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Arbitrary Lagrangian–Eulerian Local Discontinuous Galerkin Method for Linear Convection–Diffusion Equations

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Abstract
In this paper, we present and analyze an arbitrary Lagrangian–Eulerian local discontinuous Galerkin (ALE-LDG) method for one-dimensional linear convection–diffusion problems. The semi-discrete ALE-LDG method is shown to preserve L^2-stability and sub-optimal $(k + \frac{1}{2})$ convergence rate, when piecewise polynomials of degree k on the reference cell are used and Lax–Friedrichs flux is taken for the convection term. In addition, we also discuss three specific fully discrete ALE-LDG schemes, in which implicit–explicit Runge–Kutta (IMEX) time-marching is applied. With the aid of scaling arguments and the standard energy analysis, we prove that the corresponding fully discrete schemes are stable provided the time step $\tau \leq \tau_0$, where the positive constant τ_0 is independent of the mesh size h but depends on the convection and diffusion coefficients, the polynomial degree, and the moving grid function. Under the time step restriction, we obtain quasi-optimal error estimate in space and optimal convergence rate in time for the fully discrete schemes. Numerical examples are also given to illustrate our theoretical results.

Keywords Convection–diffusion problems · Arbitrary Lagrangian–Eulerian local discontinuous Galerkin method · Implicit–explicit Runge–Kutta scheme · Stability · Error estimates

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1 Introduction

In this paper, we consider the arbitrary Lagrangian–Eulerian local discontinuous Galerkin (ALE-LDG) method for one-dimensional linear convection–diffusion equations

$$\begin{aligned} \partial_t u + c\partial_x u - d\partial_{xx} u = 0, &\quad (x,t) \in [a,b] \times (0,T], \\ u(x,0) = u_0(x), &\quad x \in [a,b], \\ u(a,t) = u(b,t), &\quad t \in [0,T]. \end{aligned} \tag{1.1}$$

Here $d > 0$ is the diffusion coefficient and we assume the velocity $c > 0$. We pay attention to the smooth solution of (1.1). For simplicity of presentation, we will give detailed analysis only for the Eq. (1.1). We remark that there is no essential difficulty to extend the analysis and results to the problem with a source term.

The discontinuous Galerkin (DG) method is a finite element method employing discontinuous basis functions. It was first introduced to solve the neutron transport equation by Reed and Hill [32]. The first a priori error estimate for DG method has been proven by Lesaint and Raviart [28]. Later, Cockburn, Shu et al. carried out a major development of the method, in which they constructed a framework of Runge–Kutta DG (RKDG) method for nonlinear conservation laws [8,10–12]. The local discontinuous Galerkin (LDG) method is developed to solve partial differential equations (PDEs) with higher order derivatives. Motivated by the successful work of Bassi and Rebay [4], Cockburn and Shu constructed the first LDG method for convection–diffusion equations [13]. The main idea of LDG methods is to rewrite the equations as a first order system, then apply the DG method with carefully selecting numerical fluxes in the system. Optimal a priori error estimates of the semi-discrete LDG method for convection–diffusion problems were obtained in [6]. DG methods became very popular due to the strong stability, high-order accuracy, parallelization capability and conservation properties. For more details of DG methods, we refer to [9,14–16,34,41] and the references therein.

In many applications, such as aeroelastic computations of wings (c.f. [33]) and star-formations and galaxies in astrophysics (c.f. [26]), grid deformation methods maintaining accuracy are usually desirable. One popular technique is the arbitrary Lagrangian–Eulerian (ALE) method, which combines the advantages of the traditional Lagrangian and Eulerian descriptions (see [17]). In the literatures, there have been a number of works about the implementation and applications of DG methods in the ALE framework, e.g. [18,27,29,30,35]. For the ALE method, the geometric conservation law (GCL) is of particular importance, which has been analyzed by Guillard and Farhat in [20]. Recently, Klingenberg et al. carried out an arbitrary Lagrangian–Eulerian discontinuous Galerkin (ALE-DG) method for conservation laws [24], in which they defined local affine linear mappings to connect the current and next time level cells. Thus the ALE-DG method has the local structure as traditional DG methods on static grids. Moreover, it was shown that the ALE-DG method satisfies the GCL condition for any Runge–Kutta method and maintains almost all good features of the RKDG methods on static grids, such as the L^2 stability, high order accuracy, the local maximum principle, and so on. The ALE-DG method has also been extended to Hamilton–Jacobi equations [25], conservation laws on moving simplex meshes [19] and hyperbolic equations involving δ-singularities [22]. Superconvergence of the ALE-DG method for linear hyperbolic equation was analyzed in [36].

In this paper, we carry on developing the ALE-LDG method to solve convection–diffusion equations, which combines LDG methods with the ALE framework suggested in [24]. The piecewise linear mesh velocity is used and only mild Lipschitz continuity of the mesh movement function is required. For convection–diffusion equations which are not convection-dominated, implicit or semi-implicit time discretization is a natural consideration to overcome the small time step caused by stability restrictions. Balázsová and collaborators studied the stability of the ALE space-time DG method [3] and Kirk et al. presented an analysis of a space-time hybridizable DG method [23]. It is well-known that the space-time DG method results in more degrees-of-freedom than traditional time-stepping approaches. In this paper, we would like to consider the implicit–explicit (IMEX) schemes, which handles the convection term explicitly and could be more efficient for problems with a nonlinear convection term. The IMEX time discretizations are usually applied on the differential system including both stiff (often higher order spatial derivatives but linear) and non-stiff (low order derivatives but non-linear) terms. The IMEX methods are easier to implement than fully implicit schemes when the convection term is nonlinear and allow much larger time step for stability-preserving than explicit approaches. For different purposes, there are different IMEX methods. We refer the reader to [1,2,7,21,31,37], in which [37] is an extrapolated space-time DG method, [2,21] are multistep methods and the rest are Runge–Kutta (RK) type IMEX schemes. In [37], Vlasák and collaborators analyzed the extrapolated space-time DG method for nonlinear convection–diffusion problems and derived a priori error estimates. Calvo, Frutos and Novo used a Fourier analysis to study the stability of the IMEX RK method for linear convection–diffusion equations in [7]. The time step restriction $\tau \leq \tau_0$ was given to ensure stability, where τ_0 depends on the values of c and d. Besides, Xia, Xu and Shu explored the semi-implicit spectral deferred correction (SDC) time discretization coupled with LDG schemes [43], which are efficient for solving PDEs with higher order spatial derivatives. Here, we consider the ALE-LDG method coupled with three specific RK type IMEX schemes displayed in [1,7].

For fully discretized DG methods on static grids, there are already some theoretical analysis in the literature. Zhang and Shu have analyzed the second and third order RKDG methods for conservation laws [44–46]. They obtained stability and optimal (or suboptimal) error estimate in the L^2-norm, when solutions are sufficiently smooth. Very recently, a unified framework to investigate the L^2-norm stability of RKDG methods for the linear hyperbolic equations is proposed in [42]. Performances of many popular RKDG schemes are carefully explored. In addition, in [48] the stability and error estimates of the ALE-DG methods for linear conservation laws with RK time-marching schemes was established, where the energy technique and scaling arguments play an important role in the analysis. When comes to the fully discretized LDG methods on static grids, we refer the reader to [38–40,47]. For linear convection–diffusion problems [38,47], the stability and optimal error estimates of IMEX schemes coupled with LDG methods for advection–diffusion problems are obtained under the condition $\tau \leq \tau_0$, where τ_0 is independent of the spacial mesh size h and proportional to d/c^2. In this paper, we will explore similar stability results for the ALE-LDG method with IMEX schemes for solving convection–diffusion problems, especially the relation with the grid movement.

The first purpose of our work is to construct the ALE-LDG method for convection–diffusion equations and present a concrete analysis of stability as well as a priori error estimate for the semi-discrete ALE-LDG scheme. Our analysis indicates that the proposed ALE-LDG method is L^2 stable for the piecewise polynomials space with any degree k. Additionally, we obtain sub-optimal $(k + \frac{1}{2})$ convergence order with Lax–Friedrichs fluxes used for the convection term. The second contribution of our work is to study the stability and error estimates for the fully discretized ALE-LDG method, where three specific RK type IMEX time-marching schemes are applied. Compared with the work on static grids [38], the line of our analysis is similar but the process is complicated. It is more technical for the moving grids since each local cell varies with time. The energy analysis and scaling arguments are still the main strategies in our work. We prove that the corresponding fully discretized ALE-LDG schemes are stable with the time step restriction $\tau \leq \tau_0$, where τ_0 is a positive constant independent of the spacial mesh size h but involved with the moving grid function, the polynomial degree, coefficients of the convection and diffusion terms. To clearly show the main ideas of the error estimate, we only present the detailed proof for the first order in time fully discrete scheme. The quasi-optimal error estimate in space and optimal convergence order in time are established under the condition $\tau \leq \tau_0$. The proof and conclusion can be extended to the second and third order fully discrete schemes.

The rest of the paper is organized as follows. In Sect. 2, we present the semi-discrete ALE-LDG scheme for the linear convection–diffusion problems. Information about the ALE framework and some properties of the semi-discrete ALE-LDG scheme are also given. Section 3 shows three specific fully discrete ALE-LDG schemes as well as the stability. In Sect. 4, error estimates of fully discrete schemes are established. We show some numerical examples to verify our findings in Sect. 5. Section 6 is devoted to the concluding remarks.

2 Semi-discrete ALE-LDG Method

In this section, we shall present and analyze the semi-discrete ALE-LDG method.

2.1 The ALE Framework

To derive the semi-discrete ALE-LDG method, we start with introducing some notations about the ALE framework. Assume that the distribution of the mesh has been known at any time level t_n, $n = 0, 1, \ldots, M$, i.e.,

$$[a, b] = \bigcup_{j=1}^{N} \left[x^n_{j-\frac{1}{2}},\ \ x^n_{j+\frac{1}{2}} \right].$$

Then a local time-dependent straight line connecting the current and next time level points can be defined,

$$x_{j-\frac{1}{2}}(t) = x^n_{j-\frac{1}{2}} + \omega_{j-\frac{1}{2}}(t - t_n), \quad \forall t \in [t_n, t_{n+1}], \tag{2.1}$$

where

$$\omega_{j-\frac{1}{2}} = \frac{x^{n+1}_{j-\frac{1}{2}} - x^n_{j-\frac{1}{2}}}{t_{n+1} - t_n}. \tag{2.2}$$

Let $K_j(t) = [x_{j-\frac{1}{2}}(t), x_{j+\frac{1}{2}}(t)]$ denote the time-dependent cells and $h_j(t) = x_{j+\frac{1}{2}}(t) - x_{j-\frac{1}{2}}(t)$. In addition, we use the notation h to stand for the global length,

$$h = \max_{t\in[0,T]} \max_{1\leq j\leq N} h_j(t).$$

Suppose the mesh is quasi-uniform in the sense that

$$h \leq C_m h_j(t), \quad \forall j = 1, 2, \ldots, N, \quad t \in [0, T], \tag{2.3}$$

where C_m is a positive constant and independent of h and t. Obviously, we have the following property,

$$h_j(t) = \left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right)(t - t_n) + h_j(t_n) > 0, \quad t \in [t_n, t_{n+1}]. \tag{2.4}$$

Now we introduce the grid velocity field for all $t \in [t_n, t_{n+1}]$,

$$\omega(x,t) = \omega_{j+\frac{1}{2}} \frac{x - x_{j-\frac{1}{2}}(t)}{h_j(t)} + \omega_{j-\frac{1}{2}} \frac{x_{j+\frac{1}{2}}(t) - x}{h_j(t)}, \quad \forall x \in K_j(t). \tag{2.5}$$

Note that

$$\partial_x(\omega(x,t)) = \frac{\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}}{h_j(t)} = \frac{h_j'(t)}{h_j(t)}, \tag{2.6}$$

which solely dependents on t. For simplicity, we denote $(\partial_x\omega)(t) = \partial_x(\omega(x,t))$. Moreover, $\omega(x,t)$ satisfies the following assumptions as that in [48].

- There exists a positive constant C_w, independent of h, such that

$$\max_{(x,t)\in[a,b]\times[0,T]} |\omega(x,t)| \le C_w;$$

- There exists a positive constant C_{wx}, independent of h, such that

$$\max_{(x,t)\in[a,b]\times[0,T]} |\partial_x(\omega(x,t))| \le C_{wx}. \tag{2.7}$$

For any $K_j(t)$, $t \in [t_n, t_{n+1}]$, a time-dependent linear mapping will be defined,

$$\chi_j : [-1,1] \longrightarrow K_j(t), \quad \xi \mapsto \chi_j(\xi,t) := \frac{h_j(t)}{2}(\xi + 1) + x_{j-\frac{1}{2}}(t). \tag{2.8}$$

Particularly, we have

$$\partial_t(\chi_j(\xi,t)) = \omega(\chi_j(\xi,t),t), \quad \forall(\xi,t) \in [-1,1] \times [t_n, t_{n+1}],$$

and

$$\omega(\chi_j(\xi,t),t) = \omega_{j+\frac{1}{2}} \frac{\xi+1}{2} + \omega_{j-\frac{1}{2}} \frac{1-\xi}{2}, \tag{2.9}$$

which means that the grid velocity function solely dependents on the space variable ξ on the reference cell. With the aid of mapping (2.8), the approximation space for any $t \in [t_n, t_{n+1}]$ is defined by

$$V_h(t) = \{v \in L^2([a,b]) : v(\chi_j(\cdot,t)) \in P^k([-1,1]), \quad j = 1,2,\dots,N\},$$

where $P^k([-1,1])$ is the space of polynomials of degree at most k on $[-1,1]$. Denote the broken Sobolev space

$$H_h^s(t) = \{v : v(\chi_j(\cdot,t)) \in H^s([-1,1]), \quad j = 1,2,\dots,N\},$$

where $H^s([-1,1])$ is the usual Sobolev space for any integer $s \ge 0$. As in general, we set

$$(v,r)_{K_j(t)} = \int_{K_j(t)} vr dx, \quad \|v\|_{K_j(t)} = \sqrt{(v,v)_{K_j(t)}},$$

and

$$\{\!\{v\}\!\}_{j-\frac{1}{2}} = \frac{1}{2}\left(v^+_{j-\frac{1}{2}} + v^-_{j-\frac{1}{2}}\right), \quad [\![v]\!]_{j-\frac{1}{2}} = v^+_{j-\frac{1}{2}} - v^-_{j-\frac{1}{2}}.$$

Here $v^-_{j-\frac{1}{2}}$ and $v^+_{j-\frac{1}{2}}$ stand for the left and right limits of v at the point $x_{j-\frac{1}{2}}(t)$, respectively. Summing over all the elements, we denote

$$(v, r) = \sum_{j=1}^{N} (v, r)_{K_j(t)}, \quad \|v\|^2 = \sum_{j=1}^{N} \|v\|^2_{K_j(t)}, \quad [\![v]\!]^2 = \sum_{j=1}^{N} [\![v]\!]^2_{j-\frac{1}{2}}.$$

Suppose $\Gamma_h(t)$ is the union of all element interface points, then we can define the L^2-norm on $\Gamma_h(t)$ by

$$\|v\|_{\Gamma_h(t)} = \left[\sum_{j=1}^{N} \left(\left| v^+_{j-\frac{1}{2}} \right|^2 + \left| v^-_{j+\frac{1}{2}} \right|^2 \right) \right]^{1/2}.$$

In what follows, the inverse inequalities will be used in the analysis,

$$h\|\partial_x v\| \le \mu_1 \|v\|, \quad h^{\frac{1}{2}} \|v\|_{\Gamma_h(t)} \le \mu_2 \|v\|, \tag{2.10}$$

for $v \in V_h(t)$, where μ_1 and μ_2 are positive constants and independent of v and h. Denote $\mu = \max\{\mu_1, \mu_2^2\}$, which increases with the degree of polynomials. For more details of the inverse property, we refer the reader to [5].

