AFFINE PERIPLECTIC BRAUER ALGEBRAS

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ABSTRACT. We formulate Nazarov-Wenzl type algebras $\widehat{P}_d^-$ for the representation theory of the periplectic Lie superalgebras $\mathfrak{p}(n)$. We establish a Arakawa-Suzuki type functor to provide a connection between $\mathfrak{p}(n)$-representations and $\widehat{P}_d^-$-representations. We also consider various tensor product representations for $\widehat{P}_d^-$. The periplectic Brauer algebra A_d developed by Moon is a quotient of $\widehat{P}_d^-$. In particular, actions induced by Jucys-Murphy elements can be reobtained under the tensor product representation of $\widehat{P}_d^-$. Moreover, a Poincare-Birkhoff-Witt type basis for $\widehat{P}_d^-$ is obtained.

1. Introduction

1.1. The periplectic Lie superalgebra $\mathfrak{p}(n)$ is a superanalogue of the orthogonal or symplectic Lie algebra preserving an odd non-degenerate symmetric or skew-symmetric bilinear form. In the past three decades, there have been some related studies on $\mathfrak{p}(n)$-representation theory, see, e.g., [Sch], [PS], [Go], [Se1], [Ch] and [Co]. However, the representation theory of the periplectic Lie superalgebra is still not well understood. One of the main reasons is that many classical and traditional methods in representation theory are not applicable. In particular, the center of its universal enveloping algebra fails to provide us with information about the blocks in the respective categories (cf. [Go]).

1.2. In recent years, diagrammatically defined algebras naturally have appeared in numerous areas of mathematics. In particular, their representation theory have recently aroused much interest. The very first exposition in representation theory was studied by Brauer in his celebrated paper [Br] where the Brauer algebras was formulated as centralizers:

$$Br_d(\delta) \longrightarrow \mathrm{End}_G(V^{\otimes d}),$$

where $G \subset GL(V)$ is the orthogonal or symplectic group. The method of establishing a link between representation theories via diagram algebras have attracted considerable attention and offers new perspectives in representation theory.

Brauer's theory has been known to admit a generalization in the $\mathbb{Z}_2$-graded setting (see, e.g., [BSR]). In particular, Moon introduced and studied a Brauer type algebra A_d in [Mo] for the first time by giving generators and relations of A_d. Furthermore, the connection between tensor product representations of $\mathfrak{p}(n)$ and A_d were established. Moon used Bergman's diamond Lemma to prove that the dimension of A_d is identical to the Brauer algebra $Br_d(0)$ and noticed that the generators and relations of A_d bear resemblance to $Br_d(0)$. Later on, Kujawa and Tharp provided a diagrammatically defined algebras in [KT], which gave a uniform method to study the algebras of Brauer and Moon simultaneously.

In a recent article [Co], Coulembier studied the *periplectic Brauer algebra*, which is exactly the algebra A_d discovered by Moon, for the invariant theory for $\mathfrak{p}(n)$. As an application, the blocks in the category of finite dimensional weight modules over $\mathfrak{p}(n)$ has been determined.

1.3. The *degenerate affine Nazarov-Wenzl algebra* $We(d)$, introduced in [Na, Section 4], can be regarded as an affine analog of the Brauer algebra $Br_d(0)$. In particular, $Br_d(0)$ can be regarded as a homomorphic image of $We(d)$, where the Jucys-Murphy elements in $Br_d(0)$ appeared as the images of the polynomial generators in $We(d)$.

The main purpose of this paper is to study the affine version of periplectic Brauer algebras. We first formulate the definition of the *affine periplectic Brauer algebra* $\widehat{P}_d^-$ by generators and relations, together with an action of $\widehat{P}_d^-$ on $M \otimes V^{\otimes d}$, where M is an arbitrary $\mathfrak{p}(n)$-module and $V = \mathbb{C}^{n|n}$. In particular, the construction allows us to define an Arakawa-Suzuki type functor $\mathcal{F}$ from the category of $\mathfrak{p}(n)$-modules to the category of $\widehat{P}_d^-$-modules; a similar approach appears also in [B+9, Section 4], where the properties and applications of similar functors are studied. In the second part, we establish a homomorphism from $\widehat{P}_d^-$ to the periplectic Brauer algebra A_d; in particular, the Jucys-Murphy elements of A_d given in [Co] are precisely the images of certain polynomial generators of $\widehat{P}_d^-$. Finally, we give a Poincare-Birkhoff-Witt type basis for $\widehat{P}_d^-$.

1.4. This paper is organized as follows. In Section 2, some basic definition and notation for Lie superalgebras $\mathfrak{gl}(n|n)$ and $\mathfrak{p}(n)$ are introduced. Subsequently, we gives an explicit construction of a Casimir-like element C. The definition of the affine periplectic Brauer algebra $\widehat{P}_d^-$ and a $\mathfrak{p}(n)$-analogue of Arakawa-Suzuki type functor $\mathcal{F}$ are established in Section 3. Particularly, we evaluate the functor $\mathcal{F}$ at various $\mathfrak{p}(n)$-modules in Section 4. Finally, a Poincare-Birkhoff-Witt type basis for $\widehat{P}_d^-$ is given in Section 5.

1.5. Throughout the paper, the symbols $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{Z}_+$ stand for the sets of all, positive and non-negative integers, respectively. All vector spaces, algebras, tensor products, et cetera, are over the field of complex numbers $\mathbb{C}$. For given nonegative integers m, n, we define a partial ordered set $I(m|n) := \{1 < 2 < \cdots < m < \overline{1} < \overline{2} < \cdots < \overline{n}\}$. For a supersapce $M = M_{\overline{0}} \oplus M_{\overline{1}}$ and a given homogenous element $m \in M_\chi$ ($\chi \in \mathbb{Z}_2$), we let $\overline{m} = \chi$.

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2. Lie superalgebras $\mathfrak{gl}(m|n)$ and $\mathfrak{p}(n)$

2.1. **Basic setting and notation for $\mathfrak{gl}(n|n)$ and $\mathfrak{p}(n)$.** Let $m, n \in \mathbb{Z}_+$. Let $\mathbb{C}^{m|n}$ be the complex superspace of superdimension $(m|n)$. The *general linear Lie superalgebra* $\mathfrak{gl}(m|n)$ may be realized as $(m+n)\times(m+n)$ complex matrices:

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.1}$$

where A, B, C and D are respectively $m\times m, m\times n, n\times m, n\times n$ matrices. Let $\mathrm{Str} : \mathfrak{gl}(m|n) \to \mathbb{C}$ denote the supertrace function given by $\mathrm{Str}(X) = \mathrm{tr}\, A - \mathrm{tr}\, D$.

In the rest of this paper, we fix $m = n$ and let $I := I(n|n) = \{1, 2, \cdots, n, \overline{1}, \overline{2}, \cdots, \overline{n}\}$ and let $I^0 := \{1, 2, \cdots, n\}$. Let $\{e_i\}_{i\in I}$ be the basis of the standard $\mathfrak{gl}(n|n)$-representation $V := \mathbb{C}^{n|n}$, where $(\mathbb{C}^{n|n})_{\bar{0}} = \sum_{i\in I^0} \mathbb{C}e_i$ and $(\mathbb{C}^{n|n})_{\bar{1}} = \sum_{i\in I\setminus I^0} \mathbb{C}e_i$. For each $\bar{i} \in I\backslash I^0$, we set $\bar{\bar{i}} = i$.

