphic line bundle can be realized as quantizations of a class of quantizable functions.

Our notion of quantizable functions is a vast generalization of the previous notion of quantizable functions (or polarization-preserving functions) in geometric quantization (see e.g. [17]), which can only produce first order differential operators. In Section 5, we will see that quantizable functions in our sense also arise from Hamiltonian G-actions on the Kähler manifold X. More precisely, we will show that images of *quantum moment maps* are all examples of first order quantizable functions in Theorem 5.8. From this, we obtain a Lie algebra homomorphism from the Lie algebra $\mathfrak{g}$ of G to the space of quantizable functions. When X is a flag variety G/B, this reproduces the Lie algebra representation in the Borel-Weil-Bott Theorem [4].

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2. Quantizable functions via Fedosov quantization

Recall that a *deformation quantization* of a symplectic manifold (X,ω) is a formal deformation of the commutative algebra $(C^\infty(X),\cdot)$ equipped with pointwise multiplication to a noncommutative one $(C^\infty(X)[[\hbar]],\star)$ equipped with a *star product* of the following form

$$f\star g = fg+\sum_{i\geq 1}\hbar^i\cdot C_i(f,g),$$

where each $C_i(-,-)$ is a bi-differential operator, so that the leading order of noncommutativity is a constant multiple of the Poisson bracket $\{-,-\}$ associated to ω:

$$C_1(f,g)-C_1(g,f)=\frac{d}{d\hbar}\Big|_{\hbar=0}(f\star_\hbar g-g\star_\hbar f)=\frac{\sqrt{-1}}{2}\{f,g\}. \tag{2.1}$$

In [8], Fedosov gave a beautiful geometric construction of deformation quantizations on symplectic manifolds.

2.1. Fedosov quantization of a Kähler manifold.

We will focus on Wick type star products on Kähler manifolds. The Kähler form on a Kähler manifold X will always be written as

$$\omega=\omega_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta,$$

where we adopt the convention that $\omega^{\bar\gamma\alpha}\omega_{\alpha\bar\beta}=\delta^{\bar\gamma}_{\bar\beta}$.

We consider the following Weyl bundles on X:

$$\mathcal{W}_X:=\widehat{\mathrm{Sym}}\,T^*X,\quad \overline{\mathcal{W}}_X:=\widehat{\mathrm{Sym}}\,\overline{T^*X},$$
$$\mathcal{W}_{X,\mathbb{C}}:=\mathcal{W}_X\otimes_{C^\infty_X}\overline{\mathcal{W}}_X=\widehat{\mathrm{Sym}}\,T^*X_{\mathbb{C}}.$$

To give explicit expressions of these bundles, we let $(z^1, \cdots, z^n)$ be a local holomorphic coordinate system on X, use $dz^i, d\bar{z}^j$'s to denote 1-forms in $\mathcal{A}_X^\bullet$ and use $y^i, \bar{y}^j$ to denote sections in $\mathcal{W}_{X,\mathbb{C}}$. The Kähler form enables us to define a (non-commutative) fiberwise Wick product on $\mathcal{W}_{X,\mathbb{C}}$:

$$\alpha \star \beta := \sum_{k\geq 0} \frac{\hbar^k}{k!} \cdot \omega^{i_1\bar{j}_1} \cdots \omega^{i_k\bar{j}_k} \cdot \frac{\partial^k \alpha}{\partial y^{i_1} \cdots \partial y^{i_k}} \frac{\partial^k \beta}{\partial \bar{y}^{j_1} \cdots \partial \bar{y}^{j_k}}. \tag{2.2}$$

Throughout this paper, we denote by ∇ the Levi-Civita connection on X, and its natural extension to the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}$. Its curvature can be written as a bracket:

$$\nabla^2 = \frac{1}{\hbar}[R_\nabla, -]_\star,$$

where $R_\nabla = R_{i\bar{j}k\bar{l}} dz^i \wedge d\bar{z}^j \otimes y^k \bar{y}^l \in \mathcal{A}_X^2(\mathcal{W}_{X,\mathbb{C}})$.

A natural filtration on these Weyl bundles is defined by polynomial degrees. For instance, $(\overline{\mathcal{W}_X})_{\leq N}$ denotes the sum of anti-holomorphic monomials of degree $\leq N$. The *symbol map*

$$\sigma : \mathcal{A}_X^\bullet(\mathcal{W}_{X,\mathbb{C}})[[\hbar]] \to \mathcal{A}_X^\bullet[[\hbar]]. \tag{2.3}$$

is defined by setting all $y^i, \bar{y}^j$'s to zero.

Definition 2.1. We will use the notation $\mathcal{W}_{p,q}$ to denote the component $\mathrm{Sym}^p\, T^*X \otimes_{C_X^\infty} \mathrm{Sym}^q\, \overline{T^*X}$ of $\mathcal{W}_{X,\mathbb{C}}$; sections of this subbundle are said to be *of type* (p, q). There are 4 natural operators acting as derivations on $\mathcal{A}_X^\bullet(\mathcal{W}_{X,\mathbb{C}})$:

$$\delta^{1,0} a = dz^i \wedge \frac{\partial a}{\partial y^i}, \quad \delta^{0,1} a = d\bar{z}^j \wedge \frac{\partial a}{\partial \bar{y}^j},$$

as well as

$$(\delta^{1,0})^* a = y^k \cdot \iota_{\partial_{z^k}} a, \quad (\delta^{0,1})^* a = \bar{y}^j \cdot \iota_{\partial_{\bar{z}^j}} a.$$

We define the operators $(\delta^{1,0})^{-1}$ and $(\delta^{0,1})^{-1}$ by normalizing $(\delta^{1,0})^*$ and $(\delta^{1,0})^*$ respectively:

$$(\delta^{1,0})^{-1} := \frac{1}{p_1 + p_2} (\delta^{1,0})^* \text{ on } \mathcal{A}_X^{p_1,q_1}(\mathcal{W}_{p_2,q_2}),$$
$$(\delta^{0,1})^{-1} := \frac{1}{q_1 + q_2} (\delta^{0,1})^* \text{ on } \mathcal{A}_X^{p_1,q_1}(\mathcal{W}_{p_2,q_2}).$$

The following equality will be needed later:

$$\mathrm{id} - \pi_{0,*} = \delta^{1,0} \circ (\delta^{1,0})^{-1} + (\delta^{1,0})^{-1} \circ \delta^{1,0}, \tag{2.4}$$

where $\pi_{0,*}$ denotes the natural projection from $\mathcal{A}_X^\bullet(\mathcal{W}_{X,\mathbb{C}})$ to $\mathcal{A}_X^{0,\bullet}(\overline{\mathcal{W}}_X)$. We also define the fiberwise de Rham differential as $\delta := \delta^{1,0} + \delta^{0,1}$.

Definition 2.2. A connection on the formal Weyl bundle $\mathcal{W}_{X,\mathbb{C}}[[\hbar]]$ of the form

$$D = \nabla - \delta + \frac{1}{\hbar}[I, -]_\star$$

is called a *Fedosov connection* if $D^2 = 0$. Here ∇ is the Levi-Civita connection, and $I \in \mathcal{A}_X^1(\mathcal{W}_{X,\mathbb{C}})[[\hbar]]$ is a 1-form valued section of $\mathcal{W}_{X,\mathbb{C}}[[\hbar]]$. This connection can be extended to a differential on $\mathcal{A}_X^\bullet(\mathcal{W}_{X,\mathbb{C}})$.

Notation 2.3. Let ∇ be the Levi-Civita connection. We define the following operator

$$\tilde{\nabla}^{1,0} := (\delta^{1,0})^{-1} \circ \nabla^{1,0};$$

$\tilde{\nabla}^{0,1}$ is similarly defined.

For later computations, we need the following:

Lemma 2.4. *For any $k \geq 0$, given any $\alpha \in \Gamma(X, \mathrm{Sym}^k \overline{TX})$, there exists a unique $\tilde{\alpha}$ such that $(\nabla^{1,0} - \delta^{1,0})(\tilde{\alpha}) = 0$ and that $\pi_{0,*}(\tilde{\alpha}) = \alpha$.*

Proof. For $k \geq 0$, let $\alpha_k := (\tilde{\nabla}^{1,0})^k(\alpha)$, and define $\tilde{\alpha}$ by

$$\tilde{\alpha} := \sum_{k \geq 0} \alpha_k.$$

Clearly, $\pi_{0,*}(\tilde{\alpha}) = \alpha$. According to our construction, we then have $\alpha_{k+1} = (\delta^{1,0})^{-1}\left(\nabla^{1,0}\alpha_k\right)$ because $\delta^{1,0}(\nabla^{1,0}\alpha_k) = 0$ for all $k \geq 0$. Applying equation (2.4), we obtain

$$\begin{aligned}\delta^{1,0}\alpha_{k+1} &= \delta^{1,0} \circ (\delta^{1,0})^{-1}\left(\nabla^{1,0}\alpha_k\right)\\ &= \delta^{1,0} \circ (\delta^{1,0})^{-1}\left(\nabla^{1,0}\alpha_k\right) + (\delta^{1,0})^{-1} \circ \delta^{1,0}\left(\nabla^{1,0}\alpha_k\right) = \nabla^{1,0}\alpha_k.\end{aligned}$$

Thus we get

$$\begin{aligned}(\nabla^{1,0} - \delta^{1,0})(\tilde{\alpha}) &= -\delta^{1,0}(\alpha_0) + \nabla^{1,0}(\alpha_0) - \delta^{1,0}(\alpha_1) + \nabla^{1,0}(\alpha_1) - \delta^{1,0}(\alpha_2) \cdots\\ &= -\delta^{1,0}(\alpha_0) = 0.\end{aligned}$$

□

The main result in Fedosov's approach to deformation quantization is summarized in the following

Theorem 2.5 (Fedosov [8])**.** *There exist Fedosov connections on the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}[[\hbar]]$. Furthermore, for every formal smooth function $f \in C^\infty(X)[[\hbar]]$, there is a unique flat section O_f of the Weyl bundle with $\sigma(O_f) = f$. The associated deformation quantization (or star product) is defined by the formula*

$$O_f \star O_g = O_{f\star g}.$$

In [6], we showed that a class of Fedosov connections can be obtained by quantizing Kapranov's L_∞ structure on a Kähler manifold [12]:

Theorem 2.6 (Theorems 2.17 and 2.25 in [6])**.** *Let α be a representative of a formal cohomology class in $\hbar H^2_{dR}(X)[[\hbar]]$ of type $(1,1)$. Then there exists a solution of the Fedosov equation of the form $I_\alpha = I + J_\alpha \in \mathcal{A}^{0,1}_X(\mathcal{W}_{X,\mathbb{C}})$:*

$$\nabla I_\alpha - \delta I_\alpha + \frac{1}{\hbar} I_\alpha \star I_\alpha + R_\nabla = \alpha. \tag{2.5}$$

We denote the corresponding Fedosov connection by $D_\alpha := \nabla - \delta + \frac{1}{\hbar}[I_\alpha, -]_\star$. The deformation quantization associated to D_α is a Wick type star product whose Karabegov form is $\frac{1}{\hbar}(\omega - \alpha)$.