Based on the definition of the finite element space, the transport equation is satisfied, which has been proven in [24] and plays an important role to obtain the ALE-LDG scheme.

Lemma 2.1 *Suppose u is a sufficiently smooth function, then for all $v \in V_h(t)$, the transport equation holds*

$$\frac{d}{dt}(u, v)_{K_j(t)} = (\partial_t u, v)_{K_j(t)} + (\partial_x(\omega u), v)_{K_j(t)}, \quad \forall j = 1, \dots, N. \tag{2.11}$$

2.2 The Semi-discrete ALE-LDG Scheme

Following the standard procedure of constructing LDG method, we obtain an equivalent first-order system of Eq. (1.1)

$$\partial_t u + \partial_x \left(cu - \sqrt{d} q \right) = 0, \quad q - \sqrt{d} \partial_x u = 0. \tag{2.12}$$

Then, multiply the above equations by test functions $v, r \in V_h(t)$, respectively. For the first equation in (2.12), we apply the transport Eq. (2.11) to obtain

$$\frac{d}{dt}(u, v)_{K_j(t)} + \left(\partial_x \left(g(\omega, u) \right), v \right)_{K_j(t)} - \sqrt{d} (\partial_x q, v)_{K_j(t)} = 0,$$

where $g(\omega, u) = (c - \omega)u$. After integrating by parts with respect to x, we obtain the semi-discrete ALE-LDG scheme: find $u_h, q_h \in V_h(t)$ such that for all test functions $v, r \in V_h(t)$, there hold

$$\frac{d}{dt}(u_h, v)_{K_j(t)} = \mathcal{A}_j(\omega, u_h, v) - \sqrt{d} \mathcal{L}^+_j(q_h, v), \tag{2.13}$$

$$(q_h, r)_{K_j(t)} = -\sqrt{d} \mathcal{L}^-_j(u_h, r), \tag{2.14}$$

where

$$\mathcal{A}_j(\omega, v, r) = (g(\omega, v), \partial_x r)_{K_j(t)} - \hat{g}(\omega, v)_{j+\frac{1}{2}} r^-_{j+\frac{1}{2}} + \hat{g}(\omega, v)_{j-\frac{1}{2}} r^+_{j-\frac{1}{2}}, \tag{2.15}$$

$$\mathcal{L}^\pm_j(v, r) = (v, \partial_x r)_{K_j(t)} - v^\pm_{j+\frac{1}{2}} r^-_{j+\frac{1}{2}} + v^\pm_{j-\frac{1}{2}} r^+_{j-\frac{1}{2}}, \tag{2.16}$$

and $\hat{g}(\omega, v)_{j-\frac{1}{2}}$ is chosen as the Lax–Friedrichs flux,

$$\hat{g}(\omega, v)_{j-\frac{1}{2}} = \left(c - \omega_{j-\frac{1}{2}}\right)\{\!\{v\}\!\}_{j-\frac{1}{2}} - \frac{\alpha}{2}[\![v]\!]_{j-\frac{1}{2}}, \quad \alpha = \max_{[a,b]\times[t_n, t_{n+1}]} |c - \omega|.$$

When $\omega(x, t) = 0$, the numerical flux reduces to the Lax–Friedrichs flux on fixed grids. For the initial discretization, a natural way is to choose $u_h(x, 0) = P_h^- u_0(x)$, where $P_h^- u_0(x)$ is the Gauss–Radau projection of $u_0(x)$ and will be defined in Sect. 2.3. Applying the transport Eq. (2.11) again, we arrive at the equivalent form of the equality (2.13),

$$(\partial_t u_h, v)_{K_j(t)} + \mathcal{B}_j(\omega, u_h, v) + \sqrt{d}\mathcal{L}^+_j(q_h, v) = 0, \tag{2.17}$$

with

$$\mathcal{B}_j(\omega, v, r) =(\partial_x(\omega v r), 1)_{K_j(t)} - c(v, \partial_x r)_{K_j(t)} + \hat{g}(\omega, v)_{j+\frac{1}{2}} r^-_{j+\frac{1}{2}} - \hat{g}(\omega, v)_{j-\frac{1}{2}} r^+_{j-\frac{1}{2}}. \tag{2.18}$$

To satisfy the stability analysis, we also need the following equality

$$\sum_{j=1}^{N} \mathcal{A}_j(\omega, v, r) = \sum_{j=1}^{N} (\partial_x \omega)(t)(v, r)_{K_j(t)} - \mathcal{D}(\omega, v, r), \tag{2.19}$$

where

$$\mathcal{D}(\omega, v, r) = ((c - \omega)\partial_x v, r) + \sum_{j=1}^{N} \left(c - \omega_{j+\frac{1}{2}}\right)[\![v]\!]_{j+\frac{1}{2}}\{\!\{r\}\!\}_{j+\frac{1}{2}} + \sum_{j=1}^{N} \frac{\alpha}{2}[\![v]\!]_{j+\frac{1}{2}}[\![r]\!]_{j+\frac{1}{2}}.$$

This can be obtained by integrating (2.15) by parts. In the end, we introduce some notations for simplicity. Summing up the operators (2.15)–(2.16) and (2.18) over $j = 1, \ldots, N$, we define $\mathcal{L}^\pm(v, r) = \sum_{j=1}^{N} \mathcal{L}^\pm_j(v, r)$ and

$$\mathcal{A}(\omega, v, r) = \sum_{j=1}^{N} \mathcal{A}_j(\omega, v, r), \quad \mathcal{B}(\omega, v, r) = \sum_{j=1}^{N} \mathcal{B}_j(\omega, v, r). \tag{2.20}$$

2.3 Properties of the Semi-discrete ALE-LDG Scheme

In this subsection, some properties of the operators (2.20) will be listed first. The detail of the proof is omitted to save space, and similar analysis can be found in [38,48].

Lemma 2.2 *Suppose $\mathcal{A}$ and $\mathcal{D}$ are defined by (2.20) and (2.19), respectively, then for any v, $r \in V_h(t)$ and $t \in [t_n, t_{n+1}]$, there hold*

$$|\mathcal{A}(\omega, v, r)| \leq \alpha \left(\|\partial_x r\| + \sqrt{2\mu h^{-1}}[\![r]\!]\right) \|v\|, \tag{2.21}$$

$$|\mathcal{D}(\omega, v, r)| \leq \alpha \left(\|\partial_x v\| + \sqrt{2\mu h^{-1}}[\![v]\!]\right) \|r\|. \tag{2.22}$$

Moreover, we have the following property,

$$\mathcal{A}(\omega, v, v) = -\sum_{j=1}^{N} \frac{\alpha}{2} [\![v]\!]_{j+\frac{1}{2}}^2 + \sum_{j=1}^{N} \frac{(\partial_x \omega)(t)}{2} \|v\|_{K_j(t)}^2. \tag{2.23}$$

Lemma 2.3 *Suppose $\mathcal{B}$ is defined by (2.18), then for any $v \in V_h(t)$ and $t \in [t_n, t_{n+1}]$, there hold*

$$\mathcal{B}(\omega, v, v) = \frac{\alpha}{2} [\![v]\!]^2 - \sum_{j=1}^{N} \frac{\omega_{j+\frac{1}{2}}}{2} [\![v^2]\!]_{j+\frac{1}{2}}. \tag{2.24}$$

Lemma 2.4 *Suppose $\mathcal{L}^{\pm}$ are defined by (2.20), then for any $v, r \in H_h^1(t)$,*

$$\mathcal{L}^-(v, v) = -\frac{1}{2} [\![v]\!]^2, \tag{2.25}$$

$$\mathcal{L}^-(v, r) = -\mathcal{L}^+(r, v). \tag{2.26}$$

Lemma 2.5 *Suppose $u_h, q_h \in V_h(t)$ are the numerical solutions of the scheme (2.13)–(2.14), then*

$$\|\partial_x u_h\| + \sqrt{2\mu h^{-1}} [\![u_h]\!] \le \frac{C_\mu}{\sqrt{d}} \|q_h\|, \tag{2.27}$$

where C_μ is a positive constant, which is independent of h but may depend on the inverse constant μ.

Next, we will show the L^2 stability of the semi-discrete scheme (2.13)–(2.14). The proof has a character similar in spirit to the stability analysis in [24].

Theorem 2.6 *Let (u_h, q_h) be the numerical solution of the semi-discrete scheme (2.13)–(2.14), then we have for any $t \in [0, T]$,*

$$\|u_h(\cdot, t)\|^2 + 2\int_0^t \|q_h(\cdot, \tau)\|^2 d\tau \le \|u_h(\cdot, 0)\|^2.$$

Proof Take $v = u_h$ in the scheme (2.17) to obtain

$$(\partial_t u_h, u_h)_{K_j(t)} + \mathcal{B}_j(\omega, u_h, u_h) + \sqrt{d}\mathcal{L}_j^+(q_h, u_h) = 0. \tag{2.28}$$

For the first term, the transport Eq. (2.11) with $u = u_h^2$ and $v = 1$ yields

$$(\partial_t u_h, u_h)_{K_j(t)} = \frac{1}{2}\frac{d}{dt}(u_h, u_h)_{K_j(t)} - \frac{1}{2}\left(\partial_x(\omega(u_h)^2), 1\right)_{K_j(t)}.$$

Then by the definition (2.16) and (2.14), we have

$$\begin{aligned}\sqrt{d}\mathcal{L}_j^+(q_h, u_h) &= \sqrt{d}\left(-\mathcal{L}_j^-(u_h, q_h) - q_h^+ u_h^-\big|_{j+\frac{1}{2}} + q_h^+ u_h^-\big|_{j-\frac{1}{2}}\right) \\ &= (q_h, q_h)_{K_j(t)} - \sqrt{d}\left(q_h^+ u_h^-\big|_{j+\frac{1}{2}} - q_h^+ u_h^-\big|_{j-\frac{1}{2}}\right).\end{aligned}$$

Collecting the above equalities together and summing up the formulation over $j = 1, \ldots, N$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_a^b (u_h)^2 dx + \int_a^b (q_h)^2 dx + \sum_{j=1}^{N} \mathcal{F}_j = 0,$$

where

$$\mathcal{F}_j = \mathcal{B}_j(\omega, u_h, u_h) - \frac{1}{2}\left(\partial_x(\omega(u_h)^2), 1\right)_{K_j(t)} - \sqrt{d}\left(q_h^+ u_h^-\big|_{j+\frac{1}{2}} - q_h^+ u_h^-\big|_{j-\frac{1}{2}}\right).$$

It follows by using the periodic boundary condition and property (2.24)

$$\sum_{j=1}^{N} \mathcal{F}_j = \frac{\alpha}{2}[\![u_h]\!]^2 \geq 0.$$

Thus the proof is completed. □

In the following, we will present error estimates for the semi-discrete ALE-LDG scheme (2.13)–(2.14). As usual, we introduce two projections first. The L^2 projection $P_h u$ of u is defined by

$$(P_h u, v)_{K_j(t)} = (u, v)_{K_j(t)}, \quad \forall v \in V_h(t). \tag{2.29}$$

For $v(\chi(\cdot, t)) \in P^{k-1}([-1, 1])$ and $k \geq 1$, define the Gauss-Radau projection

$$\left(P_h^- u, v\right)_{K_j(t)} = (u, v)_{K_j(t)}, \quad P_h^- u\left(x_{j+\frac{1}{2}}^-(t)\right) = u\left(x_{j+\frac{1}{2}}^-(t)\right), \tag{2.30}$$

$$\left(P_h^+ u, v\right)_{K_j(t)} = (u, v)_{K_j(t)}, \quad P_h^+ u\left(x_{j-\frac{1}{2}}^+(t)\right) = u\left(x_{j-\frac{1}{2}}^+(t)\right). \tag{2.31}$$

Let $Q_h u$ be either $P_h u$ or $P_h^{\pm} u$. Similar to the well known results in [5], the projections satisfy

$$\|\eta\| + h^{1/2}\|\eta\|_{\Gamma_h(t)} + h\|\partial_x \eta\| \leq C h^{k+1}, \quad \forall u \in H^{k+1}([a, b]), \tag{2.32}$$

where $\eta = u - Q_h u$. The positive constant C depends on u and its derivatives, but it is independent of h. In addition, the following properties are also satisfied,

$$\mathcal{L}_j^+\left(v - P_h^+ v, r\right) = 0, \quad \mathcal{L}_j^-(v - P_h^- v, r) = 0, \quad \forall v \in H_h^1(t), \ \forall r \in V_h(t), \tag{2.33}$$

$$\partial_t(Q_h u) + \omega \cdot \partial_x(Q_h u) = Q_h(\partial_t u) + Q_h(\omega \cdot \partial_x u), \tag{2.34}$$

where $\mathcal{L}_j^{\pm}$ is defined by (2.16) and the second equality is proven in [24]. Now we are ready to provide the suboptimal error estimate by using the Lax–Friedrichs flux for the convection term.

