Segmented strings from a different angle

David Vegh*
CMSA, Harvard University, Cambridge, MA 02138, USA

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Segmented strings in flat space are piecewise linear classical string solutions. Kinks between the segments move with the speed of light and their worldlines form a lattice on the worldsheet. This idea can be generalized to AdS_3 where the embedding is built from AdS_2 patches. The construction provides an exact discretization of the non-linear string equations of motion.

This paper computes the area of segmented strings using cross-ratios constructed from the kink vectors. The cross-ratios can be expressed in terms of either left-handed or right-handed variables. The string equation of motion in AdS_3 is reduced to that of an integrable time-discretized relativistic Toda-type lattice. Positive solutions yield string embeddings that are unique up to global transformations. In the appendix, the Poincaré target space energy is computed by integrating the worldsheet current along kink worldlines and a formula is derived for the integrated scalar curvature of the embedding.

I. INTRODUCTION

Strings moving in anti-de Sitter spacetime are interesting for many reasons. Strings lie at the heart of the AdS/CFT correspondence [1–3]. Understanding string theory enables us to study the correspondence beyond the gravity approximation. An open string ending on the boundary is dual to a Wilson loop in the boundary theory [4, 5]. Strings moving on a fixed asymptotically AdS background are among the simplest holographic non-equilibrium systems [6–8]. Finally, the string worldsheet with the induced metric can be thought of as a two-dimensional toy model for gravity.

In this paper, we consider classical strings in AdS_3 described by the Nambu-Goto action. The theory has the remarkable property of being integrable. It can be reduced to the two-dimensional generalized sinh-Gordon theory [9–14]. The string embedding can be reconstructed by solving an auxiliary linear problem.

The analog of a straight string in flat space is an embedding $AdS_2 \subset AdS_3$ with a constant surface normal vector. More complex string solutions can be constructed by gluing AdS_2 patches with different normal vectors. At the joints between adjacent segments, the string embedding contains kinks that move with the speed of light. On the worldsheet, their worldlines form a quad lattice as seen in FIG. 1. Each square in the figure is an AdS_2 patch with a constant normal vector. The kink collision events will be called kink vertices. Note that in the sinh-Gordon picture segmented strings are special solutions, because the generalized sinh-Gordon equation degenerates into the Liouville equation [15].

In [15, 16], the basic motion of segmented strings has been analyzed. The technique is ideally suited for numerical simulations, because the discretization is exact. This means that there are no numerical errors that otherwise may accumulate over time [17].

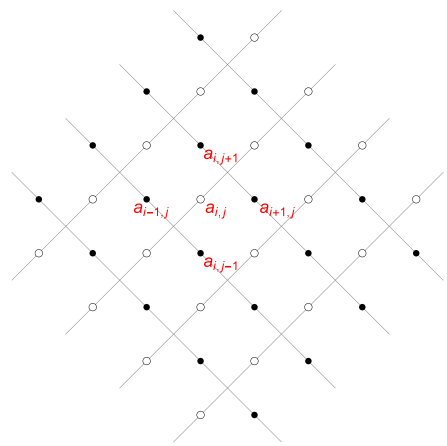

FIG. 1: Kink worldlines form a rectangular lattice on the string worldsheet. The field a_{ij} lives on the edges (black or white dots depending on edge orientation).

The present paper computes various properties of the string, including its area, energy, and scalar curvature. The area can be expressed in terms of left (or right) variables a_{ij} (or $\tilde{a}_{ij}$) where i and j label the edges of the kink lattice. We argue that *classical string motion in AdS_3 satisfies the following equation of motion of a discrete-time relativistic Toda-type lattice*

$$\frac{1}{a_{ij}-a_{i,j+1}}+\frac{1}{a_{ij}-a_{i,j-1}}=\frac{1}{a_{ij}-a_{i+1,j}}+\frac{1}{a_{ij}-a_{i-1,j}}$$

This is the main result of the paper. The field a_{ij} is in some sense holographic. As we will see, its value is related to the (retarded or advanced) Poincaré time when the kink corresponding to the (ij) edge would reach the AdS boundary if there were no other kinks on the worldsheet. Furthermore, we show how segmented string embeddings are obtained (up to a global $SL(2)$ transformation) from certain "positive" solutions of the Toda-type theory.

In the next section, we discuss the basics of segmented strings in AdS_3. Section III computes the area of a sin-

*Electronic address: dvegh@cmsa.fas.harvard.edu

gle patch that is bounded by four kink lines. Section IV computes the total area and the new equations of motion. Finally, reconstruction of the string from the Toda soluton is discussed. In the appendix, the string energy on the Poincaré patch and the scalar curvature of the worldsheet are computed.

II. SEGMENTED STRINGS

Let us recall that the (universal cover of the) surface

$$\vec{Y}\cdot\vec{Y} \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -L^2 \tag{1}$$

gives an embedding of AdS_3 into $\mathbb{R}^{2,2}$. L denotes the AdS_3 radius that we henceforth set to one. A function $\vec{Y}(z,\bar{z})$ maps the string worldsheet into this target space. The equations of motion supplemented by the Virasoro constraints are[1]

$$\begin{aligned} \partial\bar{\partial}\vec{Y} - (\partial\vec{Y}\cdot\bar{\partial}\vec{Y})\vec{Y} = 0 \\ \partial\vec{Y}\cdot\partial\vec{Y} = \bar{\partial}\vec{Y}\cdot\bar{\partial}\vec{Y} = 0 \end{aligned} \tag{2}$$

where the scalar product is again that of $\mathbb{R}^{2,2}$. A normal vector to the string can be defined by

$$N_a = \frac{\epsilon_{abcd}Y^b\partial Y^c\bar{\partial}Y^d}{\partial\vec{Y}\cdot\bar{\partial}\vec{Y}}$$

It satisfies $\vec{N}\cdot\vec{Y} = \vec{N}\cdot\partial\vec{Y} = \vec{N}\cdot\bar{\partial}\vec{Y} = 0$ and $\vec{N}\cdot\vec{N} = 1$. The simplest solution of (2) has a constant normal vector. It is the AdS_3 analog of an infinitely long straight string in flat spacetime.