Throughout this paper, let

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

denote the *periplectic Lie superalgebra* $\mathfrak{p}(n)$ defined in [Ka]. Recall that $\mathfrak{g}$ admits a $\mathbb{Z}$-gradation $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ where $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}} = \mathfrak{gl}(n)$, $\mathfrak{g}_{-1} \cong \bigwedge^2(\mathbb{C}^{n*})$, and $\mathfrak{g}_{+1} \cong S^2(\mathbb{C}^n)$ as $\mathfrak{g}_0$-modules. The standard matrix realization is given by

$$\mathfrak{p}(n) = \left\{ \begin{pmatrix} A & U \\ L & -A^t \end{pmatrix}, \text{where } U \text{ is symmetric and } L \text{ is skew-symmetric} \right\} \subset \mathfrak{gl}(n|n).$$

It is well-known that $\mathfrak{gl}(n|n) \cong \mathfrak{g} \oplus \mathfrak{g}^*$ as $\mathfrak{g}$-modules, and the $\mathbb{Z}$-gradation of $\mathfrak{p}(n)$ is compatible with the $\mathbb{Z}_2$-grading of $\mathfrak{gl}(n|n)$.

2.2. **A Casimir-like element.** As explained in the introduction of [B+9], one of the difficulties in the study of the representation theory of $\mathfrak{p}(n)$ is the lack of quadratic Casimir elements in $U(\mathfrak{p}(n))$. However, a key observation is that the embedding $\mathfrak{p}(n) \subset \mathfrak{gl}(n|n) \cong \mathfrak{g} \oplus \mathfrak{g}^*$ allows one to construct a Casimir-like element, see [B+9, Section 4.1]. As a result, some of the classical approaches can be applied after suitable modifications if necessary.

Proposition 2.1. [B+9, Lemma 4.1.2], [Co] *Let $\{x_i\}$ be a basis for $\mathfrak{g}$ and let $\{x_i^*\}$ be the dual basis for $\mathfrak{g}^*$ with respect to the supertrace form given by $\mathrm{Str}(x_i^* x_j) = \delta_{ij}$. Then we have $\overline{x_i} = \overline{x_i^*}$, for all $1 \le i \le \dim \mathfrak{g}$. Moreover, let*

$$C = \sum_{i=1}^{\dim \mathfrak{g}} x_i \otimes x_i^* \in \mathfrak{g} \otimes \mathfrak{gl}(n|n). \tag{2.2}$$

Then for each $\mathfrak{g}$-module M, we have $C \in \mathrm{End}_{\mathfrak{g}}(M \otimes V)$.

Proof. Set $\ell = \dim \mathfrak{g}$. For each $1 \le a, b \le \ell$, write $[x_a, x_b] = \sum_{q=1}^{\ell} C_{ab}^q x_q$. It is a well-known fact that the supertrace form is an even supersymmetric invariant bilinear form on $\mathfrak{gl}(n|n)$, and hence we have $\overline{x_i} = \overline{x_i^*}$, $\forall 1 \le i \le \ell$.

Since $\mathrm{Str}(x_k \sum_{q=1}^{\ell} C_{qa}^b (-1)^{\overline{x_q}} x_q^*) = C_{ka}^b = (-1)^{\overline{x_b^*}} \mathrm{Str}([x_k, x_a] x_b^*) = (-1)^{\overline{x_b^*}} \mathrm{Str}(x_k [x_a, x_b^*])$ for all $1 \le a, b, k \le \ell$, we have that $[x_a, x_b^*] = \sum_{q=1}^{\ell} C_{qa}^b (-1)^{\overline{x_b^*} + \overline{x_q}} x_q^*$, for all $1 \le a, b \le \ell$. Now fix $1 \le k \le \ell$ and homogenous elements $m \in M$, $v \in V$. By using the fact that $C_{ij}^q = (-1)^{1 + \overline{x_i} \cdot \overline{x_j}} C_{ji}^q$, together with the fact that $C_{ij}^q \neq 0$ implies $\overline{x_i} + \overline{x_j} = \overline{x_q}$, we have

$$C(x_k(m \otimes v))) = (\sum_{i=1}^{\ell} x_i \otimes x_i^*)(x_k m \otimes v + (-1)^{\overline{x_k}\cdot\overline{m}} m \otimes x_k v)$$
$$= \sum_{i=1}^{\ell} \{((-1)^{(\overline{x_k}+\overline{m})\overline{x_i^*}}[x_i, x_k]m \otimes x_i^* v) + ((-1)^{\overline{x_k m}+\overline{x_i^*}\cdot\overline{m}} x_i m \otimes [x_i^*, x_k]v)\} + x_k(C(m \otimes v))$$
$$= \sum_{i,j=1}^{\ell} \{((-1)^{(\overline{x_k}+\overline{m})\overline{x_j^*}} C_{jk}^i x_i m \otimes x_j^* v) + ((-1)^{\overline{x_k m}+\overline{x_i^*}\cdot\overline{m}+\overline{x_i^*}\cdot\overline{x_k}+1+\overline{x_j^*}+\overline{x_i^*}} C_{jk}^i x_i m \otimes x_j^* v)\} + x_k(C(m \otimes v))$$
$$= \sum_{i,j=1}^{\ell} \{((-1)^{(\overline{x_k}+\overline{m})\overline{x_j^*}} C_{jk}^i + (-1)^{\overline{x_k m}+\overline{x_i^*}\cdot\overline{m}+\overline{x_i^*}\cdot\overline{x_k}+1+\overline{x_j^*}+\overline{x_i^*}} C_{jk}^i)(x_i m \otimes x_j^* v)\} + x_k(C(m \otimes v))$$
$$= x_k(C(m \otimes v)).$$

The last equality follows from the following fact:

If $C_{jk}^i \neq 0$ then we have $\overline{x_j} + \overline{x_k} + \overline{x_i} \equiv 0 \pmod 2$, and so

$$(\overline{x_k} + \overline{m})\overline{x_j^*} \equiv (\overline{x_k} + \overline{m})(\overline{x_i} + \overline{x_k}) \equiv \overline{x_k} + \overline{x_i^*}\overline{x_k} + \overline{x_k m} + \overline{x_i^*}\overline{m}$$
$$\equiv \overline{x_j} + \overline{x_i} + \overline{x_i^*}\overline{x_k} + \overline{x_k m} + \overline{x_i^*}\overline{m} \pmod 2.$$

□

We decompose $\mathfrak{gl}(n|n)$ into a direct sum of $\mathfrak{g}$-submodules $\mathfrak{g}$ and $\mathfrak{g}^*$ as follows:

$$\mathfrak{gl}(n|n) = \mathfrak{g} \oplus \mathfrak{g}^* = \mathfrak{g} \oplus \left\{ \begin{pmatrix} A & U \\ L & A^t \end{pmatrix} \middle| \; U^t = -U, \quad L^t = L \right\}.$$

Let $\ell = \dim \mathfrak{g}$. A choice of basis $\{x_i\}$ for $\mathfrak{g}$ and its dual basis $\{x_i^*\}$ for $\mathfrak{g}^*$ is given as follows with respect to the supertrace form $\langle x_i, x_j^* \rangle := \mathrm{Str}(x_j^* x_i)$.

$$\{x_i\}_{i=1}^{\ell} := \left\{ E_{st} := \begin{pmatrix} e_{st} & 0 \\ 0 & -e_{ts} \end{pmatrix} \right\} \bigcup \left\{ X_{st} := \begin{pmatrix} 0 & e_{st} + e_{ts} \\ 0 & 0 \end{pmatrix} \right\} \bigcup$$
$$\left\{ X_{ss} := \begin{pmatrix} 0 & e_{ss} \\ 0 & 0 \end{pmatrix} \right\} \bigcup \left\{ Y_{st} := \begin{pmatrix} 0 & 0 \\ e_{st} - e_{ts} & 0 \end{pmatrix} \right\},$$

$$\{x_j^*\}_{j=1}^{\ell} := \left\{ E_{st}^* := \frac{1}{2} \begin{pmatrix} e_{ts} & 0 \\ 0 & e_{st} \end{pmatrix} \right\} \bigcup \left\{ X_{st}^* := -\frac{1}{2} \begin{pmatrix} 0 & 0 \\ e_{ts} + e_{st} & 0 \end{pmatrix} \right\} \bigcup$$
$$\left\{ X_{ss}^* := -\begin{pmatrix} 0 & 0 \\ e_{ss} & 0 \end{pmatrix} \right\} \bigcup \left\{ Y_{st}^* := -\frac{1}{2} \begin{pmatrix} 0 & e_{st} - e_{ts} \\ 0 & 0 \end{pmatrix} \right\},$$

where $1 \leq s \neq t \leq n$. Here e_{st} denotes the elementary $n \times n$ matrix with (s,t)-entry 1 and zero elsewhere. Throughout this article, we shall consider the element C with respect to the above choice of bases.

