

## AN APPROACH TO THE FAITH-MENAL CONJECTURE

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**ABSTRACT.** The Faith-Menal conjecture is one of the three main open conjectures on  $QF$  rings. It says that every right noetherian and left  $FP$ -injective ring is  $QF$ . In this paper, it is proved that the conjecture is true if every nonzero complement left ideal of the ring  $R$  is not small (or not singular). Several known results are then obtained as corollaries.

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### 1. Introduction

Throughout this paper rings are associative with identity. For a subset  $X$  of a ring  $R$ , the left annihilator of  $X$  in  $R$  is  $\mathbf{l}(X) = \{r \in R: rx = 0 \text{ for all } x \in X\}$ . For any  $a \in R$ , we write  $\mathbf{l}(a)$  for  $\mathbf{l}(\{a\})$ . Right annihilators are defined analogously. We write  $J$ ,  $Z_l$ ,  $Z_r$ ,  $S_l$  and  $S_r$  for the Jacobson radical, the left singular ideal, the right singular ideal, the left socle and the right socle of  $R$ , respectively.  $I \subseteq^{ess} R_R$  means that  $I$  is an essential right ideal of  $R$ .  $f = c \cdot$  (resp.,  $f = \cdot c$ ) means that  $f$  is a left (resp., right) multiplication map by the element  $c \in R$ .

Recall that a ring  $R$  is *quasi-Frobenius* ( $QF$ ) if  $R$  is one-sided noetherian and one-sided self-injective. There are three famous conjectures on  $QF$  rings (see [11]). One of them is the Faith-Menal conjecture, which was raised by Faith and Menal in [4]. A ring  $R$  is called *right Johns* if  $R$  is right noetherian and every right ideal of  $R$  is a right annihilator.  $R$  is called *strongly right Johns* if the matrix ring  $M_n(R)$  is right Johns for all  $n \geq 1$ . In [7], Johns used a false result of Kurshan [8, Theorem 3.3] to show that right Johns rings are right artinian. In [3], Faith and Menal gave a counter example to show that right Johns rings may not be right artinian. They characterized strongly right Johns rings as right noetherian and left  $FP$ -injective rings (see [4, Theorem 1.1]). Recall that a ring  $R$  is called *left  $FP$ -injective* if every  $R$ -homomorphism from a submodule  $N$  of a free left  $R$ -module  $F$  to  ${}_R R$  can be

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extended to one from  $F$  to  ${}_R R$ .

In this paper, we apply McCoy's Lemma (see Lemma 2.1) to show that Faith-Menal conjecture is true if every nonzero complement left ideal of  $R$  is not small (or not singular). A left ideal  $I$  of  $R$  is said to be a *complement* if it is maximal with respect to the property that  $I \cap K = 0$  for some left ideal  $K$  of  $R$ . A left ideal  $I$  of  $R$  is *small* if, for any proper left ideal  $K$  of  $R$ ,  $I + K \neq R$ .

## 2. Results

Let  $R$  be a ring. An element  $a \in R$  is called *regular* if there exists an element  $b \in R$  such that  $a = aba$ .  $R$  is called *regular* if every element of  $R$  is regular.

**Lemma 2.1.** (McCoy's Lemma) *Let  $R$  be a ring and  $a, c \in R$ . If  $b = a - aca$  is a regular element of  $R$ , then  $a$  is also a regular element of  $R$ .*

**Proof.** This follows easily from the definition.  $\square$

A ring  $R$  is called *left  $P$ -injective* if every homomorphism from a principal left ideal  $Rt$  to  $R$  can be extended to one from  ${}_R R$  to  ${}_R R$ .

**Proposition 2.2.** *Suppose  $R$  is a right noetherian and left  $P$ -injective ring. Then  $J = Z_l$  is a nilpotent right annihilator, and  $\mathbf{l}(J)$  is essential both as a left and as a right ideal of  $R$ .*

**Proof.** Since  $R$  is left  $P$ -injective,  $J = Z_l$  (see [11, Theorem 5.14]). By [5, Theorem 2.7],  $J$  is nilpotent and  $\mathbf{l}(J)$  is essential both as a left and as a right ideal of  $R$ . Since  $\mathbf{l}(J)$  is an essential left ideal of  $R$ ,  $\mathbf{rl}(J) \subseteq Z_l = J$ . Thus  $J = \mathbf{rl}(J)$  is a right annihilator.  $\square$