Segmented strings are obtained by gluing worldsheet patches that have constant normal vectors [15, 16]. At the edges of the patches the string "breaks": on a fixed timeslice the embedding contains a kink that moves with the speed of light.

Normal vectors change whenever two kinks collide. The collision on the worldsheet is shown in FIG. 2. Worldsheet time increases towards the top. The kink worldlines are indicated by two intersecting lines. Before the collision, the string consists of three segments A, B, C that are characterized by three normal vectors: $\vec{A}$, $\vec{B}$, $\vec{C}$. We require $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{C} = 1$. This ensures that the kinks move on null geodesics.

After the collision, the new normal vector for the middle string piece is given by the collision formula [15, 16]

$$\vec{B}' = -\vec{B} + 4\frac{\vec{A}+\vec{C}}{(\vec{A}+\vec{C})^2} \tag{3}$$

[1] In the $SL(2)$ WZW model, spacetime points are $g \in SL(2,\mathbb{R})$ group elements. Classical solutions are given by $g = g_1(z)g_2(\bar{z})$. In this paper, we consider an ordinary sigma model and set the NS-NS fields (other than the metric) to zero. Thus, classical solutions will not have such a simple product structure.

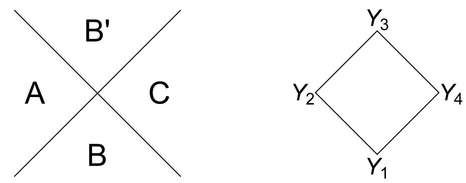

FIG. 2: *Left:* Four AdS_2 patches on the worldsheet around a kink vertex. The lines are lightlike worldlines of two colliding kinks. The constant normal vectors of the for regions are $\vec{A}$, $\vec{B}$, $\vec{B}'$, and $\vec{C}$. The collision formula computes any one of these vectors from the other three. *Right:* A single AdS_2 patch and its four vertices. The dual collision formula calculates any one of these vertices from the other three.

One can check that $\vec{A}\cdot\vec{B}' = \vec{B}'\cdot\vec{C} = 1$. This means that after the collision the kinks still move with the speed of light in AdS_3, only the directions change.

Note that $\vec{A}+\vec{C} \propto \vec{B}+\vec{B}'$. The collision formula computes any one of the four vectors from the other three by an appropriate relabeling. Further collisions between other pairs of kinks can be computed by repeated applications of the formula.

A. Dual description

There is an internal $SO(2,2)$ symmetry that acts on the variables (see [13])

$$q_1 = \vec{Y},\ \ q_2 = e^{-\alpha}\bar{\partial}\vec{Y},\ \ q_3 = e^{-\alpha}\partial\vec{Y},\ \ q_4 = \vec{N}$$

Here $\vec{Y} \in \mathbb{R}^{2,2}$ is a point in target space, $\vec{N}$ is the normal vector, and the sinh-Gordon field is defined by

$$e^{2\alpha} = \frac{1}{2}\partial\vec{Y}\cdot\bar{\partial}\vec{Y}$$

The symmetry treats spacetime points and normal vectors on the same footing. Therefore, we expect to have a dual description of segmented strings in terms of points in target space instead of normal vectors. Without proof we present here the evolution equation directly in terms of the kink vertices

$$\vec{Y}_3 = -\vec{Y}_1 - 4\frac{\vec{Y}_2+\vec{Y}_4}{(\vec{Y}_2+\vec{Y}_4)^2}$$

Here Y_i are the four vertices of a single patch as in the right of FIG. 2. This equation is dual to (3). Note the sign change in the equation. Products of vertices that are connected by a kink line are constrained, similarly to adjacent normal vectors. For instance,

$$\vec{Y}_1\cdot\vec{Y}_2 = -1.$$

This ensures that the kink vertices $\vec{Y}_1$ and $\vec{Y}_2$ are connected by a null vector in $\mathbb{R}^{2,2}$.

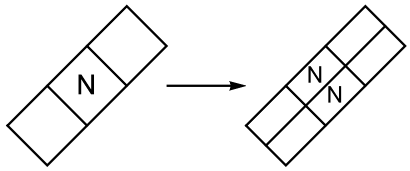

FIG. 3: Cutting patches in half by adding a zero kink. The patches inherit the normal vectors. For instance, in the new lattice two patches will have the same normal vector $\vec{N}$.

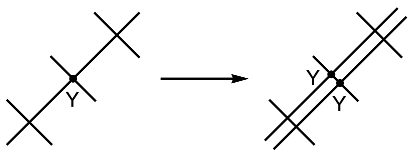

FIG. 4: Splitting a kink into two. In the new lattice, neighboring vertices are mapped to the same point in the target space. For instance, two dots will be mapped to the same $\vec{Y} \in \mathbb{R}^{2,2}$ point in target space.

B. Equivalent descriptions

A segmented string solution can be given by assigning normal vectors to faces in a square lattice. (In the dual picture, position vectors are assigned to the vertices.) The map is not one-to-one. In fact, a physical string embedding can be described by different vector lattices. There are two basic operations that preserve the physical string, but modify the lattice of vectors:

- *Adding zero strength kinks.* One can always add an extra kink line to the lattice, see FIG. 3. This cuts a series of patches in half. On the other hand, the string embedding remains the same if the kink strength is zero: the string segment will not break at the location of the would-be kink.

- *Splitting kinks.* This operation replaces a kink by a "composite" kink, see FIG. 4. The new patches in between have zero area and thus the string embedding is still the same.

The two operations are dual under the $SL(2)$ transformation of the previous section. At the graphical level this can be seen by placing the dual vertices in the middle of the faces and rotating the edges by 90°.

Smooth string solutions can be obtained by considering a continuum limit with weaker and weaker kinks. Even though segmented strings have no diffeomorphism or Weyl symmetries, the redundancies discussed above will form the basis of the worldsheet conformal symmetry.

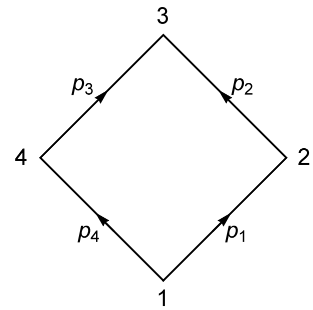

FIG. 5: A single AdS_2 patch of the worldsheet. The four edges are the kink worldlines where the normal vector jumps. In $\mathbb{R}^{2,2}$ these are straight null lines with direction vectors $\vec{p}_i$.