3. Affine Periplectic Brauer algebras

3.1. **Generators and Relations.** In this section, we give the definition of the affine periplectic Brauer algebra $\widehat{P}_d^-$, which is the main object studied in this article. It can be regarded as an affine analogue of the periplectic Brauer algebra A_d discussed in [Co, Mo] (also known as the marked Brauer algebra in [KT]), or a periplectic analogue of the affine Nazarov-Wenzel algebra studied in [ES, Na].

Definition 3.1. Let $d \in \mathbb{N}$. The *affine periplectic Brauer algebra*, denoted by $\widehat{P}_d^-$, is the associative algebra over $\mathbb{C}$ generated by the elements

$$s_a,\ \varepsilon_a,\ y_j, \qquad 1 \le a \le d-1,\ 1 \le j \le d,$$

subject to the relations (for all $1 \le a, b, c \le d-1$ and $1 \le i, j \le d$):

$(P.1)$ $s_a^2 = 1$
$(P.2)$ (a) $s_a s_b = s_b s_a$ for $|a-b| > 1$
(b) $s_c s_{c+1} s_c = s_{c+1} s_c s_{c+1}$
(c) $s_a y_i = y_i s_a$ for $i \notin \{a, a+1\}$
$(P.3)$ $\varepsilon_a^2 = 0$
$(P.4)$ $\varepsilon_1 y_1^k \varepsilon_1 = 0$ for $k \in \mathbb{N}$
$(P.5)$ (a) $s_a \varepsilon_b = \varepsilon_b s_a$ and $\varepsilon_a \varepsilon_b = \varepsilon_b \varepsilon_a$ for $|a-b| > 1$
(b) $\varepsilon_a y_i = y_i \varepsilon_a$ for $i \notin \{a, a+1\}$
(c) $y_i y_j = y_j y_i$
$(P.6)$ (a) $\varepsilon_a s_a = -\varepsilon_a = -s_a \varepsilon_a$
(b) $s_c \varepsilon_{c+1} \varepsilon_c = -s_{c+1} \varepsilon_c$ and $\varepsilon_c \varepsilon_{c+1} s_c = \varepsilon_c s_{c+1}$
(c) $\varepsilon_{c+1} \varepsilon_c s_{c+1} = -\varepsilon_{c+1} s_c$ and $s_{c+1} \varepsilon_c \varepsilon_{c+1} = s_c \varepsilon_{c+1}$
(d) $\varepsilon_{c+1} \varepsilon_c \varepsilon_{c+1} = -\varepsilon_{c+1}$ and $\varepsilon_c \varepsilon_{c+1} \varepsilon_c = -\varepsilon_c$
$(P.7)$ $s_a y_a - y_{a+1} s_a = \varepsilon_a - 1$ and $y_a s_a - s_a y_{a+1} = -\varepsilon_a - 1$
$(P.8)$ (a) $\varepsilon_a (y_a - y_{a+1}) = \varepsilon_a$
(b) $(y_a - y_{a+1}) \varepsilon_a = -\varepsilon_a$

Note that the relations without the appearance of any y_j are precisely the defining relations of the *periplectic Brauer algebra* A_d studied in [Co, Mo]. On the other hand, these relations appeared in [ES, Definition 2.1] (see also [Na]) by setting all $w_k = 0$ there, except for few differences in signs.

3.2. **An Arakawa-Suzuki type functor $\mathcal{F}$.** Let M be an arbitrary $\mathfrak{g}$-module. Denote by $C_{M,V} \in \mathrm{End}_{\mathfrak{g}}(M \otimes V)$ the corresponding image of the Casimir-like element C (see Proposition 2.1) in the action $C \curvearrowright M \otimes V$. Denote by $s \in \mathrm{End}_{\mathfrak{g}}(V^{\otimes 2})$ the super-permutation

$$s(e_a \otimes e_b) = (-1)^{\overline{e_a e_b}} e_b \otimes e_a, \quad \text{for } a, b \in I. \tag{3.1}$$

Furthermore, define $\varepsilon \in \mathrm{End}_{\mathfrak{g}}(V^{\otimes 2})$ by

$$\varepsilon = 2C_{V,V} - s. \tag{3.2}$$

Lemma 3.2.

$$\varepsilon(e_a \otimes e_b) = (e_a, e_b) \sum_{i=1}^{n} (e_i \otimes e_{\bar{i}} - e_{\bar{i}} \otimes e_i) = \begin{cases} 0, & \text{if } a \neq \bar{b}, \\ \sum_{i \in I} (-1)^{\overline{e_i}} (e_i \otimes e_{\bar{i}}), & \text{if } a = \bar{b}. \end{cases}$$

By abusing notation, we define the following elements in $\mathrm{End}(M\otimes V^{\otimes d})$:

$$s_a=\mathrm{Id}^{\otimes a}\otimes s\otimes \mathrm{Id}^{\otimes d-a-1},\quad \varepsilon_a=\mathrm{Id}^{\otimes a}\otimes e\otimes \mathrm{Id}^{\otimes d-a-1},\quad y_j=2C_{M\otimes V^{\otimes j-1},V}\otimes \mathrm{Id}^{\otimes d-j},\tag{3.3}$$

for $1\le a\le d-1$, $1\le j\le d$. It shall be clear from the context when we regard these elements as elements in $\widehat{P}_d^-$ and when as elements in $\mathrm{End}(M\otimes V^{\otimes d})$.

Moreover, for $1\le i<j\le d$ and homogenous $m\in M$, we define $\Omega_{i,j}\in \mathrm{End}(M\otimes V^{\otimes d})$ by

$$\begin{aligned}
&\Omega_{i,j}(m\otimes e_{t_1}\otimes e_{t_2}\otimes\cdots\otimes e_{t_d})\\
&\quad:=2\sum_{k=1}^{\dim\mathfrak{g}}(-1)^{\overline{x_k}(\sum_{s=1}^{i-1}\overline{e_{t_s}}+\overline{m})+\overline{x_k}(\sum_{s=1}^{j-1}\overline{e_{t_s}}+\overline{m})}(m\otimes e_{t_1}\otimes\cdots\otimes x_ke_{t_i}\otimes\cdots\otimes x_k^*e_{t_j}\otimes\cdots\otimes e_{t_d}),\\
&\Omega_{0,j}(m\otimes e_{t_1}\otimes e_{t_2}\otimes\cdots\otimes e_{t_d})\\
&\quad:=2\sum_{k=1}^{\dim\mathfrak{g}}(-1)^{\overline{x_k}(\sum_{s=1}^{j-1}\overline{e_{t_s}})}(x_km\otimes e_{t_1}\otimes e_{t_2}\otimes\cdots\otimes x_k^*e_{t_j}\otimes\cdots\otimes e_{t_d}).
\end{aligned}$$

We may observe that

$$y_j=\sum_{0\le k\le j-1}\Omega_{k,j},\tag{3.4}$$

for all $1\le j\le d$.

Lemma 3.3. *The elements s_a, ε_a, y_j defined by (3.3) commute with the natural $\mathfrak{g}$-action on $M\otimes V^{\otimes d}$. In other words, they belong to $\mathrm{End}_{\mathfrak{g}}(M\otimes V^{\otimes d})$.*

Proof. For ε_a, the statement follows from Lemma 3.2. For s_a, a similar operator is defined in [CW, Section 5.1], which is known to commutes with the action of $\mathfrak{gl}(n|n)$ and hence commute with the action of $\mathfrak{g}$. For y_j, replacing M by $M\otimes V^{\otimes j-1}$ in Proposition 2.1, and the lemma follows. □

Proposition 3.4. *The elements $\{s_a,\varepsilon_a,y_j\,|\,1\le a\le d-1,\ 1\le j\le d\}$ in $\mathrm{End}_{\mathfrak{g}}(M\otimes V^{\otimes d})$ satisfy the relations (P.1)–(P.3) and (P.5)–(P.8).*

Proof. We check by definition that these relations hold in $\mathrm{End}_{\mathfrak{g}}(M\otimes V^{\otimes d})$. Relations $(P.1)$, $(P.2)$, $(P.5)$(a), $(P.5)$(c), $(P.6)$ are straightforward to check, where some of them can be found in [Co, Mo]; $(P.3)$ is an immediate consequence of Lemma 3.2; $(P.8)$(a) and $(P.8)$(b) are somehow involved and they are verified in Appendixes A.2 and A.3.