The term I in the connection D_α is obtained by repeatedly taking covariant exterior derivatives to the curvature tensor (see [6] for more details). A key property is that I is *uniformly of polynomial degree* 1 *in* $\overline{\mathcal{W}}_X$, and we can decompose $I = \sum_{k\geq 2} I_k$ according to the polynomial degrees in the holomorphic Weyl bundle $\mathcal{W}_X$. In particular, the first term is given by

$$I_2 = (\delta^{1,0})^{-1}\left(R_{i\bar{j}k\bar{l}}dz^i \wedge d\bar{z}^j \otimes y^k \bar{y}^l\right).$$

The term J_α is given as follows: Let φ be a (locally defined) function such that $\partial\bar{\partial}\varphi = \alpha$. Then we set $J_\alpha := -\sum_{k\geq 1}(\tilde{\nabla}^{1,0})^k(\bar{\partial}\varphi)$. Thus J_α is independent of the choice of φ and uniformly of degree 0 in $\overline{\mathcal{W}}_X$.

Remark 2.7. The operator δ can also be written as a bracket. Accordingly, the Fedosov connection D_α can be written as

$$D_\alpha = \nabla + \frac{1}{\hbar}[\gamma_\alpha, -]_\star,$$

where $\gamma_\alpha = \omega_{i\bar{j}}(d\bar{z}^j \otimes y^i - dz^i \otimes \bar{y}^j) + I_\alpha$. The flatness of D_α is then equivalent to the following Fedosov equation:

$$\nabla\gamma_\alpha + \frac{1}{\hbar}\gamma_\alpha \star \gamma_\alpha + R_\nabla = -\omega + \alpha. \tag{2.6}$$

If the term α in the Fedosov equation (2.5) is a polynomial in $\hbar$, we call the Fedosov connection and the associated star product *admissible*. In this paper, we will only consider admissible Fedosov connections. It was shown in [6] that the associated star products of these Fedosov connections must be of Wick type. Comparing to several previous Fedosov constructions of Wick type star products [3,14,20], there are several nice properties of our Fedosov connections D_α which will play important roles in this paper:

- First of all, the Karabegov form of the associated star product can be read off from the Fedosov equation (2.6).
- Secondly, if the formal $(1,1)$-form α is only a polynomial in $\hbar$, then the term I_α in the Fedosov connection D_α is also a polynomial in $\hbar$. This enables us to evaluate $\hbar$ at any complex number without divergence issues. This only works on Kähler manifolds and is significantly different from Fedosov's original construction.
- Lastly, it was shown in [6] that for a (local) holomorphic function f, its associated flat section O_f is only a section of the holomorphic Weyl bundle $\mathcal{W}_X$. (This fact is independent of the closed formal $(1,1)$-form α).

2.2. **Quantizable functions.**

To define quantizable functions, we need the following:

Definition 2.8. We define a *weight* on $\mathcal{W}_{X,\mathbb{C}}[[\hbar]]$ by assigning weights on its generators:

$$|y^i| = 0, \ |\bar{y}^j| = 2, \ |\hbar| = 2. \tag{2.7}$$

This weight is compatible with the fiberwise Wick product $\star$. It is clear that a section of $\mathcal{W}_{X,\mathbb{C}}$ is of finite weight if and only if it is both a polynomial in $\hbar$ and $\bar{y}^j$'s. There is an associated increasing filtration on the Weyl bundle. Explicitly, we let $(\mathcal{W}_{X,\mathbb{C}}[[\hbar]])_N$ denote sums of monomials with weight $\leq N$.

Remark 2.9. A section of the formal Weyl bundle lives in a finite filtration component if and only if it lives in $\mathrm{Sym}^\bullet \overline{TX}^* \otimes \mathcal{W}_X[\hbar]$.

Remark 2.10. This weight is different from the one in [8], although both are compatible with the fiberwise Wick product. The weight we just defined is a polarized version, namely, only anti-holomorphic terms in $\mathcal{W}_{X,\mathbb{C}}$ have non-zero weights.

Admissible Fedosov connections D_α of polynomial degrees in $\hbar$ have the following nice property:

Lemma 2.11. *Suppose D_α is an admissible Fedosov connection which is of degree k as a polynomial in $\hbar$. Then for any $N \geq 0$, we have*

$$D_\alpha\left((\mathcal{W}_{X,\mathbb{C}}[[\hbar]])_N\right) \subset (\mathcal{W}_{X,\mathbb{C}}[[\hbar]])_{N+2k}.$$

Definition 2.12. A formal function $f \in C^\infty(X)[[\hbar]]$ is called *quantizable* if the associated flat section O_f lives in a finite filtration component of $\mathcal{W}_{X,\mathbb{C}}[[\hbar]]$, or equivalently $O_f \in \mathrm{Sym}^\bullet \overline{TX}^* \otimes \mathcal{W}_X[\hbar]$.

Example 2.13. Every (local) holomorphic function f is quantizable for any degree k and any Karabegov form. Explicitly, the flat section associated to f is given by

$$O_f = \sum_{k\geq 0} (\tilde{\nabla}^{1,0})^k(f),$$

We have seen in Lemma 2.22 that $D_\alpha^{1,0}(O_f) = 0$. On the other hand, we have

$$D_\alpha^{0,1}(O_f) = \sum_{k\geq 0} (\tilde{\nabla}^{1,0})^k(\bar{\partial} f) = 0,$$

since f is holomorphic. Thus $O_f \in (\mathcal{W}_{X,\mathbb{C}}[[\hbar]])_0 = \mathcal{W}_X$, making f a formal quantizable function.

A natural question is whether there are examples of quantizable formal functions other than holomorphic ones. We will answer this question by giving a class of such formal functions in the following proposition, which will play an important role in later sections.

Proposition 2.14. *Let $\alpha = \sum_{i\geq 1} \hbar^i \alpha_i$ be a formal closed differential form of type $(1,1)$ and $\sum_{i\geq 1} \hbar^i \rho_i$ be a potential of α (i.e., $\partial\bar{\partial}\rho_i = \alpha_i$), and let ρ be a potential of ω. Then the (locally defined) formal functions $u_k = \frac{\partial}{\partial z^k}\left(\rho - \sum_{i\geq 1} \hbar^i \rho_i\right)$ satisfy the following two properties:*

1. *The anti-holomorphic terms in O_{u_k} have degree at most 1, i.e., O_{u_k} is a section of $\mathcal{W}_X \otimes (\overline{\mathcal{W}_X})_{\leq 1}$.*
2. *The terms in O_{u_k} which live in $\overline{\mathcal{W}}_X$ (which we call "terms of purely anti-holomorphic type") are given by*

$$u_k + \omega_{k\bar{m}}\bar{y}^m. \tag{2.8}$$

Hence if the Fedosov connection D_α is admissible (i.e., α is a polynomial in $\hbar$), then these u_k's are formal quantizable functions.

Proof. For simplicity, we will prove the case where α only has one term, namely, $\alpha = \hbar\alpha_1$; the general case can be proven similarly. Recall that O_{u_k} is the unique solution of the iterative equation:

$$O_{u_k} = u_k + \delta^{-1} \circ \left(\nabla + \frac{1}{\hbar}[I_\alpha, -]_\star \right) (O_{u_k}).$$

Observe that if a monomial A does not live in $\mathcal{A}_X^\bullet(\overline{\mathcal{W}}_X)$, then $\nabla A + \frac{1}{\hbar}[I_\alpha, A]_\star$ does not have terms living in $\mathcal{A}_X^\bullet(\overline{\mathcal{W}}_X)$. So we can prove the theorem by an induction on the weights of "terms of purely anti-holomorphic type" in O_{u_k}.

The terms in O_{u_k} of weight 1 are given by

$$\frac{\partial^2 \rho}{\partial \bar{z}^l \partial z^k} \bar{y}^l = \omega_{k\bar{l}} \bar{y}^l.$$

We know from the iterative equation that the weight 2 terms are given by

$$\delta^{-1} \circ \nabla^{0,1}(\omega_{k\bar{l}} \bar{y}^l),$$

which vanish since the Levi-Civita connection is compatible with both the symplectic form and the complex structure. The next terms are

$$\begin{aligned}
&\delta^{-1} \left(\nabla^{0,1} \left(-\hbar \frac{\partial \rho_1}{\partial z^k} \right) + \frac{1}{\hbar} \left[-\hbar \frac{\partial^2 \rho_1}{\partial \bar{z}^n \partial z^m} d\bar{z}^n \otimes y^m, \omega_{k\bar{l}} \bar{y}^l \right]_\star \right) \\
=&\delta^{-1} \left(-\hbar \frac{\partial^2 \rho_1}{\partial z^k \partial \bar{z}^l} d\bar{z}^l - \hbar \frac{\partial^2 \rho_1}{\partial \bar{z}^n \partial z^m} d\bar{z}^n \omega_{k\bar{l}} \omega^{m\bar{l}} \right) = 0.
\end{aligned}$$

Thus the weight 3 terms of purely anti-holomorphic type in O_{u_k} vanish. This argument can be generalized to all such terms of higher weights. $\square$

Remark 2.15. These formal functions are generalizations of holomorphic partial derivatives of a Kähler potential $\rho = \sum_i z^i \bar{z}^i$ on flat spaces $\mathbb{C}^n$:

$$f_i = \frac{\partial \rho}{\partial z^i} = \bar{z}^i,$$

which are also quantizable functions.

Given an admissible Fedosov connection D_α, as it is only a polynomial in $\hbar$ instead of a formal power series, we can evaluate D_α at $\hbar = 1/k$ for any non-zero complex number $k \in \mathbb{C} \setminus \{0\}$. The resulting connection will be denoted as $D_{\alpha,k}$:

$$D_{\alpha,k} = \nabla - \delta + k \cdot [I_{\alpha,k}, -]_{\star_k},$$

which acts as a differential on the Weyl bundle $\mathcal{A}_X^\bullet(\mathcal{W}_{X,\mathbb{C}})$. Since all terms in D_α (and also its evaluation $D_{\alpha,k}$) increase the degrees in $\mathrm{Sym}^\bullet \overline{TX}^*$ by 1, we obtain the following sub-complex:

$$\left(\mathrm{Sym}^\bullet \overline{TX}^* \otimes \mathcal{W}_X, D_{\alpha,k} \right).$$

A nice property of this sub-complex is that it is closed under the fiberwise star product $\star_k$. We are now ready to define (non-formal) quantizable functions of a fixed level k:

Definition 2.16. A flat section of $\mathrm{Sym}^\bullet \overline{TX}^* \otimes \mathcal{W}_X$ under the Fedosov connection $D_{\alpha,k}$ is called a *(non-formal) quantizable function of level k*. It is clear that quantizable functions of level k form a sheaf on X, and we let $\mathcal{C}^\infty_{\alpha,k}$ and $C^\infty_{\alpha,k}(X)$ denote the sheaf and the space of global quantizable functions respectively.

Remark 2.17. In general, we add a subscript k in a symbol to denote the evaluation $\hbar = 1/k$. For instance, the fiberwise product and the associated bracket in the above formula all take the complex value $\hbar = 1/k$.

Remark 2.18. The fact that the differential $D_{\alpha,k}$ is well-defined relies heavily on the property that the Fedosov connections D_α are quantizations of Kapranov's L_∞ structure [12]. General Fedosov connections cannot be evaluated at arbitrary non-zero complex values because they are power series in $\hbar$ and there is no convergence in general.

Next we describe some simple properties of non-formal Fedosov connections and the corresponding quantizable functions. First of all, we consider the weight defined by the degrees of anti-holomorphic terms in $\mathrm{Sym}^\bullet \overline{TX}^*$ and the associated increasing filtration on the bundle $\mathrm{Sym}^\bullet \overline{TX}^* \otimes \mathcal{W}_X$. From the explicit formula of the non-formal Fedosov connection $D_{\alpha,k}$, it is easy to see that it does not increase the degree in $\mathrm{Sym}^\bullet \overline{TX}^*$, and thus preserves the above increasing filtration. In particular, there is an associated increasing filtration on the (non-formal) quantizable functions. We let $(\mathcal{C}^\infty_{\alpha,k})_N$ denote those quantizable functions whose anti-holomorphic terms in $\mathcal{W}_{X,\mathbb{C}}$ have degrees at most N.