III. AREA OF A SINGLE PATCH

What is the string area in terms of the discrete data that defines segmented strings? Let us first focus on a single patch with a constant normal vector, see FIG. 5. The boundary of the worldsheet patch consists of four kink lines. In the target space, these are mapped to straight null lines (a consequence of the Virasoro constraints). Let us denote the four vertices of the patch by $\vec{V}_i \in \mathbb{R}^{2,2}$. We have $(\vec{V}_i)^2 = -1$. Let us define the lightlike direction vectors as in the figure

$$\vec{p}_1 = \vec{V}_2 - \vec{V}_1 \qquad \vec{p}_2 = \vec{V}_3 - \vec{V}_2$$
$$\vec{p}_3 = \vec{V}_3 - \vec{V}_4 \qquad \vec{p}_4 = \vec{V}_4 - \vec{V}_1 \tag{4}$$

We have

$$(\vec{p}_i)^2 = 0 \qquad \text{and} \qquad \vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4$$

The latter equation can be interpreted as "momentum conservation" during the scattering of two massless scalar particles with initial and final momenta $\vec{p}_{1,2}$ and $\vec{p}_{3,4}$, respectively. The area of the patch is analogous to a scattering amplitude that is invariant under the $SO(2,2)$ isometry group of AdS_3. The only independent invariants are the Mandelstam variables $s = (\vec{p}_1 + \vec{p}_2)^2$ and $u = (\vec{p}_1 - \vec{p}_4)^2$. The patch area is then

$$A_{\text{patch}} = L^2 \mathcal{F}\left(\frac{u}{s}\right)$$

where L is the AdS_3 radius (henceforth set to one) and $\mathcal{F}(x)$ is a dimensionless function to be determined.

Let us consider an $\mathrm{AdS}_2 \subset \mathrm{AdS}_3$ patch with normal vector $N = (0,0,0,1)$. Points on the surface are of the form $X = (x_{-1}, x_0, x_1, 0)$ with $x_{-1}^2 + x_0^2 - x_1^2 = 1$. Let us fix a parameter $a \in (0,1)$ and consider four points

$$\begin{aligned}
\vec{V}_1 &= (a, -\sqrt{1-a^2}, \quad 0, \quad 0) \\
\vec{V}_2 &= (a^{-1}, \quad 0, \sqrt{-1+a^{-2}}, \ 0) \\
\vec{V}_3 &= (a, \ \sqrt{1-a^2}, \quad 0, \quad 0) \\
\vec{V}_4 &= (a^{-1}, \quad 0, -\sqrt{-1+a^{-2}}, 0)
\end{aligned}$$

These points satisfy $(\vec{V}_i)^2 = -1$. It is easy to check that the corresponding difference vectors from (4) indeed satisfy $(\vec{p}_i)^2 = 0$. Cusps on the patch boundaries move along these lightlike vectors. Let us now compute the area of this patch. Define

$$\vec{X}_0(\tau,\sigma) = [(1-\sigma)\vec{V}_1 + \sigma\vec{V}_2](1-\tau) + \tau\vec{V}_3$$

and $X = X_0/|X_0|$. Thus, $X(\tau,\sigma)$ parametrizes half of the patch if $\sigma,\tau \in (0,1)$. After a lengthy calculation, the induced metric g gives

$$\sqrt{-g} = \frac{2(1-a^2)(1-\tau)}{[1-4(1-a^2)(1-\sigma)(1-\tau)\tau]^{\frac{3}{2}}}$$

Integrating with respect to τ and σ gives the area of half of the patch. From this we get

$$A_{\text{patch}} = -4\log a$$

For our patch, the ratio of the Mandelstam variables is given by $s/u = -a^2$. Combining these results fixes $\mathcal{F}(x)$ and we get the covariant formula

$$A_{\text{patch}} = \log\left[\frac{(\vec{p}_1 - \vec{p}_4)^2}{(\vec{p}_1+\vec{p}_2)^2}\right]^2 \tag{5}$$

Positiveness of the area requires

$$|\vec{p}_1 - \vec{p}_4| > |\vec{p}_1 + \vec{p}_2|$$

This constrains the possible values of $\vec{p}_i$.

A. Helicity spinors

In the spinor helicity formalism, one exhibits lightlike momentum vectors as products of spinors. We define

$$\begin{aligned}\sigma^\mu &= (1, -i\sigma_2, \sigma_1, \sigma_3)\\ p_{a\dot{a}} &= \sigma^\mu_{a\dot{a}} p_\mu\end{aligned}$$

Since $p^2 = \det(p_{a\dot{a}}) = 0$, we can write

$$p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

Since $\lambda_a \to s\lambda_a$, $\tilde{\lambda}_{\dot{a}} \to \frac{1}{s}\tilde{\lambda}_{\dot{a}}$ does not change $p_{a\dot{a}}$, there is a new gauge redundancy in this description. The spinors can be chosen to be real in $\mathbb{R}^{2,2}$.

Let us now define the $SO(2,2)$ invariants,

$$\begin{aligned}\langle \lambda_i, \lambda_j\rangle &= \epsilon_{ab}\lambda_1^a \lambda_2^b\\ \left[\tilde{\lambda}_i, \tilde{\lambda}_j\right] &= \epsilon_{\dot{a}\dot{b}}\tilde{\lambda}_1^{\dot{a}}\tilde{\lambda}_2^{\dot{b}}\end{aligned}$$

and consider two vectors $\vec{p}$ and $\vec{q}$ with the decomposition

$$\begin{aligned}p_{a\dot{a}} &= \lambda_a \tilde{\lambda}_{\dot{a}}\\ q_{a\dot{a}} &= \kappa_a \tilde{\kappa}_{\dot{a}}\end{aligned}$$

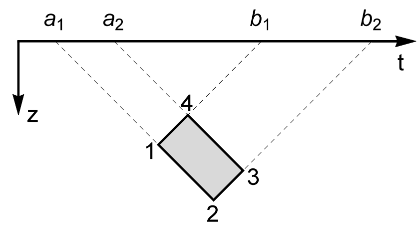

FIG. 6: Computing the area of a single rectangular patch of AdS_2. The figure shows the Poincaré patch. The t-axis on the top is the boundary. The string patch (gray rectangle) can be specified by four points on the boundary: $a_{1,2}$ & $b_{1,2}$. The four vertices of the rectangle are connected to these points by the null lines (dashed).