Consider $(P.5)$(b). The proof is similar to [ES, Lemma 8.4]. If $i\le a+1$ then it obviously holds. Assume that $i>a+1$. Since the parity of the basis element $x_j\in\mathfrak{g}$ and its dual basis element $x_j^*\in\mathfrak{g}^*$ must be the same, we may conclude that

$$\Omega_{s,i}\circ\varepsilon_a=\varepsilon_a\circ\Omega_{s,i},\ \text{for all } s\ne a,a+1.$$

Consequently, $(P.5)$(b) is equivalent to

$$(\Omega_{a,i}+\Omega_{a+1,i})\circ\varepsilon_a=\varepsilon_a\circ(\Omega_{a,i}+\Omega_{a+1,i}).$$

The last equality follows from Lemma A.2 and $(P.5)$(b) is verified.

We now check $(P.7)$. Let $m \in M \otimes V^{\otimes a-1}$ and $v \in V^{\otimes d-a-1}$ be homogenous elements. Then for all $i, j \in I$, we have

$$\begin{aligned}
&(y_{a+1}(s_a(m \otimes e_i \otimes e_j \otimes v))) = y_{a+1}((-1)^{\overline{e_i e_j}} m \otimes e_j \otimes e_i \otimes v) \\
&= \sum_k ((-1)^{\overline{e_i e_j} + (\overline{m} + \overline{e_j})\overline{x_k}} x_k (m \otimes e_j) \otimes x_k^* e_i \otimes v) \\
&= \sum_k ((-1)^{\overline{e_i e_j} + (\overline{m} + \overline{e_j})\overline{x_k}} x_k m \otimes e_j \otimes x_k^* e_i \otimes v) + \sum_k ((-1)^{\overline{e_i e_j} + \overline{e_j x_k}} m \otimes x_k e_j \otimes x_k^* e_i \otimes v) \\
&= s_a y_a (m \otimes e_i \otimes e_j \otimes v) + \sum_k ((-1)^{\overline{e_i e_j} + \overline{e_j x_k}} m \otimes x_k e_j \otimes x_k^* e_i \otimes v) \\
&= s_a y_a (m \otimes e_i \otimes e_j \otimes v) + m \otimes ((-1)^{\overline{e_i e_j}} 2 C_{V,V}(e_j \otimes e_i)) \otimes v. \\
&= s_a y_a (m \otimes e_i \otimes e_j \otimes v) + m \otimes ((s + \varepsilon) \circ s(e_i \otimes e_j)) \otimes v. \\
&= (s_a y_a + \mathrm{Id} + \varepsilon_a)(m \otimes e_i \otimes e_j \otimes v),
\end{aligned}$$

where the last equality comes from $(P.6)$(a). The other equality of $(P.7)$ can be proved in a similar method. □

Lemma 3.5. *For any highest weight $\mathfrak{g}$-module M and any $k \in \mathbb{N}$, the operator $\varepsilon_1 y_1^k \varepsilon_1$ acts as zero on $M \otimes V^{\otimes d}$.*

Proof. By relations $(P.8)$(a), $(P.5)$(c) and $(P.3)$, we have

$$\varepsilon_1 y_2^2 \varepsilon_1 = \varepsilon_1 y_2 (y_2 \varepsilon_1) = \varepsilon_1 y_2 (y_1 \varepsilon_1 + \varepsilon_1) = \varepsilon_1 y_1 (y_1 \varepsilon_1 + \varepsilon_1) + \varepsilon_1 (y_1 \varepsilon_1 + \varepsilon_1) = \varepsilon_1 y_1^2 \varepsilon_1 + 2 \varepsilon_1 y_1 \varepsilon_1.$$

On the other hand, by $(P.8)$(b), $(P.5)$(c) and $(P.3)$, we have

$$\varepsilon_1 y_2^2 \varepsilon_1 = (\varepsilon_1 y_2) y_2 \varepsilon_1 = (\varepsilon_1 y_1 - \varepsilon_1) y_2 \varepsilon_1 = (\varepsilon_1 y_1 - \varepsilon_1) y_1 \varepsilon_1 - (\varepsilon_1 y_1 - \varepsilon_1)\varepsilon_1 = \varepsilon_1 y_1^2 \varepsilon_1 - 2 \varepsilon_1 y_1 \varepsilon_1.$$

Therefore, $\varepsilon_1 y_1 \varepsilon_1 = 0$. The general case follows from a similar argument and induction on k. □

Remark 3.6. One may actually exclude $(P.4)$ from the defining relations. We keep it here so that one can compare with the relations in [ES, Definition 2.1].

As a consequence, we have the following $\mathfrak{p}(n)$ analogue of the Arakawa-Suzuki functor; cf. [AS].

Theorem 3.7. *Let M be a highest weight $\mathfrak{p}(n)$-module. Then we have the following right action of $\widehat{P}_d^-$ on $M \otimes V^{\otimes d}$:*

$$v \cdot s_a := s_a(v), \quad v \cdot \varepsilon_a := \varepsilon_a(v), \quad v \cdot y_j := y_j(v), \tag{3.5}$$

for all $v \in M \otimes V^{\otimes d}$, $1 \le a \le d-1$, $1 \le j \le d$. In other words, there is an algebra homomorphism

$$\Psi_M : \widehat{P}_d^- \longrightarrow \mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})^{\mathrm{opp}}. \tag{3.6}$$

Proof. By Lemma 3.3, Proposition 3.4, and Lemma 3.5, all defining relations of $\widehat{P}_d^-$ are preserved under Ψ_M. □

Let $\widehat{P}_d^-$-mod denotes the category of $\widehat{P}_d^-$-modules. Observe that Theorem 3.7 defines a functor $\mathcal{F}(\bullet) := \bullet \otimes V^{\otimes d}$ from the category of highest weight $\mathfrak{g}$-modules to $\widehat{P}_d^-$-mod.

4. Tensor product representations

In this section, we evaluate the functor $\mathcal{F}$ in Theorem 3.7 at various $\mathfrak{g}$-modules $M = V^{\otimes k}$ for $k \in \mathbb{Z}_+$ to give a connection between tensor product representations of $\mathfrak{p}(n)$ and $\widehat{P}_d^-$.

4.1. **Periplectic Brauer algebras.** We start with the simplest case where $M = \mathbb{C}$, the trivial representation. It turns out that $\Psi_{\mathbb{C}}(\widehat{P}_d^-)$ is in fact the periplectic Brauer algebra in [Co, Mo].

Firstly we recall the Brauer (d,d)-diagram realization of A_d in [Co, KT, Mo]. A Brauer (d,d)-diagram is a graph with two rows of d vertices $\{1,2,\ldots,d\}$ and $\{\overline{1},\overline{2},\ldots,\overline{d}\}$, i above $\overline{i}$ for all $1 \le i \le d$, and d-edges such that each vertex is connected to precisely one edge. That is, Brauer (d,d)-diagrams correspond to all partitions of $2d$-dots into pairs. The set $\mathcal{G}(d)$ of all Brauer (d,d)-diagrams forms a basis of Brauer algebra originally defined by Brauer in [Br].