Similar to formal Fedosov quantization, we obtain a (non-formal) star product:

Proposition 2.19. *For every $k \in \mathbb{C} \setminus \{0\}$, the sheaf $\mathcal{C}^\infty_{\alpha,k}$ is closed under the star product $\star_k$ defined via the Fedosov connection $D_{\alpha,k}$. This star product is compatible with the above filtration in the sense that $(\mathcal{C}^\infty_{\alpha,k})_{N_1} \star_k (\mathcal{C}^\infty_{\alpha,k})_{N_2} \subset (\mathcal{C}^\infty_{\alpha,k})_{N_1+N_2}$.*

Proof. Since the fiberwise Wick product $\star_k$ is compatible with the connection $D_{\alpha,k}$, the product of two flat sections is still flat. The statement about the filtration is obvious. □

We now give some examples of quantizable functions.

Example 2.20. On the flat space $\mathbb{C}^n$ equipped with the standard Kähler form, every polynomial in $\mathbb{C}[z^1, \bar{z}^1, \cdots, z^n, \bar{z}^n]$ is a quantizable function.

Example 2.21. We can construct a class of quantizable functions by evaluating formal quantizable functions at $\hbar = 1/k$. First of all, there exists the following morphism of bundles:

$$\mathrm{Sym}^\bullet \overline{TX}^* \otimes \mathcal{W}_X[\hbar] \to \mathrm{Sym}^\bullet \overline{TX}^* \otimes \mathcal{W}_X$$

by taking the evaluation $\hbar = 1/k$. It is easy to see that this map preserves the Wick products and Fedosov connections on both sides. In particular, by taking cohomology, we obtain the following morphism of sheaves of algebras:

$$ev_k : \mathcal{C}^\infty_q[\hbar] \to \mathcal{C}^\infty_{\alpha,k}. \tag{2.9}$$

Since holomorphic functions are quantizable for any level k, the sheaf of quantizable functions gives a quantum $\mathcal{O}_X$-module.

In Section 5, we will see another class of quantizable functions arising from symmetries of the Kähler manifold X.

There is a type decomposition by the 1-forms in $D_{\alpha,k}$, in which the $(1,0)$-component is independent of k and explicitly given by

$$D^{1,0}_{\alpha,k} = \nabla^{1,0} - \delta^{1,0}.$$

This implies the following:

Lemma 2.22. *Let γ be a quantizable function with respect to the Fedosov connection $D_{\alpha,k}$. Then γ is determined by its components in $\overline{\mathcal{W}}_X$.*

Proof. Let $\gamma_{0,*}$ denote the components of γ in $\overline{\mathcal{W}}_X$. By Lemma 2.4, the section

$$\tilde{\gamma} := \sum_{k\geq 0}(\tilde{\nabla}^{1,0})^k(\gamma_{0,*})$$

must be annihilated by $D^{1,0}_{\alpha,k}$. Thus

$$D^{1,0}_{\alpha,k}(\gamma - \tilde{\gamma}) = 0.$$

According to the construction, $(\gamma-\tilde{\gamma})_{0,*} = 0$. Suppose $\gamma - \tilde{\gamma} \neq 0$, the terms in $D^{1,0}_{\alpha,k}(\gamma-\tilde{\gamma})$ of the lowest degree in $\mathcal{W}_X$ is equal to $\delta^{1,0}(\gamma-\tilde{\gamma})$ and cannot vanish, which is a contradiction. □

In formal Fedosov quantization, there is a stronger statement, namely, a flat section is uniquely determined by its symbol, which gives a one-to-one correspondence between flat sections and formal smooth functions. It is natural to ask if this still holds for non-formal Fedosov connections and quantizable functions. The following example gives a negative answer to this question.

Example 2.23. We consider the Fedosov connection D_α where $\alpha = \hbar \cdot \omega$. The term J_α in this connection is explicitly given by

$$J_\alpha = -\hbar \cdot \sum_{l\geq 0}(\tilde{\nabla}^{1,0})^k\left(\omega_{i\bar{j}}d\bar{z}^j \otimes y^i\right) = -\hbar \cdot \omega_{i\bar{j}}d\bar{z}^j \otimes y^i.$$

If we take the evaluation $\hbar = 1$, then the non-formal Fedosov connection can be written explicitly as

$$D_{\alpha,k=1} = \nabla - \delta^{1,0} + [I, -]_{\star_k}.$$

We now construct a non-trivial flat section whose symbol actually vanishes. We claim that the local section $\omega_{i\bar{m}}\bar{y}^m$ is flat under $D_{\alpha,k}$. Firstly,

$$D^{1,0}_{\alpha,k}(\omega_{i\bar{m}}\bar{y}^m) = \nabla^{1,0}(\omega_{i\bar{m}}\bar{y}^m) - \delta^{1,0}(\omega_{i\bar{m}}\bar{y}^m) = 0.$$

On the other hand, the fact that $\nabla^{0,1}(\omega_{i\bar{m}}\bar{y}^m) = 0$ implies the vanishing

$$D^{0,1}_{\alpha,k}(\omega_{i\bar{m}}\bar{y}^m) = 0.$$

Thus $\omega_{i\bar{m}}\bar{y}^m$ is a local quantizable function whose symbol vanishes.

It seems then that the name "quantizable functions" might not be so appropriate, since the symbol of a non-trivial flat section might vanish. This name can be justified in two ways. First of all, for a big enough k, the above uniqueness statement remains true:

Proposition 2.24. *For $|k| >> 0$, a flat section γ under the Fedosov connection $D_{\alpha,k}$ is uniquely determined by its symbol $\sigma(\gamma)$.*

Proof. Suppose γ and $\tilde{\gamma}$ are two flat sections under $D_{\alpha,k}$ with the same symbol, such that $\xi = \gamma - \tilde{\gamma}$ is non-trivial. By Lemma 2.22, ξ is determined by $\xi_{0,*}$. Let $\xi_{0,l}$ be the term in $\xi_{0,*}$ of the highest degree in $\mathrm{Sym}^\bullet \overline{TX}^*$. Then we have $l > 0$, since otherwise, $\sigma(\xi)$ will involve a term of degree $(0,0)$, i.e., a holomorphic function, which contradicts the fact that γ and $\tilde{\gamma}$ have the same symbols.

Let φ be a potential of α, i.e., $\partial\bar{\partial}\varphi = \alpha$. Recall that the term J_α in the Fedosov connection D_α is given by $-\sum_{m\geq 1}(\tilde{\nabla}^{1,0})^m(\bar{\partial}\varphi)$. An explicit computation gives that the term in $D^{0,1}_{\alpha,k}(\xi)$ of type $\mathcal{A}^{0,1}_X(\mathrm{Sym}^{l-1}\overline{TX}^*)$ can be written as

$$-\delta^{0,1}(\xi_{0,l}) - [\tilde{\nabla}^{1,0}\,(\bar{\partial}\varphi)_{\hbar=1/k}, \xi_{0,1}]_{\star_k} = \left(-\delta^{0,1} + O\left(\frac{1}{k}\right)\right)(\xi_{0,1}).$$

Since the leading term is non-zero, if $|k|$ is big enough, we obtain the non-vanishing of $D^{0,1}_{\alpha,k}(\xi)$, which contradicts the flatness of ξ. □

Therefore, for $|k| >> 0$, quantizable functions are in a one-to-one correspondence with a sub-class of smooth functions in $C^\infty(X)$, which, by abuse of notations, will also be called *quantizable functions*. The Fedosov connection $D_{\alpha,k}$ defines a non-formal deformation of this sub-class of smooth functions. Moreover, these functions are, by construction "close enough" to holomorphic ones since their associated flat sections have finite degrees in $\overline{TX}^*$. We will see in the next section that these functions form a sheaf of twisted holomorphic differential operators.

The second justification is that, as we will show in Corollary 3.5, the evaluation map (2.9) is sheaf-theoretically surjective. In other words, for any level k, every quantizable function of level k can be locally obtained from a formal quantizable function.

3. Sheaves of twisted differential operators from quantizable functions

We first give a brief review of the notion of twisted differential operators, following the notes [9].

Definition 3.1. A filtered ring A is called *almost commutative* if grA is commutative.

Definition 3.2. A *sheaf of twisted differential operators* (TDO for short) on X is a positively filtered sheaf $\mathcal{D}$ of almost commutative algebras together with an isomorphism

$$\psi_{\mathcal{D}} : gr\mathcal{D} \to \mathrm{Sym}^* \mathcal{T}X$$

of Poisson algebras, where $\mathcal{T}X$ denotes the holomorphic tangent sheaf on X.

Remark 3.3. The Poisson bracket on $\mathrm{Sym}^* \mathcal{T}X$ is the natural extension of the Lie bracket on $\mathcal{T}X$. In particular, we have the maps

$$\{-,-\} : \mathrm{Sym}^m \mathcal{T}X \times \mathrm{Sym}^n \mathcal{T}X \to \mathrm{Sym}^{m+n-1} \mathcal{T}X.$$

The main result of this section is that the sheaf $\mathcal{C}^\infty_{\alpha,k}$ of non-formal quantizable functions of level k is a TDO.

3.1. **A filtration on quantizable functions.**

Consider the natural increasing filtration $(\mathcal{C}^\infty_{\alpha,k})_0 \subset (\mathcal{C}^\infty_{\alpha,k})_1 \subset \cdots$ on $\mathcal{C}^\infty_{\alpha,k}$ by degrees of anti-holomorphic terms in $\mathcal{W}_{X,\mathbb{C}}$. Let $\widetilde{\mathcal{C}^\infty_{\alpha,k}}$ denote the associated graded sheaf. Suppose α is a local flat section under $D_{\alpha,k}$ which lives in $\widetilde{(\mathcal{C}^\infty_{\alpha,k})}_N(U)$. Let $\alpha_N \in (\mathcal{W}_{X,\mathbb{C}})_N$ denote the leading term with respect to this filtration. Lemma 2.4 says that it must be of the form

$$\alpha_N = \sum_{k\geq 0} (\tilde{\nabla}^{1,0})^k(\alpha_{0,N}),$$

and the following vanishing holds:

$$\nabla^{0,1}(\alpha_{0,N}) = 0.$$

Proposition 3.4. *For any $N \geq 0$, there is the following isomorphism of $\mathcal{O}_X$-modules:*

$$\begin{aligned} \psi : \widetilde{(\mathcal{C}^\infty_{\alpha,k})}_N &\to \mathrm{Sym}^N \mathcal{T}X \\ \alpha &\mapsto \alpha_{0,N} \lrcorner (\omega^{-1})^N \end{aligned}$$

Proof. We have seen that $\nabla^{0,1}(\alpha_N) = 0$. Since ω is parallel with respect to ∇, it follows that the image $\alpha_{0,N}\lrcorner(\omega^{-1})^N$ is $\bar{\partial}$-closed. Since the flat section corresponding to a holomorphic function does not contain any terms in $\overline{\mathcal{W}}_X$, the map ψ is a morphism of $\mathcal{O}_X$-modules.