Their product is expressed as

$$\vec{p}\cdot\vec{q} = \langle\lambda,\kappa\rangle[\tilde{\lambda},\tilde{\kappa}]$$

This allows us to write (5) in the form

$$A_{\text{patch}} = \log\left(\frac{\langle\lambda_1,\lambda_4\rangle[\tilde{\lambda}_1,\tilde{\lambda}_4]}{\langle\lambda_1,\lambda_2\rangle[\tilde{\lambda}_1,\tilde{\lambda}_2]}\right)^2$$

"Momentum conservation" can be written as

$$\sum_{i=1}^4 \lambda_i^a \tilde{\lambda}_i^{\dot{a}} = 0$$

where i runs over the four edges around the patch. This can be used to cast the area formula in the form

$$A_{\text{patch}} = 2\log\left|\frac{\langle\lambda_1,\lambda_4\rangle\langle\lambda_2,\lambda_3\rangle}{\langle\lambda_1,\lambda_2\rangle\langle\lambda_3,\lambda_4\rangle}\right| \tag{6}$$

Note that this expression contains only "left-handed" variables. There is a similar formula with only "right-handed" $\tilde{\lambda}$ spinors. Let us stress that the spinors cannot take on arbitrary values because the area must be non-negative.

B. Global variables

Clearly, formula (6) does not depend on the modulus of λ_i, i.e. it is gauge-invariant. By defining $|\lambda_i|e^{i\alpha_i} := \lambda_i^1 + i\lambda_i^2$ we get

$$A_{\text{patch}} = 2\log\left|\frac{\sin(\alpha_1-\alpha_4)\sin(\alpha_2-\alpha_3)}{\sin(\alpha_1-\alpha_2)\sin(\alpha_3-\alpha_4)}\right| \tag{7}$$

For a given AdS_2 with a fixed normal vector, the four angles fully specify the four (infinite) straight kink lines in $\mathbb{R}^{2,2}$. Changing an angle means that we move the corresponding line on AdS_2. Note that the area diverges whenever two adjacent angles are equal. This corresponds to a configuration where a kink collision takes place on the UV boundary of AdS.

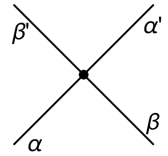

FIG. 7: Two kinks crossing. The (left) angles change from α, β to α', β'. If the collision point is specified in $\mathbb{R}^{2,2}$, then the angles determine the four kink vectors via (17).

When two kinks cross each other, the angles generically change, see FIG. 7. In a special case, e.g. $\alpha = \alpha'$, the β and β' lines are "zero kinks": they can be removed from the kink lattice without changing the string embedding. Such redundancies have been discussed in section II.

In order to have a better understanding of the α angles, let us compute them in Poincaré coordinates for a patch on a particular $AdS_2 \subset AdS_3$ whose normal vector is $\vec{N} = (0,0,0,1)$. Using the Poincaré parametrization of AdS_3

$$\vec{Y} = \left(\frac{t^2 - z^2 - x^2 - 1}{2z}, \frac{t}{z}, \frac{t^2 - z^2 - x^2 + 1}{2z}, \frac{x}{z} \right)$$

and setting $x = 0$, we obtain a parametrization of our AdS_2. The induced metric is

$$ds^2 = \frac{-dt^2 + dz^2}{z^2}$$

The patch in AdS_2 is bounded by $\pm 45°$ lines on the $t-z$ plane. This is shown in FIG. 6. The lines intersect the AdS_2 boundary at $t = a_1, a_2, b_1, b_2$ as in the figure. The four vertices are $V_1 = v_{11}$, $V_2 = v_{12}$, $V_3 = v_{22}$, $V_4 = v_{21}$, where

$$v_{ij} = \left(\frac{1 - a_i b_j}{a_i - b_j}, \frac{a_i + b_j}{b_j - a_i}, \frac{1 + a_i b_j}{b_j - a_i}, 0 \right).$$

From this, the lightlike boundary vectors p_i can be computed. The corresponding left angles α_i are (after a $\frac{\pi}{4}$ shift in order to have simpler expressions)

$$\begin{aligned} \tan\alpha_1 &= a_1 & \tan\alpha_2 &= b_2 \\ \tan\alpha_3 &= a_2 & \tan\alpha_4 &= b_1 \end{aligned} \tag{8}$$

The patch area is given by the formula (5) that yields

$$A_{\text{patch}} = 2 \log \left| \frac{(a_1 - b_1)(a_2 - b_2)}{(a_2 - b_1)(a_1 - b_2)} \right|$$

This expression is equal to (7).

C. Poincaré and Schwarzschild variables

Although the patch normal vectors are generically different from $(0,0,0,1)$, motivated by (8) we define the *Poincaré variables*

$$a_i := \tan\alpha_i \tag{9}$$

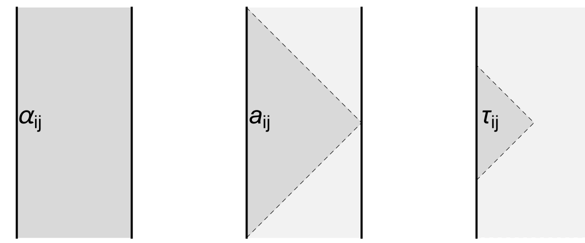

FIG. 8: The three boundary time coordinates can parametrize different (shaded) regions in AdS_2: (i) global coordinate α_{ij}, (ii) Poincaré coordinate a_{ij}, and (iii) Schwarzschild coordinate τ_{ij}.

We have seen that for the particular normal vector $\vec{N} = (0,0,0,1)$, the Poincaré variables are "holographic coordinates": they correspond to retarded and advanced times when light rays reach the boundary. Since the relationship between global and Poincaré AdS times is the same as in (9), the α variables will henceforth be called *global variables*.

