Initial Layer and Relaxation Limit of Non-Isentropic Compressible Euler Equations with Damping

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Abstract

In this paper, we study the relaxation limit of the relaxing Cauchy problem for non-isentropic compressible Euler equations with damping in multi-dimensions. We prove that the velocity of the relaxing equations converges weakly to the velocity of the relaxed equations, while other variables of the relaxing equations converge strongly to the corresponding variables of the relaxed equations. We prove that as relaxation time approaches 0, there exists an initial layer for the ill-prepared data, the convergence of the velocity is strong outside the layer; while there is no initial layer for the well-prepared data, the convergence of the velocity is strong near $t = 0$. The strong convergence rates of all variables are also estimated.

Keywords: non-isentropic Euler equation with damping, relaxation limit, ill-prepared data, initial layer, strong convergence rate

1. Introduction

In this paper, we use p, u, S, ϱ to denote the pressure, velocity, entropy and density of ideal gases respectively with the equation of gas state

$$\varrho = \varrho(p, S) := \frac{1}{\sqrt[\gamma]{A}} p^{\frac{1}{\gamma}} \exp\{-\frac{S}{\gamma}\},$$

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where $A>0$, $\gamma=\frac{C_p}{C_V}>1$ are constants. Assume the constants $\bar{p}, \bar{\varrho}, \bar{S}$ satisfy $\bar{p}>0, \bar{\varrho}>0$, $\bar{p}=A\bar{\varrho}^{\gamma}e^{\bar{S}}$. Then we study the relaxation limit of the relaxing Cauchy problem for 3D non-isentropic compressible Euler equations with damping:

$$\begin{cases} p_t+u\cdot\nabla p+\gamma p\nabla\cdot u=0,\\ \tau^2u_t+\tau^2u\cdot\nabla u+\frac{1}{\varrho}\nabla p+u=0,\\ S_t+u\cdot\nabla S=0,\\ (p,u,S)(x,0)=(p_0(x,\tau),\frac{\mathcal{U}_0(x,\tau)}{\tau},S_0(x,\tau)), \end{cases} \tag{1.1}$$

where $(p_0(x,\tau),\mathcal{U}_0(x,\tau),S_0(x,\tau))$ are small perturbations of $(\bar{p},0,\bar{S})$. In (1.1), $(p,u,S,\varrho)\to(\bar{p},0,\bar{S},\bar{\varrho})$ as $|x|\to+\infty$. τ is a small positive parameter representing the relaxation time, let $\tau\in(0,1]$. The density ϱ satisfies the equation $\varrho_t+u\cdot\nabla\varrho+\varrho\nabla\cdot u=0$ by (1.1), and $\varrho_0(x,\tau)=\varrho(p_0(x,\tau),S_0(x,\tau))$ is small perturbation of $\bar{\varrho}=\varrho(\bar{p},\bar{S})$.

The equations (1.1) are derived from

$$\begin{cases} \hat{p}_{t'}+\mathcal{U}\cdot\nabla\hat{p}+\gamma\hat{p}\nabla\cdot\mathcal{U}=0,\\ \mathcal{U}_{t'}+\mathcal{U}\cdot\nabla\mathcal{U}+\frac{1}{\hat{\varrho}}\nabla\hat{p}+\frac{1}{\tau}\mathcal{U}=0,\\ \hat{S}_{t'}+\mathcal{U}\cdot\nabla\hat{S}=0,\\ (\hat{p},\mathcal{U},\hat{S})(x,0)=(p_0(x,\tau),\mathcal{U}_0(x,\tau),S_0(x,\tau)), \end{cases} \tag{1.2}$$

with the time rescaling:

$$t=\tau t', u(x,t)=\frac{\mathcal{U}}{\tau}(x,t'), p(x,t)=\hat{p}(x,t'), \varrho(x,t)=\hat{\varrho}(x,t'), S(x,t)=\hat{S}(x,t'), \tag{1.3}$$

then $(p(x,t),u(x,t),S(x,t),\varrho(x,t))$ satisfy the equations (1.1).

(1.2) has no explicit form of its relaxed system, so we study its equivalent systerm (1.1). Note that the initial velocity of (1.1) differs from that of (1.2). For both (1.1) and (1.2), the smallness of the initial data requires $\|\mathcal{U}_0(x,\tau)\|_{H^4(\mathbb{R}^3)}$ to be small. While $u(x,0)=\frac{\mathcal{U}_0(x,\tau)}{\tau}$ in (1.1) may be large when τ is small.

Let $\tau=0$ in the relaxing equations (1.1), we formally obtain the following relaxed equations

$$\begin{cases} p_t + u\cdot\nabla p + \gamma p \nabla\cdot u = 0, \\ \frac{1}{\varrho}\nabla p + u = 0, \\ S_t + u\cdot\nabla S = 0, \\ (p,S)(x,0) = (\lim\limits_{\tau\to 0} p_0(x,\tau), \lim\limits_{\tau\to 0} S_0(x,\tau)), \end{cases} \tag{1.4}$$

where $\varrho = \varrho(p,S)$.