Theorem 2.7 *Let (u_h, q_h) be the numerical solution of the scheme (2.13)–(2.14), and (u, q) be the exact solution of Eq. (2.12). Suppose u is sufficiently smooth with bounded derivatives, then there exists a constant C, which is independent of h and u_h, such that*

$$\max_{t\in[0,T]} \|u(\cdot, t) - u_h(\cdot, t)\| + \max_{t\in[0,T]} \|q(\cdot, t) - q_h(\cdot, t)\| \leq C h^{k+\frac{1}{2}}.$$

Proof Define

$$\eta_u = u - P_h^- u, \quad \eta_q = q - P_h^+ q,$$
$$\zeta_u = u_h - P_h^- u, \quad \zeta_q = q_h - P_h^+ q.$$

Thus we have

$$e_u = u - u_h = \eta_u - \zeta_u, \quad e_q = q - q_h = \eta_q - \zeta_q.$$

Noticing that the exact solution also satisfies the scheme (2.17), we obtain the error equation, for $v, r \in V_h(t)$,

$$(\partial_t e_u, v)_{K_j(t)} + \mathcal{B}_j(\omega, e_u, v) + \sqrt{d}\mathcal{L}_j^+(e_q, v) = 0, \tag{2.35}$$

$$(e_q, r)_{K_j(t)} + \sqrt{d}\mathcal{L}_j^-(e_u, r) = 0. \tag{2.36}$$

Choosing the test function $v = \zeta_u$ in (2.35), using the transport Eq. (2.11) and the property (2.33) lead to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\zeta_u, \zeta_u)_{K_j(t)} - \frac{1}{2}\left(\partial_x(\omega\zeta_u^2), 1\right)_{K_j(t)} + \mathcal{B}_j(\omega, \zeta_u, \zeta_u)\\ = (\partial_t \eta_u, \zeta_u)_{K_j(t)} + \mathcal{B}_j(\omega, \eta_u, \zeta_u) - \sqrt{d}\mathcal{L}_j^+(\zeta_q, \zeta_u).\end{aligned} \tag{2.37}$$

In what follows, we will analyze the above equation. The property (2.24) yields

$$\sum_{j=1}^{N} -\frac{1}{2}\left(\partial_x(\omega\zeta_u^2), 1\right)_{K_j(t)} + \mathcal{B}(\omega, \zeta_u, \zeta_u) = \frac{\alpha}{2}[\![\zeta_u]\!]^2. \tag{2.38}$$

Letting $r = \zeta_q$ in (2.36) and by (2.26) and (2.33), we have

$$\sqrt{d}\mathcal{L}^+(\zeta_q, \zeta_u) = -\sqrt{d}\mathcal{L}^-(\zeta_u, \zeta_q) = (\zeta_q, \zeta_q) - (\eta_q, \zeta_q). \tag{2.39}$$

Now we can obtain the important energy equality by adding (2.37)–(2.39) together

$$\frac{1}{2}\frac{d}{dt}\|\zeta_u\|^2 + \|\zeta_q\|^2 + \frac{\alpha}{2}[\![\zeta_u]\!]^2 = a(\omega, \eta_u, \zeta_q), \tag{2.40}$$

where

$$a(\omega, \eta_u, \zeta_q) = (\eta_q, \zeta_q) + (\partial_t \eta_u, \zeta_u) + \mathcal{B}(\omega, \eta_u, \zeta_u).$$

By Young's inequality and (2.32), we get

$$(\eta_q, \zeta_q) \le \frac{1}{2}\|\eta_q\|^2 + \frac{1}{2}\|\zeta_q\|^2 \le Ch^{2k+2} + \frac{1}{2}\|\zeta_q\|^2.$$

In addition, the properties (2.33)–(2.34) of projections yield

$$\begin{aligned}(\partial_t \eta_u, \zeta_u) + \mathcal{B}(\omega, \eta_u, \zeta_u) &= \left(\partial_t u - P_h^- \partial_t u, \zeta_u\right) + \sum_{j=1}^{N} \partial_x \omega(\eta_u, \zeta_u)_{K_j(t)} + \sum_{j=1}^{N}(\omega\eta_u, \partial_x \zeta_u)_{K_j(t)}\\ &\quad + \sum_{j=1}^{N}\left(\omega\partial_x u - P_h^-(\omega\partial_x u), \zeta_u\right)_{K_j(t)} - \sum_{j=1}^{N}\hat{g}(\omega, \eta_u)_{j+\frac{1}{2}}[\![\zeta_u]\!]_{j+\frac{1}{2}}\\ &\le Ch^{k+1}\|\zeta_u\| + \sqrt{2}\alpha\|\eta_u\|_{\Gamma_h(t)}[\![\zeta_u]\!]\\ &\le C\left(h^{2k+1} + \|\zeta_u\|^2\right) + \frac{\alpha}{2}[\![\zeta_u]\!]^2.\end{aligned}$$

Here we use the similar analysis as in the proof of Theorem 2.8 in [24]. Combining the estimates above with the energy equality (2.40) leads to

$$\frac{1}{2}\frac{d}{dt}\|\zeta_u\|^2 + \frac{1}{2}\|\zeta_q\|^2 \le C\left(h^{2k+1} + \|\zeta_u\|^2\right).$$

By Gronwall's inequality and $u_h(x, 0) = P_h^- u(x, 0)$, we obtain, for all $t \in [0, T]$,

$$\|\zeta_u\|^2 + \max_{t\in[0,T]}\|\zeta_q\|^2 \le Ch^{2k+1},$$

where C is independent u_h and h. Finally, the triangle inequality is used to complete the proof. □

3 Fully Discrete Schemes

In this section, the IMEX RK methods coupled with the ALE-LDG schemes as well as the stability analysis of the fully discrete schemes will be presented. The convection part is treated explicitly and the diffusion part is treated implicitly. We would like to achieve stability under the time step restriction $\tau \leq \tau_0$, where the positive constant τ_0 is independent of h. This expectation is analogous to that in [38], which is considered on fixed grids. For simplicity, we only consider the uniform partition of the time interval $[0, T]$, namely, $M\tau = T$. The stability analysis is studied on the interval $[t_n, t_{n+1}]$. In addition, denote $K_j^n = K_j(t_n)$, $h_j^n = h_j(t_n)$, $\omega^n = \omega(x, t_n)$ and the approximation of $(u_h(t_n), q_h(t_n))$ by (u_h^n, q_h^n). For the initial value, we take $u_h^0 = P_h^- u_0(x)$.

3.1 First Order Fully Discrete Scheme

For the first order IMEX methods, we take the forward and backward Euler discretization for the explicit and implicit part, respectively. With the semi-discrete ALE-LDG scheme (2.13)–(2.14), we obtain the first order fully discrete scheme: find $u_h^{n+1}, q_h^{n+1} \in V_h(t_{n+1})$, such that for any $v^n, r^n \in V_h(t_n)$, there hold

$$\left(u_h^{n+1}, \widehat{v^n}\right)_{K_j^{n+1}} = \left(u_h^n, v^n\right)_{K_j^n} + \tau \mathcal{A}_j\left(\omega^n, u_h^n, v^n\right) - \sqrt{d}\tau \mathcal{L}_j^+\left(q_h^{n+1}, \widehat{v^n}\right), \tag{3.1}$$

$$\left(q_h^{n+1}, \widehat{r^n}\right)_{K_j^{n+1}} = -\sqrt{d}\mathcal{L}_j^-\left(u_h^{n+1}, \widehat{r^n}\right). \tag{3.2}$$

In what follows,

$$\widehat{v^n}(\chi_j(\cdot, t_{n+1})) = v^n(\chi_j(\cdot, t_n)), \tag{3.3}$$

which stands for the function mapped from K_j^n to K_j^{n+1}. Here $\chi_j(\cdot, t)$ is defined by (2.8).

Theorem 3.1 *Let u_h^{n+1} be the numerical solution of the fully discrete scheme (3.1)–(3.2), then we have*

$$\|u_h^{n+1}\| \leq \|u_h^n\|$$

under the condition $\tau \leq \tau_0$, and τ_0 is a positive constant, which is independent of h.

Proof For the first time, we obtain the energy equality. Taking $\widehat{v^n} = u_h^{n+1}$ in (3.1) leads to

$$\left(u_h^{n+1}, u_h^{n+1}\right)_{K_j^{n+1}} - \left(u_h^n, \widetilde{u_h^{n+1}}\right)_{K_j^n} = \tau \mathcal{A}_j\left(\omega^n, u_h^n, \widetilde{u_h^{n+1}}\right) - \sqrt{d}\tau \mathcal{L}_j^+\left(q_h^{n+1}, u_h^{n+1}\right), \tag{3.4}$$

where

$$\widetilde{u_h^{n+1}}(\chi_j(\cdot, t_n)) = u_h^{n+1}(\chi_j(\cdot, t_{n+1})).$$

Note that

$$\left(u_h^n, \widetilde{u_h^{n+1}}\right)_{K_j^n} = \frac{1}{2}\|u_h^n\|_{K_j^n}^2 + \frac{1}{2}\|\widetilde{u_h^{n+1}}\|_{K_j^n}^2 - \frac{1}{2}\|u_h^n - \widetilde{u_h^{n+1}}\|_{K_j^n}^2.$$

By the scaling argument with (2.4) and (2.6), we have

$$\|\widetilde{u_h^{n+1}}\|_{K_j^n}^2 = \frac{h_j^n}{h_j^{n+1}}\|u_h^{n+1}\|_{K_j^{n+1}}^2 = (1-s_2)\|u_h^{n+1}\|_{K_j^{n+1}}^2. \tag{3.5}$$

Here and in what follows,

$$s_2 = \tau(\partial_x \omega)(t_{n+1}).$$

Then summing up (3.4) over all j yields

$$\begin{aligned}\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 + \frac{1}{2}\|\widetilde{u_h^{n+1}} - u_h^n\|^2 = &-\sum_{j=1}^N \frac{s_2}{2}\|u_h^{n+1}\|_{K_j^{n+1}}^2 + \tau\mathcal{A}\big(\omega^n, u_h^n, \widetilde{u_h^{n+1}}\big)\\ &- \sqrt{d}\tau\mathcal{L}^+\big(q_h^{n+1}, u_h^{n+1}\big).\end{aligned}$$

By (2.15), (2.9) and the scaling argument, we obtain

$$\mathcal{A}\big(\omega^n, u_h^n, \widetilde{u_h^{n+1}}\big) = \mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n}, u_h^{n+1}\big). \tag{3.6}$$

Employ the property (2.23) to get

$$\begin{aligned}\tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n}, u_h^{n+1}\big) &= \tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n} - u_h^{n+1}, u_h^{n+1}\big) + \tau\mathcal{A}\big(\omega^{n+1}, u_h^{n+1}, u_h^{n+1}\big)\\ &= \tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n} - u_h^{n+1}, u_h^{n+1}\big) + \sum_{j=1}^N \frac{s_2}{2}\|u_h^{n+1}\|_{K_j^{n+1}}^2 - \frac{\alpha}{2}\tau[\![u_h^{n+1}]\!]^2.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 + \frac{1}{2}\|\widetilde{u_h^{n+1}} - u_h^n\|^2 = &-\frac{\alpha}{2}\tau[\![u_h^{n+1}]\!]^2 - \tau\|q_h^{n+1}\|^2\\ &+ \tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n} - u_h^{n+1}, u_h^{n+1}\big),\end{aligned}$$

where the property

$$-\sqrt{d}\tau\mathcal{L}^+\big(q_h^{n+1}, u_h^{n+1}\big) = \sqrt{d}\tau\mathcal{L}^-\big(u_h^{n+1}, q_h^{n+1}\big) = -\tau\|q_h^{n+1}\|^2$$

has been used due to (2.26) and (3.2). Next, by properties (2.21) and (2.27), we derive

$$\begin{aligned}\tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n} - u_h^{n+1}, u_h^{n+1}\big) &\le \alpha\tau\left(\|\partial_x\big(u_h^{n+1}\big)\| + \sqrt{2\mu h^{-1}}[\![u_h^{n+1}]\!]\right)\|\widehat{u_h^n} - u_h^{n+1}\|\\ &\le \frac{C_\mu}{\sqrt{d}}\alpha\tau\|q_h^{n+1}\|\|\widehat{u_h^n} - u_h^{n+1}\|.\end{aligned}$$

Nevertheless, the scaling argument and the quasi-uniform of the mesh (2.3) provide

$$\|\widehat{u_h^n} - u_h^{n+1}\|^2 = \sum_{j=1}^N \frac{h_j^{n+1}}{h_j^n}\|u_h^n - \widetilde{u_h^{n+1}}\|_{K_j^n}^2 \le C_m\|u_h^n - \widetilde{u_h^{n+1}}\|^2, \tag{3.7}$$

where C_m is defined by (2.3). Thus

$$\tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n} - u_h^{n+1}, u_h^{n+1}\big) \le \frac{1}{2}\|u_h^n - \widetilde{u_h^{n+1}}\|^2 + \frac{C_\mu^2\alpha^2\tau^2}{2d}C_m\|q_h^{n+1}\|^2.$$

Hence, adding the estimate to the energy equality leads to

$$\frac{1}{2}\|u_h^{n+1}\|^2-\frac{1}{2}\|u_h^n\|^2\le\left(-\tau+\frac{C_\mu^2\alpha^2\tau^2}{2d}C_m\right)\|q_h^{n+1}\|^2.$$

In the end, if $\frac{C_\mu^2\alpha^2\tau^2}{2d}C_m\le\tau$, i.e., $\tau\le\tau_0=\frac{2d}{C_\mu^2C_m\alpha^2}$, we have

$$\|u_h^{n+1}\|\le\|u_h^n\|.$$

The proof is completed. □

Remark 3.2 From the analysis of the proof, we find that τ_0 is proportional to $\frac{d}{\alpha^2C_\mu^2C_m}$, which means that time step restriction is influenced by the grid velocity function, the polynomial degree, the diffusion and convection coefficients. Moreover, for fixed grids, we have $C_m=1$ in (3.7) and $\omega(x,t)=0$, then the time step restriction is the same as that in [38].