There is a third set of variables that is easiest to see using Mikhailov's parametrization of the AdS_2 subspace [18]. Let us consider (19) in the appendix and plug in the quark motion

$$x_1(t) = \sqrt{1 + t^2} \tag{10}$$

Then, (19) gives an AdS_2 with normal vector $(0,0,1,0)$.

Let us consider a kink line on this AdS_2 subspace. Its location is specified by a global variable α. Let t_{AdS_3} denote the Poincaré AdS_3 time when this null geodesic intersects the boundary. A short calculation gives

$$t_{AdS_3} = \tan 2\alpha$$

The string endpoint on the boundary of AdS_3 is a quark that suffers constant acceleration. Its proper time is related to the AdS_3 time coordinate as $\tau = \sinh^{-1} t$. Thus, we have

$$\tau = \sinh^{-1} \tan 2\alpha = 2 \tanh^{-1} \tan\alpha \tag{11}$$

and this equation defines the *Schwarzschild variables* τ in the general case.

Let us summarize the results in this section. The global α variables are the angles of the *left* helicity spinors λ. The Poincaré and Schwarzschild fields are simply computed via (9) and (11), respectively. Different variables are related to different coordinate systems on AdS_3. This is shown in FIG. 8. Similarly, one defines right-handed fields starting from the spinors $\tilde{\lambda}$. These variables will be denoted $\tilde{\alpha}$, $\tilde{a}$, and $\tilde{\tau}$. We note that the map between the left and right fields is non-local. Finally, the area of the string can be expressed in either left or right variables, see eqn. (7). In the next section, we will compute the string action and show how the string embedding can be reconstructed from the angle variables.

IV. TOTAL AREA

The total area of the string is the sum of individual patch areas. From (7), we have

$$A = \sum_{f\in\text{patches}} \log\left(\frac{\sin(\alpha_{f_1}-\alpha_{f_4})\sin(\alpha_{f_2}-\alpha_{f_3})}{\sin(\alpha_{f_1}-\alpha_{f_2})\sin(\alpha_{f_3}-\alpha_{f_4})}\right)^2$$

where $f_{1,2,3,4}$ label the four edges around the patch f and α_i are the left angles. The action can be cast in the form,

$$A = 2\sum_{i,j}\log\left|\frac{\sin(\alpha_{i,j}-\alpha_{i+1,j})}{\sin(\alpha_{i,j}-\alpha_{i,j+1})}\right| \tag{12}$$

where i and j are coordinates on a square lattice, see FIG. 1. There is a similar formula that involves only the right-handed angles. The total area can be expressed in terms of Poincaré variables as well ($a_{ij} := \tan\alpha_{ij}$)

$$A = 2\sum_{i,j}\log\left|\frac{a_{i,j}-a_{i+1,j}}{a_{i,j}-a_{i,j+1}}\right| \tag{13}$$

Finally, in Schwarzschild variables ($\tanh\frac{\tau_{ij}}{2} := \tan\alpha_{ij}$) we have

$$A = 2\sum_{i,j}\log\left|\frac{\tanh\frac{\tau_{i,j}}{2}-\tanh\frac{\tau_{i+1,j}}{2}}{\tanh\frac{\tau_{i,j}}{2}-\tanh\frac{\tau_{i,j+1}}{2}}\right| \tag{14}$$

Note that patches are assumed to be located entirely in the bulk. Whenever a patch intersects the boundary of AdS_3, the area must be regularized.

A. Equation of motion

The Nambu-Goto action is equal to the area of the string which can be extremized by varying the left fields. The expression for the total area defines an action for a new Toda-type theory. Classical segmented string solutions yield solutions to this theory.

Let us first consider the action in Poincaré variables. The equation of motion is $\frac{\delta A}{\delta a}=0$. From (13) we have

$$\frac{1}{a_{i,j}-a_{i,j+1}}+\frac{1}{a_{i,j}-a_{i,j-1}} = \frac{1}{a_{i,j}-a_{i+1,j}}+\frac{1}{a_{i,j}-a_{i-1,j}} \tag{15}$$

The same equation is satisfied by the $\tilde{a}$ variables of the right-handed theory. The a field computed from a string solution will satisfy this equation. We recognize this equation of motion as that of a *time-discretized relativistic Toda-type lattice with a cubic Poisson bracket*, see (10.10.6) on page 442 in [19]. Note that (15) can also be thought of as a local version of the equation of motion of the discrete-time Caloger-Moser model [20].

From (14) we have another local equation

$$\frac{1}{\tanh(\tau_{i,j}-\tau_{i,j+1})}+\frac{1}{\tanh(\tau_{i,j}-\tau_{i,j-1})} = \frac{1}{\tanh(\tau_{i,j}-\tau_{i+1,j})}+\frac{1}{\tanh(\tau_{i,j}-\tau_{i-1,j})}$$

which is the same as (10.8.7) on page 440 in [19]. Similar equation follows from (12) with $\tan(x)$ functions in the denominators.

Initial conditions can be specified by setting two rows in the lattice (e.g. a_{i0} and a_{i1}).

A trivial solution is given by considering a lattice of zero kinks. For such a lattice, any two angles are the same if they lie on the same kink line,

$$a_{i,j} = \begin{cases} u(i+j) & \text{if } i+j \text{ is odd} \\ v(i-j) & \text{if } i+j \text{ is even}\end{cases}$$

This solution describes a single AdS_2 with a constant normal vector. The physical string embedding does not depend on u and v.

B. Reconstructing the embedding

The string embedding can be reconstructed from a solution of (15). There are some caveats, however.

- The symmetry group of AdS_3 is $SO(2,2) = SL(2)_L \times SL(2)_R$. The two $SL(2)$s act on the λ_a left and $\tilde{\lambda}_{\dot{a}}$ right spinors, respectively. Since left fields do not change under $SL(2)_R$, a_{ij} only determines the string embedding up to such global transformations. Similarly, $\tilde{a}_{ij}$ determines only an orbit of $SL(2)_L$.
- Note that there is a $\mathbb{Z}_2$ ambiguity in assigning a kink lattice to the lattice of black-and-white dots in FIG. 1 if their color is unknown. Thus, the symmetry between kink collision points in spacetime and AdS_2 patch normal vectors is manifest in the Toda description. As a consequence, two different embeddings may be constructed from a_{ij}.
- Not all Toda solutions correspond to string embeddings. The area of the string patches must be non-negative. This is only true for solutions satisfying

  $$(a_{i+1,j}-a_{i,j-1})(a_{ij}-a_{i+1,j-1}) > 0$$

  for the four angles around any patch (i.e. ij is a white dot in FIG. 1).