**Lemma 2.3.** [6, Lemma 5.8] *If  $R$  is a semiprime right Goldie ring, then  $R$  has DCC on right annihilators.*

**Lemma 2.4.** *Suppose  $R$  is a right noetherian and left  $P$ -injective ring such that every nonzero complement left ideal of  $R$  is not small (or not singular). Then  $R$  is right artinian.*

**Proof.** First, we prove that  $\bar{R} = R/J$  is a regular ring. Assume  $a \notin J$ . Then, since  $J = \mathbf{rl}(J) = Z_l$  by Proposition 2.2, there exists a nonzero complement left ideal  $I$  of  $R$  such that  $\mathbf{l}(a) \cap I = 0$ . We claim that there exists some  $b \in I$  such that  $ba \notin J$ . If not, then  $Ia \subseteq J$ . This implies that  $\mathbf{l}(J)Ia = 0$ . Since  $\mathbf{l}(a) \cap I = 0$ ,  $\mathbf{l}(J)I \subseteq \mathbf{l}(a) \cap I = 0$ . Thus  $I \subseteq \mathbf{rl}(J) = J = Z_l$ . So  $I$  is a small and singular left ideal of  $R$ . This is a contradiction. Since  $R$  is left  $P$ -injective, every homomorphism from

$Rba$  to  ${}_R R$  is a right multiplication map by some  $c \in R$ . Define  $f : Rba \rightarrow R$  by  $f(rba) = rb$  for all  $r \in R$ . Then  $f$  is well-defined and there exists  $0 \neq c \in R$  such that  $f = \cdot c$ . So  $b = bac$ . This implies that  $\bar{b} \in \mathbf{I}_{\bar{R}}(\bar{a} - \bar{a}c\bar{a})$ , where  $\bar{r}$  denotes  $r + J$  in  $R/J$  for any  $r \in R$ . Since  $\bar{b}\bar{a} \neq \bar{0}$ ,  $\mathbf{I}_{\bar{R}}(\bar{a})$  is properly contained in  $\mathbf{I}_{\bar{R}}(\bar{a} - \bar{a}c\bar{a})$ . If  $a - aca \in J$ , then  $\bar{a}$  is a regular element of  $\bar{R}$ . If not, let  $a_1 = a - aca$ . Then  $\mathbf{l}(a_1)$  is not essential in  ${}_R R$ . In the same way, we get  $a_2 = a_1 - a_1c_1a_1$  for some  $c_1 \in R$  and  $\mathbf{I}_{\bar{R}}(\bar{a}_1)$  is properly contained in  $\mathbf{I}_{\bar{R}}(\bar{a}_2)$ . If  $a_2 \in J$ , then  $\bar{a}_1$  is a regular element of  $\bar{R}$ . Thus, by Lemma 2.1,  $\bar{a}$  is a regular element of  $\bar{R}$ . If  $a_2 \notin J$ , we have  $a_3 = a_2 - a_2c_2a_2$  for some  $c_2 \in R$  and  $\mathbf{I}_{\bar{R}}(\bar{a}_2)$  is properly contained in  $\mathbf{I}_{\bar{R}}(\bar{a}_3)$ . Therefore we have such  $a_k \in R$  step by step,  $k = 1, 2, \dots$ . Since  $R$  is right noetherian and  $J(\bar{R}) = \bar{0}$ ,  $\bar{R}$  is a semiprime and right Goldie ring. By Lemma 2.3,  $\bar{R}$  satisfies ACC on left annihilators. So there exists some positive integer  $m$  such that  $a_m \in J$  and  $a_k = a_{k-1} - a_{k-1}c_{k-1}a_{k-1}$  for some  $c_{k-1} \in R, k = 2, 3, \dots, m$ . By Lemma 2.1,  $\bar{a}$  is a regular element of  $\bar{R}$ . Since  $\bar{a}$  is an arbitrary nonzero element of  $\bar{R}$ ,  $\bar{R}$  is a regular ring. Then  $\bar{R}$  is semisimple because  $\bar{R}$  is right noetherian. Moreover, by Proposition 2.2,  $J$  is nilpotent and so  $R$  is semiprimary. Thus  $R$  is right artinian.  $\square$

A ring  $R$  is called *left 2-injective* if every left  $R$ -homomorphism from a 2-generated left ideal  $I$  of  $R$  to  ${}_R R$  can be extended to one from  ${}_R R$  to  ${}_R R$ . It is clear that every left 2-injective ring is left  $P$ -injective, but the converse is not true (see [11, Example 2.5.]).