3.2 Second Order Fully Discrete Scheme

For the second order IMEX methods, we take the L-stable, two stages scheme given in [1]. With the semi-discrete ALE-LDG scheme (2.13)–(2.14), we obtain the second order fully discrete scheme: find $u_h^{n+1}, q_h^{n+1}\in V_h(t_{n+1})$, such that for any $v^n, r^n\in V_h(t_n)$, there hold

$$\begin{aligned}
\left(u_h^{n,1},\overline{v^n}\right)_{K_j^{n+\gamma}}&=\left(u_h^n,v^n\right)_{K_j^n}+\gamma\tau\mathcal{A}_j\left(\omega^n,u_h^n,v^n\right)-\gamma\sqrt{d}\tau\mathcal{L}_j^+\left(q_h^{n,1},\overline{v^n}\right),\\
\left(u_h^{n+1},\widehat{v^n}\right)_{K_j^{n+1}}&=\left(u_h^n,v^n\right)_{K_j^n}+\delta\tau\mathcal{A}_j\left(\omega^n,u_h^n,v^n\right)+(1-\delta)\tau\mathcal{A}_j\left(\omega^{n+\gamma},u_h^{n,1},\overline{v^n}\right) \quad (3.8)\\
&\quad-(1-\gamma)\sqrt{d}\tau\mathcal{L}_j^+\left(q_h^{n,1},\overline{v^n}\right)-\gamma\sqrt{d}\tau\mathcal{L}_j^+\left(q_h^{n+1},\widehat{v^n}\right), \quad (3.9)\\
\left(q_h^{n,1},\overline{r^n}\right)_{K_j^{n+\gamma}}&=-\sqrt{d}\mathcal{L}_j^-\left(u_h^{n,1},\overline{r^n}\right), \quad (3.10)\\
\left(q_h^{n+1},\widehat{r^n}\right)_{K_j^{n+1}}&=-\sqrt{d}\mathcal{L}_j^-\left(u_h^{n+1},\widehat{r^n}\right), \quad (3.11)
\end{aligned}$$

where $t_{n+\gamma}=t_n+\gamma\tau$, $\overline{v^n}(\chi_j(\cdot,t_{n+\gamma}))=v^n(\chi_j(\cdot,t_n))$ and $\widehat{v^n}(\chi_j(\cdot,t_{n+1}))=v^n(\chi_j(\cdot,t_n))$. In addition, $\gamma=1-\frac{\sqrt{2}}{2}$ and $\delta=1-\frac{1}{2\gamma}$.

Theorem 3.3 *Let u_h^{n+1} be the numerical solution of the fully discrete scheme (3.8)–(3.11), then we have*

$$\|u_h^{n+1}\|\le\|u_h^n\|$$

under the condition $\tau\le\tau_0$, and τ_0 is a positive constant, which is independent of h.

Proof A direct calculation from (3.8) and (3.9) yields

$$\begin{aligned}
\left(u_h^{n+1},\widehat{v^n}\right)_{K_j^{n+1}}-\left(u_h^{n,1},\overline{v^n}\right)_{K_j^{n+\gamma}}&=(\delta-\gamma)\tau\mathcal{A}_j\left(\omega^n,u_h^n,v^n\right)+(1-\delta)\tau\mathcal{A}_j\left(\omega^{n+\gamma},u_h^{n,1},\overline{v^n}\right)\\
&\quad-(1-2\gamma)\sqrt{d}\tau\mathcal{L}_j^+\left(q_h^{n,1},\overline{v^n}\right)-\gamma\sqrt{d}\tau\mathcal{L}_j^+\left(q_h^{n+1},\widehat{v^n}\right).
\end{aligned} \quad (3.12)$$

Define

$$\widetilde{u_h^{n,1}}(\chi_j(\cdot,t_n))=u_h^{n,1}(\chi_j(\cdot,t_{n+\gamma})),\quad \overline{u_h^{n+1}}(\chi_j(\cdot,t_{n+\gamma}))=u_h^{n+1}(\chi_j(\cdot,t_{n+1})).$$

Let $\overline{v^n} = u_h^{n,1}$ in (3.8) and $\widehat{v^n} = u_h^{n+1}$ in (3.12), then adding them together, we have

$$\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 + \frac{1}{2}\|u_h^{n,1} - \overline{u_h^{n+1}}\|^2 + \frac{1}{2}\|\widetilde{u_h^{n,1}} - u_h^n\|^2 = R_c + R_d, \tag{3.13}$$

where

$$\begin{aligned} R_c =& \gamma\tau\mathcal{A}\big(\omega^n, u_h^n, \widetilde{u_h^{n,1}}\big) + (\delta-\gamma)\tau\mathcal{A}\big(\omega^n, u_h^n, \widetilde{u_h^{n+1}}\big) + (1-\delta)\tau\mathcal{A}\big(\omega^{n+\gamma}, u_h^{n,1}, \overline{u_h^{n+1}}\big) \\ &- \sum_{j=1}^N \frac{(\partial_x\omega)(t_{n+\gamma})}{2}\gamma\tau\|u_h^{n,1}\|^2_{K_j^{n+\gamma}} - \sum_{j=1}^N \frac{(\partial_x\omega)(t_{n+1})}{2}(1-\gamma)\tau\|u_h^{n+1}\|^2_{K_j^{n+1}}, \\ R_d =& -\gamma\sqrt{d}\tau\mathcal{L}^+\big(q_h^{n,1}, u_h^{n,1}\big) - (1-2\gamma)\sqrt{d}\tau\mathcal{L}^+\big(q_h^{n,1}, \overline{u_h^{n+1}}\big) - \gamma\sqrt{d}\tau\mathcal{L}^+\big(q_h^{n+1}, u_h^{n+1}\big). \end{aligned}$$

Here we use the similar property

$$\begin{aligned} \|\widetilde{u_h^{n,1}}\|^2_{K_j^n} &= \frac{h_j^n}{h_j^{n+\gamma}}\|u_h^{n,1}\|^2_{K_j^{n+\gamma}} = \Big(1 - \gamma\tau(\partial_x\omega)(t_{n+\gamma})\Big)\|u_h^{n,1}\|^2_{K_j^{n+\gamma}}, \\ \|\overline{u_h^{n+1}}\|^2_{K_j^{n+\gamma}} &= \frac{h_j^{n+\gamma}}{h_j^{n+1}}\|u_h^{n+1}\|^2_{K_j^{n+1}} = \Big(1 - (1-\gamma)\tau(\partial_x\omega)(t_{n+1})\Big)\|u_h^{n+1}\|^2_{K_j^{n+1}}, \end{aligned}$$

owing to (2.4) and (2.6). Next, we will analyze R_c and R_d separately. By (2.26) and the scheme (3.10)–(3.11), we obtain

$$\begin{aligned} &\sqrt{d}\mathcal{L}^+\big(q_h^{n,1}, u_h^{n,1}\big) = -\sqrt{d}\mathcal{L}^-\big(u_h^{n,1}, q_h^{n,1}\big) = \|q_h^{n,1}\|^2, \\ &\sqrt{d}\mathcal{L}^+\big(q_h^{n,1}, \overline{u_h^{n+1}}\big) = -\sqrt{d}\mathcal{L}^-\big(\overline{u_h^{n+1}}, q_h^{n,1}\big) = -\sqrt{d}\mathcal{L}^-\big(u_h^{n+1}, \widehat{q_h^{n,1}}\big) = \big(q_h^{n+1}, \widehat{q_h^{n,1}}\big), \end{aligned}$$

where the scaling argument has been used for the second equality and $\widehat{q_h^{n,1}}(\chi_j(\cdot, t_{n+1})) = q_h^{n,1}(\chi_j(\cdot, t_{n+\gamma}))$. Similarly, we have

$$\sqrt{d}\mathcal{L}^+\big(q_h^{n+1}, u_h^{n+1}\big) = \|q_h^{n+1}\|^2.$$

It follows that

$$\begin{aligned} R_d &= -\gamma\tau\|q_h^{n,1}\|^2 - \gamma\tau\|q_h^{n+1}\|^2 - (1-2\gamma)\tau\big(q_h^{n+1}, \widehat{q_h^{n,1}}\big) \\ &\le -\frac{(4\gamma-1)\tau}{2}\|q_h^{n+1}\|^2 - \gamma\tau\|q_h^{n,1}\|^2 + \frac{(1-2\gamma)\tau}{2}\|\widehat{q_h^{n,1}}\|^2. \end{aligned} \tag{3.14}$$

Noting that $\delta - \gamma = -1$ and following the same line as in (3.6), we rewrite R_c as

$$\begin{aligned} R_c =& \gamma\tau\mathcal{A}\big(\omega^{n+\gamma}, \overline{u_h^n} - u_h^{n,1}, u_h^{n,1}\big) - \tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n} - \widehat{u_h^{n,1}}, u_h^{n+1}\big) + \gamma\tau\mathcal{A}\big(\omega^{n+\gamma}, u_h^{n,1}, u_h^{n,1}\big) \\ &+ (1-\gamma)\tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^{n,1}} - u_h^{n+1}, u_h^{n+1}\big) + (1-\gamma)\tau\mathcal{A}\big(\omega^{n+1}, u_h^{n+1}, u_h^{n+1}\big) \\ &- \sum_{j=1}^N \frac{(\partial_x\omega)(t_{n+\gamma})}{2}\gamma\tau\|u_h^{n,1}\|^2_{K_j^{n+\gamma}} - \sum_{j=1}^N \frac{(\partial_x\omega)(t_{n+1})}{2}(1-\gamma)\tau\|u_h^{n+1}\|^2_{K_j^{n+1}}, \end{aligned}$$

where

$$\overline{u_h^n}\big(\chi_j(\cdot, t_{n+\gamma})\big) = u_h^n(\chi_j(\cdot, t_n)) = \widehat{u_h^n}(\chi_j(\cdot, t_{n+1})), \ \ \widehat{u_h^{n,1}}(\chi_j(\cdot, t_{n+1})) = u_h^{n,1}(\chi_j(\cdot, t_{n+\gamma})).$$

The property (2.23) gives

$$
\begin{aligned}
R_c =& \gamma\tau\mathcal{A}\big(\omega^{n+\gamma}, \overline{u_h^n} - u_h^{n,1}, u_h^{n,1}\big) - \tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^n} - \widehat{u_h^{n,1}}, u_h^{n+1}\big) \\
&+ (1-\gamma)\tau\mathcal{A}\big(\omega^{n+1}, \widehat{u_h^{n,1}} - u_h^{n+1}, u_h^{n+1}\big) - \frac{\alpha}{2}\gamma\tau[\![u_h^{n,1}]\!]^2 - \frac{\alpha}{2}(1-\gamma)\tau[\![u_h^{n+1}]\!]^2.
\end{aligned}
$$

In addition, from (2.21) and (2.27), we have

$$
\begin{aligned}
R_c \leq& C_1\gamma\tau\|q_h^{n,1}\|\|\overline{u_h^n} - u_h^{n,1}\| + C_1\tau\|q_h^{n+1}\|\|\widehat{u_h^n} - \widehat{u_h^{n,1}}\| + C_1(1-\gamma)\tau\|q_h^{n+1}\|\|\widehat{u_h^{n,1}} - u_h^{n+1}\| \\
\leq& \frac{1}{2}\big(\|u_h^n - \widetilde{u_h^{n,1}}\|^2 + \|u_h^{n,1} - \overline{u_h^{n+1}}\|^2\big) + (C_1\gamma\tau)^2 C_m\|q_h^{n,1}\|^2 + (C_1\tau)^2 C_m\|q_h^{n+1}\|^2 \\
&+ \frac{1}{2}(C_1(1-\gamma)\tau)^2 C_m\|q_h^{n+1}\|^2,
\end{aligned} \tag{3.15}
$$

where $C_1 = \frac{\alpha C_\mu}{\sqrt{d}}$ and similar arguments to prove (3.7) are used. Consequently, adding the estimates (3.14)–(3.15) to the energy equality (3.13), we obtain

$$
\begin{aligned}
\frac{1}{2}\|u_h^{n+1}\|^2 - \frac{1}{2}\|u_h^n\|^2 \leq& \frac{((1-\gamma)^2+2)}{2}(C_1\tau)^2 C_m\|q_h^{n+1}\|^2 - \frac{(4\gamma-1)\tau}{2}\|q_h^{n+1}\|^2 \\
&+ (C_1\gamma\tau)^2 C_m\|q_h^{n,1}\|^2 + \frac{(1-2\gamma)\tau}{2}C_m\|q_h^{n,1}\|^2 - \gamma\tau\|q_h^{n,1}\|^2,
\end{aligned} \tag{3.16}
$$

Here we use the fact that

$$
\|\widehat{q_h^{n,1}}\|^2_{K_j^{n+1}} = \frac{h_j^{n+1}}{h_h^{n+\gamma}}\|q_h^{n,1}\|^2_{K_j^{n+\gamma}} \leq C_m\|q_h^{n,1}\|^2_{K_j^{n+\gamma}}.
$$

Denote each line of the right hand side in (3.16) by D_1, D_2, respectively. We find that $D_1 \leq 0$, if

$$
\tau \leq \frac{4\gamma - 1}{(1-\gamma)^2 + 2} \cdot \frac{1}{C_1^2 C_m} \approx 0.0686\frac{1}{C_1^2 C_m}.
$$

Similarly, we have $D_2 \leq 0$, if

$$
\tau \leq \frac{\gamma - \frac{1-2\gamma}{2}C_m}{\gamma^2 C_1^2 C_m}, \quad \textit{and} \quad C_m < \frac{2\gamma}{1-2\gamma} \approx 1.41.
$$

Hence, there exists a positive constant τ_0 independent of h, such that, if $\tau \leq \tau_0$, there hold

$$
\|u_h^{n+1}\| \leq \|u_h^n\|.
$$

□

3.3 Third Order Fully Discrete Scheme

For the third order IMEX method, we take the scheme presented in [7]. With the semi-discrete ALE-LDG scheme (2.13)–(2.14), we obtain the third order fully discrete scheme: find u_h^{n+1}, $q_h^{n+1} \in V_h(t_{n+1})$, such that for any $v^n \in V_h(t_n)$, there hold

$$
\begin{aligned}
\big(u_h^{n,1}, v^{n,1}\big)_{K_j^{n,1}} &= \big(u_h^n, v^n\big)_{K_j^n} + \gamma\tau\mathcal{A}_j\big(\omega^n, u_h^n, v^n\big) - \gamma\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,1}, v^{n,1}\big),\\
\big(u_h^{n,2}, v^{n,2}\big)_{K_j^{n,2}} &= \big(u_h^n, v^n\big)_{K_j^n} + \left(\frac{1+\gamma}{2} - \alpha_1\right)\tau\mathcal{A}_j\big(\omega^n, u_h^n, v^n\big) + \alpha_1\tau\mathcal{A}_j\big(\omega^{n,1}, u_h^{n,1}, v^{n,1}\big)\\
&\quad - \frac{1-\gamma}{2}\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,1}, v^{n,1}\big) - \gamma\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,2}, v^{n,2}\big),\\
\big(u_h^{n,3}, v^{n,3}\big)_{K_j^{n,3}} &= \big(u_h^n, v^n\big)_{K_j^n} + (1-\alpha_2)\tau\mathcal{A}_j\big(\omega^{n,1}, u_h^{n,1}, v^{n,1}\big) + \alpha_2\tau\mathcal{A}_j\big(\omega^{n,2}, u_h^{n,2}, v^{n,2}\big)\\
&\quad - \beta_1\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,1}, v^{n,1}\big) - \beta_2\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,2}, v^{n,2}\big) - \gamma\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,3}, v^{n,3}\big),\\
\big(u_h^{n,4}, v^{n,4}\big)_{K_j^{n,4}} &= \big(u_h^n, v^n\big)_{K_j^n} + \beta_1\tau\mathcal{A}_j\big(\omega^{n,1}, u_h^{n,1}, v^{n,1}\big) + \beta_2\tau\mathcal{A}_j\big(\omega^{n,2}, u_h^{n,2}, v^{n,2}\big)\\
&\quad + \gamma\tau\mathcal{A}_j\big(\omega^{n,3}, u_h^{n,3}, v^{n,3}\big) - \beta_1\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,1}, v^{n,1}\big) - \beta_2\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,2}, v^{n,2}\big)\\
&\quad - \gamma\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,3}, v^{n,3}\big),\\
\big(q_h^{n,*}, r^{n,*}\big)_{K_j^{n,*}} &= -\sqrt{d}\mathcal{L}_j^-\big(u_h^{n,*}, r^{n,*}\big), \quad \forall r^{n,*} \in V_h(t_{n,*}), \quad * = 1,2,3,4. \qquad (3.17)
\end{aligned}
$$

where

$$
t_{n,1} = t_n + \gamma\tau, \quad t_{n,2} = t_n + \frac{1+\gamma}{2}\tau, \quad t_{n,3} = t_{n,4} = t_n + \tau, \quad K_j^{n,*} = K_j(t_{n,*})
$$

and

$$
v^{n,*}(\chi_j(\cdot, t_{n,*})) = v^n(\chi_j(\cdot, t_n)), \quad \omega^{n,*}(\chi_j(\cdot, t_{n,*})) = \omega^n(\chi_j(\cdot, t_n)).
$$