Let $\mathcal{A}$ be the periplectic Brauer category in which objects are positive integers and morphisms are (d,d)-Brauer diagrams, see, e.g., [Co, Section 2.1]. Then

$$A_d = \mathrm{End}_{\mathcal{A}}(d).$$

We recall the generators $\widehat{s_i}$ and $\widehat{\varepsilon_i}$ for A_d. For each $1 \le i \le d-1$, denote by $\widehat{s_i} := (i,i+1) \in A_d$ the Brauer diagram with a line connecting the upper vertex i to the lower vertex $\overline{i+1}$, and with a line connecting the upper vertex $i+1$ to the lower vertex $\overline{i}$, and a line connecting j to $\overline{j}$ for each $j \neq i,i+1$. Also, let $\widehat{\varepsilon_i} := \overline{(i,i+1)} \in A_d$ corresponds to the Brauer diagram which consists only non-crossing propagating lines except for one cup and cap, connecting $\{i,i+1\}$ and $\{\overline{i},\overline{i}+1\}$, respectively. The other elements (i,j) and $\overline{(i,j)}$ in A_d are defined analogously (see, e.g., [Co, Section 2.1.5]).

In [Mo], Moon proved that there is a tensor product representation for A_d acting as a subalgebra of centralizer:

$$\psi : A_d \to \mathrm{End}_{\mathfrak{p}(n)}(V^{\otimes d}). \tag{4.1}$$

The homomorphism (4.1) is an isomorphism provided $n \ge d$ [Mo, Theorem 4.5]. It is announced by Serganova in [Se2, Theorem 3.5] that homomorphism (4.1) is surjective for all $n,d \in \mathbb{N}$.

Set $M = \mathbb{C}$, the trivial representation, in Theorem 3.7 and we have the tensor product representation of $\widehat{P}_d^-$ on $\mathbb{C} \otimes V^{\otimes d} \cong V^{\otimes d}$. In particular, by (3.3), $\Psi_{\mathbb{C}}(s_a), \Psi_{\mathbb{C}}(\varepsilon_a)$ coincide with the image of the generators $\widehat{s}_a$, $\widehat{\varepsilon}_a$ in (4.1). That is, we have $\Psi_{\mathbb{C}}(s_a) = \psi(\widehat{s}_a), \Psi_{\mathbb{C}}(\varepsilon_a) = \psi(\widehat{\varepsilon}_a)$.

Next we consider the image $\Psi_{\mathbb{C}}(y_j)$. By (3.3) again, $\Psi_{\mathbb{C}}(\Omega_{0,j}) = 0$ for all j. In particular, $\Psi_{\mathbb{C}}(y_1)$ is the zero operator. Using the defining relations in A_d [Mo, Proposition 2.1], we have the following identities for any $1 \le i < j \le d$:

$$\psi(\ \overline{(ij)}\ ) = \psi((i,i+1))\cdots\psi((j-2,j-1))\cdot\psi(\ \overline{(j-1,j)}\ )\cdot\psi((j-2,j-1))\cdots\psi((i,i+1)).$$

This implies that $\Psi_{\mathbb{C}}(\Omega_{ij}) = \psi((i,j)) + \psi(\overline{(i,j)})$, for all $1 \le i < j \le d$. By (3.4), we have

$$\Psi_{\mathbb{C}}(y_j) = \sum_{0 \le k \le j-1} \Psi_{\mathbb{C}}(\Omega_{k,j}) = \sum_{1 \le k \le j-1} \psi((k,j)) + \psi(\ \overline{(k,j)}\ ).$$

Recall that the Jucys-Murphy elements $\{z_j \,|\, 1 \le j \le d\}$ of A_d are defined in [Co, 6.1.1] by setting $z_1 = 0$ and

$$z_j := \sum_{1 \le i \le j-1} (ij) + \overline{(ij)}.$$

Our discussion above shows that $\Psi_{\mathbb{C}}(y_j) = \psi(z_j)$ for all $1 \le j \le d$.

In fact, A_d is a homomorphic image of $\widehat{P}_d^-$, where the Jucys-Murpy elements $z_j \in A_d$ are precisely the image of the polynomial generators $y_j \in \widehat{P}_d^-$. This is parallel to the same phenomenon appeared in the degenerate affine Hecke algebra and the group algebra of the symmetric group. A similar phenomenon also appeared in the degenerate affine Nazarov-Wenzel algebra and the Brauer algebra, see [Na, Section 4].

Theorem 4.1. *The map* $\pi : \widehat{P}_d^- \longrightarrow A_d$ *given by*

$$\pi(s_a) = \widehat{s}_a,\ \ \pi(\varepsilon_a) = \widehat{\varepsilon}_a,\ \ \pi(y_j) = z_j, \tag{4.2}$$

is an algebra epimorphism.

Proof. We check that the map π preserves the defining relations $(P.1)$–$(P.8)$. It suffices to check only the relations $(P.2)$(c), $(P.4)$, $(P.5)$(b), $(P.5)$(c), $(P.7)$, $(P.8)$(a), $(P.8)$(b). Relations $(P.2)$(c) and $(P.5)$(b) follow from [Co, Lemma 6.3.3]; $(P.5)$(c) follows from [Co, Lemma 6.1.2]; $(P.4)$ follows from [Co, Corollary 6.3.4]; $(P.7)$ follows from [Co, Lemma 6.3.1] together with $(P.6)$(a); $P.8$(a) and $(P.8)(b)$ follow from [Co, Lemma 6.3.1]. □

Remark 4.2. When $n \ge d$, the homomorphism (4.1) is an isomorphism [Mo, Theorem 4.5]. In this situation, the map $\pi = (\psi)^{-1} \circ \Psi_{\mathbb{C}}$ gives a simple proof of the above theorem.

4.2. **General tensor product representations.** We are concerned in this subsection with more general tensor product representations of $\widehat{P}_d^-$. Namely, we consider Ψ_M with $M = V^{\otimes m}$, $m \in \mathbb{N}$. The following is a $\mathfrak{p}(n)$-analogue of results in [Na, Section 4]. In particular, the map π_0 is exactly the map π in Theorem 4.1.

Theorem 4.3. *For each* $m \in \mathbb{Z}$*, there is an algebra homomorphism* $\pi_m : \widehat{P}_d^- \longrightarrow A_{m+d}$ *such that*

$$\pi_m(s_a) = \widehat{s}_{m+a},\ \ \pi_m(\varepsilon_a) = \widehat{\varepsilon}_{m+a},\ \ \pi_m(y_j) = z_{m+j}, \tag{4.3}$$

for each $1 \le a \le d-1$ *and* $1 \le j \le d$.

Proof. In A_d, we have $\widehat{\varepsilon}_i\, z_i^k\, \widehat{\varepsilon}_i = 0$ for all $1 \le i \le m+d$ and $k \in \mathbb{N}$ (see, e.g., [Co, Corollary 6.3.4]). It implies that defining relations of $\widehat{P}_d^-$ are all preserved under π_m for any $m \in \mathbb{Z}$. □

5. PBW basis theorem for $\widehat{P}_d^-$

In this section, we give a PBW type basis for $\widehat{P}_d^-$ by adapting the approach in [Na, Theorem 4.6].

5.1. **Regular monomials.** We first recall the notion of regular monomials defined by Nazarov in [Na, (4.18)]. Recall that A_d is a diagram algebra with basis $\mathcal{G}(d)$ of Brauer (d,d)-diagrams. In the rest of this article, adapting the notation in A_d, we set $(i, i+1) := s_i$, $\overline{(i, i+1)} := \varepsilon_i$ to be the generators of $\widehat{P}_d^-$, while the general elements (i,j) and $\overline{(i,j)}$ for $1 \le i < j \le d$ in $\widehat{P}_d^-$ are defined analogously.