Due to its fundamental importance in both application and nonlinear PDE theory, the relaxation limit problems have been attracting much attention. We survey there some results closely related to this paper.

For the relaxing isothermal compressible Euler equations with damping:

$$\begin{cases} \varrho_t + \nabla\cdot(\varrho u) = 0, \\ \varrho u_t + \varrho u\cdot\nabla u + \bar{\sigma}^2\nabla\varrho + \frac{1}{\tau}\varrho u = 0, \end{cases} \tag{1.5}$$

where $\bar{\sigma}^2 = \mathcal{R}\theta_*$ is constant. [1] proved the uniform bounds in the Sobolev space $H^s(\mathbb{R}^d), s\in\mathbb{N}, s > 1+\frac{d}{2}$, after the time rescaling, the density ϱ converges strongly to the solution of the heat equation in $C([0,T], H^{s'}(B_r))$, where $0 < s' < s$, B_r is a ball with radius r. The results of [1] was extended in [2] to more general Sobolev space of fractional order. In [3], (1.5) was studied in one dimension with BV large data away from vacuum, and it was proved in [3] that after the time rescaling, ϱ converges strongly to the solution of the heat equation in $L^2(\mathbb{R}\times[0,T])$ (global in space) by using the stream function.

For the relaxing isentropic compressible Euler equations with damping

$$\begin{cases} \varrho_t + \nabla\cdot(\varrho u) = 0, \\ \varrho u_t + \varrho u\cdot\nabla u + \nabla p + \frac{1}{\tau}\varrho u = 0, \end{cases} \tag{1.6}$$

where $p(\rho) = A\rho^\gamma$. [4] proved the uniform bounds in the Sobolev space $H^s(\mathbb{R}^d)$, $s\in\mathbb{N}, s > 1+\frac{d}{2}$, after the time rescaling, the density ϱ converges strongly to the solution of the porous media equation in $C([0,T], H^{s'}(B_r))$, where $0 < s' < s$. Similar results were obtained in the Besov space $\mathcal{B}^\sigma_{2,1}(\mathbb{R}^d), \sigma = 1+\frac{d}{2}$ (see [5]) and in the Chemin-Lerner space (see [6]).

Relaxation limit problem also appears in Euler-Poisson equations, see [7, 8, 9, 10] for weak solutions and [11, 12] for smooth solutions. It has been proved that the current density, which is the product of the electron density and electron velocity, converges weakly to that of the drift-diffusion model. If the initial data are well-prepared, the current density converges strongly to that of the drift-diffusion model (see [13]). If the initial data are ill-prepared, the authors (see [14]) proved the difference between the current density of 1D hydrodynamic model and that of the drift-diffusion model decays exponentially fast in the large time interval $[0, \frac{1}{\beta}\log(\frac{1}{\tau^\lambda})]$ with $\lambda \in (0,1), \beta > 0$. The key of the proof in [14] is that the solutions of the relaxing and relaxed equations converge to the corresponding stationary solutions exponentially fast while both stationary solutions are close to each other. As to the relaxation limit of weak solutions (see [15]) and classical solutions (see [16, 17, 18]) to non-isentropic Euler-Poisson equations, the current density converges weakly to that of the energy-transportation model or drift-diffusion model.

However, there have been no rigorous analysis of the initial layer and strong convergence of the velocity for the ill-prepared data in the above mentioned papers. A main distinction of results in this paper is that we give results on the initial layer and strong convergence of the velocity, strong convergence rates of all variables. Our main concern is the non-isentropic flow (1.1), but our results are valid for the isentropic flow (1.6) and isothermal flow (1.5) (assuming no vacuum). We show that for the ill-prepared initial data, the strong convergence of the velocity is not uniform near $t = 0$, there exists an initial layer whose thickness is $O(\tau^2)$. Outside the initial layer, the velocity of the relaxing equations converge strongly to that of the relaxed equations. Only for the well-prepared initial data, there is no initial layer, the strong convergence of the velocity holds in $[0, T]$. The key of our analysis in this paper is uniform a priori estimates with respect to τ and pointwise decay of the quantity $u + \frac{1}{\varrho}\nabla p$. Moreover, the methods in this paper can be applied to the relaxation limit problems for Euler-Poisson equations and Euler-Maxwell equations.

In this paper, all of our results are stated for the relaxing system (1.1) and the relaxed system (1.4). The first result is the following convergence result:

Theorem 1.1. *Fix an arbitrary $T \in (0, +\infty)$, suppose for all $\tau \geq 0$, the initial data for the relaxing Cauchy problem* (1.1) *satisfy $\|(p_0(x,\tau)-\bar{p}, \mathcal{U}_0(x,\tau), S_0(x,\tau)-\bar{S})\|_{H^4(\mathbb{R}^3)} \leq \epsilon_0$, ϵ_0 is sufficiently small, depends on T but not on τ. Then the problem* (1.1) *admits a unique solution (p, u, S, ϱ) in $[0, T]$ satisfying*

$$\begin{aligned} &\sum_{0\leq\ell\leq2} \left\|(\partial_t^\ell(p-\bar{p}),\ \tau\partial_t^\ell u,\ \partial_t^\ell(S-\bar{S}),\ \partial_t^\ell(\varrho-\bar{\varrho}))\right\|_{L^\infty([0,T],H^{4-\ell}(\mathbb{R}^3))} \\ &+\sum_{0\leq\ell\leq2} \left\|u\right\|_{H^\ell([0,T],H^{4-\ell}(\mathbb{R}^3))} \leq C(T,\epsilon_0), \end{aligned} \tag{1.7}$$

such that as $\tau \to 0$,

$$\begin{aligned} &(p, S, \varrho) \to (\tilde{p}, \tilde{S}, \tilde{\varrho}) \ \mathit{in}\ C([0,T], C^{2+\mu_1}(\mathcal{K}) \cap W^{3,\mu_2}(\mathcal{K})), \\ &u \rightharpoonup \tilde{u} \quad \mathit{in} \bigcap_{0\leq\ell\leq2} H^\ell([0,T], H^{4-\ell}(\mathbb{R}^3)), \end{aligned} \tag{1.8}$$

where $\mu_1 \in [0, \frac{1}{2}), \mu_2 \in [2, 6)$, $\mathcal{K}$ denotes any compact subset of $\mathbb{R}^3$, $(\tilde{p}, \tilde{u}, \tilde{S}, \tilde{\varrho})$ is the unique classical solution to the relaxed equations (1.4).

Assume $\|(p_0(x,\tau) - \lim_{\tau\to0} p_0(x,\tau), S_0(x,\tau) - \lim_{\tau\to0} S_0(x,\tau))\|_{H^3(\mathbb{R}^3)} \leq O(\tau^{\alpha_1})$, then as $\tau \to 0$,

$$\|(p-\tilde{p}, S-\tilde{S}, \varrho-\tilde{\varrho})\|_{C([0,T],C^{1+\mu_1}(\mathbb{R}^3)\cap W^{2,\mu_2}(\mathbb{R}^3))} \leq O(\tau^{\min\{1,\alpha_1\}}), \tag{1.9}$$

where $\mu_1 \in [0, \frac{1}{2}], \mu_2 \in [2, 6]$.

The main results concerned with the initial layer and strong convergence of the velocity are stated in the following theorem:

Theorem 1.2. *Let (p, u, S, ϱ) and $(\tilde{p}, \tilde{u}, \tilde{S}, \tilde{\varrho})$ be the solutions obtained in Theorem* 1.1. *For the ill-prepared data, i.e., $\lim_{\tau\to0} \left|\frac{1}{\tau}\mathcal{U}_0(x,\tau) + \frac{1}{\varrho_0(x,\tau)}\nabla p_0(x,\tau)\right|_\infty \neq 0$, there exists an initial layer $[0, t^*]$ with $t^* = C\tau^{2-\delta}$ for the velocity u, where $C > 0$, $0 < \delta < 2$, such that as $\tau \to 0$, $|u(x,t^*) - \tilde{u}(x,0)|_\infty \to 0$ and*

$$\|u-\tilde{u}\|_{C([t^*,T],\ C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))} \leq O(\tau^{\min\{1,\alpha_1\}}), \mu_1 \in [0, \tfrac{1}{2}], \mu_2 \in [2, 6]. \tag{1.10}$$

If $\delta = 0$, for any constant $C > 0$, $u(x, C\tau^2)$ does not converge to $\tilde{u}(x, 0)$.

For the well-prepared data, i.e., $\lim_{\tau\to 0}\left\|\frac{1}{\tau}\mathcal{U}_0(x,\tau)+\frac{1}{\varrho_0(x,\tau)}\nabla p_0(x,\tau)\right\|_{H^2(\mathbb{R}^3)}=0$*, assuming* $\left\|\frac{1}{\tau}\mathcal{U}_0(x,\tau)+\frac{1}{\varrho_0(x,\tau)}\nabla p_0(x,\tau)\right\|_{H^2(\mathbb{R}^3)}\le O(\tau^{\alpha_2})$*, as* $\tau\to 0$*,*

$$\|u-\tilde{u}\|_{C([0,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))}\le O(\tau^{\min\{1,\alpha_1,\alpha_2\}}),\mu_1\in[0,\tfrac{1}{2}],\mu_2\in[2,6]. \tag{1.11}$$

Remark 1.3. *(i) In the above theorems,* (p,u,S,ϱ) *depend on* τ*, while* $(\tilde{p},\tilde{u},\tilde{S},\tilde{\varrho})$ *do not. Compare* (1.8) *with* (1.9), (1.10), (1.11)*, the strong convergence with such higher regularities in* (1.8) *must be restricted to arbitrary compact subset* $\mathcal{K}$*. The strong convergence in* (1.9), (1.10), (1.11) *holds in the whole space* $\mathbb{R}^3$*, their proofs do not need compact embedding.*