In addition, γ is the middle root of $6x^3 - 18x^2 + 9x - 1 = 0$, $\beta_1 = -\frac{3}{2}\gamma^2 + 4\gamma - \frac{1}{4}$, $\beta_2 = \frac{3}{2}\gamma^2 - 5\gamma + \frac{5}{4}$, $\alpha_1 = 0.35$ and $\alpha_2 = \frac{\frac{1}{3} - 2\gamma^2 - 2\beta_2\alpha_1\gamma}{\gamma(1-\gamma)}$. Finally, we have $u_h^{n+1} = u_h^{n,4}$ and $q_h^{n+1} = q_h^{n,4}$.

In order to obtain the stability of the scheme (3.17), we first rewrite it as the following form, such that all of terms are in the same time stage.

$$
\begin{aligned}
\big(\widehat{u_h^{n,1}}, v^{n,4}\big)_{K_j^{n+1}} &= \big(\widehat{u_h^n}, v^{n,4}\big)_{K_j^{n+1}} + \gamma\tau\mathcal{A}_j\big(\omega^{n,4}, \widehat{u_h^n}, v^{n,4}\big) - \gamma\sqrt{d}\tau\mathcal{L}_j^+\big(\widehat{q_h^{n,1}}, v^{n,4}\big)\\
&\quad + (1-\gamma)s_2\big(\widehat{u_h^{n,1}}, v^{n,4}\big)_{K_j^{n+1}} - s_2\big(\widehat{u_h^n}, v^{n,4}\big)_{K_j^{n+1}},\\
\big(\widehat{u_h^{n,2}}, v^{n,4}\big)_{K_j^{n+1}} &= \big(\widehat{u_h^n}, v^{n,4}\big)_{K_j^{n+1}} + \left(\frac{1+\gamma}{2} - \alpha_1\right)\tau\mathcal{A}_j\big(\omega^{n,4}, \widehat{u_h^n}, v^{n,4}\big) + \alpha_1\tau\mathcal{A}_j\big(\omega^{n,4}, \widehat{u_h^{n,1}}, v^{n,4}\big)\\
&\quad - \frac{1-\gamma}{2}\sqrt{d}\tau\mathcal{L}_j^+\big(\widehat{q_h^{n,1}}, v^{n,4}\big) - \gamma\sqrt{d}\tau\mathcal{L}_j^+\big(\widehat{q_h^{n,2}}, v^{n,4}\big)\\
&\quad + \frac{1-\gamma}{2}s_2\big(\widehat{u_h^{n,2}}, v^{n,4}\big)_{K_j^{n+1}} - s_2\big(\widehat{u_h^n}, v^{n,4}\big)_{K_j^{n+1}},\\
\big(u_h^{n,3}, v^{n,4}\big)_{K_j^{n+1}} &= \big(\widehat{u_h^n}, v^{n,4}\big)_{K_j^{n+1}} + (1-\alpha_2)\tau\mathcal{A}_j\big(\omega^{n,4}, \widehat{u_h^{n,1}}, v^{n,4}\big) + \alpha_2\tau\mathcal{A}_j\big(\omega^{n,4}, \widehat{u_h^{n,2}}, v^{n,4}\big)\\
&\quad - \beta_1\sqrt{d}\tau\mathcal{L}_j^+\big(\widehat{q_h^{n,1}}, v^{n,4}\big) - \beta_2\sqrt{d}\tau\mathcal{L}_j^+\big(\widehat{q_h^{n,2}}, v^{n,4}\big) - \gamma\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,3}, v^{n,4}\big)\\
&\quad - s_2\big(\widehat{u_h^n}, v^{n,4}\big)_{K_j^{n+1}},\\
\big(u_h^{n,4}, v^{n,4}\big)_{K_j^{n+1}} &= \big(\widehat{u_h^n}, v^{n,4}\big)_{K_j^{n+1}} + \beta_1\tau\mathcal{A}_j\big(\omega^{n,4}, \widehat{u_h^{n,1}}, v^{n,4}\big) + \beta_2\tau\mathcal{A}_j\big(\omega^{n,4}, \widehat{u_h^{n,2}}, v^{n,4}\big)\\
&\quad + \gamma\tau\mathcal{A}_j\big(\omega^{n,4}, u_h^{n,3}, v^{n,4}\big) - \beta_1\sqrt{d}\tau\mathcal{L}_j^+\big(\widehat{q_h^{n,1}}, v^{n,4}\big) - \beta_2\sqrt{d}\tau\mathcal{L}_j^+\big(\widehat{q_h^{n,2}}, v^{n,4}\big)\\
&\quad - \gamma\sqrt{d}\tau\mathcal{L}_j^+\big(q_h^{n,3}, v^{n,4}\big) - s_2\big(\widehat{u_h^n}, v^{n,4}\big)_{K_j^{n,4}},
\end{aligned}
$$

where $\widehat{u_h^{n,*}}(\chi_j(\cdot,t_{n+1})) = u_h^{n,*}(\chi_j(\cdot,t_{n,*}))$ and $s_2 = (\partial_x\omega)(t_{n+1})\tau$. Moreover, the scaling arguments with (2.4) and (2.6) are used, e.g.,

$$\left(u_h^{n,1}, v^{n,1}\right)_{K_j^{n,1}} = \frac{h_j(t_{n,1})}{h_j(t_{n+1})}\left(\widehat{u_h^{n,1}}, v^{n,4}\right)_{K_j^{n+1}} = 1-(1-\gamma)s_2\left(\widehat{u_h^{n,1}}, v^{n,4}\right)_{K_j^{n+1}}.$$

Following the same line as in [38], we introduce some notations

$$\begin{aligned}
\mathbb{E}_1 u_h &= \widehat{u_h^{n,1}} - \widehat{u_h^{n}}, & \mathbb{E}_2 u_h &= \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}} + \widehat{u_h^{n}},\\
\mathbb{E}_3 u_h &= 2u_h^{n,3} + \widehat{u_h^{n,2}} - 3\widehat{u_h^{n,1}}, & \mathbb{E}_4 u_h &= u_h^{n,4} - u_h^{n,3},\\
\mathbb{E}_{31} u_h &= u_h^{n,3} + \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}, & \mathbb{E}_{32} u_h &= u_h^{n,3} - \widehat{u_h^{n,1}}.
\end{aligned}$$

Then some algebraic manipulations give

$$\left(\mathbb{E}_l u_h, v^{n,4}\right) = F_l\left(u_h, v^{n,4}\right) + G_l\left(q_h, v^{n,4}\right) + R_l\left(u_h, v^{n,4}\right), \quad l = 1,2,3,4, \tag{3.18}$$

where

$$\begin{aligned}
F_1\left(u_h, v^{n,4}\right) &= \gamma\tau\mathcal{A}\left(\omega^{n,4}, \widehat{u_h^{n}}, v^{n,4}\right),\\
F_2\left(u_h, v^{n,4}\right) &= \left(\frac{1-3\gamma}{2} - \alpha_1\right)\tau\mathcal{A}\left(\omega^{n,4}, \widehat{u_h^{n}}, v^{n,4}\right) + \alpha_1\tau\mathcal{A}\left(\omega^{n,4}, \widehat{u_h^{n,1}}, v^{n,4}\right),\\
F_3\left(u_h, v^{n,4}\right) &= \left(\frac{1-5\gamma}{2} - \alpha_1\right)\tau\mathcal{A}\left(\omega^{n,4}, \widehat{u_h^{n}}, v^{n,4}\right) + \left(2-2\alpha_2+\alpha_1\right)\tau\mathcal{A}\left(\omega^{n,4}, \widehat{u_h^{n,1}}, v^{n,4}\right)\\
&\quad + 2\alpha_2\tau\mathcal{A}\left(\omega^{n,4}, \widehat{u_h^{n,2}}, v^{n,4}\right),\\
F_4\left(u_h, v^{n,4}\right) &= \left(\alpha_2-\beta_2-\gamma\right)\tau\mathcal{A}\left(\omega^{n,4}, \widehat{u_h^{n,1}}, v^{n,4}\right) + \left(\beta_2-\alpha_2\right)\tau\mathcal{A}\left(\omega^{n,4}, \widehat{u_h^{n,2}}, v^{n,4}\right)\\
&\quad + \gamma\tau\mathcal{A}\left(\omega^{n,4}, u_h^{n,3}, v^{n,4}\right)
\end{aligned}$$

and

$$\begin{aligned}
G_1\left(q_h, v^{n,4}\right) &= -\gamma\sqrt{d}\tau\mathcal{L}^+\left(\widehat{q_h^{n,1}}, v^{n,4}\right),\\
G_2\left(q_h, v^{n,4}\right) &= -\gamma\sqrt{d}\tau\mathcal{L}^+\left(\widehat{q_h^{n,2}} - 2\widehat{q_h^{n,1}}, v^{n,4}\right) - \frac{1-\gamma}{2}\sqrt{d}\tau\mathcal{L}^+\left(\widehat{q_h^{n,1}}, v^{n,4}\right),\\
G_3\left(q_h, v^{n,4}\right) &= -2\gamma\sqrt{d}\tau\mathcal{L}^+\left(q_h^{n,3}, v^{n,4}\right) - 2\left(1-\beta_1-\frac{\gamma}{2}\right)\sqrt{d}\tau\mathcal{L}^+\left(\widehat{q_h^{n,2}} - 2\widehat{q_h^{n,1}}, v^{n,4}\right)\\
&\quad - 2\left(\frac{9}{4} - \frac{11}{4}\gamma - \beta_1\right)\sqrt{d}\tau\mathcal{L}^+\left(\widehat{q_h^{n,1}}, v^{n,4}\right),\\
G_4\left(q_h, v^{n,4}\right) &= 0.
\end{aligned}$$

Besides, we have the extra terms involved with $\omega_x(t_{n+1})$,

$$\begin{aligned}
R_1\left(u_h, v^{n,4}\right) &= \sum_{j=1}^{N} s_2\left[(1-\gamma)\left(\widehat{u_h^{n,1}}, v^{n,4}\right)_{K_j^{n+1}} - \left(\widehat{u_h^{n}}, v^{n,4}\right)_{K_j^{n+1}}\right],\\
R_2\left(u_h, v^{n,4}\right) &= \sum_{j=1}^{N} s_2\left[\frac{1-\gamma}{2}\left(\widehat{u_h^{n,2}}, v^{n,4}\right)_{K_j^{n+1}} - 2(1-\gamma)\left(\widehat{u_h^{n,1}}, v^{n,4}\right)_{K_j^{n+1}} + \left(\widehat{u_h^{n}}, v^{n,4}\right)_{K_j^{n+1}}\right],
\end{aligned}$$

$$R_3\big(u_h, v^{n,4}\big) = \sum_{j=1}^{N} s_2\left[\frac{(1-\gamma)}{2}\big(\widehat{u_h^{n,2}}, v^{n,4}\big)_{K_j^{n+1}} - 3(1-\gamma)\big(\widehat{u_h^{n,1}}, v^{n,4}\big)_{K_j^{n+1}}\right],$$
$$R_4\big(u_h, v^{n,4}\big) = 0,$$

owing to the moving grid.

3.3.1 Energy Equations for the Third Order Scheme

Similar to the fixed grids case [38], we take the test functions $v^{n,4} = \widehat{u_h^{n,1}}, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}, u_h^{n,3}$ and $2u_h^{n,4}$ in (3.18) for $l = 1, 2, 3, 4$, respectively. Add them together to obtain the following energy equality

$$\big\|u_h^{n,4}\big\|^2 - \big\|\widehat{u_h^n}\big\|^2 + \Lambda = \Phi_c + \Psi_d + \Upsilon_\omega, \tag{3.19}$$

where

$$\begin{aligned}
\Lambda &= \big\|\mathbb{E}_4 u_h\big\|^2 + \frac{1}{2}\big\|\mathbb{E}_{31} u_h\big\|^2 + \frac{1}{2}\big\|\mathbb{E}_{32} u_h\big\|^2 + \frac{1}{2}\big\|\mathbb{E}_2 u_h\big\|^2 + \frac{1}{2}\big\|\mathbb{E}_1 u_h\big\|^2,\\
\Phi_c &= F_1\big(u_h, \widehat{u_h^{n,1}}\big) + F_2\big(u_h, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}\big) + F_3\big(u_h, u_h^{n,3}\big) + F_4\big(u_h, 2u_h^{n,4}\big),\\
\Psi_d &= G_1\big(q_h, \widehat{u_h^{n,1}}\big) + G_2\big(q_h, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}\big) + G_3\big(q_h, u_h^{n,3}\big) + G_4\big(q_h, 2u_h^{n,4}\big),\\
\Upsilon_\omega &= R_1\big(u_h, \widehat{u_h^{n,1}}\big) + R_2\big(u_h, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}\big) + R_3\big(u_h, u_h^{n,3}\big) + R_4\big(u_h, 2u_h^{n,4}\big).
\end{aligned}$$

Note that one part of the stability is provided by Λ. Next we will analyze the remaining terms one by one.