Next we show that ψ is an isomorphism. The injectivity of ψ is obvious. To show its surjectivity, we consider the formal functions u_j's in Proposition 2.14. From the explicit formula in the proposition, we obtain degree 1 quantizable functions after taking the evaluation. Explicitly, we have $ev_k(O_{u_j}) \in \widetilde{(\mathcal{C}^\infty_{\alpha,k})}_1$. Moreover, the leading term of $ev_k(O_{u_j})$ is $\omega_{j\bar{l}}\bar{y}^l$, which implies that $\psi(ev_k(O_{u_j})) = \partial_{y^j}$. This finishes the proof of this proposition since $\mathrm{Sym}^N \mathcal{T}X$ is locally generated by ∂_{y^j}'s as an $\mathcal{O}_X$-module. ∎

Corollary 3.5. *The evaluation map* (2.9) *in Example 2.21 is surjective as a morphism of sheaf of algebras.*

Proof. We only need to show that locally every quantizable function can be obtained by taking evaluation of formal quantizable functions. We use an induction via the filtration on $\mathcal{C}^\infty_{\alpha,k}$: for $N = 0$, we have the isomorphism $(\widetilde{\mathcal{C}^\infty_{\alpha,k}})_0 \cong \mathcal{O}_X$, and Example 2.13 implies that $(\widetilde{\mathcal{C}^\infty_{\alpha,k}})_0$ lives in the image of ψ. Suppose $\gamma \in (\widetilde{\mathcal{C}^\infty_{\alpha,k}})_N$ for some $N > 0$. By Proposition 3.4, there exists a holomorphic function f and indices $i_1, \cdots, i_N$, such that $O_f \star_k O_{u_1} \star \cdots \star O_{u_N} - \gamma \in (\widetilde{\mathcal{C}^\infty_{\alpha,k}})_{N-1}$. Applying the induction hypothesis finishes the proof. ∎

Theorem 3.6. *For any α and level k, the sheaf $\mathcal{C}^\infty_{\alpha,k}$ of quantizable functions (under the Fedosov connection $D_{\alpha,k}$) forms a sheaf of twisted differential operators (TDO) on X.*

Proof. We first show that the associated product on $gr\mathcal{D}$ is commutative. Let $O_f \in \mathcal{D}_k \setminus \mathcal{D}_{k-1}$ and $O_g \in \mathcal{D}_l \setminus \mathcal{D}_{l-1}$; equivalently, the highest degrees of anti-holomorphic terms in O_f and O_g are k and l respectively. It is clear that $O_f \star O_g, O_g \star O_f \in \mathcal{D}_{l+k} \setminus \mathcal{D}_{l+k-1}$, and they have the same monomials of the highest anti-holomorphic degree. So we have $[O_f, O_g]_\star \in \mathcal{D}_{l+k-1}$, and hence the associated graded grD is commutative.

To show that ψ respects the Poisson structure on both sides, we only need to compute the bracket

$$[-,-] : \mathcal{D}^1 \times \mathcal{D}^0 \to \mathcal{D}^0.$$

We take any holomorphic function f on U and take its associated flat section O_f, and consider $ev_k(O_{u_j}) \in \mathcal{D}^1(U)$. It is then clear that $[O_{u_j}, O_f]_\star \in \mathcal{D}^0$, which must correspond to a holomorphic function given by its symbol. A simple computation shows that it is precisely given by

$$[\omega_{j\bar{l}}\bar{y}^l, \frac{\partial f}{\partial z^m} y^m]_\star = \frac{\partial f}{\partial z^j}.$$

This identifies the Poisson brackets on $gr\mathcal{D}$ and $\mathrm{Sym}^\bullet(\mathcal{T}X)$. □

3.2. Characteristic classes of the TDO given by quantizable functions.

For every TDO on a complex algebraic variety, there is a characteristic class which lives in $H^1(X, \Omega^{\geq 1})$. When X is a complex manifold, this cohomology group is isomorphic to $H^1(X, \Omega^1_{cl})$, where Ω_{cl} denotes the subsheaf of Ω^1_X which is closed under ∂.

We briefly recall the characteristic class of a TDO $\mathcal{D}$ on a complex manifold X, and refer to [9] for more details. As a TDO, there is an increasing filtration $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \subset \mathcal{D}$ on $\mathcal{D}$, in which $\mathcal{D}_1$ is locally isomorphic to $\mathcal{O}_X \oplus \mathcal{T}_X$. Let $\{U_i\}$ be an open covering of X such that over each intersection $U_i \cap U_j$, there is an isomorphism

$$\mathcal{D}_1(U_i) \to \mathcal{D}_1(U_j),$$

which must be of the form

$$f \mapsto f, \quad \xi \mapsto \xi + \alpha_{ij}(\xi), \tag{3.1}$$

where $\xi \in \Omega^1(U_i \cap U_j)$. In particular, on a Kähler manifold, if we choose the $(1,0)$-forms α_{ij} to be ∂-closed. Then the Cech cohomology class $\{\alpha_{ij}\} \in H^1(X, \Omega^1)$ is the characteristic class of the TDO $\mathcal{D}$.

For the sheaf $\mathcal{C}^\infty_{\alpha,k}$ of quantizable functions, we first choose an open covering $\{U_i\}$ of X by balls, and fix a potential ρ_i on each U_i. Then on each U_i, the isomorphism $\mathcal{O}_X(U_i) \oplus \mathcal{T}_X(U_i) \cong \mathcal{D}_1(U_i)$ is given explicitly by

$$f \mapsto f, \quad \partial_{z^j} \mapsto O_k\left(\frac{\partial \rho_i}{\partial z^j}\right)$$

for any holomorphic function f on U_i. On each intersection $U_i \cap U_j$, we choose a $(1,0)$-form

$$\alpha_{ij} := \partial(\rho_i - \rho_j),$$

which clearly satisfies equation (3.1). We have $\bar{\partial}(\alpha_{ij}) = 0$ since ρ_i's are all potentials of the same closed $(1,1)$-form, and $\partial(\alpha_{ij}) = 0$ since $\partial^2 = 0$. Thus we obtain the Cech cohomology representative of the characteristic class of $\mathcal{C}^\infty_{\alpha,k}$:

$$\{\alpha_{ij}\} \in H^1(X, \Omega^1_{cl}).$$

On the other hand, a standard argument in complex geometry shows that the Dolbeault representative of $\{\alpha_{ij}\}$ is exactly given by $\bar{\partial}\partial\rho_i$ which gives a global closed differential form of type $(1,1)$. This exactly coincides with the Karabagov form.

To summarize, we obtain following:

Theorem 3.7. *The characteristic class of the TDO $\mathcal{C}^{\infty}_{\alpha,k}$ is given by $[\omega - \alpha]$.*

4. QUANTIZING FUNCTIONS AS HOLOMORPHIC DIFFERENTIAL OPERATORS

In this section, we will see how quantizable functions, which come from deformation quantization, "act on" geometric quantization. More precisely, let X be a Kähler manifold X equipped with a prequantum line bundle L, meaning that there is a connection ∇_L on L whose curvature is $\nabla_L^2 = i\omega$. Then we will show that quantizable functions act on the holomorphic sections of $L^{\otimes k}$ as *holomorphic differential operators*. This gives a non-formal quantization of a large subspace of smooth functions on X.

Let k be any integer, we consider the sheaf $\mathcal{C}^{\infty}_{\alpha,k}$ of quantizable functions with level k and $\alpha := -R_{i\bar{j}k\bar{l}}\omega^{k\bar{l}}dz^i \wedge d\bar{z}^j$. Throughout this section, we fix this α whose associated star product is exactly the *Berezin-Toeplitz quantization* with Karabegov form $k(\omega - \alpha)$. In this case, we will prove that the sheaf $\mathcal{C}^{\infty}_{\alpha,k}$ of quantizable functions is isomorphic to the sheaf $\mathcal{D}(L^{\otimes k})$ of holomorphic differential operators on $L^{\otimes k}$ (as TDOs).

The main idea comes from our previous work [5], where we extended Fedosov's method from algebras to modules. More concretely, we will construct a sheaf of modules over the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}$ with a compatible flat Fedosov connection whose associated cohomology is exactly $H^0(X, L^{\otimes k})$. The quantizable functions then act on this module sheaf, preserving the flat sections corresponding to holomorphic sections of $L^{\otimes k}$. We will also explain a generalization to holomorphic differential operators on any line bundle as quantization of functions.

We begin with the linear algebra of the Bargmann-Fock action:

Definition 4.1. We define an action of a monomial $f = z^{\alpha_1} \cdots z^{\alpha_k}\bar{z}^{\beta_1} \cdots \bar{z}^{\beta_l} \in \mathcal{W}_{\mathbb{C}^n}$ on $s \in \mathcal{F}_{\mathbb{C}^n} := \mathbb{C}[[z^1, \cdots, z^n]][[\hbar]]$ by

$$f \circledast s := (-\hbar)^l \frac{\partial}{\partial z^{\beta_1}} \circ \cdots \circ \frac{\partial}{\partial z^{\beta_l}} \circ m_{z^{\alpha_1} \cdots z^{\alpha_k}}(s), \tag{4.1}$$

where $m_{z^{\alpha_1} \cdots z^{\alpha_k}}$ denotes the multiplication by $z^{\alpha_1} \cdots z^{\alpha_k}$. It is known that

$$f \circledast (g \circledast s) = (f \star g) \circledast s.$$

Thus equation (4.1) defines an action of the Wick algebra $\mathcal{W}_{\mathbb{C}^n}$ on $\mathcal{F}_{\mathbb{C}^n}$, known as the *Bargmann-Fock representation* (or the *Wick normal ordering* in physics literature).

The Kähler form on X enables us to define the fiberwise Bargmann-Fock action, making the holomorphic Weyl bundle $\mathcal{W}_X$ a sheaf of $\mathcal{W}_{X,\mathbb{C}}$-modules. Explicitly, a monomial in $\mathcal{W}_{X,\mathbb{C}}$ acts as a differential operator on $\mathcal{W}_X$ as follows:

$$y^{i_1} \cdots y^{i_k}\bar{y}^{j_1} \cdots \bar{y}^{j_l} \mapsto (-\hbar)^l \, \omega^{p_1\bar{j}_1} \cdots \omega^{p_l\bar{j}_l} \frac{\partial}{\partial y^{p_1}} \circ \cdots \frac{\partial}{\partial y^{p_l}} \circ m_{y^{i_1} \cdots y^{i_k}} \tag{4.2}$$

Definition 4.2. For every $k > 0$, we define the *level k Bargmann-Fock sheaf* $\mathcal{F}_{L^{\otimes k}}$ by twisting $\mathcal{W}_X$ with tensor powers of the prequantum line bundle L:

$$\mathcal{F}_{L^{\otimes k}} := \mathcal{W}_X \otimes_{\mathcal{O}_X} L^{\otimes k}.$$

The following proposition shows that the Fedosov connection can be extended to a flat connection on $\mathcal{F}_{L^{\otimes k}}$ in a compatible way.

Proposition 4.3. *Let $\nabla_{L^{\otimes k}}$ denote the Chern connection on $L^{\otimes k}$ whose curvature is $k \cdot \omega$. Then the connection*
$$(\nabla + k \cdot \gamma_{\alpha,k} \circledast_k) \otimes 1 + 1 \otimes \nabla_{L^{\otimes k}}$$
on the Bargmann-Fock sheaf $\mathcal{F}_{L^{\otimes k}}$ is flat and compatible with the Fedosov connection $D_{\alpha,k}$ on $\mathcal{W}_{X,\mathbb{C}}$. Thus, by abuse of notation, we also denote the above connection on $\mathcal{F}_{L^{\otimes k}}$ by $D_{\alpha,k}$.