(ii) In Theorem 1.2 *for ill-prepared data,* $\delta\neq 0$ *implies that the thickness of the initial layer is* $O(\tau^2)$*. The strong convergence of* u *is not uniform near* $t=0$*,* $u|_{t=0}\neq\tilde{u}|_{t=0}$*,* $u(x,Ct^2)$ *does not converge to* $\tilde{u}(x,0)$*, while* $u(x,Ct^{2-\delta})$ *converges to* $\tilde{u}(x,0)$ *when* $\delta>0$*. Simply, for any fixed small number* $t_*>0$*,* $u(x,t)\to\tilde{u}(x,t)$ *in* $C([t_*,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))$*,* $\mu_1\in[0,\frac{1}{2}],\mu_2\in[2,6]$*.*

(iii) (p,S,ϱ) *converge strongly to* $(\tilde{p},\tilde{S},\tilde{\varrho})$ *in* $[0,T]$*, but for ill-prepared data,* (p_t,S_t,ϱ_t) *may not converge strongly to* $(\tilde{p}_t,\tilde{S}_t,\tilde{\varrho}_t)$ *near* $t=0$*, they may have an initial layer (see (5.3)). In this paper, we do not have enough regularity of* $\nabla\cdot(v-\tilde{v})$ *to prove the initial layer and strong convergence of* p_t,S_t,ϱ_t*.*

(iv) If one replaces the initial data $(p_0(x,\tau),\frac{1}{\tau}\mathcal{U}_0(x,\tau),S_0(x,\tau))$ *in* (1.1) *with* $(p_0(x),\frac{1}{\tau}\mathcal{U}_0(x),S_0(x))$*, where* $(p_0(x),\mathcal{U}_0(x),S_0(x))$ *are independent of* τ*, then for the well-prepared data,* $p_0(x)\equiv const$ *and* $\mathcal{U}_0(x)\equiv 0$ *(equilibrium states); while for the ill-prepared data,* $p_0(x)\neq const$ *or* $\mathcal{U}_0(x)\neq 0$ *(non-equilibrium states).*

In the following, we give more comments on Theorem 1.1 and Theorem 1.2. If $\tau>0$ is fixed, the global existence of classical solutions to the equations (1.2) is proved in [19, 20, 21]. However, for the relaxation limit problem in this paper, $\tau>0$ is variant and approaches 0, so we need the uniform existence of the solutions and uniform regularities (1.7) which are different from [19, 20, 21]. The uniform a priori estimates with respect to τ produce the convergence results. In order to have enough regularities to treat the initial layer of the velocity,

$(p_0(x,\tau)-\bar{p}, \mathcal{U}_0(x,\tau), S_0(x,\tau)-\bar{S})$ are required to be in $H^4(\mathbb{R}^3)$ for all $\tau \geq 0$.

The uniform a priori estimates for the equations (1.1) imply the uniform regularities (1.7). Passing to the limit, we have the convergence results (1.8).

Let us give some comments and remarks on the initial layer as follows. The asymptotic expansions of the solutions to (1.1) give us some indication of the initial layer. We illustrate this as follows: assume that the initial data have asymptotic expansion

$$(p(x,0), u(x,0), S(x,0), \varrho(x,0)) = \sum_{m\geq 0} \tau^{2m}(p_0^m, u_0^m, S_0^m, \varrho_0^m),$$

and solutions of the equations (1.1) have the asymptotic expansion

$$(p(x,t), u(x,t), S(x,t), \varrho(x,t)) = \sum_{m\geq 0} \tau^{2m}(p^m(x,t), u^m(x,t), S^m(x,t), \varrho^m(x,t)),$$

then the leading order profiles satisfy the equations

$$\begin{cases} \partial_t p^0 + u^0 \cdot \nabla p^0 + \gamma p^0 \nabla \cdot u^0 = 0, \\ u^0 + \frac{1}{\varrho_0} \nabla p^0 = 0, \\ \partial_t S^0 + u^0 \cdot \nabla S^0 = 0, \\ (p^0, S^0)(x,0) = (p_0^0, S_0^0), \end{cases} \tag{1.12}$$

if the initial velocity is well-prepared, i.e., $u_0^0 = -\frac{1}{\varrho_0^0}\nabla p_0^0$, where $\varrho^0 = \varrho(p^0, S^0)$. Note that in this case, $u^0 + \frac{1}{\varrho^0}\nabla p^0 = 0$ matches $u_0^0 + \frac{1}{\varrho_0^0}\nabla p_0^0 = 0$.

However, if the initial velocity is ill-prepared, i. e., $u_0^0 \neq -\frac{1}{\varrho_0^0}\nabla p_0^0$, we may assume that the velocity has the asymptotic expansion

$$u(x,t) = \sum_{m\geq 0} \tau^{2m}(u^m(x,t) + \hat{u}^m(x,z)), \ z = \frac{t}{\tau^2},$$

then the leading order profile of the initial layer correction $\hat{u}$ satisfies the equation

$$\begin{cases} \partial_z \hat{u}^0 + \hat{u}^0 = 0, \\ \hat{u}^0(x,0) = u_0^0 + \frac{1}{\varrho_0^0}\nabla p_0^0. \end{cases} \tag{1.13}$$

Then $\hat{u}(x,z) = (u_0^0 + \frac{1}{\varrho_0^0}\nabla p_0^0)e^{-z}$. Thus, the difference between u^0 and $-\frac{1}{\varrho^0}\nabla p^0$ decays exponentially within the initial layer.