In the following we sketch the procedure for rebuilding the string solution. Let us fix a spacetime point $\vec{X}\in\mathbb{R}^{2,2}$ that will correspond to a particular kink collision event in FIG. 1. The four angles around the vertex are

$$\alpha_{ij},\ \alpha_{i+1,j},\ \alpha_{i,j+1},\ \alpha_{i+1,j+1} \tag{16}$$

For any one of these angles, the corresponding kink vector is computed to be

$$\vec{p} \propto \begin{pmatrix} -X_0 & + & X_2\sin 2\alpha & + & X_1\cos 2\alpha \\ X_{-1} & - & X_1\sin 2\alpha & + & X_2\cos 2\alpha \\ X_2 & - & X_0\sin 2\alpha & + & X_{-1}\cos 2\alpha \\ -X_1 & + & X_{-1}\sin 2\alpha & + & X_0\cos 2\alpha \end{pmatrix} \tag{17}$$

for which $p^2 = 0$. Two adjacent kink vectors, e.g. $\vec{p}_{ij}$ and $\vec{p}_{i+1,j}$, define an AdS_2 with a constant normal vector that contains the points $\vec{X} + \lambda\vec{p}_{ij}$ and $\vec{X} + \lambda'\vec{p}_{i+1,j}$ for any λ and λ'. Let us pick two adjacent angles α and β from (16). The corresponding kink vectors span an AdS_2 patch with normal vector

$$\vec{N}(\alpha,\beta) = \tfrac{1}{\sin(\alpha-\beta)}\times$$
$$\times\begin{pmatrix} X_0\cos(\alpha-\beta) - X_1\cos(\alpha+\beta) - X_2\sin(\alpha+\beta) \\ -X_{-1}\cos(\alpha-\beta) - X_2\cos(\alpha+\beta) + X_1\sin(\alpha+\beta) \\ -X_2\cos(\alpha-\beta) - X_{-1}\cos(\alpha+\beta) + X_0\sin(\alpha+\beta) \\ X_1\cos(\alpha-\beta) - X_0\cos(\alpha+\beta) - X_{-1}\sin(\alpha+\beta) \end{pmatrix}$$

We have seen that once the location of a kink vertex is fixed in spacetime, the four angles around it fully specify the four kink vectors. Similarly, if a normal vector is known, four angles fully specify the four boundaries of the corresponding patch. The boundary edges then intersect each other at new kink vertices and the kink vectors around those can also be computed. This procedure can be repeated until the entire worldsheet embedding is recovered.

V. DISCUSSION

Let us consider space-like string embeddings in anti-de Sitter spacetime. A smooth open string that ends on a curve $\mathcal{C}$ in the boundary can be approximated by another string that ends on a zigzag line in the boundary whose segments are lightlike and which itself is sufficiently close to $\mathcal{C}$ [13][2]. In the case of Lorentzian embeddings, a Lorentzian zigzag worldline constitutes a singular limit, because the boundary quark sitting on the endpoint of the string would radiate off an infinite amount of energy at the turning points. If the quark velocity cannot jump, how can smooth strings be approximated by strings that are described by discrete data? A solution is provided by segmented strings [15, 16]. In this case only the acceleration of the quark jumps whenever kinks enter or leave the string. Kinks between the segments move with the speed of light and (between collisions) their velocities are constant vectors in the embedding $\mathbb{R}^{2,2}$ spacetime. When kinks collide, the new normal vector to the string is given by the collision formula (3).

In this paper, we have computed the area of segmented strings in terms of cross-ratios of helicity spinors. These spinors arise from the decomposition of the kink vectors. The string area equals the Nambu-Goto action which we have expressed purely in terms of left (or right) angle variables. We have argued that the time evolution of the segmented string can be described by the evolution equation of a discrete-time Toda-type lattice.

[2] This is probably best visualized by imagining a soap bubble (minimal surface) stretching on a zigzag wire.

The Toda-type lattice contains only left- or right-handed fields that are exchanged by a parity transformation ($X_2 \to -X_2$ in $\mathbb{R}^{2,2}$). On the other hand, the sigma model has manifest parity symmetry and is probably best written as a constrained sum of a left and a right Toda-type lattice. For each AdS_2 patch on the worldsheet, there is a "momentum constraint" that involves spinors of both handedness

$$\sum_{i=1}^{4} \lambda_i^a \tilde{\lambda}_i^{\dot{a}} = 0 \tag{18}$$

The meaning of this equation is that the boundary of a patch is a closed loop in spacetime. (There is a similar constraint for every kink collision vertex as well.) The left and right variables are therefore not independent: they are "classically entangled". It would be interesting to relate the Toda-type lattice to a matrix model perhaps via a (relativistic) Calogero-Moser theory.

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Appendix

A. String energy

In this section, we compute the energy of the string on the Poincaré patch of AdS_3. The metric is

$$ds^2 = \frac{-dt^2 + dx^2 + dz^2}{z^2}$$

Let us consider the action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

where X^μ are embedding coordinates, h_{ab} is the worldsheet metric, and $a, b \in \{\tau, \sigma\}$. One defines the worldsheet currents of target space energy-momentum

$$P^a_\mu = -\frac{1}{2\pi\alpha'}\sqrt{-h}\, h^{ab} G_{\mu\nu}\partial_b X^\nu$$

From the equation of motion it follows that

$$\partial_a P^a_\mu - \Gamma^\kappa_{\mu\lambda}\partial_a X^\lambda P^a_\kappa = 0$$

Defining $p^a_\mu = \frac{P^a_\mu}{\sqrt{-h}}$, and substituting the induced metric $g_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$ for h_{ab}, this can be written as

$$\nabla_a p^a_\mu - \Gamma^\kappa_{\mu\lambda}\partial_a X^\lambda p^a_\kappa = 0$$