3.3.2 Analysis of the Diffusion Part Ψ_d

We first analyze Ψ_d, which is related to the diffusion part. For simplicity, introduce the notation

$$\mathbf{W}^\top = \big(\widehat{q_h^{n,1}}, \widehat{q_h^{n,2}} - 2\widehat{q_h^{n,1}}, q_h^{n,3}\big).$$

With the property (2.26), the scaling argument and the scheme (3.17), we have

$$\begin{aligned}
\sqrt{d}\mathcal{L}^+\big(\widehat{q_h^{n,1}}, \widehat{u_h^{n,1}}\big) &= -\sqrt{d}\mathcal{L}^-\big(\widehat{u^{n,1}}, \widehat{q_h^{n,1}}\big) = -\sqrt{d}\mathcal{L}^-\big(u^{n,1}, q_h^{n,1}\big)\\
&= \big(q_h^{n,1}, q_h^{n,1}\big) = \sum_{j=1}^{N} \frac{h_j^{n,1}}{h_j^{n+1}}\big(\widehat{q_h^{n,1}}, \widehat{q_h^{n,1}}\big)_{K_j^{n+1}}.
\end{aligned}$$

Similarly, we can obtain

$$\sqrt{d}\mathcal{L}^+\big(\widehat{q_h^{n,l}}, \widehat{u_h^{n,*}}\big) = -\sqrt{d}\mathcal{L}^-\big(\widehat{u^{n,*}}, \widehat{q_h^{n,l}}\big) = \sum_{j=1}^{N} \frac{h_j^{n,*}}{h_j^{n+1}}\big(\widehat{q_h^{n,*}}, \widehat{q_h^{n,l}}\big)_{K_j^{n+1}}, \tag{3.20}$$

for any $l, * = 1, 2, 3$ and 4. Here $h_j^{n,*} = h_j(t_{n,*})$. In addition, the definitions (2.4) and (2.6) give

$$\frac{h_j^{n,*}}{h_j^{n+1}} = 1 - \frac{t_{n+1} - t_{n,*}}{\tau} s_2, \quad * = 1, 2, 3, 4.$$

Here $s_2 = (\partial_x \omega)(t_{n+1})\tau$. Thus Ψ_d turns out to be

$$\Psi_d = -\tau \int_a^b \mathbf{W}^\top \mathbb{F} \mathbf{W} dx + \sum_{j=1}^{N} s_2 \Psi_{d,1},$$

where

$$\mathbb{F} = \begin{pmatrix} \gamma & \frac{1-\gamma}{4} & \frac{9-11\gamma}{4} - \beta_1 \\ \frac{1-\gamma}{4} & \gamma & 1-\beta_1-\frac{\gamma}{2} \\ \frac{9-11\gamma}{4} - \beta_1 & 1-\beta_1-\frac{\gamma}{2} & 2\gamma \end{pmatrix},$$

and

$$\begin{aligned}\Psi_{d,1} =& \frac{(3\gamma-1)(1-\gamma)}{2}\tau \|\widehat{q_h^{n,1}}\|^2_{K_j^{n+1}} + \frac{\gamma(1-\gamma)}{2}\tau \|\widehat{q_h^{n,2}} - 2\widehat{q_h^{n,1}}\|^2_{K_j^{n+1}} \\ &+ \frac{(1-\gamma)(1-5\gamma)}{4}\tau \left(\widehat{q_h^{n,2}} - 2\widehat{q_h^{n,1}}, \widehat{q_h^{n,1}}\right)_{K_j^{n+1}}.\end{aligned}$$

Apply Young's inequality and the bound (2.7) to obtain

$$\sum_{j=1}^{N} s_2 \Psi_{d,1} \le \frac{(17\gamma-5)(1-\gamma)}{8} C_{wx}\tau^2 \|\widehat{q_h^{n,1}}\|^2 + \frac{(9\gamma-1)(1-\gamma)}{8} C_{wx}\tau^2 \|\widehat{q_h^{n,2}} - 2\widehat{q_h^{n,1}}\|^2.$$

If

$$\frac{(17\gamma-5)(1-\gamma)}{8} C_{wx}\tau^2 \le \frac{\gamma}{10}\tau, \quad \frac{(9\gamma-1)(1-\gamma)}{8} C_{wx}\tau^2 \le \frac{\gamma}{10}\tau, \tag{3.21}$$

then we have

$$\Psi_d \le -\tau \int_a^b \mathbf{W}^\top \left(\mathbb{F} - \frac{\gamma}{10}\mathbb{I}\right) \mathbf{W} dx, \tag{3.22}$$

where $\mathbb{I}$ is the identity matrix.

Remark 3.4 We remark that compared with the analysis on fixed grids, the main difference lies in the equality (3.20), which makes the process more complicated. Furthermore, there is no restriction (3.21) on the fixed grid.

3.3.3 Analysis of the Convection Part Φ_c

The proceeding for the analysis of Φ_c is similar to the case on fixed grids in [38]. However, it is more technical for the moving grids due to the different sizes of the spatial step in different time levels. In the following, define C to represent a positive constant, which is independent of h, τ and u_h^n, but may depends on C_{wx} and γ. Each occurrence may have a different value.

Lemma 3.5 *There exists a positive constant C_*, independent of h and τ, such that*

$$\Phi_c + \Upsilon_\omega \le C\tau \sum_{l=0}^{3} \|\widehat{u_h^{n,l}}\|^2 + C_*\tau\Lambda + \frac{\gamma}{4}\tau \int_a^b \mathbf{W}^\top \mathbf{W} dx,$$

where $\widehat{u_h^{n,0}} = \widehat{u_h^n}$.

Proof With analogous arguments in [38], we can rewrite Φ_c as $\Phi_c = \sum_{i=1}^4 \Phi_{c,i}$, where

$$\begin{aligned}
\Phi_{c,1} &= \gamma\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,1}}, \widehat{u_h^{n,1}}\big) + \frac{3\gamma-1}{2}\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}\big)\\
&\quad + \frac{5(1-\gamma)}{2}\tau\mathcal{A}\big(\omega^{n,4}, u_h^{n,3}, u_h^{n,3}\big),\\
\Phi_{c,2} &= 2\big(\beta_2-\alpha_2-\gamma\big)\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,1}}, u_h^{n,4}-u_h^{n,3}\big) - \gamma\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,1}} - \widehat{u_h^{n}}, \widehat{u_h^{n,1}}\big),\\
\Phi_{c,3} &= 2\big(\beta_2-\alpha_2\big)\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}, u_h^{n,4}-u_h^{n,3}\big) + \alpha_1\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,1}} - \widehat{u_h^{n}}, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}\big)\\
&\quad + \frac{1-3\gamma}{2}\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}} + \widehat{u_h^{n}}, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}\big),\\
\Phi_{c,4} &= 2\gamma\tau\mathcal{A}\big(\omega^{n,4}, u_h^{n,3}, u_h^{n,4}-u_h^{n,3}\big) + 2\beta_2\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}} + \widehat{u_h^{n}}, u_h^{n,3}\big)\\
&\quad + \big(\alpha_1 + 2\beta_2 - \frac{1-5\gamma}{2}\big)\tau\mathcal{A}\big(\omega^{n,4}, \widehat{u_h^{n,1}} - \widehat{u_h^{n}}, u_h^{n,3}\big) - \frac{5-9\gamma}{2}\tau\mathcal{A}\big(\omega^{n,4}, u_h^{n,3} - \widehat{u_h^{n,1}}, u_h^{n,3}\big).
\end{aligned}$$

By the property (2.23), we have

$$\begin{aligned}
\Phi_{c,1} &= -\frac{\gamma}{2}\alpha\tau[\![\widehat{u_h^{n,1}}]\!]^2 - \frac{3\gamma-1}{4}\alpha\tau[\![\widehat{u_h^{n,2}} - \widehat{u_h^{n,1}}]\!]^2 - \frac{5(1-\gamma)}{4}\alpha\tau[\![u_h^{n,3}]\!]^2\\
&\quad + \sum_{j=1}^{N} s_2\left[\frac{\gamma}{2}\big\|\widehat{u_h^{n,1}}\big\|^2_{K_j^{n+1}} + \frac{3\gamma-1}{4}\big\|\widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}\big\|^2_{K_j^{n+1}} + \frac{5(1-\gamma)}{4}\big\|u_h^{n,3}\big\|^2_{K_j^{n+1}}\right].
\end{aligned}$$

On the other hand, some direct calculations give

$$\begin{aligned}
\Upsilon_\omega &= -\sum_{j=1}^{N} s_2\left[\frac{\gamma}{2}\big\|\widehat{u_h^{n,1}}\big\|^2_{K_j^{n+1}} + \frac{3\gamma-1}{4}\big\|\widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}\big\|^2_{K_j^{n+1}} + \frac{5(1-\gamma)}{4}\big\|u_h^{n,3}\big\|^2_{K_j^{n+1}}\right]\\
&\quad + \sum_{j=1}^{N} s_2\left[\frac{1}{2}\big\|\mathbb{E}_1 u_h\big\|^2_{K_j^{n+1}} + \frac{1}{2}\big\|\mathbb{E}_2 u_h\big\|^2_{K_j^{n+1}} + \frac{1-\gamma}{4}\big\|\mathbb{E}_{31} u_h\big\|^2_{K_j^{n+1}}\right]\\
&\quad + \sum_{j=1}^{N} s_2\left[(1-\gamma)\big\|\mathbb{E}_{32} u_h\big\|^2_{K_j^{n+1}} - \frac{1-\gamma}{2}\big\|\widehat{u_h^{n,2}} - \widehat{u_h^{n,1}}\big\|^2_{K_j^{n+1}} - \big\|\widehat{u_h^{n}}\big\|^2_{K_j^{n+1}}\right].
\end{aligned}$$

Thus we can obtain

$$\Phi_{c,1} + \Upsilon_\omega \le C\tau\sum_{l=0}^{3}\big\|\widehat{u_h^{n,l}}\big\|^2.$$

Here we use the fact that $s_2 = (\partial_x\omega)(t_{n+1})\tau$ and the bound (2.7). For $\Phi_{c,i}$, $i = 2, 3, 4$, denote the maximum of the absolute value of all coefficients in $\Phi_{c,i}$ by C_γ. Then apply the equivalent form (2.19), the estimate (2.21)–(2.22), the property (2.27) and the scaling argument to yield

$$\begin{aligned}
\Phi_{c,2} &= \Theta_2 - 2\big(\beta_2-\alpha_2-\gamma\big)\tau\mathcal{D}\big(\omega^{n,4}, \widehat{u_h^{n,1}}, \mathbb{E}_4 u_h\big) - \gamma\tau\mathcal{A}\big(\omega^{n,4}, \mathbb{E}_1 u_h, \widehat{u_h^{n,1}}\big)\\
&\le \Theta_2 + 2C_\gamma C_1 C_m\tau\big\|\widehat{q_h^{n,1}}\big\|(2\Lambda)^{\frac{1}{2}},
\end{aligned}$$

where $C_1 = \frac{\alpha C_\mu}{\sqrt{d}}$ and

$$\Theta_2 = 2\big(\beta_2 - \alpha_2 - \gamma\big)\sum_{j=1}^{N} s_2\big(\widehat{u_h^{n,1}}, \mathbb{E}_4 u_h\big)_{K_j^{n+1}}.$$

Similarly,

$$\Phi_{c,3} \le \Theta_3 + 2C_\gamma C_1 C_m \tau \big\|\widehat{q_h^{n,2}} - 2\widehat{q_h^{n,1}}\big\|(2\Lambda)^{\frac{1}{2}},$$
$$\Phi_{c,4} \le \Theta_4 + 2C_\gamma C_1 C_m \tau \big\|q_h^{n,3}\big\|(2\Lambda)^{\frac{1}{2}},$$

and

$$\Theta_3 = 2(\beta_2 - \alpha_2)\sum_{j=1}^{N} s_2\big(\widehat{u_h^{n,2}} - 2\widehat{u_h^{n,1}}, \mathbb{E}_4 u_h\big)_{K_j^{n+1}},$$

$$\Theta_4 = 2\gamma\sum_{j=1}^{N} s_2\big(u_h^{n,3}, \mathbb{E}_4 u_h\big)_{K_j^{n+1}}.$$

Moreover, we find that

$$\begin{aligned}\sum_{i=2}^{4}\Theta_i &= \sum_{j=1}^{N} s_2\Big[2\big(\beta_2-\alpha_2\big)\big(\mathbb{E}_1 u_h + \mathbb{E}_2 u_h, \mathbb{E}_4 u_h\big)_{K_j^{n+1}} + 2\gamma\big(\mathbb{E}_{32} u_h, \mathbb{E}_4 u_h\big)_{K_j^{n+1}}\Big]\\ &\le 2C_{wx}\tau C_\gamma\Lambda.\end{aligned}$$

Here the second step use (2.7). Consequently, combine the estimates together to derive

$$\begin{aligned}\Phi_c + \Upsilon_\omega \le& C\tau\sum_{l=0}^{3}\big\|\widehat{u_h^{n,l}}\big\|^2 + 2C_{wx}C_\gamma\tau\Lambda\\ &+ 2C_\gamma C_1 C_m\tau\big(\big\|\widehat{q_h^{n,1}}\big\| + \big\|\widehat{q_h^{n,2}} - 2\widehat{q_h^{n,1}}\big\| + \big\|q_h^{n,3}\big\|\big)(2\Lambda)^{\frac{1}{2}}\\ \le& C\tau\sum_{l=0}^{3}\big\|\widehat{u_h^{n,l}}\big\|^2 + \left(2C_{wx}C_\gamma + \frac{6\big(2C_1C_\gamma C_m\big)^2}{\gamma}\right)\tau\Lambda + \frac{\gamma}{4}\tau\int_a^b \mathbf{W}^\top\mathbf{W}dx.\end{aligned} \tag{3.23}$$

Define $C_* = 2C_{wx}C_\gamma + \frac{24\left(C_1 C_\gamma C_m\right)^2}{\gamma}$, then the proof is finished. □

3.3.4 Stability of the Third Order Fully Discrete Scheme

In light of the estimates (3.22)–(3.23), the energy equality (3.19) turns out to be

$$\begin{aligned}\big\|u_h^{n,4}\big\|^2 - \big\|\widehat{u_h^n}\big\|^2 + \Lambda &\le C\tau\sum_{l=0}^{3}\big\|\widehat{u_h^{n,l}}\big\|^2 + C_*\tau\Lambda - \tau\int_a^b \mathbf{W}^\top\big(\mathbb{F} - \frac{\gamma}{10}\mathbb{I} - \frac{\gamma}{4}\mathbb{I}\big)\mathbf{W}dx\\ &\le C\tau\sum_{l=0}^{3}\big\|\widehat{u_h^{n,l}}\big\|^2 + C_*\tau\Lambda,\end{aligned}$$

since the principal minor determinants of $\mathbb{F} - \frac{\gamma}{10}\mathbb{I} - \frac{\gamma}{4}\mathbb{I}$ are all positive. Under the restriction (3.21) and if

$$C_*\tau \leq 1,$$

we have

$$\|u_h^{n,4}\|^2 - \|\widehat{u_h^n}\|^2 \leq C\tau \sum_{l=0}^{3} \|\widehat{u_h^{n,l}}\|^2.$$

Along the similar arguments, we can prove

$$\|\widehat{u_h^{n,l}}\| \leq C\|\widehat{u_h^n}\| \leq C\|u_h^n\|, \quad l = 1, 2, 3,$$

which is provided by $\tau \leq \tau_0$, and the positive constant τ_0 is independent of h. Here the similar property (3.7) is used. In the end, we conclude the stability results in the following theorem.