A monomial $u \in \widehat{P}_d^-$ in s_a, ε_b, y_i is called *regular* if

$$u = y_1^{i_1} y_2^{i_2} \cdots y_d^{i_d} \cdot \gamma \cdot y_1^{j_1} y_2^{j_2} \cdots y_d^{j_d}, \tag{5.1}$$

where $\gamma \in \mathcal{G}(d)$ such that

$$\text{if } k \in \{r_1, \dots r_q\} \text{ then } i_k = 0, \tag{5.2}$$

$$\text{if } j_t \neq 0 \text{ then } t \in \{\overline{r'_1}, \dots \overline{r'_q}\}, \tag{5.3}$$

where $r_1, r_2, \dots, r_q \in \{1, 2, \dots, d\}$ (resp. $\overline{r'_1}, \dots \overline{r'_q} \in \{\overline{1}, \overline{2}, \dots, \overline{d}\}$) are all upper (resp. lower) right ends of horizontal edges of γ. The following theorem is the main result in this section.

Theorem 5.1. *The set of all regular monomials forms a basis for $\widehat{P}_d^-$.*

The rest of this section is devoted to the proof of Theorem 5.1. We first equip $\widehat{P}_d^-$ with a filtration by setting degrees of the generators as follows:

$$\deg(s_a) = \deg(\varepsilon_a) = 0, \ \deg(y_j) = 1.$$

Denote by w_a the image of y_a in the corresponding graded algebra $\mathrm{gr}\widehat{P}_d^-$. The following identities in $\mathrm{gr}\widehat{P}_d^-$ come from $(P.2)$(c) and $(P.7)$:

$$\tau w_a \tau^{-1} = w_{\tau(a)}, \quad \tau \in \mathfrak{S}_d. \tag{5.4}$$

By (5.4) and $(P.4)$, it follows that

$$\overline{(ij)} w_i^k \overline{(ij)} = 0, \tag{5.5}$$

for all $k \in \mathbb{N}$ and $i \neq j$. Again, by (5.4), $(P.2)$(c), $(P.5)$(b), $(P.7)$, $(P.8)$(a) and $(P.8)$(b), we obtain the following identities:

$$\overline{(ij)} w_a = w_a \overline{(ij)}, \ \text{for } a \neq i, j, \tag{5.6}$$

$$\overline{(ij)}(w_i - w_j) = 0 = (w_i - w_j)\overline{(ij)}, \ \text{for } i \neq j. \tag{5.7}$$

The identities (5.4), (5.5), (5.6) and (5.7) provide all ingredients for the following lemma.

Lemma 5.2. *$\widehat{P}_d^-$ is spanned by the set of all regular monomials.*

Proof. The proof of this lemma is obtained by calculation on (d, d)-Brauer diagrams, which is quite similar to that given in [Na, Lemma 4.4] and [Na, Lemma 4.5], and we omit the details. □

To prove Theorem 5.1, it suffices to prove the linear independence of regular monomials. By induction on degrees, it suffices to prove the linear independence of those terms having *maximal degree* in a linear combination of regular monomials. This can be obtained by a similar method as employed in the proof of [Na, Lemma 4.8]. The major difference is the possible occurrence of minus signs, which does not affect the validity of the proof. A heuristic argument, the details of which we omit, will be given below. The key point for understanding arguments in the proof of [Na, Lemma 4.8] is to use dots tracing from the upper row and the lower row.

As had been explained in the proof of [Na, Lemma 4.8], we may only consider a linear combination $F \in \widehat{P}_d^-$ of regular monomials satisfying the following two assumptions:

(L1) $F = \sum_{k=1}^{n} \alpha_k F_k$, where $\alpha_k \in \mathbb{C}\backslash\{0\}$, F_k are regular monomials, and $\deg F_k = m \in \mathbb{N}$ for all k.

(L2) the number of horizontal edges of corresponding graphs in these terms are the same, say $2r$, for some $r \in \mathbb{Z}_+$.

We proceed our argument by considering the tensor product representation π_m in Theorem 4.3. Consider the distinguished set $\mathcal{G} \subseteq \mathcal{G}(m+d)$ consisting of $(m+d, m+d)$-Brauer diagram Γ satisfying the following three conditions:

(i) Γ has exactly $2r$ horizontal edges.
(ii) There are no vertical edges in Γ of the form $\{k, \overline{k}\}$ with $1 \le k \le m$.
(iii) There are no horizontal edges in Γ of the form $\{k, l\}$ or $\{\overline{k}, \overline{l}\}$, with $1 \le k, l \le m$.

Let $F = \sum_{k=1}^{n} \alpha_k F_k$ satisfy the hypotheses (L1) and (L2) above. Suppose that $F = 0$, then definitely we have $\pi_m(F) = 0$ in A_{m+d} for any $m \in \mathbb{Z}_+$. Let $\mathbb{C}\mathcal{G}$ be the subspace of A_{m+d} spanned by $\mathcal{G}$. Recall the *leading terms* defined in [Na, Section 4] are those terms of the following form:

$$\Pi_{\ell=1}^{d}(N_{\ell,1}, m+\ell)\cdots(N_{\ell,i_\ell}, m+\ell)\cdot\gamma\cdot\Pi_{\ell=1}^{d}(N'_{\ell,1}, m+\ell)\cdots(N'_{\ell,j_\ell}, m+\ell),$$

where $N_{\ell,1}, N_{\ell,2}, \ldots, N_{\ell,i_\ell}$ and $N'_{\ell,1}, N'_{\ell,2}, \ldots, N'_{\ell,j_\ell}$ $(\ell = 1, 2, \ldots, d)$ run over the set of all permutations of $\{1, 2, \ldots, m\}$. As had been observed in the proof of [Na, Lemma 4.8], the projection p_m of A_{m+d} on $\mathbb{C}\mathcal{G}$ classifies all terms in the expression of $\pi_m(F_k)$ into leading and non-leading terms.

Furthermore, p_m preserves all information of leading terms of each F_k, and all shape of the non-leading terms do not coincide with that of leading terms. This forces that all α_k must be zero. Theorem 5.1 follows from this observation.

Along with the proof of Theorem 5.1, we obtain a $\mathfrak{p}(n)$ counterpart of [Na, Theorem 4.7].

Theorem 5.3. *We have*

$$\bigcap_{m\ge 0} \mathrm{Ker}(\pi_m) = 0.$$

We conclude this article by a similar result in [Na, Corollary 4.9]. Recall that the *degenerate affine Hecke algebra H_d* (see e.g. [Dr]) is generated by the group algebra of the symmetric group $\mathbb{C}[\mathfrak{S}_d]$ and the polynomial generators $v_1, v_2, \ldots, v_d$ subject to the relations

(1) $s_i v_j = v_j s_i$ for $j \notin \{i, i+1\}$,
(2) $s_i v_i - v_{i+1} s_i = -1, \qquad s_i v_{i+1} - v_i s_i = 1.$

The following corollary can be observed from the defining relations of $\widehat{P}_d^-$.

Corollary 5.4. *The map $s_a \mapsto s_a$, $\varepsilon_a \mapsto 0$, $y_j \mapsto v_j$ defines a homomorphism from $\widehat{P}_d^-$ to H_d.*

Appendix A.

A.1. We need some preparations to prove the relations $(P.8)$ in Proposition 3.4. Note that we may assume the relations $(P.1)$–$(P.3)$, $(P.5)$(a), $(P.5)$(c), $(P.6)$–$(P.7)$ hold in $\mathrm{End}_{\mathfrak{g}}(M \otimes V^{\otimes d})$, since their proofs do not involve $(P.8)$.

Lemma A.1. *For each $v \in V^{\otimes i-1}$ and $w \in V^{\otimes d-(i+1)}$, we have*

$$\sum_{k\in I}(-1)^{\overline{e_k}} m \otimes v \otimes \left(x_j e_k \otimes e_{\overline{k}} + (-1)^{\overline{x_j e_k}} e_k \otimes x_j e_{\overline{k}}\right) \otimes w = 0, \tag{A.1}$$

$$\varepsilon_i \left(m \otimes v \otimes (x_j e_a \otimes e_b + (-1)^{\overline{x_j e_a}} e_a \otimes x_j e_b) \otimes w\right) = 0. \tag{A.2}$$

Proof. Observe that $\varepsilon(V\otimes V)$ is a trivial $\mathfrak{g}$-module, and (A.1) follows.