**Lemma 2.5.** [12, Corollary 3] *If  $R$  is a left 2-injective ring satisfying ACC on left annihilators, then  $R$  is QF.*

**Theorem 2.6.**  *$R$  is QF if and only if  $R$  is right noetherian, left 2-injective and every nonzero complement left ideal of  $R$  is not small (or not singular).*

**Proof.** “ $\Rightarrow$ ”. This is obvious.

“ $\Leftarrow$ ”. By Lemma 2.4,  $R$  is right artinian. So  $R$  has ACC on left annihilators. Thus Lemma 2.5 implies that  $R$  is QF.  $\square$

Since every left *FP-injective* ring is left 2-injective, we have

**Corollary 2.7.** *The Faith-Menal conjecture is true if every nonzero complement left ideal of  $R$  is not small (or not singular).*

A ring  $R$  is called *left CS* (resp. *left min-CS*) if every left ideal (resp. minimal left ideal) is essential in a direct summand of  ${}_R R$ . The left CS condition is also equivalent to saying that every complement left ideal of  $R$  is a direct summand of  ${}_R R$ . So if  $R$  is left CS, then every nonzero complement left ideal is neither small

nor singular. But the converse is not true (see Example 2.11).  $R$  is called left  $C2$  if every left ideal that is isomorphic to a direct summand of  ${}_R R$  is also a direct summand of  ${}_R R$ . Every left  $P$ -injective ring is left  $C2$  (see [11, Proposition 5.10]). A left  $CS$  and left  $C2$  ring is called a *left continuous* ring.

**Theorem 2.8.** *Suppose  $R$  is a right noetherian, left  $P$ -injective, and left min- $CS$  ring such that every nonzero complement left ideal is not small (or not singular). Then  $R$  is  $QF$ .*

**Proof.** By Lemma 2.4,  $R$  is right artinian. So  $R$  is a left  $GPF$  ring (i.e.,  $R$  is left  $P$ -injective, semiperfect, and  $S_l \subseteq^{ess} {}_R R$ ). Then  $R$  is left Kasch and  $S_r = S_l$  by [11, Theorem 5.31]. Thus  $Soc(Re)$  is simple for each local idempotent  $e$  of  $R$  (see [11, Lemma 4.5]). By [11, Lemma 3.1] the dual,  $(eR/eJ)^* \cong l(J)e = S_r e = S_l e = Soc(Re)$  is simple for every local idempotent  $e$  of  $R$ . Since  $R$  is semiperfect, each simple right  $R$ -module is isomorphic to  $eR/eJ$  for some local idempotent  $e$  of  $R$  (see [1, Theorem 27.10]). Thus  $R$  is right mininjective by [11, Theorem 2.29]. Since a left  $P$ -injective ring is left mininjective,  $R$  is a two-sided mininjective and right artinian ring. So  $R$  is  $QF$  by [13, Theorem 2.5].  $\square$

**Corollary 2.9.** [2, Theorem 2.21] *If  $R$  is right noetherian, left  $CS$  and left  $P$ -injective, then  $R$  is  $QF$ .*

**Corollary 2.10.** [10, Theorem 3.2] *The following are equivalent for a ring  $R$ .*

- (1)  $R$  is  $QF$ .
- (2)  $R$  is a right Johns and left  $CS$  ring.

**Example 2.11.** *There is a left min- $CS$  ring  $S$  satisfying every nonzero complement left ideal of  $S$  is neither small nor singular. But it is not left  $CS$ .*

**Proof.** Let  $k$  be a division ring and  $V_k$  be a right  $k$ -vector space of infinite dimension. Take  $R = \text{End}(V_k)$ , then  $R$  is regular but not left self-injective (see [9, Example 3.74B]). Let  $S = M_{2 \times 2}(R)$ , then  $S$  is also regular. This implies  $S$  is left  $P$ -injective and  $J(S) = Z_l(S) = 0$ . Thus  $S$  is left  $C2$  and every nonzero complement left ideal of  $S$  is neither small nor singular. And every minimal left ideal of  $S$  is a direct summand of  ${}_S S$ . So  $S$  is left min- $CS$ . But  $S$  is not left  $CS$ . For if  $S$  is left  $CS$ , then  $S$  is left continuous. Thus  $R$  is left self-injective by [11, Theorem 1.35]. This is a contradiction.  $\square$

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