The above arguments of (1.12) and (1.13) indicate the relationship between the existence of initial layer and a class of initial data on a formal level. We are not concerned with the asymptotic expansions in this paper (as to asymptotic expansion analysis in relaxation limit problem for Euler-Poisson equations, see [22, 23, 24]; for Euler-Maxwell equations, see [25, 26]) and only focus on rigorous analysis of the initial layer and relaxation limit of the relaxing equations (1.1).

The relaxation limit is a singular limit, since $(p, -\frac{1}{\varrho}\nabla p, S, \varrho)$ converge strongly to $(\tilde{p}, \tilde{u}, \tilde{S}, \tilde{\varrho})$, instead of $u \to \tilde{u}$. In order to measure the difference between u and $-\frac{1}{\varrho}\nabla p$, we introduce a quantity:

$$\eta = u + \tfrac{1}{\varrho}\nabla p\ . \tag{1.14}$$

The pointwise decay of η outside the initial layer is the key to the strong convergence of the velocity. η satisfies the following transport equations with damping and forcing

$$\eta_t + u\cdot\nabla\eta + \frac{1}{\tau^2}\eta = \textit{forcing terms}, \tag{1.15}$$

where $\tau\cdot[\textit{forcing terms}]$ is bounded uniformly with respect to τ, the damping effect becomes stronger as τ decreases.

Also, η satisfies another equation:

$$\eta = \tau^2(u_t + u\cdot\nabla u). \tag{1.16}$$

Then the equations (1.15) and (1.16) produce the following estimates respectively:

$$\begin{cases} |\eta|_\infty^2 \le C\|\eta\|_{H^2(\mathbb{R}^3)}^2 \le C\|\eta|_{t=0}\|_{H^2(\mathbb{R}^3)}^2 \exp\{-\frac{t}{\tau^2}\} + C\tau^2, \\ \int_0^T \|\eta_t\|_{H^1(\mathbb{R}^3)}^2\,\mathrm{d}x \le C\tau^2. \end{cases} \tag{1.17}$$

Therefore, for the well-prepared data, $\lim\limits_{\tau\to 0}\eta(x,t) = 0$ for any $t\in[0,T]$, there is no discrepancy between $\eta(x,t)|_{t>0}$ and $\eta(x)|_{t=0} = u_0 + \frac{1}{\varrho_0}\nabla p_0$, thus there is no initial layer, $u \to \tilde{u}$ strongly in $C([0,T], C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3)), \mu_1 \in [0,\frac{1}{2}], \mu_2\in[2,6]$.

While for the ill-prepared data, $\lim\limits_{\tau\to 0}\eta(x)|_{t=0}\neq 0$, while $\lim\limits_{\tau\to 0}\eta(x,t_*)\to 0$ pointwisely for any fixed small number $t_*>0$, thus the discrepancy between $\eta|_{t=0}$ and $\eta|_{t=t_*}$ make the initial layer of the velocity exist. Within $[0,t_*]$, η must decreases rapidly, $\frac{1}{\varrho}\nabla p$ is uniform bounded, then u changes dramatically. In $[t_*,T]$, $\eta\to 0$ strongly, $u\to\tilde{u}$ strongly. Near $t=0$, the behavior of η is not uniform with respect to τ, $\eta(x,C\tau^2)$ does not converge to 0, while $\eta(x,C\tau^{2-\delta})$ converges to 0 when $\delta>0$. Correspondingly, $u|_{t=0}\neq\tilde{u}|_{t=0}$, $u(x,C\tau^2)$ does not converge to $\tilde{u}(x,0)$, while $u(x,C\tau^{2-\delta})$ converges to $\tilde{u}(x,0)$ when $\delta>0$.

Finally, in order to prove that the thickness of the initial layer is $O(\tau^2)$, we have the following equation of $\eta(x,z)$ by rescaling the time variable $z=\frac{t}{\tau^2}$:

$$\partial_z\eta(x,z)+u(x,z)\cdot\nabla\eta(x,z)+\eta(x,z)=\textit{new forcing terms}, \tag{1.18}$$

where $\frac{1}{\tau}\cdot u(x,z)$ and $\frac{1}{\tau}\cdot[\textit{new forcing terms}]$ are bounded uniformly with respect to τ. (1.18) produces the estimate $\|\eta(x,z)\|_{L^2(\mathbb{R}^3)}>0$ when τ is small.

The uniform a priori estimates for the relaxing equations are hard to obtain for the bounded domain Ω with fixed boundary $\partial\Omega$, due to the characteristic boundary condition $u\cdot n|_{\partial\Omega}=0$. However, our results can be extended without difficulties to the periodic domains $\mathbb{T}^3$ due to the convenience of periodic boundary conditions. The results are the same except that $\mathbb{R}^3$ and $\mathcal{K}$ in the theorems need to be replaced by $\mathbb{T}^3$, there is nothing new in methodology.