Consider (A.2).

$$\varepsilon_i\left(m\otimes v\otimes(x_je_a\otimes e_b+(-1)^{\overline{x_je_a}}e_a\otimes x_je_b)\otimes w\right) \tag{A.3}$$

$$=\varepsilon_i\left(m\otimes v\otimes((x_je_a,e_b)e_{\bar{b}}\otimes e_b+(-1)^{\overline{x_je_a}}e_a\otimes(x_je_b,e_a)e_{\bar{a}})\otimes w\right) \tag{A.4}$$

$$=m\otimes v\otimes\left(((x_je_a,e_b)+(-1)^{\overline{x_je_a}}(x_je_b,e_a))\sum_{k\in I}(-1)^{\overline{e_k}}e_k\otimes e_{\bar{k}}\right)\otimes w=0. \tag{A.5}$$

The equation (A.5) follows from the following two facts

(i) $(-1)^{\overline{x_j}+\overline{e_b}}=(-1)^{\overline{e_a}+1}$ and so $(-1)^{\overline{x_je_b}}=(-1)^{\overline{x_je_a}}$, if $(x_je_b,e_a)\neq 0$.

(ii) $(x_je_b,e_a)=-(-1)^{\overline{x_je_b}}(x_je_a,e_b)$.

□

The following lemma is an analog of [ES, Lemma 8.4].

Lemma A.2. *For $k>i+1$ and $i>0$, we have*

$$(\Omega_{i,k}+\Omega_{i+1,k})\varepsilon_i=0=\varepsilon_i(\Omega_{i,k}+\Omega_{i+1,k}). \tag{A.6}$$

Proof. We first prove (A.6) for $i=1$ and $k=3$. We may observe that

$$\begin{aligned}
&((\Omega_{1,3}+\Omega_{2,3})\varepsilon_1)\,(m\otimes e_a\otimes e_{\bar{a}}\otimes e_c)\\
&=(\Omega_{1,3}+\Omega_{2,3})\sum_{k\in I}(-1)^{\overline{e_k}}m\otimes e_k\otimes e_{\bar{k}}\otimes e_c\\
&=\sum_j\sum_{k\in I}(-1)^{\overline{e_k}}m\otimes x_je_k\otimes e_{\bar{k}}\otimes x_j^*e_c+\sum_j\sum_{k\in I}(-1)^{\overline{e_k}+\overline{x_je_k}}m\otimes e_k\otimes x_je_{\bar{k}}\otimes x_j^*e_c\\
&=\sum_j\sum_{k\in I}(-1)^{\overline{e_k}}m\otimes\left(x_je_k\otimes e_{\bar{k}}+\sum_{k\in I}(-1)^{\overline{x_je_k}}e_k\otimes x_je_{\bar{k}}\right)\otimes x_j^*e_c,
\end{aligned}$$

which is zero by Lemma A.1

We now consider the right hand side of identity (A.6) as follows:

$$\begin{aligned}
&(\varepsilon_1(\Omega_{1,3}+\Omega_{2,3}))\,(m\otimes e_a\otimes e_b\otimes e_c)\\
&=\varepsilon_1\left(\sum_j(-1)^{\overline{x_jm}}m\otimes\left(x_je_a\otimes e_b+(-1)^{\overline{e_ax_j}}e_a\otimes x_je_b\right)\otimes(-1)^{\overline{x_j^*}(\overline{e_a}+\overline{e_b}+\overline{m})}x_j^*e_c\right),
\end{aligned}$$

which is zero by Lemma A.1.

Similar calculations hold for general i and k by using Lemma A.1. This completes the proof. □

A.2. **Proof of equation** $(P.8)$**(a).** This subsection is devoted to the proof for $(P.8)$(a).

Lemma A.3. *We have*

$$\sum_{k\in I}(-1)^{\overline{e_k}}x_i^*e_k\otimes e_{\bar{k}}-\sum_{k\in I}(-1)^{\overline{e_k}+\overline{x_ie_k}}e_k\otimes x_i^*e_{\bar{k}}=0, \tag{A.7}$$

for all $1\leq i\leq \dim\mathfrak{g}$.

Proof. The proof is completed by the following calculations.

(i) Set $s \in I^0$ and $x_i^* := E_{ss}^*$ in (A.7), we obtain

$$\frac{1}{2}\left(e_s \otimes e_{\overline{s}} + e_{\overline{s}} \otimes e_s - e_{\overline{s}} \otimes e_s - e_s \otimes e_{\overline{s}}\right) = 0.$$

(ii) Set $s, t \in I^0$ with $s \neq t$, and $x_i^* := E_{ts}^*$ in (A.7), we obtain

$$\frac{1}{2}\left((e_s \otimes e_{\overline{t}} - e_{\overline{t}} \otimes e_s) - (-e_{\overline{t}} \otimes e_s + e_s \otimes e_{\overline{t}})\right) = 0.$$

(iii) Set $s, t \in I^0$ with $s \neq t$, and $x_i^* := Y_{st}^*$ in (A.7), we obtain

$$\frac{-1}{2}\left((-e_s \otimes e_t - e_t \otimes e_s) - (-e_t \otimes e_s - e_s \otimes e_t)\right) = 0.$$

(iv) Set $s, t \in I^0$ with $s \neq t$, and $x_i^* := X_{st}^*$ in (A.7), we obtain

$$\frac{-1}{2}\left((e_{\overline{s}} \otimes e_{\overline{t}} - (-1)(-1)e_{\overline{t}} \otimes e_{\overline{s}}) + (e_{\overline{t}} \otimes e_{\overline{s}} - e_{\overline{s}} \otimes e_{\overline{t}})\right) = 0.$$

(v) Set $s \in I^0$ and $x_i^* := X_{ss}^*$ in (A.7), we obtain

$$-(e_{\overline{s}} \otimes e_{\overline{s}} - (-1)(-1)e_{\overline{s}} \otimes e_{\overline{s}}) = 0.$$

□

Lemma A.4. (*Equation* $(P.8)(a)$) $\varepsilon_a(y_a - y_{a+1}) = \varepsilon_a$, *for all* $1 \leq a \leq d-1$.

Proof. Let $m \in M \otimes V^{\otimes a-1}$ and $v \in V^{\otimes d-a-1}$ be homogenous elements. Then for all $i, j \in I$ we have the following calculation.

$$\begin{aligned}
&((y_a - y_{a+1})\varepsilon_a)(m \otimes e_i \otimes e_j \otimes v) && \text{(A.8)}\\
&= 2\delta_{i,\overline{j}}(y_a - y_{a+1})\left(m \otimes \sum_{k\in I}(-1)^{\overline{e_k}} e_k \otimes e_{\overline{k}} \otimes v\right) && \text{(A.9)}\\
&= 2\delta_{i,\overline{j}}\Big(\sum_j x_j m \otimes \sum_{k\in I}(-1)^{\overline{e_k}+\overline{x_j}\overline{m}} x_j^* e_k \otimes e_{\overline{k}} \otimes v - \sum_j x_j m \otimes \sum_{k\in I}(-1)^{\overline{e_k}+\overline{x_j}(\overline{m}+\overline{e_k})} e_k \otimes x_j^* e_{\overline{k}} \otimes v && \text{(A.10)}\\
&- \sum_j m \otimes \sum_{k\in I}(-1)^{\overline{e_k}+\overline{x_j e_k}} x_j e_k \otimes x_j^* e_{\overline{k}} \otimes v\Big) = -2\delta_{i,\overline{j}} \sum_j m \otimes \sum_{k\in I}(-1)^{\overline{e_k}+\overline{x_j e_k}} x_j e_k \otimes x_j^* e_{\overline{k}} \otimes v. && \text{(A.11)}
\end{aligned}$$

The last equation follows from Lemma A.3. We now observe that the last term above can be rewritten as follows:

$$-2\delta_{i,\overline{j}} \sum_j m \otimes \sum_{k\in I}(-1)^{\overline{e_k}+\overline{x_j e_k}} x_j e_k \otimes x_j^* e_{\overline{k}} \otimes v = -\delta_{i,\overline{j}} m \otimes \left(2C_{V,V}(\varepsilon(e_i \otimes e_j))\right) \otimes v. \tag{A.12}$$

By definition of ε in (3.2) and $(P.6)$(a), we have $2C_{V,V} \circ \varepsilon = (\varepsilon + s) \circ \varepsilon = s \circ \varepsilon = -\varepsilon$ in $\mathrm{End}_{\mathfrak{g}}(V^{\otimes 2})$. This completes the proof. □

A.3. **Proof of equation (P.8)(b).** In this section, we prove the equation (P.8)(b).