Theorem 3.6 *Let u_h^{n+1} be the numerical solution of the third order fully discrete scheme* (3.17)*, then we have*

$$\|u_h^{n+1}\|^2 \leq (1 + C\tau)\|u_h^n\|^2$$

under the condition $\tau \leq \tau_0$, and τ_0 is a positive constant, which is independent of h.

Remark 3.7 From the proof of Theorem 3.6, we know that C depends on C_{wx} and γ, where C_{wx} is the upper bound of $|\partial_x(\omega(x,t))|$ and γ comes from the coefficients of the third order IMEX scheme. The results indicate the relations of the stability and error estimates with the grid functions. It is consistent with the results on static grids where $\partial_x(\omega(x,t)) = 0$ and $\|u_h^{n+1}\| \leq \|u_h^n\|$. We can reduce the impact of C on stability by manipulating the suitable grid movement function.

4 Error Estimates for the Fully Discrete Scheme

In this section, error estimates for the fully discrete schemes will be shown by the aid of stability analysis. To construct the error equation conveniently, we first rescale the system (2.12) by a time-dependent coordinate transformation $\chi = \chi(\xi, t)$ in (2.8). For simplicity, define $v^*(\xi, t) = v(\chi(\xi, t), t)$ for any function $v(x, t)$. Thus by the chain rule, we have

$$\partial_t u^* = \partial_t u + \partial_x u \partial_t x, \quad \partial_\xi u^* = \partial_x u \partial_\xi x, \quad \partial_{\xi\xi} u^* = (\partial_\xi x)^2 \partial_{xx} u,$$

where $\partial_\xi x = \frac{h_j(t)}{2}$ and $\partial_t x = \omega^*(\xi, t)$. The Eq. (1.1) in $K_j(t)$, $t \in [t_n, t_{n+1}]$ turns out to be

$$\partial_t u^* + \frac{2}{h_j(t)}(c - \omega^*)\partial_\xi u^* - d\left(\frac{2}{h_j(t)}\right)^2 \partial_{\xi\xi} u^* = 0, \quad (\xi, t) \in [-1, 1] \times [t_n, t_{n+1}]. \tag{4.1}$$

In addition, by (2.5), (2.8) and (2.4), we obtain

$$\partial_\xi \omega^* = \frac{1}{2}\left(\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}\right) = \frac{h_j^{'}(t)}{2}. \tag{4.2}$$

It follows from (4.1) and (4.2) that,

$$\partial_t\left(u^*h_j(t)\right)+\partial_\xi\left(2(c-\omega^*)u^*\right)-\frac{4d}{h_j(t)}\partial_{\xi\xi}u^*=0.$$

Now we derive the equivalent form of (2.12) on the reference cell,

$$\partial_t U+\partial_\xi(aU)-\frac{2\sqrt{d}}{h_j(t)}\partial_\xi Q=0,\quad Q=\frac{2\sqrt{d}}{h_j(t)}\partial_\xi U, \tag{4.3}$$

where

$$U(\xi,t)=u^*h_j(t),\quad a(\xi,t)=\frac{2(c-\omega^*)}{h_j(t)}.$$

4.1 Error Equation for the First Order Fully Discrete Scheme

Denote $u^n=u(x,t_n)$ and $q^n=q(x,t_n)$ for any time level n. We present the following lemma to describe the local truncation error in time.

Lemma 4.1 *Let (u,q) be the exact solution of Eq. (2.12). Suppose u is sufficiently smooth with bounded derivatives, then for any $v^n\in V_h(t_n)$ and $1\le j\le N$, there holds,*

$$\left(u^{n+1},\widehat{v^n}\right)_{K_j^{n+1}}=\left(u^n,v^n\right)_{K_j^n}+\tau\mathcal{A}_j\left(\omega^n,u^n,v^n\right)-\sqrt{d}\tau\mathcal{L}_j^+\left(q^{n+1},\widehat{v^n}\right)+\left(\varepsilon_1^n,v^n\right)_{K_j^n}, \tag{4.4}$$

where $\widehat{v_h^n}$ is defined by (3.3), ε_1^n is the local truncation error in time and $\|\varepsilon_1^n\|_{K_j^n}=\mathcal{O}(\tau^2)$ for any j and n.

Proof By the Taylor expansion with Lagrange form of the remainder, we obviously have

$$\begin{aligned}U(\xi,t+\tau)&=U(\xi,t)+\tau\partial_t U(\xi,t)+\frac{\tau^2}{2}\partial_{tt}U(\xi,t_1)\\&=U(\xi,t)-\tau\partial_\xi(aU)(\xi,t)+2\sqrt{d}\tau\frac{\partial_\xi Q(\xi,t)}{h_j(t)}+\frac{\tau^2}{2}\partial_{tt}U(\xi,t_1),\end{aligned} \tag{4.5}$$

where we use the relation (4.3) and $t_1\in(t,t+\tau)$. Denote $b(\xi,t)=\frac{\partial_{\xi\xi}U(\xi,t)}{h_j^2(t)}$, then by Taylor expansion again, we get

$$b(\xi,t+\tau)=b(\xi,t)+\tau\partial_t b(\xi,t_2),\quad t_2\in(t,t+\tau),$$

which yields that

$$\frac{\partial_\xi Q(\xi,t+\tau)}{h_j(t+\tau)}=\frac{\partial_\xi Q(\xi,t)}{h_j(t)}+2\sqrt{d}\tau\partial_t b(\xi,t_2).$$

Substituting into (4.5) leads to

$$\begin{aligned}U(\xi,t+\tau)=U(\xi,t)-\tau\partial_\xi(aU)(\xi,t)+2\sqrt{d}\tau\frac{\partial_\xi Q(\xi,t+\tau)}{h_j(t+\tau)}\\+\frac{\tau^2}{2}\partial_{tt}U(\xi,t_1)-4d\tau^2\partial_t b(\xi,t_2).\end{aligned}$$

Here

$$\partial_{tt} U = \partial_{tt} u^* h_j + 2\partial_t u^* h_j^{'}, \quad \partial_t b = \frac{\partial_{\xi\xi t} u^* h_j - \partial_{\xi\xi} u^* h_j^{'}}{(h_j)^2}.$$

Let $t = t_n$. Along the same arguments in [24], we obtain

$$\left(u^{n+1}, \widehat{v^n}\right)_{K_j^{n+1}} = \left(u^n, v^n\right)_{K_j^n} + \tau \mathcal{A}_j\left(\omega^n, u^n, v^n\right) - \sqrt{d}\tau \mathcal{L}_j^+\left(q^{n+1}, \widehat{v^n}\right) + \left(\varepsilon_1^n, v_h^n\right)_{K_j^n},$$

where $\|\varepsilon_1^n\|_{K_j^n} = \mathcal{O}(\tau^2)$ for any j and n. □

Denote the error between the exact and numerical solution of the first scheme (3.1)–(3.2) by $e_u^n = u(x, t_n) - u_h^n$ and $e_q^n = q(x, t_n) - q_h^n$ for any stage n. Subtract (3.1) from (4.4) to obtain the first error equation

$$\left(e_u^{n+1}, \widehat{v^n}\right)_{K_j^{n+1}} = \left(e_u^n, v^n\right)_{K_j^n} + \tau \mathcal{A}_j\left(\omega^n, e_u^n, v^n\right) - \sqrt{d}\tau \mathcal{L}_j^+\left(e_q^{n+1}, \widehat{v^n}\right) + \left(\varepsilon_1^n, v^n\right)_{K_j^n}.$$

Noting the fact that the exact solution also satisfies the scheme (3.2), we get the second error equation

$$\left(e_q^{n+1}, \widehat{r^n}\right)_{K_j^{n+1}} = -\sqrt{d}\mathcal{L}_j^-\left(e_u^{n+1}, \widehat{r^n}\right), \quad \forall \widehat{r^n} \in V_h(t_{n+1}).$$

By convention, let

$$\zeta_u^n = u_h^n - P_h^- u^n, \quad \eta_u^n = u^n - P_h^- u^n, \quad \zeta_q^n = q_h^n - P_h^+ q^n, \quad \eta_q^n = q^n - P_h^+ q^n,$$

where $P_h^- u^n$ and $P_h^+ q^n$ are projections defined by (2.30)–(2.31). Thus the errors can be divided into $e_u^n = \eta_u^n - \zeta_u^n$ and $e_q^n = \eta_q^n - \zeta_q^n$, which implies that

$$\begin{aligned} \left(\zeta_u^{n+1}, \widehat{v^n}\right)_{K_j^{n+1}} &= \left(\zeta_u^n, v^n\right)_{K_j^n} + \tau \mathcal{A}_j\left(\omega^n, \zeta_u^n, v^n\right) + \mathcal{H}_j\left(\eta_u, v^n\right) \\ &\quad - \sqrt{d}\tau \mathcal{L}_j^+\left(\zeta_q^{n+1}, \widehat{v^n}\right), \end{aligned} \tag{4.6}$$

$$\left(\zeta_q^{n+1}, \widehat{r^n}\right)_{K_j^{n+1}} = \left(\eta_q^{n+1}, \widehat{r^n}\right)_{K_j^{n+1}} - \sqrt{d}\mathcal{L}_j^-\left(\zeta_u^{n+1}, \widehat{r^n}\right), \tag{4.7}$$

where

$$\mathcal{H}_j(\eta_u, v^n) = \left(\eta_u^{n+1}, \widehat{v^n}\right)_{K_j^{n+1}} - \left(\eta_u^n, v^n\right)_{K_j^n} - \tau \mathcal{A}_j\left(\omega^n, \eta_u^n, v^n\right) - \left(\varepsilon_1^n, v^n\right)_{K_j^n}, \tag{4.8}$$

and the property (2.33) is used. In addition, some estimates for the projection error will be given without proof, and similar analysis can be found in [48].

Lemma 4.2 *Assume u is sufficiently smooth, then there exists a constant $C > 0$, independent of h, τ and n, such that*

$$\|\eta_*^n\| + h^{1/2}\|\eta_*^n\|_{\Gamma_h(t_n)} + h\|\partial_x \eta_*^n\| \le Ch^{k+1}, \quad for \ * = u,\ q, \tag{4.9}$$

$$\left(\eta_u^{n+1}, \widehat{v^n}\right) - \left(\eta_u^n, v^n\right) \le C\tau h^{k+1}\|v^n\|, \ \forall v^n \in H_h^1(t_n). \tag{4.10}$$

Lemma 4.3 *Assume $\mathcal{A}$ is defined by (2.15), then for any $r^n \in V_h(t_n)$, there hold*

$$|\mathcal{A}(\omega^n, \eta_u^n, r^n)| \le \mu C_{wx}\|\eta_u^n\|\|r^n\| + \sqrt{2}\alpha\|\eta_u^n\|_{\Gamma_h(t_n)}[\![r^n]\!], \tag{4.11}$$

$$|\mathcal{A}(\omega^n, \eta_u^n, r^n)| \le \mu C_{wx}\|\eta_u^n\|\|r^n\| + 2\alpha\mu h^{-\frac{1}{2}}\|\eta_u^n\|_{\Gamma_h(t_n)}\|r^n\|. \tag{4.12}$$

We present an estimate for $\mathcal{H}_j(\eta_u, v^n)$ to end this subsection. Applying the estimates (4.9)–(4.11) and the local truncation error (4.4), we have

$$\sum_{j=1}^{N} \mathcal{H}_j\big(\eta_u, v^n\big) \le C\big(\tau h^{k+1} + \tau^2\big)\|v^n\| + C\alpha\tau h^{k+\frac{1}{2}}[\![v^n]\!]. \tag{4.13}$$

4.2 Error Estimates for the First Order Fully Discrete Scheme

In this subsection, we will show the error estimates for the first order fully discrete scheme (3.1).