Proof. Let s be a section of $\mathcal{W}_X$. Then we have
$$\begin{aligned}
R_\nabla \circledast_k s &= R_{i\bar{j}p\bar{q}} dz^i \wedge d\bar{z}^j \otimes y^p \bar{y}^q \circledast_k s \\
&= R_{i\bar{j}p\bar{q}} dz^i \wedge d\bar{z}^j \otimes \bar{y}^q \circledast_k (y^p \cdot s) \\
&= R_{i\bar{j}p\bar{q}} dz^i \wedge d\bar{z}^j \otimes \left(-\frac{1}{k}\right) \omega^{l\bar{q}} \frac{\partial}{\partial y^l}(y^p \cdot s) \\
&= \frac{1}{k}\left(\nabla^2(s) - R_{i\bar{j}p\bar{q}} \cdot \omega^{p\bar{q}} dz^i \wedge d\bar{z}^j \circledast_k s\right) \\
&= \frac{1}{k}\left(\nabla^2(s) + \alpha \circledast_k s\right).
\end{aligned}$$
Note that we have used the definition of the Bargmann-Fock action. Flatness of the connection $D_{\alpha,k}$ follows from a straightforward computation:
$$\begin{aligned}
D_{\alpha,k}^2 =& \nabla^2 \otimes 1 + \left((k\nabla\gamma_{\alpha,k} + k^2 \cdot \gamma_{\alpha,k} \star_k \gamma_{\alpha,k}) \circledast_k\right) \otimes 1 + 1 \otimes \nabla_{L^{\otimes k}}^2 \\
=& (k\nabla\gamma_{\alpha,k} + k^2 \cdot \gamma_{\alpha,k} \star_k \gamma_{\alpha,k} + k \cdot R_\nabla - \alpha) \circledast_k + 1 \otimes \nabla_{L^{\otimes k}}^2 \\
=& -k \cdot \omega + k \cdot \omega = 0.
\end{aligned}$$
Here we have used equation (2.6) and the prequantum condition that $\nabla_{L^{\otimes k}}^2 = k \cdot \omega$. To see that compatibility between Fedosov connections, let ξ and s be sections of $\mathcal{W}_{X,\mathbb{C}}$ and $\mathcal{F}_{L^{\otimes k}}$ respectively. Then we have
$$\begin{aligned}
D_{\alpha,k}(\xi \circledast_k s) =& \nabla(\xi) \circledast_k s + (-1)^{|\xi|} \xi \circledast_k \nabla(s) + k\gamma_{\alpha,k} \circledast_k (\xi \circledast_k s) \\
=& \nabla(\xi) \circledast_k s + (-1)^{|\xi|} \xi \circledast_k \nabla(s) + k[\gamma_{\alpha,k}, \xi]_{\star_k} \circledast_k s) + (-1)^{|\xi|} \xi \circledast_k (k\gamma_{\alpha,k} \circledast_k s) \\
=& D_{\alpha,k}(\xi) \circledast_k s + (-1)^{|\alpha|} \xi \circledast_k D_{\alpha,k}(s).
\end{aligned}$$
□

In Fedosov quantization, we have a ono-to-one correspondence between smooth functions and flat sections of the Weyl bundle. We have an analogous isomorphism theorem for the Bargmann-Fock sheaves:

Theorem 4.4. *For any open set $U \subset X$, the space of flat sections of $\mathcal{F}_{L^{\otimes k}}$ under the connection $D_{\alpha,k}$ is canonically isomorphic to holomorphic sections of $L^{\otimes k}$.*

Proof. For any open set $U \subset X$, the symbol map
$$\sigma : \Gamma(U, \mathcal{F}_{L^{\otimes k}}) \to \Gamma(U, L^{\otimes k}).$$
is defined (as before) by setting all y^i's to zero. We write a section $s \in \Gamma(U, \mathcal{F}_{L^{\otimes k}})$ in local coordinates as $s = \sum_J s_J y^J \otimes e_{L^k}$, where J runs over all holomorphic multi-indices and e_{L^k} is a local holomorphic frame of $L^{\otimes k}$. Then
$$D_{\alpha,k}(s) = D_{\alpha,k}\Big(\sum_{|J|\geq 0} s_J y^J \otimes e_{L^k}\Big)$$

$$= d_X(s_0) \otimes e_{L^k} + \sum_{|J|>0} k \cdot \gamma_{\alpha,k} \circledast_k (s_J y^J \otimes e_{L^k}) + (\sum_{|J|\geq 0} s_J y^J) \otimes \nabla_L(e_{L^k})$$
$$= \big(\partial_X(s_0) + \bar{\partial}_X(s_0)\big) \otimes e_{L^k} + \sum_{|J|>0} k \cdot \gamma_{\alpha,k} \circledast_k (s_J y^J \otimes e_{L^k}) + (\sum_{|J|\geq 0} s_J y^J) \otimes \nabla_L^{1,0}(e_{L^k}).$$

Here d_X denotes the de Rham differential on X. By analyzing the type of the part $\gamma_{\alpha,k}$ in the Fedosov connection, it is easy to see that if $\bar{\partial} s_0 \neq 0$, then we must have $D_{\alpha,k}(s) \neq 0$. Thus, the symbols of flat sections $\Gamma^{flat}(U, \mathcal{F}_{L^{\otimes k}})$ must lie in $H^0(U, L^{\otimes k})$. This induces the map

$$\sigma : \Gamma^{flat}(U, \mathcal{F}_{L^{\otimes k}}) \to H^0(X, L^{\otimes k}). \tag{4.3}$$

To show the surjectivity of this map, we first find a flat section of the Bargmann-Fock sheaf $\mathcal{F}_{L^{\otimes k}}$ whose symbol is e_{L^k}. Suppose that the hermitian metric of $L^{\otimes k}$ is given locally by $\langle e_{L^k}, e_{L^k} \rangle = e^{k\cdot\rho}$. The connection $\nabla_{L^{\otimes k}}$ can then be written explicitly as

$$\nabla_{L^{\otimes k}}(e_{L^k}) = k \cdot \partial\rho \otimes e_{L^k},$$

and the prequantum condition implies that $\bar{\partial}\partial\rho = \omega$. We define a local section of the holomorphic Weyl bundle $\mathcal{W}_X$ by $\beta := \sum_{k\geq 1} (\tilde{\nabla}^{1,0})^k(\rho)$. It is clear the section $e^{k\cdot\beta} \otimes e_{L^k}$ of $\mathcal{F}_{L^{\otimes k}}$ has symbol e_{L^k}. The following proposition, which will be proved in Appendix A, says that this section is indeed flat under $D_{k,\alpha}$:

Proposition 4.5. *The section $e^{k\cdot\beta} \otimes e_{L^k}$ is closed under the Fedosov connection, i.e.,*

$$D_{\alpha,k}(e^{k\cdot\beta} \otimes e_{L^k}) = 0.$$

The surjectivity of the symbol map (4.3) now follows from this proposition since any local holomorphic section of $L^{\otimes k}$ is of the form $f \cdot e_{L^k}$ for some holomorphic function f. We simply take the section of $\mathcal{F}_{L^{\otimes k}}$ to be $O_f \cdot \big(e^{k\cdot\beta} \otimes e_{L^k}\big)$, which is obviously the image of $f \cdot e_{L^k}$.

For injectivity, suppose there is nonzero section $s \in \Gamma(U, \mathcal{F}_{L^{\otimes k}})$ such that $\sigma(s) = 0$. Writing $s = \sum_{|J|=i>0}^{\infty} s_J y^J$, then the lowest degree term of $D_{\alpha,k}(s)$ is given by $\delta(\sum_{|J|=i} s_J y^J) \neq 0$ which implies the non-vanishing of $D_{\alpha,k}(s)$. $\square$

Theorem 4.6. *Suppose that X is a Kähler manifold equipped with a prequantum line bundle L. Let $\alpha := -R_{i\bar{j}k\bar{l}}\omega^{k\bar{l}} dz^i \wedge d\bar{z}^j$ (so that the Karabegov form coincides with that of the Berezin-Toeplitz quantization of X). Then for any positive integer k, there is a natural isomorphism*

$$\varphi : \mathcal{C}^\infty_{\alpha,k} \to \mathcal{D}(L^{\otimes k})$$

from the sheaf of algebras of level k quantizable functions to the sheaf of holomorphic differential operators on $L^{\otimes k}$. Furthermore, this isomorphism is compatible with the filtration on quantizable functions and that on differential operators by orders, and hence gives an isomorphism of TDOs.

Proof. We first define the map φ by showing that quantizable functions act on the space of holomorphic sections of $L^{\otimes k}$ as differential operators. We have seen that level k quantizable functions correspond to flat sections of $\mathcal{W}_{X,\mathbb{C}}$ under $D_{\alpha,k}$, and holomorphic sections of $L^{\otimes k}$ correspond to flat sections of $\mathcal{F}_{L^{\otimes k}}$. Since the flat connections on these two bundles are compatible, the outcome of this action is also flat and thus correspond to a holomorphic section of $L^{\otimes k}$. As to the compatibility of filtrations on both sides, it is clear since

$\bar{y}^j$ acts as the differential $-\omega^{i\bar{j}}\frac{\partial}{\partial y^i}$ in the Bargmann-Fock action, and thus the filtration on both sides is preserved by φ. The locality is clear from the construction.

We will perform local computations to show that φ is an isomorphism of sheaves. For the injectivity of φ, suppose f is a quantizable function such that $\varphi(f) = 0$ as a differential operator. To show that $f = 0$, we use induction on the anti-holomorphic degrees of f. Suppose $f \in (\mathcal{C}^\infty_{\alpha,k})_0$, or equivalently, f is a holomorphic function (recall that there is a natural increasing filtration $(\mathcal{C}^\infty_{\alpha,k})_0 \subset (\mathcal{C}^\infty_{\alpha,k})_1 \subset \cdots$ on $\mathcal{C}^\infty_{\alpha,k}$ by degrees of anti-holomorphic terms in $\mathcal{W}_{X,\mathbb{C}}$). Then $\varphi(f)$ simply acts by multiplication and thus $f = 0$. For the induction step, suppose $f \in (\mathcal{C}^\infty_{\alpha,k})_m$ and $\varphi(f) = 0$. We take any local flat section s of $\mathcal{F}_{L^{\otimes k}}$, and any non constant holomorphic function g. Then we have

$$O_f \circledast_k (O_g \cdot s) = [O_f, O_g]_\star \circledast_k s + O_g \circledast_k (O_f \circledast_k s) = [O_f, O_g]_{\star_k} \circledast_k s = 0,$$

which implies that $\varphi(f \star_k g - g \star_k f) = 0$. Since O_g contains only monomials in $\mathcal{W}_X$ and the bracket $[-,-]_{\star_k}$ kills at least one $\bar{y}^j$'s in O_f, we see that $[O_f, O_g]_{\star_k} \in (\mathcal{C}^\infty_{\alpha,k})_{m-1}$, and we have $f \star_k g - g \star_k f = 0$ by the induction hypothesis. This implies that O_f is also a section of $\mathcal{W}_X$ and has to be zero.

Next we show the surjectivity of φ. With respect to the choice of the holomorphic frame e_{L^k}, the holomorphic differential operators are generated by holomorphic functions and ∂_{z^i}'s. Let $e^{k\cdot\beta} \otimes e_{L^k}$ be the section of $\mathcal{F}_{L^{\otimes k}}$ as in the proof of Theorem 4.4. We consider the function u_i defined in Propossition 2.14, and claim that $O_{u_i} \circledast_k (e^{k\cdot\beta} \otimes e_{L^k}) = 0$. By Theorem 4.4, we only need to show that its symbol vanishes.