The rest of this paper is organized as follows: In Section 2, we reformulate the equations into appropriate forms and derive the equations of η. In Section 3, we prove uniform a priori estimates for the relaxing equations. In Section 4, we prove the uniqueness of the relaxed equations and the relaxation limit of the the relaxing equations. In Section 5, we estimate the strong convergence rates of the pressure, entropy and density. In Section 6, we study the initial layer and strong convergence of the velocity, and then we prove the thickness of the initial layer for the ill-prepared data.

2. Preliminaries

In this section, we will reformulate the equations (1.1) into appropriate forms, define the energy functionals and derive the equations of the quantity η.

For the relaxing equations (1.1) together with their initial data $(p_0, u_0, S_0, \varrho_0)$ and constants $\bar{p}, \bar{S}, \bar{\varrho}$, we introduce the constants:

$$k_1 = \sqrt{\tfrac{1}{\gamma \bar{\varrho} \bar{p}}}, \quad k_2 = \sqrt{\tfrac{\gamma \bar{p}}{\bar{\varrho}}},$$

define the variables:

$$\xi = p - \bar{p}, \quad v = \tfrac{1}{k_1} u, \quad \phi = S - \bar{S}, \quad \zeta = \varrho - \bar{\varrho},$$

where $(\xi, v, \phi, \zeta) \to (0,0,0,0)$ as $|x| \to +\infty$.

In order to prove the uniform a priori estimates, similar to the symmetrization used in [27, 20], we symmetrize the equations (1.1) into the following form:

$$\begin{cases} \xi_t + k_2 \nabla \cdot v = -\gamma k_1 \xi \nabla \cdot v - k_1 v \cdot \nabla \xi, \\ \tau^2 v_t + k_2 \nabla \xi + v = -k_1 \tau^2 v \cdot \nabla v + \tfrac{1}{k_1}(\tfrac{1}{\bar{\varrho}} - \tfrac{1}{\varrho}) \nabla \xi, \\ \phi_t = -k_1 v \cdot \nabla \phi, \\ (\xi, v, \phi)(x, 0) = (p_0(x,\tau) - \bar{p}, \tfrac{1}{k_1 \tau} \mathcal{U}_0(x,\tau), S_0(x,\tau) - \bar{S}), \end{cases} \tag{2.1}$$

where $\varrho = \zeta + \bar{\varrho} = \varrho(\xi + \bar{p}, \phi + \bar{S})$ and ζ satisfies the equation $\zeta_t + k_1 v \cdot \nabla \zeta + k_1 \varrho \nabla \cdot v = 0$ by the equation of gas state.

Let $\tau = 0$ in the relaxing equations (2.1), we formally obtained the following relaxed equations, which are equivalent to the relaxed equations (1.4).

$$\begin{cases} \xi_t + k_2 \nabla \cdot v = -\gamma k_1 \xi \nabla \cdot v - k_1 v \cdot \nabla \xi, \\ k_2 \nabla \xi + v = \tfrac{1}{k_1}(\tfrac{1}{\bar{\varrho}} - \tfrac{1}{\varrho}) \nabla \xi, \\ \phi_t = -k_1 v \cdot \nabla \phi, \\ (\xi, \phi)(x, 0) = (\lim\limits_{\tau \to 0} p_0(x,\tau) - \bar{p}, \lim\limits_{\tau \to 0} S_0(x,\tau) - \bar{S}), \end{cases} \tag{2.2}$$

where $\varrho = \zeta + \bar{\varrho} = \varrho(\xi + \bar{p}, \phi + \bar{S})$.

In order to use the energy method to derive uniform a priori estimates with respect to τ, we define the following generic energy functional $\mathcal{E}[\xi](t)$ and three specific energy functionals $\mathcal{E}_X[\xi](t), \mathcal{E}_1[\xi](t), \mathcal{E}_2[v](t)$:

Definition 2.1. *Define*

$$
\begin{aligned}
&\mathcal{E}[\xi](t) := \sum_{0\le\ell\le2, 0\le\ell+|\alpha|\le4} \|\partial_t^\ell \mathcal{D}^\alpha \xi(t)\|^2_{L^2(\mathbb{R}^3)},\\
&\mathcal{E}_X[\xi](t) := \sum_{0\le\ell\le2, 0<\ell+|\alpha|\le4} \|\partial_t^\ell \mathcal{D}^\alpha \xi\|^2_{L^2(\mathbb{R}^3)},\\
&\mathcal{E}_1[\xi](t) := \mathcal{E}[\xi](t) - \sum_{0\le\ell\le2, \ell+|\alpha|=4} \int_{\mathbb{R}^3} \tfrac{\xi}{p}(\partial_t^\ell \mathcal{D}^\alpha \xi)^2 \,\mathrm{d}x,\\
&\mathcal{E}_2[v](t) := \mathcal{E}[v](t) + \sum_{0\le\ell\le2, \ell+|\alpha|=4} \int_{\mathbb{R}^3} (\tfrac{\varrho}{\bar\varrho}-1)|\partial_t^\ell \mathcal{D}^\alpha v|^2 \,\mathrm{d}x.
\end{aligned}
\tag{2.3}
$$

Moreover, we use the following two notations: $\mathcal{E}[\xi, v](t) := \mathcal{E}[\xi](t) + \mathcal{E}[v](t)$, $\mathcal{E}[\xi, v, \phi, \zeta](t) := \mathcal{E}[\xi, v](t) + \mathcal{E}[\phi](t) + \mathcal{E}[\zeta](t)$.