Theorem 4.4 *Let the sufficiently smooth function u be the exact solution of Eq.* (1.1) *and* u_h^n *be the numerical solution of the fully discrete scheme* (3.1)*. Then we have the following error estimate*

$$\max_{n\tau\le T} \|u(x, t_n) - u_h^n\| \le C\big(h^{k+\frac{1}{2}} + \tau\big),$$

under the restriction $\tau \le \tau_0$*, where* $\tau_0 > 0$ *is a constant independent of h and the positive constant C is independent of h,* τ*, n and* u_h*.*

Proof With the analogous arguments used for the stability analysis, we take the test function $\widehat{v^n} = \zeta_u^{n+1}$ in the error equation (4.6) to derive the energy identity

$$\begin{aligned}\frac{1}{2}\|\zeta_u^{n+1}\|^2 - \frac{1}{2}\|\zeta_u^n\|^2 = &-\frac{1}{2}\|\widetilde{\zeta_u^{n+1}} - \zeta_u^n\|^2 - \frac{\tau}{2}\alpha[\![\zeta_u^{n+1}]\!]^2 + \tau\mathcal{A}\big(\omega^{n+1}, \widehat{\zeta_u^n} - \zeta_u^{n+1}, \zeta_u^{n+1}\big)\\ &- \sqrt{d}\tau\mathcal{L}^+\big(\zeta_q^{n+1}, \zeta_u^{n+1}\big) + \sum_{j=1}^{N}\mathcal{H}_j\big(\eta_u, \widetilde{\zeta_u^{n+1}}\big)\end{aligned} \tag{4.14}$$

For the estimate of $\mathcal{A}(\omega^{n+1}, \widehat{\zeta_u^n} - \zeta_u^{n+1}, \zeta_u^{n+1})$, we need the similar property as that in (2.27)

$$\|\partial_x\big(\zeta_u^{n+1}\big)\| + \sqrt{2\mu h^{-1}}[\![\zeta_u^{n+1}]\!] \le \frac{C_\mu}{\sqrt{d}}\big(\|\zeta_q^{n+1}\| + \|\eta_q^{n+1}\|\big).$$

Then the bound (2.21) yields

$$\begin{aligned}\tau\mathcal{A}\big(\omega^{n+1}, \widehat{\zeta_u^n} - \zeta_u^{n+1}, \zeta_u^{n+1}\big) &\le \frac{C_\mu}{\sqrt{d}}\alpha\tau\big(\|\zeta_q^{n+1}\| + \|\eta_q^{n+1}\|\big)\|\zeta_u^{n+1} - \widehat{\zeta_u^n}\|\\ &\le \frac{\tau}{2}\|\zeta_q^{n+1}\|^2 + \frac{\tau}{2}\|\eta_q^{n+1}\|^2 + \frac{C_\mu^2\alpha^2}{d}C_m\tau\|\widetilde{\zeta_u^{n+1}} - \zeta_u^n\|^2,\end{aligned} \tag{4.15}$$

where we use the similar property (3.7). From the property (2.26) and error Eq. (4.7) , we obtain

$$\begin{aligned}-\sqrt{d}\tau\mathcal{L}^+\big(\zeta_q^{n+1}, \zeta_u^{n+1}\big) &= \sqrt{d}\tau\mathcal{L}^-\big(\zeta_u^{n+1}, \zeta_q^{n+1}\big)\\ &= -\tau\|\zeta_q^{n+1}\|^2 + \tau\big(\eta_q^{n+1}, \zeta_q^{n+1}\big)\\ &\le -\frac{\tau}{2}\|\zeta_q^{n+1}\|^2 + \frac{\tau}{2}\|\eta_q^{n+1}\|^2.\end{aligned} \tag{4.16}$$

Moreover, applying the estimate (4.13), we have

$$\sum_{j=1}^{N}\mathcal{H}_j\big(\eta_u,\widetilde{\zeta_u^{n+1}}\big)\le C\tau\big(h^{k+1}+\tau\big)\big\|\widetilde{\zeta_u^{n+1}}\big\|+C\alpha\tau h^{k+\frac{1}{2}}[\![\zeta_u^{n+1}]\!]$$
$$\le\frac{\tau}{4}\big\|\zeta_u^{n+1}\big\|^2+\frac{\alpha}{2}\tau[\![\zeta_u^{n+1}]\!]^2+C\tau\big(h^{2k+1}+\tau^2\big).\tag{4.17}$$

Here the similar property (3.7) is used. Combining the estimates (4.15)–(4.17) and (4.9) together, the energy Eq. (4.14) turns out to be

$$\frac{1}{2}\big\|\zeta_u^{n+1}\big\|^2-\frac{1}{2}\big\|\zeta_u^{n}\big\|^2\le\frac{\tau}{4}\big\|\zeta_u^{n+1}\big\|^2+\left(\frac{C_\mu^2\alpha^2}{d}C_m\tau-\frac{1}{2}\right)\big\|\widetilde{\zeta_u^{n+1}}-\zeta_u^n\big\|^2+C\tau\big(h^{2k+1}+\tau^2\big).$$

If the time step satisfies

$$\frac{C_\mu^2\alpha^2}{d}C_m\tau-\frac{1}{2}\le 0,\quad and\quad \tau\le 1,$$

there holds

$$\big\|\zeta_u^{n+1}\big\|^2\le C\big\|\zeta_u^{n}\big\|^2+C\tau\big(h^{2k+1}+\tau^2\big).$$

Consequently, the special initial condition $u_h(x,0)=P_h^-u(x,0)$ and Gronwall's inequality provide

$$\|\zeta_u^{n+1}\|^2\le C\big(h^{2k+1}+\tau^2\big),$$

which completes the proof by the triangle inequality. □

Along the same arguments, we can obtain the error estimates for the second fully discrete scheme (3.9) as well as the third order scheme (3.17). The results are listed in the following without detailed proof.

Theorem 4.5 *Let the sufficiently smooth function u be the exact solution of Eq.(1.1). Then we have the following error estimate*

$$\max_{n\tau\le T}\big\|u(x,t_n)-u_h^n\big\|\le C\big(h^{k+\frac{1}{2}}+\tau^l\big),$$

here $l=2,3$ for u_h^n is the numerical solution of the second order scheme (3.9) and third order scheme (3.17), respectively. The time step satisfies $\tau\le\tau_0$, where $\tau_0>0$ is a constant independent of h and the positive constant C is independent of h, τ, n and u_h.

Remark 4.6 For general smoothly moving domain problems, the analysis and results can not be applied directly. One technical difficulty is that the grid function should approximate the movement of the domain accurately, which needs to define a suitable high order mesh velocity in space and time other than the piecewise linear functions in the boundary elements.

5 Numerical Results

In this section, we will show the performance of the ALE-LDG method coupled with the second and third order IMEX time-marching schemes (3.8)–(3.11) and (3.17) for linear convection–diffusion equations. The program for testing examples for the first order fully

Table 1 Accuracy test for $c = 1$ with fixed boundary, $T = 10$

	N	$d = 1$		$d = 0.1$		$d = 0.01$	
		L^2 error	Order	L^2 error	Order	L^2 error	Order
P^1	40	2.64E−05	–	1.37E−01	–	3.98E−02	–
	80	6.59E−06	2.00	2.99E−02	2.20	1.00E−02	1.99
	160	1.64E−06	2.01	7.47E−03	2.00	2.52E−03	1.99
	320	4.07E−07	2.01	1.87E−03	2.00	6.30E−04	2.00
	640	1.01E−07	2.00	4.66E−04	2.00	1.58E−04	2.00
P^2	20	1.29E−05	–	2.10E−02	–	1.37E−03	–
	40	1.77E−06	2.86	2.69E−03	2.97	1.70E−04	3.02
	80	2.33E−07	2.93	3.42E−04	2.97	2.10E−05	3.01
	160	2.99E−08	2.97	4.32E−05	2.99	2.61E−06	3.01
	320	3.78E−09	2.98	5.42E−06	2.99	3.26E−07	3.00

discrete scheme (3.1) is similar, we omit it here to save space. In all tests, the periodic boundary conditions are used. For simplicity, we only consider the uniform partition of the time interval $[0, T]$, namely, $t_n = n\tau$.

Example 5.1 We first test the problem with the fixed boundary

$$\begin{aligned} \partial_t u + c\partial_x u - d\partial_{xx} u = 0, \quad (x, t) \in [0, 2\pi] \times (0, T], \\ u(x, 0) = \sin(x), \quad x \in [0, 2\pi], \end{aligned} \tag{5.1}$$

and the moving grid function

$$x_{j+\frac{1}{2}}(t_n) = x_{j+\frac{1}{2}}(0) + \frac{0.04}{\pi^2} \sin(t_n)\big(x_{j+\frac{1}{2}}(0) - 2\pi\big)x_{j+\frac{1}{2}}(0), \quad j = 0, 1, \ldots, N,$$

which starts with a uniform grid, $x_{j+\frac{1}{2}}(0) = jh$, $h = \frac{2\pi}{N}$. The exact solution is $u(x, t) = e^{-dt} \sin(x - ct)$.

The finite element space is piecewise linear and piecewise quadratic polynomials for the second and third order fully discrete schemes, respectively. In the test, we take $\tau = h$ except for $d = 0.01$, in which $\tau = 0.36h$ is used for the second order fully discrete scheme (3.8)–(3.11) and $\tau = 0.2h$ is used for the third order scheme (3.17). The L^2 error as well as the rates of convergence are summarized in Table 1, which indicates that both schemes give optimal orders of accuracy.

We also show accuracy test in time of the schemes (3.8)–(3.11) and (3.17) for the problem (5.1). In this test, the space of piecewise polynomials of degree k is used for the kth order time discretization such that the error of time discretization is dominant. We take $N = 320$, $T = 10$ and $c = 1$ in the computation. Errors in L^2-norm and orders of accuracy are listed in Table 2, where optimal orders of accuracy in time are observed.

Example 5.2 Next we consider the problem

$$\begin{aligned} &\partial_t u + c\partial_x u - d\partial_{xx} u = 0, \quad (x, t) \in [0, 1] \times (0, T], \\ &u(x, 0) = \frac{1}{2} \sin(\pi(2x - 1)), \quad x \in [0, 1]. \end{aligned}$$

Table 2 Accuracy test in time for schemes (3.8)–(3.11) and (3.17)

τ	$d = 1$		$d = 0.5$		$d = 0.1$	
	L^2 error	Order	L^2error	Order	L^2 error	Order
0.2	4.45E−05	–	4.40E−03	–	8.50E−01	–
0.1	1.12E−05	1.99	1.10E−03	2.00	5.02E−02	4.08
0.05	2.78E−06	2.01	2.73E−04	2.01	1.25E−02	2.00
0.025	6.90E−07	2.01	6.81E−05	2.00	3.13E−03	2.00
0.0125	1.72E−07	2.00	1.70E−05	2.00	7.82E−04	2.00
0.2	3.75E−06	–	1.96E−04	–	5.80E−03	–
0.1	4.98E−07	2.91	2.60E−05	2.92	7.39E−04	2.97
0.05	6.42E−08	2.96	3.34E−06	2.96	9.36E−05	2.98
0.025	8.16E−09	2.98	4.25E−07	2.98	1.19E−05	2.98
0.0125	1.03E−09	2.99	5.40E−08	2.98	1.52E−06	2.96

Table 3 Accuracy test for the moving grid function (5.2)

	N	$c = 1$		$c = 0.1$		$c = 0.01$	
		L^2 error	Order	L^2error	Order	L^2 error	Order
P^1	20	1.77E−02	–	8.20E−04	–	6.99E−04	–
	40	4.34E−03	2.03	1.98E−04	2.05	1.71E−04	2.03
	80	1.07E−03	2.01	4.86E−05	2.02	4.23E−05	2.02
	160	2.66E−04	2.01	1.21E−05	2.01	1.05E−05	2.01
	320	6.61E−05	2.01	3.00E−06	2.01	2.62E−06	2.00
P^2	20	2.27E−03	–	1.78E−04	–	1.61E−04	–
	40	3.37E−04	2.75	2.44E−05	2.87	2.21E−05	2.86
	80	4.60E−05	2.87	3.20E−06	2.93	2.91E−06	2.92
	160	6.01E−06	2.94	4.12E−07	2.96	3.74E−07	2.96
	320	7.69E−07	2.97	5.23E−08	2.98	4.74E−08	2.98

The grid movement is no longer obtained from a smooth function, which is taken as

$$x_{j+\frac{1}{2}}(t_{n+1}) = x_{j+\frac{1}{2}}(t_n) + (-1)^n 0.1 h x_{j+\frac{1}{2}}(0)\big(x_{j+\frac{1}{2}}(0) - 1\big), \quad j = 0, 1, \ldots, N, \tag{5.2}$$

where $x_{j+\frac{1}{2}}(0) = jh$ and $h = \frac{1}{N}$. The finite element space is piecewise linear and piecewise quadratic polynomials for the second and third order fully discrete schemes, respectively. In this test, we take $d = 0.1$, $T = 1$ and $\tau = 2h$. Table 3 shows the performance of fully discrete schemes (3.8)–(3.11) and (3.17). In this case, the optimal convergence rate is also observed from Table 3.

Example 5.3 We consider the problem with moving boundary

$$\begin{aligned} \partial_t u + c\partial_x u - d\partial_{xx} u = 0, &\quad (x, t) \in [\sin(t), \sin(t) + 2\pi] \times (0, T], \\ u(x, 0) = \sin(x), &\quad x \in [0, 2\pi], \end{aligned}$$

Table 4 Accuracy test for $c = 1$ with moving boundary, $T = 10$

	N	$d = 1$		$d = 0.1$		$d = 0.01$	
		L^2 error	Order	L^2error	Order	L^2 error	Order
P^1	40	5.37E−05	–	4.58E−00	–	1.74E−02	–
	80	1.36E−05	1.98	8.54E−02	5.75	4.50E−03	1.95
	160	3.40E−06	2.00	2.14E−02	2.00	1.15E−03	1.97
	320	8.49E−07	2.00	5.35E−03	2.00	2.90E−04	1.98
	640	2.12E−07	2.00	1.34E−03	2.00	7.29E−04	1.99
P^2	20	2.23E−05	–	1.05E−01	–	1.37E−03	–
	40	3.39E−06	2.72	1.34E−02	2.97	1.70E−04	3.01
	80	4.68E−07	2.85	1.71E−03	2.96	2.11E−05	3.01
	160	6.12E−08	2.93	2.16E−04	2.99	2.63E−06	3.01
	320	7.84E−09	2.96	2.72E−05	2.99	3.28E−07	3.00

and the moving grid $x_{j+\frac{1}{2}}(t_n) = x_{j+\frac{1}{2}}(0) + \sin(t_n)$, which begins with a uniform grid as that in Example 5.1. Notice that the computational domain does not follow the physical domain exactly. The exact solution is $u(x, t) = e^{-dt} \sin(x - ct)$.

The performance of the fully discrete schemes (3.8)–(3.11) and (3.17) are shown in Table 4. In the computation, we limit $\tau = h$ except for $d = 0.01$, where $\tau = 0.12h$ is taken to ensure the stability. The numerical results in Table 4 reveal that both schemes arrive at the optimal orders of accuracy. Moreover, compared with Table 1 when $d = 0.01$, we find that the time step restriction is also influenced by the moving grid function.

6 Conclusion

In this paper, we have presented an ALE-LDG method for one-dimensional linear convection–diffusion problems. The ALE framework is suggested by [24]. We have shown that the semi-discrete ALE-LDG method satisfies L^2 stability and sub-optimal error estimate, when the Lax-Friedrichs flux is taken for the convection term. Moreover, we also discussed three specific fully discrete ALE-LDG schemes and the time discretization is IMEX RK approaches. The scaling argument plays an important role in our work, which has been used to analyze quantities caused by the time-dependent cells, approximation space and velocity grid field. We have proven that three fully discrete schemes are stable under the time step restriction $\tau \leq \tau_0$, where τ_0 is a positive constant and independent of the mesh size h, but depends on the convection and diffusion coefficients, the polynomial degree, and the moving grid function. With the time step condition, the quasi-optimal error estimate in space and optimal convergence order in time for the corresponding fully discrete schemes have been established. We also gave numerical examples to verify our theoretical results. The ALE-LDG method, stability analysis and error estimates can be extended to convection–diffusion problems with a nonlinear convection term. The analysis of the fully discrete ALE-LDG scheme in the two dimensional case is more technical and will be considered in future.

Availability of Data and Materials The datasets generated during the current study are available from the corresponding author on reasonable request. They support our published claims and comply with field standards.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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