For this purpose, notice that, as shown in Proposition 2.14, every term in O_{u_i} has degree at most 1 in the anti-holomorphic $\bar{y}^i$'s. Thus we only need terms in O_{u_i} of type $(0,0)$, $(0,1)$ and $(1,1)$ to find the symbol $\sigma\left(O_{u_i} \circledast_k (e^{k\cdot\beta} \otimes e_{L^k})\right)$. For this, we recall that

$$O_{u_i} = \left(\frac{\partial\rho}{\partial z^i} + 2\sqrt{-1}\omega_{i\bar{k}}\bar{y}^k + 2\sqrt{-1}\frac{\partial\omega_{i\bar{j}}}{\partial z^k}y^k\bar{y}^j\right) + \frac{1}{k}\frac{\partial\rho_1}{\partial z^i} + \cdots;$$

here ρ and ρ_1 are local potentials for ω and the Ricci form respectively. The dots denote those terms which contribute trivially to the symbol. Then we compute:

$$\begin{aligned}
&\sigma\left(O_{u_i} \circledast_k (e^{k\cdot\beta} \otimes e_{L^k})\right)\\
=&\left(\frac{\partial\rho}{\partial z^i} + \frac{1}{k}\frac{\partial\rho_1}{\partial z^i} + 2\sqrt{-1}\sigma\left([\omega_{i\bar{m}}\bar{y}^m, \beta]_{\star_k}\right) + 2\sqrt{-1}\frac{1}{k}\frac{\partial\omega_{i\bar{j}}}{\partial z^k}\omega^{k\bar{j}}\right) \cdot (e^{k\cdot\beta} \otimes e_{L^{\otimes k}})\\
=&\frac{\partial\rho}{\partial z^i} + 2\sqrt{-1}[\omega_{i\bar{m}}\bar{y}^m, \frac{\partial\rho}{\partial z^j}y^j]_{\star_k} + \frac{1}{k}\left(\frac{\partial\rho_1}{\partial z^i} + \frac{\partial\omega_{i\bar{j}}}{\partial z^k}\omega^{k\bar{j}}\right)\\
=&0,
\end{aligned}$$

where we have used the basic fact in Kähler geometry that

$$\frac{\partial\rho_1}{\partial z^i} + \frac{\partial\omega_{i\bar{j}}}{\partial z^k}\omega^{k\bar{j}} = 0$$

in the second equality.

On the other hand, any local flat section of $\mathcal{F}_{L^{\otimes k}}$ can be written as $O_g \cdot \left(e^{k\cdot\beta} \otimes e_{L^k}\right)$ for some holomorphic function g. We have

$$\begin{aligned} O_{u_i} \circledast_k (O_g \cdot e^{k\cdot\beta} \otimes e_{L^k}) &= [O_{u_i}, O_g]_{\star_k} \circledast_k (e^{k\cdot\beta} \otimes e_{L^k}) + O_g \cdot (O_{u_i} \circledast_k e^{k\cdot\beta} \otimes e_{L^k}) \\ &= [O_{u_i}, O_g]_{\star_k} \circledast_k (e^{k\cdot\beta} \otimes e_{L^k}) \\ &= -\frac{1}{k} O_{\frac{\partial g}{\partial z^i}} \cdot (e^{k\cdot\beta} \otimes e_{L^k}). \end{aligned}$$

In the last equality we have used Lemma 4.7 below. This shows that $\varphi(u_i) = -\frac{1}{k} \cdot \partial_{z^i}$ under the holomorphic frame e_{L^k}. The proof of the theorem is now completed. □

Lemma 4.7. *For any holomorphic function g, there is the following equality:*

$$[O_{u_i}, O_g]_{\star_k} = -\frac{1}{k} \cdot O_{\frac{\partial g}{\partial z^i}}.$$

Equivalently, taking bracket of flat sections of holomorphic functions with O_{u_i} is equivalent to taking the partial derivative $-\frac{1}{k}\partial_{z^i}$ to smooth functions.

Proof. It is clear that the bracket $[O_{u_i}, O_g]_{\star_k}$ is still a flat section. Thus we only need to find its symbol. Since g is a local holomorphic function, O_g only contains terms in $\mathcal{W}_X$ and we only need the purely anti-holomorphic terms of O_{u_i} to compute $\sigma\left([O_{u_i}, O_g]_{\star_k}\right)$:

$$\sigma\left([O_{u_i}, O_g]_{\star_k}\right) = [2\sqrt{-1}\omega_{i\bar{j}}\bar{y}^j, \frac{\partial g}{\partial z^m} y^m]_{\star_k} = 2\sqrt{-1} \cdot \omega_{i\bar{j}} \cdot \frac{\sqrt{-1}}{2k}(-1)\omega^{m\bar{j}} \cdot \frac{\partial g}{\partial z^m} = -\frac{1}{k} \cdot \frac{\partial g}{\partial z^i}.$$

□

Theorem 4.6 says that the sheaf $\mathcal{C}^\infty_{\alpha,k}$ consists of exactly those functions which can be quantized to differential operators on the Hilbert spaces $H^0(X, L^{\otimes k})$. This produces a non-formal deformation of the classical multiplication. This isomorphism also gives local generators of $\mathcal{C}^\infty_{\alpha,k}$.

Corollary 4.8. *For any open set $U \subset X$ isomorphic to an open ball in $\mathbb{C}^n$, quantizable functions on U are generated by holomorphic functions and the functions u_i's defined in Proposition 2.14.*

All the above computations and results can be generalized to the following situation: Assume that the Karabegov form is admissible and *integral*, namely,

$$\omega_\hbar = \frac{1}{\hbar}(\omega - \alpha) \in \frac{1}{\hbar} \cdot \left(H^{1,1}(X) \cap H^2(X, \mathbb{Z})\right)[\hbar]. \tag{4.4}$$

Then we can take the evaluation $\hbar = 1$, and twist $\mathcal{W}_X$ by a line bundle to obtain a Bargmann-Fock bundle which admits a Fedosov flat connection. Thus the quantizable functions associated to these Karabegov forms can be quantized to holomorphic differential operators on a line bundle. Conversely, holomorphic differential operator on any holomorphic line bundle can be identified with a class of quantizable functions.

Remark 4.9. The construction here can be generalized straightforwardly to the non-abelian case: we can twist the Weyl bundle with any holomorphic vector bundle E over the Kähler manifold X, and identify holomorphic differential operators $\mathcal{D}(E, E)$ with a subspace of $C^\infty(X, \mathrm{End}(E))$. By combining this result with the BV quantization method in [6,10], we can give another proof of the trace formula for differential operators for Kähler manifolds as first introduced in [7].

4.1. **Quantizable functions in geometric quantization.**

In geometric quantization, the prequantum operator Q_f associated to a smooth function f on sections of k-th tensor power of the prequantum line bundle L is defined as

$$Q_f := \frac{\sqrt{-1}}{2\pi \cdot k}\nabla^k_{X_f} + f,$$

where ∇^k denotes the connection on the line bundle $L^{\otimes k}$. For a holomorphic section $s \in H^0(X, L^{\otimes k})$, the output $Q_f(s)$ is in general not a holomorphic section of $L^{\otimes k}$.

Definition 4.10. A smooth function $f \in C^\infty(X)$ is called *quantizable in the sense of geometric quantization* if the operator Q_f preserves the Hilbert space $H^0(X, L^{\otimes k})$.

It is clear from the definition that Q_f is a differential operator. Theorem 4.6 implies that there exists a quantizable function whose action can be identified with Q_f. So our notion of quantizable functions is a vast generalization of the previous notion of quantizable functions (or *polarization-preserving functions*) in geometric quantization. In particular, we can obtain higher order differential operators from them. Furthermore, we get a subspace of smooth functions closed under the star product.

5. EXAMPLES: FIRST ORDER QUANTIZABLE FUNCTIONS FROM SYMMETRIES

In this section, we give a class of examples of (first order) quantizable functions arising from symmetries on Kähler manifolds. It is known that symmetries of symplectic manifolds is encoded in moment maps. More precisely, let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Let (X, ω) be a symplectic manifold which admits a Hamiltonian G-action. For every $g \in \mathfrak{g}$, let V_g denote the vector field associated to the infinitesimal action, whose action on smooth functions can be expressed as a Poisson bracket $\mathcal{L}_{V_g} = \{\mu(g), -\}$, where $\mu : \mathfrak{g} \to C^\infty(X)$ is the classical moment map. A quantized notion of the moment map, called *the quantum moment map*, was introduced in [21].

Definition 5.1. Let X, G be as above. Suppose $(C^\infty(X)[[\hbar]], \star)$ is a G-invariant deformation quantization of X. Then a *quantum moment map* is a homomorphism of Lie algebras

$$\mu_\hbar : \mathfrak{g} \to C^\infty(X)[[\hbar]],$$

such that for every $g \in \mathfrak{g}$, we have the equality $\mathcal{L}_{V_g} = [\mu_\hbar(g), -]_\star$ for formal smooth functions $C^\infty(X)[[\hbar]]$. Here the Lie bracket on the right hand side is the one associated to the star product $\star$. Explicitly, for $g, h \in \mathfrak{g}$, we require

$$\mu_\hbar([g,h]) = \mu_\hbar(g) \star \mu_\hbar(h) - \mu_\hbar(h) \star \mu_\hbar(g).$$

We will focus on the case when X is a Kähler manifold and the G-action *also preserves the complex structure*. For later computations, we will fix a basis $\{g_i\}_{i=1}^{\dim \mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$, and let $\mathcal{L}_i$ and ι_i denote respectively the Lie derivative and contraction associated to the vector field V_{g_i}. These operators can all be naturally extended to $\mathcal{A}^\bullet_X(\mathcal{W}_{X,\mathbb{C}})$.

We would like to show that the image of the quantum moment map are all first order quantizable functions. To start with, recall that the Fedosov connection D_F is of the form

$$D_F = \nabla + \frac{1}{\hbar}[I, -]_\star,$$

where $[-,-]_\star$ is the Lie bracket associated to the fiberwise star product. A simple computation shows that $[D_F-\nabla,\iota_i]=\frac{1}{\hbar}[\iota_i(I),-]_\star$: let $\alpha\in\mathcal{A}^*(X),\beta\in\mathcal{W}_{X,\mathbb{C}}$. Then we have

$$\begin{aligned}&[D_F-\nabla,\iota_i](\alpha\otimes\beta)\\ =&(D_F-\nabla)(\iota_i(\alpha)\otimes\beta)+\iota_i\circ(D_F-\nabla)(\alpha\otimes\beta)\\ =&\frac{1}{\hbar}[I,\iota_i(\alpha)\otimes\beta]_\star+\iota_i\left(\frac{1}{\hbar}[I,\alpha\otimes\beta]_\star\right)\\ =&\frac{1}{\hbar}\Big(I\star(\iota_i(\alpha)\otimes\beta)-(-1)^{|\alpha|-1}(\iota_i(\alpha)\otimes\beta)\star I+\iota_i(I\star(\alpha\otimes\beta)-(-1)^{|\alpha|}(\alpha\otimes\beta)\star I)\Big)\\ =&\frac{1}{\hbar}\Big(\iota_i(I)\star(\alpha\otimes\beta)-(-1)^{2|\alpha|}(\alpha\otimes\beta)\star\iota_i(I)\Big)\\ =&\frac{1}{\hbar}[\iota_i(I),\alpha\otimes\beta]_\star\end{aligned}$$

We will give an explicit expression of the operators $\mathbb{A}_i:=\mathcal{L}_i-[D_F,\iota_i]$ on the Weyl bundle. First, we have the following lemma.