Lemma A.5. *We have*

$$\varepsilon(x_i^* e_p \otimes e_q - (-1)^{\overline{x_i e_p}} e_p \otimes x_i^* e_q) = 0, \tag{A.13}$$

for all $1 \le i \le dim\mathfrak{g}$ *and* $p, q \in I$.

Proof. Recall that $\varepsilon(e_p \otimes e_q) = 0$, if both $p, q \in I \backslash I^0$, or both $p, q \in I^0$. Therefore, the proof reduces to the following calculations:

(i) Set $s \in I^0$ and $x_i^* := E_{ss}^*$ in (A.13), and assume that $p = s$, $q = \overline{j}$ for some $j \in I^0$ with $j \neq s$. Then we obtain
$$\varepsilon\left(e_s \otimes e_{\overline{j}}\right) = 0.$$

(ii) Set $s \in I^0$ and $x_i^* := E_{ss}^*$ in (A.13), and assume that $p = j$, $q = \overline{s}$ for some $j \in I^0$ with $j \neq s$. Then we obtain
$$\varepsilon\left(e_j \otimes e_{\overline{s}}\right) = 0.$$

(iii) Set $s \in I^0$ and $x_i^* := E_{ss}^*$ in (A.13), and assume that $p = s$, $q = \overline{s}$ (resp. $p = \overline{s}$, $q = s$). Then we obtain
$$\varepsilon\left(e_s \otimes e_{\overline{s}} - e_s \otimes e_{\overline{s}}\right) = \varepsilon(0) = 0.$$
$$(\text{resp. } \varepsilon\left(e_{\overline{s}} \otimes e_s - e_{\overline{s}} \otimes e_s\right) = \varepsilon(0) = 0)$$

(iv) Set $s, t \in I^0$, $s \neq t$ and $x_i^* := E_{ts}^*$ in (A.13), and assume that $p = t$, $q = \overline{j}$. Then we obtain
$$\varepsilon\left(e_s \otimes e_{\overline{j}} - \left(\delta_{j,s} e_t \otimes e_{\overline{t}}\right)\right) = 0.$$

(v) Set $s, t \in I^0$, $s \neq t$ and $x_i^* := E_{ts}^*$ in (A.13), and assume that $p = \overline{s}$, $q = j$. Then we obtain
$$\varepsilon\left(e_{\overline{t}} \otimes e_j - \left(\delta_{j,t} e_{\overline{s}} \otimes e_s\right)\right) = 0.$$

(vi) Set $s, t \in I^0$, $s \neq t$ and $x_i^* := E_{ts}^*$ in (A.13), and assume that $p = j$, $q = \overline{s}$. Then we obtain
$$\varepsilon\left(\delta_{j,t} e_s \otimes e_{\overline{s}} - e_{\overline{j}} \otimes e_{\overline{t}}\right) = 0.$$

(vi) Set $s, t \in I^0$, $s \neq t$ and $x_i^* := E_{ts}^*$ in (A.13), and assume that $p = \overline{j}$, $q = t$. Then we obtain
$$\varepsilon\left(\delta_{j,s} e_{\overline{t}} \otimes e_t - e_{\overline{j}} \otimes e_s\right) = 0.$$

(vii) Set $s, t \in I^0$, $s \neq t$ and $x_i^* := Y_{st}^*$ in (A.13), and assume that $p, q \in I \backslash I^0$. Then we obtain
$$\begin{aligned}&\varepsilon\left(\delta_{\overline{p},t} e_s \otimes e_q - (-1) e_p \otimes \delta_{\overline{q},t} e_s\right) - \varepsilon\left(\delta_{\overline{p},s} e_t \otimes e_q - (-1) e_p \otimes \delta_{\overline{q},s} e_t\right)\\ &= \left(\delta_{\overline{p},s}\delta_{\overline{q},t} - \delta_{t,\overline{q}}\delta_{s,\overline{p}}\right) \sum_{k \in I} (-1)^{\overline{e_k}} e_k \otimes e_{\overline{k}} = 0.\end{aligned}$$

(viii) Set $s, t \in I^0$, $s \neq t$ and $x_i^* := X_{st}^*$ in (A.13), and assume that $p, q \in I^0$. Then we obtain
$$\begin{aligned}&\varepsilon\left(\delta_{p,t} e_{\overline{s}} \otimes e_q - e_p \otimes \delta_{q,t} e_{\overline{s}}\right) + \varepsilon\left(\delta_{p,s} e_{\overline{t}} \otimes e_q - e_p \otimes \delta_{q,s} e_{\overline{t}}\right)\\ &= \left(\delta_{q,s}\delta_{p,t} - \delta_{s,p}\delta_{t,q} + \delta_{p,s}\delta_{q,t} - \delta_{p,t}\delta_{q,s}\right) \sum_{k \in I} (-1)^{\overline{e_k}} e_k \otimes e_{\overline{k}} = 0.\end{aligned}$$

(viii) Set $s \in I^0$ and $x_i^* := X_{ss}^*$ in (A.13), and assume that $p, q \in I^0$. Then we obtain
$$\varepsilon\left(\delta_{p,s} e_{\overline{p}} \otimes e_q - \delta_{q,s} e_p \otimes e_{\overline{q}}\right) = \left(\delta_{p,s}\delta_{p,q} - \delta_{q,s}\delta_{p,q}\right) \sum_{k \in I} (-1)^{\overline{e_k}} e_k \otimes e_{\overline{k}} = 0.$$

□

Lemma A.6. *(Equation $(P.8)(b)$)* $(y_a - y_{a+1})\varepsilon_a = -\varepsilon_a$, *for all* $1 \le a \le d-1$.

Proof. Let $m \in M \otimes V^{\otimes a-1}$ and $v \in V^{\otimes d-a-1}$ be homogenous elements. Then for all $i, j \in I$ we have the following calculation.

$$(\varepsilon_a(y_a - y_{a+1}))(m \otimes e_i \otimes e_j \otimes v) \tag{A.14}$$

$$= 2\varepsilon_a\Big(\sum_k x_k m \otimes (-1)^{\overline{x_k m}} x_k^* e_i \otimes e_j \otimes v - \sum_k (-1)^{\overline{x_k}(\overline{m}+\overline{e_i})} x_k m \otimes e_i \otimes x_k^* e_j \otimes v \tag{A.15}$$

$$- \sum_k (-1)^{\overline{x_k e_i}} m \otimes x_k e_i \otimes x_k^* e_j \otimes v\Big) \tag{A.16}$$

$$= -2\varepsilon_a \sum_k (-1)^{\overline{x_k e_i}} m \otimes x_k e_i \otimes x_k^* e_j \otimes v. \tag{A.17}$$

The last equation follows from Lemma A.5. We now observe that the last term above can be rewritten as follows:

$$-2\varepsilon_a \sum_k (-1)^{\overline{x_k e_i}} m \otimes x_k e_i \otimes x_k^* e_j \otimes v = -m \otimes (\varepsilon(2C_{V,V}(e_i \otimes e_j))) \otimes v. \tag{A.18}$$

By definition of ε in (3.2) and $(P.6)$(a), we have $\varepsilon \circ 2C_{V,V} = \varepsilon \circ (\varepsilon + s) = \varepsilon \circ s = \varepsilon$ in $\mathrm{End}_{\mathfrak{g}}(V^{\otimes 2})$. This completes the proof. □

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