It is easy to get the following lemma states that $\mathcal{E}[\xi](t)$ and $\mathcal{E}_1[\xi](t)$ are equivalent, $\mathcal{E}[v](t)$ and $\mathcal{E}_2[v](t)$ are equivalent.

Lemma 2.2. *For any fixed $T \in (0, +\infty), \tau \in [0, 1]$, if*

$$
\sup_{0\le t\le T} \mathcal{E}[\xi, \tau v, \phi, \zeta](t) \le \epsilon,
$$

where $0 < \epsilon \ll 1$, then there exist $c_1 > 0, c_2 > 0$ such that for any $t \in [0, T]$,

$$
c_1\mathcal{E}[\xi](t) \le \mathcal{E}_1[\xi](t) \le c_2\mathcal{E}[\xi](t), \quad c_1\mathcal{E}[v](t) \le \mathcal{E}_2[v](t) \le c_2\mathcal{E}[v](t). \tag{2.4}
$$

Remark 2.3. *Different from the symmetrization* (2.1)*, there is an straight way to symmetrize* (1.1)*, which is*

$$
\begin{cases}
\xi_t + \gamma k_1 p \nabla\cdot v = -k_1 v\cdot\nabla\xi,\\
\tau^2 v_t + v + \frac{1}{k_1\varrho}\nabla\xi = -\tau^2 k_1 v\cdot\nabla v,\\
\phi_t + k_1 v\cdot\nabla\phi = 0,
\end{cases}
\tag{2.5}
$$

with the weighted energy functionals:

$$
\mathcal{E}_\xi[\xi] := \sum_{\ell,\alpha}\int_{\mathbb{R}^3} \tfrac{\bar p}{p}(\partial_t^\ell\partial_x^\alpha\xi)^2, \quad \mathcal{E}_v[v] := \sum_{\ell,\alpha}\int_{\mathbb{R}^3} \tfrac{\varrho}{\bar\rho}(\partial_t^\ell\partial_x^\alpha v)^2.
$$

However, if we use (2.5) *to prove uniform a priori estimates, the uniqueness and convergence rates, we have to introduce many different weighted energy functionals into this paper. So we still use* (2.1) *because* (2.1) *makes the proofs in this paper a little easier.*

Finally, we define the quantity η precisely and derive its equations. Define

$$\eta(x,t) = v(x,t) + \tfrac{1}{k_1\varrho(x,t)}\nabla\xi(x,t). \tag{2.6}$$

Note that here $\eta = \frac{1}{k_1}(u + \frac{1}{\varrho}\nabla p)$, which differs from η introduced in the introduction (see (1.14)) up to a constant coefficient $\frac{1}{k_1}$. Actually, the coefficient can be any positive number, η defined in (2.6) makes our calculation easier, while η introduced in (1.14) is convenient to represent our ideas.

Differentiate (2.6) wit respect to t, we have

$$\eta_t = v_t + \tfrac{1}{k_1\varrho}\nabla\xi_t - \tfrac{\zeta_t}{k_1\varrho^2}\nabla\xi,$$

then insert the evolution equations of v, ξ, ζ into the above equation, we get the following transport equation with damping and forcing for η:

$$\eta_t + k_1 v\cdot\nabla\eta + \tfrac{1}{\tau^2}\eta = -\tfrac{1}{\varrho}(\nabla v)\nabla\xi - \tfrac{\gamma-1}{\varrho}\nabla\xi\nabla\cdot v - \tfrac{\gamma p}{\varrho}\nabla(\nabla\cdot v), \tag{2.7}$$

where (∇v) is a matrix, $\tau\cdot[R.H.S.\ of(2.7)]$ is bounded uniformly with respect to τ, 'R.H.S.' is the abbreviation for 'right hand side'.

Besides the equation (2.7), η satisfies another equation:

$$\eta = \tau^2(v_t + k_1 v\cdot\nabla v), \tag{2.8}$$

In the rest of this paper, we will use the following notations: $X \lesssim Y$ denotes the estimate $X \le CY$ for some implied constant $C > 0$ which may be different line by line. $[A, B]$ is the commutator of A and B.

3. Uniform A Priori Estimates for the Relaxing Equations

In this section, we derive unform a priori estimates for the relaxing Cauchy problem (2.1). In order to discuss the initial layer and strong convergence of the velocity, we need to estimate the higher order time derivatives.

The following lemma gives uniform a priori estimate for $\mathcal{E}_1[\xi](t)+\tau^2\mathcal{E}_2[v](t)$, which is equivalent to $\mathcal{E}[\xi](t)+\tau^2\mathcal{E}[v](t)$.