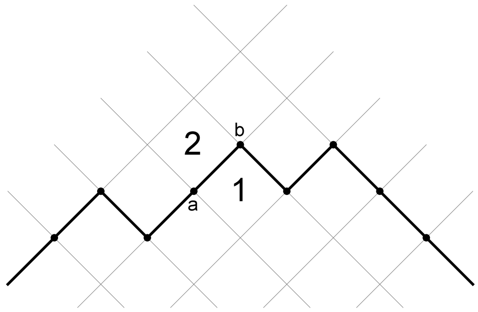

FIG. 9: Integration of spacetime currents along lightlike patch boundaries on the worldsheet. Kinks are indicated by thin lines. The thick line shows a possible path of integration. Contributions to the energy corresponding to elementary lightlike segments can be computed (e.g. $E_{1,2}$ for the segment between a and b) and the sum gives the total energy of the string.

where ∇ is the covariant derivative with respect to g_{ab}. Note that the target space index μ is only a spectator when the derivative is taken. The second term is

$$\Gamma^{\kappa}_{\mu\lambda}\partial_a X^{\lambda} p^a_{\kappa} = \frac{1}{2} G_{\nu\lambda,\mu}\partial_a X^{\lambda}\partial_b X^{\nu} g^{ab}$$

If $G_{\nu\lambda}$ is independent of X^{μ} for some μ, then $\zeta^{\alpha} = \delta^{\alpha}_{\mu}$ is a Killing vector. Then $\zeta^{\alpha}\nabla_a p^a_{\alpha} = 0$ and one can define the conserved quantity

$$E_{\zeta} = -\int d\sigma \zeta^{\alpha} P^{\tau}_{\alpha}$$

that satisfies $\partial_{\tau} E_{\zeta} = 0$. We are going to use $\zeta = \partial_t$ in the following.

Energy expressions are typically complicated (see [16]). In order to simplify the results, we perform the integration on the worldsheet along a path that consists of lightlike patch boundaries (instead of constant τ slices).

In terms of the Poincaré coordinates, a single AdS_2 patch is a contracting and expanding semi-circle. Let x_1 denote the path of the quark in the boundary

$$x_1(t) = X_1 + \sqrt{R_1^2 + (t - T_1)^2}$$

This is a hyperbola, parametrized by X_1, T_1, and R_1. The subscript indicates the patch, see FIG. 9. Using Mikhailov's result, the string embedding is given by

$$\begin{aligned} t(\tau, z) &= \tau + \frac{z}{\sqrt{1 - x_1'(\tau)^2}} \\ z(\tau, z) &= z \\ x(\tau, z) &= x_1(\tau) + \frac{z x_1'(\tau)}{\sqrt{1 - x_1'(\tau)^2}} \end{aligned} \tag{19}$$

Here τ plays the role of retarded time. The relationship between the normal vector $\vec{N}$ and the parameters of the hyperbola are

$$(T_1,\, X_1,\, R_1) = \left(\frac{-N_0}{N_{-1} + N_2},\ \frac{-N_1}{N_{-1} + N_2},\ \frac{1}{|N_{-1} + N_2|} \right).$$

The induced metric on the worldsheet is

$$g = \frac{1}{z^2\sqrt{R_1^2 + (\tau - T_1)^2}} \begin{pmatrix} \frac{z^2 - R_1^2}{\sqrt{R_1^2 + (\tau - T_1)^2}} & -R_1 \\ -R_1 & 0 \end{pmatrix}$$

The current takes the form

$$p^a{}_{\mu}(\tau, z) = \frac{1}{2\pi\alpha'} \begin{pmatrix} -\frac{R_1^2 + (\tau - T_1)^2}{R_1^2} & \frac{\sqrt{R_1^2 + (\tau - T_1)^2}}{R_1} & \frac{\sqrt{R_1^2 + (\tau - T_1)^2}(\tau - T_1)}{R_1^2} \\ -\frac{z\left(R_1(\tau - T_1) + z\sqrt{R_1^2 + (\tau - T_1)^2}\right)}{R_1^3} & \frac{z^2}{R_1^2} - 1 & \frac{z\left(z(\tau - T_1) + R_1\sqrt{R_1^2 + (\tau - T_1)^2}\right)}{R_1^3} \end{pmatrix}$$

We want to integrate P^{τ}_t along the lightlike boundary between the patch (T_1, X_1, R_1) and another patch (T_2, X_2, R_2). This translates to integrating over z at a fixed τ. The interpretation of τ is that the kink between the two patches reaches the boundary at Poincaré time $t = \tau$ (unless it collides with other kinks).

The value of τ can be computed by requiring that at this time the quark velocity is continuous

$$\partial_{\tau} x_1(\tau, 0) = \partial_{\tau} x_2(\tau, 0)$$

from which

$$\tau = \frac{R_2 T_1 + R_1 T_2}{R_1 + R_2}$$

Finally, the contribution of the kink to the total energy is given by the integral

$$E_{1,2} = \int_{z_a}^{z_b} dz \sqrt{-g}\, p^{\tau}_t = \frac{\frac{1}{z_b} - \frac{1}{z_a}}{2\pi\alpha'} \sqrt{1 + \left(\frac{T_1 - T_2}{R_1 - R_2}\right)^2}$$

where z_a and z_b are the z coordinates of the points where the kink is created and annihilated, respectively. Note that the formula is symmetric under $1 \leftrightarrow 2$ as it should be. Furthermore, X_1 and X_2 have dropped out. Note that for the correct null cusp limit one takes $T_1 \to T_2$ before $R_1 \to R_2$. The total energy of the string is given by the sum of all $E_{i,j}$ along a zigzag path on the patch boundaries, see FIG. 9.

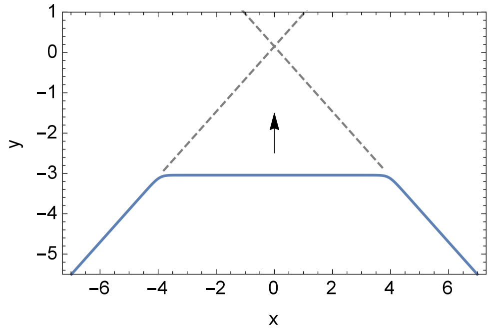

FIG. 10: Collision of two smooth kinks in Minkowski spacetime. The left and right segments are static semi-infinite lines. The middle segment moves with a constant velocity. The velocity changes sign at the collision.