Lemma 5.2. *For every $1\le i\le\dim(G)$, the operator $\mathcal{L}_i-[D_F,\iota_i]$ is linear over $\mathcal{A}^\bullet(X)$; equivalently, we have $\mathcal{L}_i-[D_F,\iota_i]\in\Gamma(X,\mathrm{End}(\mathcal{W}_{X,\mathbb{C}}))$.*

Proof. Let $s\in\mathcal{A}^\bullet(X,\mathcal{W}_{X,\mathbb{C}})$, and let $\alpha\in\mathcal{A}^k(X)$. Then

$$\begin{aligned}&(\mathcal{L}_i-[D_F,\iota_i])(\alpha\wedge s)\\ =&\mathcal{L}_i(\alpha)\wedge s+\alpha\wedge\mathcal{L}_i(s)-D_F\circ\iota_i(\alpha\wedge s)-\iota_i\circ D_F(\alpha\wedge s)\\ =&\mathcal{L}_i(\alpha)\wedge s+\alpha\wedge\mathcal{L}_i(s)-D_F(\iota_i(\alpha)\wedge s+(-1)^k\cdot\alpha\wedge\iota_i(s))\\ &-\iota_i(d_X(\alpha)\wedge s+(-1)^k\cdot\alpha\wedge D_F(s))\\ =&\mathcal{L}_i(\alpha)\wedge s+\alpha\wedge\mathcal{L}_i(s)-d_X(\iota_i(\alpha))\wedge s+(-1)^k\cdot\iota_i(\alpha)\wedge D_F(s)\\ &-(-1)^k\cdot d_X(\alpha)\wedge\iota_i(s)-\alpha\wedge D_F(\iota_i(s))\\ &-\iota_i(d_X(\alpha))\wedge s+(-1)^k\cdot d_X(\alpha)\wedge\iota_i(s)-(-1)^k\iota_i(\alpha)\wedge D_F(s)-\alpha\wedge\iota_i(D_F(s))\\ =&\alpha\wedge\mathcal{L}_i(s)-\alpha\wedge([D_F,\iota_i](s)).\end{aligned}$$

□

In a similar way, we can show that the operator $\mathcal{L}_i-[\nabla,\iota_i]$ is linear over $\mathcal{A}^\bullet(X)$. More importantly, we will show that this operator on the Weyl bundle $\mathcal{W}_{X,\mathbb{C}}$ can be expressed as a bracket with respect to the Wick product. To do so, we need the following well-known result on the equivalence between the fiberwise Wick product and Weyl product:

Lemma 5.3. *Let $\star_{MW}$ denote the Moyal-Weyl product and $\star_{Wick}$ denote the Wick product on $\mathcal{W}_{\mathbb{C}^n}$. Then the following map is an isomorphism between these two algebras:*

$$e^S:(\mathcal{W}_{\mathbb{C}^n},\star_{MW})\to(\mathcal{W}_{\mathbb{C}^n},\star_{Wick}).$$

Here $S=\hbar\cdot\omega^{i\bar{j}}\frac{\partial}{\partial z^i}\frac{\partial}{\partial\bar{z}^j}$.

Lemma 5.4. *There exists a section s_i of the Weyl bundle such that $\mathcal{L}_i-[D_F,\iota_i]$ can be written as a bracket:*

$$\mathcal{L}_i-[D_F,\iota_i]=\frac{1}{\hbar}[s_i,-]_\star. \tag{5.1}$$

Proof. It is clear that $[D_F - \nabla, \iota] = [[I, -]_\star, \iota_i] = [\iota_i(I), -]_\star$. Thus we only need to show that $\mathcal{L}_i - [\nabla, \iota_i]$ can be written as a bracket with respect to the Wick product on $\mathcal{W}_{M,\mathbb{C}}$.

We will first show that $\mathcal{L}_i - [\nabla, \iota_i]$ can be expressed as a bracket with respect to the fiberwise Moyal-Weyl product $\star_{MW}$. For the explicit local computations, we will use real coordinates on X and $\mathcal{W}_{X,\mathbb{C}}$. Since both $\mathcal{L}_i$ and ∇ preserves the types of sections in $\mathcal{W}_{X,\mathbb{C}}$, we only need to compute $(\mathcal{L}_i - [\nabla, \iota_i])\,(y^\alpha)$ for a local section y^α of TX^*. Let $f_i = \mu(g_i)$, where $\mu : \mathfrak{g} \to C^\infty(X)$ is the classical moment map. Then the vector field associated to g_i is $V_{g_i} = \frac{\partial f_i}{\partial x^j}\omega^{jk}\frac{\partial}{\partial x^k}$. Then

$$\begin{aligned}[\nabla, \iota_i](y^\alpha) =&\iota_i(\nabla y^\alpha) = \iota_i(\Gamma^\alpha_{\beta\gamma} dx^\beta \otimes y^\gamma)\\ =&\iota_i(dx^\beta)\cdot \Gamma^\alpha_{\beta\gamma} y^\gamma = \frac{\partial f_i}{\partial x^j}\omega^{j\beta}\cdot \Gamma^\alpha_{\beta\gamma} y^\gamma;\end{aligned}$$

here $\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel symbols of ∇. On the other hand, using Cartan's formula, we have

$$\begin{aligned}\mathcal{L}_i(dx^\alpha) =&[d_X, \iota_i](dx^\alpha) = d_X(\iota_i(dx^\alpha))\\ =&d_X\left(\frac{\partial f_i}{\partial x^j}\omega^{j\alpha}\right) = \left(\frac{\partial^2 f_i}{\partial x^k \partial x^j}\omega^{j\alpha} + \frac{\partial f_i}{\partial x^j}\frac{\partial \omega^{j\alpha}}{\partial x^k}\right) dx^k.\end{aligned}$$

Since $\mathcal{L}_i - [\nabla, \iota_i]$ is linear over $\mathcal{A}^\bullet(X)$, via the above computations, we can write it as

$$\mathcal{L}_i - [\nabla, \iota_i] = \left(\frac{\partial^2 f_i}{\partial x^\gamma \partial x^\beta}\omega^{\beta\alpha} + \frac{\partial f_i}{\partial x^\beta}\frac{\partial \omega^{\beta\alpha}}{\partial x^\gamma} - \frac{\partial f_i}{\partial x^j}\omega^{j\beta}\cdot\Gamma^\alpha_{\beta\gamma}\right)\partial_{x^\alpha}\otimes y^\gamma \in \Gamma(X, TX\otimes TX^*).$$

We use the symplectic form to lift the subscript in ∂_{x^α} in the above section, and obtain

$$\begin{aligned}s_i =& \left(\frac{\partial^2 f_i}{\partial x^\gamma \partial x^\beta}\omega^{\beta\alpha} + \frac{\partial f_i}{\partial x^\beta}\frac{\partial \omega^{\beta\alpha}}{\partial x^\gamma} - \frac{\partial f_i}{\partial x^j}\omega^{j\beta}\cdot\Gamma^\alpha_{\beta\gamma}\right)\omega_{\alpha\xi} y^\xi \otimes y^\gamma \in \Gamma(X, TX^*\otimes TX^*)\\ =& \left(\frac{\partial^2 f_i}{\partial x^\gamma \partial x^\xi} + \frac{\partial f_i}{\partial x^\beta}\frac{\partial \omega^{\beta\alpha}}{\partial x^\gamma}\omega_{\alpha\xi} - \frac{\partial f_i}{\partial x^j}\omega^{j\beta}\cdot\Gamma^\alpha_{\beta\gamma}\omega_{\alpha\xi}\right) y^\xi \otimes y^\gamma \in \Gamma(X, TX^*\otimes TX^*).\end{aligned}$$

Then the operator $\mathcal{L}_i - [\nabla, \iota_i]$ can be expressed as a bracket if and only if s_i is symmetric in the indices ξ and γ. The first term clearly satisfies this symmetry. For the second and third terms, we can locally choose Darboux coordinates. Then the second term vanishes since they are derivatives of $\omega_{\beta\alpha}$. The third term has the desired symmetry since any symplectic connection in any Darboux coordinates satisfies the condition that

$$\Gamma^\alpha_{\beta\gamma}\omega_{\alpha\xi}$$

is symmetric in all three indices β, γ and ξ. Thus, there exists a section γ_i of the Weyl bundle such that for any section $\alpha \in \Gamma(X, \mathcal{W}_{X,\mathbb{C}})$ (not necessarily flat), there is

$$(\mathcal{L}_i - [\nabla, \iota_i])(\alpha) = [\gamma_i, \alpha]_{\star_{MW}}.$$

Using the equivalence between the Weyl product and the Wick product on $\mathcal{W}_{X,\mathbb{C}}$ in Lemma 5.3, we see that

$$(\mathcal{L}_i - [\nabla, \iota_i])(\alpha) = [\gamma_i, \alpha]_{\star_{MW}} = e^{-S}[e^S(\gamma_i), e^S(\alpha)]_\star$$

This implies that $\mathcal{L}_i - [\nabla, \iota_i] = e^{-S}\circ\left([e^S(\gamma), -]_\star\right)\circ e^S$. On the other hand, we have

$$\begin{aligned}[e^S(\gamma_i), -]_\star =&e^S\circ(\mathcal{L}_i - [\nabla, \iota_i])\circ e^{-S}(\alpha)\\ =&(\mathcal{L}_i - [\nabla, \iota_i])\circ e^S\circ e^{-S}(\alpha)\end{aligned}$$

$$=(\mathcal{L}_i - [\nabla, \iota_i])(\alpha),$$

where we used the fact that both $\mathcal{L}_i$ and ∇ commute with the operator S. □

Proposition 5.5. *Let $\alpha \in \Gamma^{flat}(X, \mathcal{W}_{X,\mathbb{C}}[[\hbar]]) \cong C^\infty(X)[[\hbar]]$ be any flat section of the Weyl bundle, which corresponds to a formal function on X. Then we have*

$$s_i \star \alpha - \alpha \star s_i = \mathcal{L}_i(\alpha).$$

In other words, quantum Hamiltonian symmetries $\mathcal{L}_i$ on formal functions can be expressed as brackets with s_i.

Proof. From Lemma 5.4, we have

$$s_i \star \alpha - \alpha \star s_i = (\mathcal{L}_i - [D_F, \iota_i])(\alpha) = \mathcal{L}_i(\alpha) - \iota_i(D_F(\alpha)) = \mathcal{L}_i(\alpha).$$

□

We give the following lemma, which we will need later:

Lemma 5.6. *If the G-action on X preserves both the symplectic and complex structures (i.e. holomorphic isometries), then the Fedosov connection D_F is G-invariant. In particular, for all $1 \leq i \leq \dim(G)$, we have*

$$[\mathcal{L}_i, D_F] = 0. \tag{5.2}$$

Proof. Recall that the Fedosov connection is of the explicit form $D_F = \nabla + \frac{1}{\hbar}[I, -]_\star$. The Levi-Civita connection ∇ obviously commutes with the G-action. On the other hand, the components of the term I in the Fedosov connection arises by iteratively applying the operators $(\delta^{1,0})^{-1}$ and $\nabla^{1,0}$ to the curvature operator ∇^2 and the Ricci curvature. The result now follows because all of these commute with the G-action. □

Lemma 5.7. *Locally, by adding formal smooth functions (the constant terms in $\mathcal{W}_{X,\mathbb{C}}[[\hbar]]$) to the sections s_i's, we obtain flat sections of $\mathcal{W}_{X,\mathbb{C}}[[\hbar]]$ under the Fedosov connection D_F. When the first de Rham cohomology group $H^1_{dR}(X)$ vanishes, these flat sections corresponds to the images of quantum moment maps under the isomorphism $C^\infty(X)[[\hbar]] \cong \Gamma^{flat}(X, \mathcal{W}_{X,\mathbb{C}}[[\hbar]])$.*

Proof. Using Lemma 5.4, we have

$$\frac{1}{\hbar}[[s_i, -]_\star, D_F] = ([\mathcal{L}_i, D_F] - [[D_F, \iota_i], D_F]) = -([[D_F, \iota_i], D_F]) = 0;$$

here we used equation (5.2) in the above lemma in the second equality. On the other hand, there is

$$\frac{1}{\hbar}[[s_i, -]_\star, D_F] = \pm\frac{1}{\hbar}[D_F(s_i), -]_\star = \pm\frac{1}{\hbar}[D_F(s_i), -]_\star = 0.$$

It follows that for every $1 \leq i \leq \dim(G)$, we have $D_F(s_i) \in \mathcal{A}^1(X)$ (since $\mathcal{A}^\bullet(X)$ is the center of $\mathcal{A}^\bullet_X(\mathcal{W}_{X,\mathbb{C}})$). Moreover, these 1-forms must be closed since $D_F^2(s_i) = d_X(D_F(s_i)) = 0$. The first statement follows because locally we can take anti-derivatives of closed 1-forms. Globally, vanishing of the first de Rham cohomology group also implies the existence of the anti-derivatives we need, and these are images of quantum moment maps by Proposition 5.5. □

To summarize, we have

Theorem 5.8. *The image of quantum moment maps are first order quantizable functions. Thus, for any α and k, there exists a Lie algebra homomorphism*

$$S : \mathfrak{g} \to C^\infty_{\alpha,k}(X),$$

where the Lie bracket on the right hand side is induced by the star product $\star_k$.

Proof. We only need to show that every section s_i has a uniformly bounded degree in $\overline{\mathcal{W}}_X$. This follows from some simple observations on its defining equation (5.1). First of all, the component in s_i corresponding to the operator $\mathcal{L}_i - [\nabla, \iota_i]$ must be of degree 2 and lives in $TX \otimes \overline{TX}$ (since $\mathcal{L}_i$ is a derivation with respect to the classical product on the Weyl bundle). On the other hand, the term I in the Fedosov connection satisfies our desired finiteness property by its construction, and so does the term in s_i corresponding to $[D_F - \nabla, \iota_i]$. Hence, we conclude that s_i has a uniformly bounded degree in $\overline{\mathcal{W}}_X$. □

Appendix A. Proof of Proposition 4.5

Notice that the Karabegov form in this situation is $\omega - \alpha = \omega - \hbar \cdot R^k_{i\bar{j}k} dz^i \wedge d\bar{z}^j$. Recall that, after Theorem 2.6, we write term I in the Fedosov connection as $I = \sum_{i\geq 2} I_i$. Let us write each I_n explicitly as

$$I_n = R^j_{i_1\cdots i_n,\bar{l}}\, \omega_{j\bar{k}} d\bar{z}^l \otimes (y^{i_1}\cdots y^{i_n}\bar{y}^k).$$

Lemma A.1. *We have $(J_\alpha)_n = -(n+1)\hbar \cdot R^i_{ii_1\cdots i_n,\bar{l}} d\bar{z}^l \otimes y^{i_1}\cdots y^{i_n}$*

Proof. The proof is by induction on n. For $n = 1$, we have

$$(J_\alpha)_1 = (\delta^{1,0})^{-1}\left(-\hbar \cdot R^k_{i\bar{j}k} dz^i \wedge d\bar{z}^j\right) = -2\hbar \cdot \left(\frac{1}{2} R^k_{i\bar{j}k} d\bar{z}^j \otimes y^i\right).$$

Then, by the induction hypothesis for $n-1$, we have

$$\nabla^{1,0}(J_\alpha)_{n-1} = \nabla^{1,0}\left(-n\hbar \cdot R^i_{ii_1\cdots i_{n-1},\bar{l}} d\bar{z}^l \otimes y^{i_1}\cdots y^{i_{n-1}}\right).$$

On the other hand,

$$\begin{aligned}&\nabla^{1,0}\left(n\hbar \cdot R^j_{i_1\cdots i_n,\bar{l}} d\bar{z}^l \otimes y^{i_1}\cdots y^{i_n} \otimes \partial_{y^j}\right)\\ =&(n+1)\cdot n\hbar \cdot R^j_{i_1\cdots i_{n+1},\bar{l}} dz^{i_{n+1}} \wedge d\bar{z}^l \otimes y^{i_1}\cdots y^{i_n} \otimes \partial_{y^j}.\end{aligned}$$

Since $\nabla^{1,0}$ is compatible with the contraction between TX and T^*X, the above computation shows that

$$(J_\alpha)_n = (\delta^{1,0})^{-1}(\nabla^{1,0}(J_\alpha)_{n-1}) = -(n+1)\hbar \cdot R^{i_1}_{i_1\cdots i_{n+1},\bar{l}} d\bar{z}^l \otimes y^{i_1}\cdots y^{i_n} y^{i_{n+1}}.$$

□

Lemma A.2. *The section β satisfies $D_{\alpha,k}(\beta) = -\omega_{i\bar{j}} d\bar{z}^j \otimes y^i - \partial\rho$.*

Proof. The function ρ satisfies the condition that $\bar{\partial}\partial(\rho) = \omega$. Recall that $\beta = \sum_{k\geq 1}(\tilde{\nabla}^{1,0})^k(\rho)$, and it is easy to check that $\sigma\left(D_{\alpha,k}(\beta)\right) = \sigma(-\delta(\tilde{\nabla}^{1,0}(\rho)) = -\partial\rho$. On the other hand, the following computation shows that $-\omega_{i\bar{j}} d\bar{z}^j \otimes y^i - \partial\rho$ is closed under $D_{\alpha,k}$:

$$D_{\alpha,k}(-\omega_{i\bar{j}} d\bar{z}^j \otimes y^i - \partial\rho)$$

$$\begin{aligned}&=\nabla(-\omega_{i\bar{j}}d\bar{z}^j\otimes y^i)-\delta(-\omega_{i\bar{j}}d\bar{z}^j\otimes y^i)+k\cdot[I_\alpha,-2\omega_{i\bar{j}}d\bar{z}^j\otimes y^i]_{\star_k}-\bar{\partial}\partial\rho\\&=\delta(\omega_{i\bar{j}}d\bar{z}^j\otimes y^i)-\bar{\partial}\partial\rho=\omega_{i\bar{j}}dz^i\wedge d\bar{z}^j-\omega=0.\end{aligned}$$

Here we have used the fact that ω is parallel with respect to ω. Since β is a section of the holomorphic Weyl bundle $\mathcal{W}_X$, so is its differential $D_{\alpha,k}(\beta)\in A^1_X(\mathcal{W}_X)$. Furthermore, we have $D^{1,0}_{\alpha,k}(\beta)=-\rho$, which implies that

$$\gamma:=D_\alpha(\beta)+\omega_{i\bar{j}}d\bar{z}^j\otimes y^i+\partial\rho\in\mathcal{A}^{0,1}_X(\mathcal{W}_X).$$

Suppose γ does not vanish. Then $\delta(\gamma)\neq 0$ which implies the non-vanishing of $D_{\alpha,k}(\gamma)$. This is a contradiction. □

Lemma A.3. *We have* $(I+J_\alpha)\circledast_k(e^{k\cdot\beta})=\left(\sum_{n\geq 2}\tilde{R}^*_n(k\cdot\beta)\right)\circledast_k e^{k\cdot\beta}=k[I,k\cdot\beta]_\star\circledast_k e^{k\cdot\beta}$.

Proof. For every $n\geq 2$, there is the following straightforward computation:

$$\begin{aligned}&k\cdot I_n\circledast_k(e^{k\cdot\beta}\otimes e_{L^k})\\&=-2\sqrt{-1}\cdot R^j_{i_1\cdots i_n,\bar{l}}\omega_{j\bar{k}}d\bar{z}^l\otimes(y^{i_1}\cdots y^{i_n}\bar{y}^k)\circledast_k(e^{k\cdot\beta}\otimes e_{L^k})\\&=-2\sqrt{-1}\cdot R^j_{i_1\cdots i_n,\bar{l}}\omega_{j\bar{k}}d\bar{z}^l\otimes\left(\frac{\omega^{i\bar{k}}}{2\sqrt{-1}}\frac{\partial}{\partial y^i}\right)(y^{i_1}\cdots y^{i_n}e^{k\cdot\beta}\otimes e_{L^k})\\&=R^i_{i_1\cdots i_n,\bar{l}}d\bar{z}^l\otimes y^1\cdots y^n\frac{\partial(k\cdot\beta))}{\partial y^i}\cdot(e^{k\cdot\beta}\otimes e_{L^k})+n\cdot R^i_{ii_1\cdots i_{n-1},\bar{l}}d\bar{z}^l\otimes y^{i_1}\cdots y^{i_{n-1}}\cdot(e^{k\cdot\beta}\otimes e_{L^k})\\&=\left(\tilde{R}^*_n(k\cdot\beta)+n\cdot R^i_{ii_1\cdots i_{n-1},\bar{l}}d\bar{z}^l\otimes y^{i_1}\cdots y^{i_{n-1}}\right)\cdot(e^{k\cdot\beta}\otimes e_{L^k})\\&=\left(\tilde{R}^*_n(k\cdot\beta)-(J_\alpha)_{n-1}\right)\circledast_k(e^{k\cdot\beta}\otimes e_{L^k}).\end{aligned}$$

□

Summarizing the above computations, we have

$$\begin{aligned}&D_{\alpha,k}(e^{k\cdot\beta}\otimes e_{L^k})\\&=(\nabla+k\cdot\gamma_\alpha\circledast_k)(e^{k\cdot\beta}\otimes e_{L^k})+e^{k\cdot\beta}\otimes\nabla_{L^{\otimes k}}(e_{L^k})\\&=\left(\nabla(k\cdot\beta)+k\cdot\omega_{i\bar{j}}(d\bar{z}^j\otimes y^i-dz^i\otimes\bar{y}^j)\circledast_k+k(I+J_\alpha)\circledast_k\right)(e^{k\cdot\beta}\otimes e_{L^k})+e^{k\cdot\beta}\otimes(k\partial\rho\cdot e_{L^k})\\&=\left(\nabla(k\cdot\beta)+k\cdot\omega_{i\bar{j}}d\bar{z}^j\otimes y^i+k\cdot\partial\rho\right)(e^{k\cdot\beta}\otimes e_{L^k})\\&\quad+k(-\omega_{i\bar{j}}dz^i\otimes\bar{y}^j+I+J_\alpha)\circledast_k(e^{k\cdot\beta}\otimes e_{L^k})\\&=\left(\nabla(k\cdot\beta)+k\cdot\omega_{i\bar{j}}d\bar{z}^j\otimes y^i+k\cdot\partial\rho+(-\omega_{i\bar{j}})dz^i(-\omega^{k\bar{j}})\frac{\partial(k\cdot\beta)}{\partial y^k}+k[I,k\cdot\beta]_{\star_k}\right)\\&\quad\circledast_k(e^{k\cdot\beta}\otimes e_{L^k})\\&=\left(\nabla(k\cdot\beta)+k[I,k\cdot\beta]_{\star_k}+k\cdot\omega_{i\bar{j}}d\bar{z}^j\otimes y^i+k\cdot\partial\rho-\delta^{1,0}(k\cdot\beta)\right)\circledast_k(e^{k\cdot\beta}\otimes e_{L^k})\\&=k\cdot\left(D_{\alpha,k}(\beta)+\omega_{i\bar{j}}d\bar{z}^j\otimes y^i+\partial\rho\right)\circledast_k(e^{k\cdot\beta}\otimes e_{L^k})\\&=0.\end{aligned}$$

This completes the proof of Proposition 4.5.

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