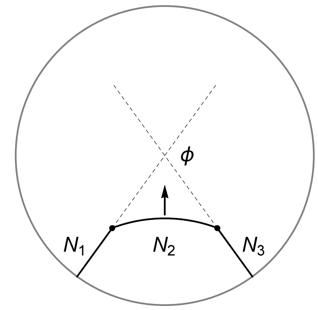

FIG. 11: Collision of two kinks (dots in the figure) in global AdS_3. The figure shows a constant time slice before the collision. Patches N_1 and N_3 are static straight lines in these coordinates. The angle between them is denoted ϕ. The length of the N_2 piece decreases and vanishes at the time of collision. After the collision, its velocity is flipped. The process is time-reversion symmetric.

B. Scalar curvature

Let us consider a static string that hangs from the boundary of AdS_3. The induced metric on the worldsheet is that of AdS_2 and the Ricci scalar is $R = -2$. This is the only intrinsic curvature invariant that one can compute in two dimensions. If the string endpoint is perturbed, kinks (or waves in the continuum case) will travel down the string. However, the scalar curvature does not change. The reason for this is simple and best understood through a flat space analogy. Consider, for instance, a cube and cut out the top and bottom faces.

What's left is the four adjacent side faces that can be unwrapped and arranged on a plane. The four edges are the analogs of the kinks that move in the same direction.

The induced metric on the four faces is clearly flat. Only when kinks *collide* can the curvature differ from the constant value. In this section, we compute the integrated Ricci scalar at collision points.

Since kink collisions happen at single points, the background curvature can be neglected. Thus, one can analyze the problem in 2+1 dimensional Minkowski space with coordinates $x^{0,1,2}$. In order to handle the divergence in the curvature at the collision point, we smoothen the step functions corresponding to kinks.

A string with two smooth kinks colliding on it is given by the embedding

$$\begin{aligned} x^0 &= \frac{(2+A^2)(\sigma^+ + \sigma^-)}{2\sqrt{2}} - \frac{A^2(\tanh\epsilon\sigma^- + \tanh\epsilon\sigma^+)}{2\sqrt{2}\epsilon} \\ x^1 &= \frac{(2-A^2)(\sigma^+ - \sigma^-)}{2\sqrt{2}} - \frac{A^2(\tanh\epsilon\sigma^- - \tanh\epsilon\sigma^+)}{2\sqrt{2}\epsilon} \\ x^2 &= -\frac{A(\log\cosh\epsilon\sigma^- + \log\cosh\epsilon\sigma^+)}{\epsilon} \end{aligned}$$

Here σ^+ and σ^- are lightcone coordinates on the worldsheet. A parametrizes the kink strengths and ϵ is related to the smoothness of the step functions.

The induced metric on the worldsheet has components

$$\begin{aligned} g_{+-} &= -\left(1 - \tfrac{A^2}{2}\tanh\epsilon\sigma^-\tanh\epsilon\sigma^+\right)^2 \\ g_{++} &= g_{--} = 0 \end{aligned}$$

The Ricci scalar of the induced metric is

$$R = -\frac{2A^2\epsilon^2}{(1-\frac{A^2}{2}\tanh\epsilon\sigma^-\tanh\epsilon\sigma^+)^4(\cosh\epsilon\sigma^-)^2(\cosh\epsilon\sigma^+)^2}$$

Integrating this with respect to σ^- and σ^+ over the entire worldsheet, we get

$$\int d^2\sigma\sqrt{-g}\,R = -16\tanh^{-1}\frac{A^2}{2}$$

Note that ϵ has dropped out and thus the result is finite in the $\epsilon\to\infty$ limit that corresponds to sharp kinks. In this limit, $R = 0$ away from the collision vertex at $\sigma^+ = \sigma^- = 0$. Thus, it is enough to integrate in an infinitesimally small neighborhood of the origin. The result is then a local feature which generalizes to AdS_3.

We would like to eliminate A from the expression and replace it with a more natural quantity. The angle between the two static string pieces can be computed

$$\tan\frac{\phi}{2} = \frac{2\sqrt{2}A}{A^2-2}$$

Going back to AdS_3, we can set up a similar collision using the three patches N_1, N_2, N_3

$$\begin{aligned} \vec{N}_1 &= \left(\quad 0, \quad 0, \quad \cos\frac{\phi}{2}, \quad \sin\frac{\phi}{2}\right) \\ \vec{N}_2 &= \left(-\tan\frac{\phi}{2}, \ 0, \ \left(\cos\frac{\phi}{2}\right)^{-1}, \ 0\right) \\ \vec{N}_3 &= \left(\quad 0, \quad 0, \quad \cos\frac{\phi}{2}, -\sin\frac{\phi}{2}\right) \end{aligned}$$

The angle ϕ can be computed using the scalar product between normal vectors

$$\cos\phi = \vec{N}_1 \cdot \vec{N}_3$$

and from this, A can be determined. The final result

$$\int_{\text{vertex}} \sqrt{-g}\, R = 8 \log \cos \frac{\phi}{2}$$

This is the integrated Ricci scalar around a kink collision point where the string piece corresponding to the middle patch N_2 vanishes. Note that the formula does not depend on N_2. Generic normal vectors form a three-dimensional space. However, the middle patch N_2 is constrained by $\vec{N}_1 \cdot \vec{N}_2 = \vec{N}_2 \cdot \vec{N}_3 = 1$ and thus the allowed values form a one-dimensional subspace. Motion on this subspace corresponds to global AdS_3 time translations. This is a symmetry of the system that preserves the curvature.

The four angle variables around the vertex in the kink lattice are:

$$\alpha_1 = -\alpha_3 = \frac{\phi}{4}, \quad \alpha_2 = \frac{\pi}{2} - \frac{\phi}{4}, \quad \alpha_4 = \frac{\pi}{2} + \frac{\phi}{4}$$

If $\phi = 0$, then the angles do not change at the vertex (i.e. $\alpha_1 = \alpha_3$ and $\alpha_2 = \alpha_4$) and it's clear that they describe the collision of two zero kinks.

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