Lemma 3.1. *For any fixed $T\in(0,+\infty),\tau\in[0,1]$, if*

$$\sup_{0\le t\le T}\mathcal{E}[\xi,\tau v,\phi,\zeta](t)\le\epsilon,$$

where $0<\epsilon\ll 1$, then there exists a constant $C_1>0$ such that for any $t\in[0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1[\xi](t)+\tau^2\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2[v](t)+2\mathcal{E}_2[v](t)\le C_1\sqrt{\epsilon}(\mathcal{E}_X[\xi](t)+\mathcal{E}[v](t)). \tag{3.1}$$

Proof. Let $\xi\cdot(2.1)_1+v\cdot(2.1)_2$, integrate in $\mathbb{R}^3$, note that $\int\limits_{\mathbb{R}^3}\nabla\cdot(\xi v)\,\mathrm{d}x=0$, then we get

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\xi|^2+\tau^2|v|^2\,\mathrm{d}x+2\int\limits_{\mathbb{R}^3}|v|^2\,\mathrm{d}x\\
&=\int\limits_{\mathbb{R}^3}2\gamma k_1 v\cdot\nabla(\xi^2)-2k_1\xi v\cdot\nabla\xi-2k_1\tau^2 v\cdot\nabla v\cdot v+\tfrac{2}{k_1}(\tfrac{1}{\bar\varrho}-\tfrac{1}{\varrho})\nabla\xi\cdot v\,\mathrm{d}x\\
&\lesssim\sqrt{\epsilon}\|\nabla\xi\|_{L^2(\mathbb{R}^3)}\|v\|_{L^2(\mathbb{R}^3)}+\sqrt{\epsilon}\|v\|^2_{L^2(\mathbb{R}^3)}\lesssim\sqrt{\epsilon}(\mathcal{E}_X[\xi](t)+\mathcal{E}[v](t)).
\end{aligned}\tag{3.2}$$

Let $\partial_t^\ell\mathcal{D}^\alpha\xi\cdot\partial_t^\ell\mathcal{D}^\alpha(2.1)_1+\partial_t^\ell\mathcal{D}^\alpha v\cdot\partial_t^\ell\mathcal{D}^\alpha(2.1)_2$, where $0\le\ell\le 2, 1\le\ell+|\alpha|\le 4$, integrate in $\mathbb{R}^3$, note that $\int\limits_{\mathbb{R}^3}\nabla\cdot(\partial_t^\ell\mathcal{D}^\alpha\xi\partial_t^\ell\mathcal{D}^\alpha v)\,\mathrm{d}x=0$, then we get

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\partial_t^\ell\mathcal{D}^\alpha\xi|^2+\tau^2|\partial_t^\ell\mathcal{D}^\alpha v|^2\,\mathrm{d}x+2\int\limits_{\mathbb{R}^3}|\partial_t^\ell\mathcal{D}^\alpha v|^2\,\mathrm{d}x\\
&=\int\limits_{\mathbb{R}^3}-2\gamma k_1(\partial_t^\ell\mathcal{D}^\alpha\xi)\partial_t^\ell\mathcal{D}^\alpha(\xi\nabla\cdot v)-2k_1(\partial_t^\ell\mathcal{D}^\alpha\xi)\partial_t^\ell\mathcal{D}^\alpha(v\cdot\nabla\xi)\\
&\quad-2k_1\tau^2(\partial_t^\ell\mathcal{D}^\alpha v)\cdot\partial_t^\ell\mathcal{D}^\alpha(v\cdot\nabla v)+\tfrac{2}{k_1}(\partial_t^\ell\mathcal{D}^\alpha v)\cdot\partial_t^\ell\mathcal{D}^\alpha[(\tfrac{1}{\bar\varrho}-\tfrac{1}{\varrho})\nabla\xi]\,\mathrm{d}x:=I_1.
\end{aligned}\tag{3.3}$$

When $1\le\ell+|\alpha|\le 3$, it is easy to check that $I_1\lesssim\sqrt{\epsilon}(\mathcal{E}_X[\xi](t)+\mathcal{E}[v](t))$.

When $\ell+|\alpha|=4$, we estimate the quantity $I_1-\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}\frac{\xi}{p}(\partial_t^\ell\mathcal{D}^\alpha\xi)^2\,\mathrm{d}x+$

$\tau^2 \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\mathbb{R}^3} (\frac{\varrho}{\bar{\varrho}} - 1)|\partial_t^\ell \mathcal{D}^\alpha v|^2 \,\mathrm{d}x$, then

$$\begin{aligned}
& I_1 - \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\mathbb{R}^3} \frac{\xi}{p} (\partial_t^\ell \mathcal{D}^\alpha \xi)^2 \,\mathrm{d}x + \tau^2 \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\mathbb{R}^3} (\frac{\varrho}{\bar{\varrho}} - 1)|\partial_t^\ell \mathcal{D}^\alpha v|^2 \,\mathrm{d}x \\
&= -2\gamma k_1 \int\limits_{\mathbb{R}^3} (\partial_t^\ell \mathcal{D}^\alpha \xi)\xi \nabla \cdot (\partial_t^\ell \mathcal{D}^\alpha v) \,\mathrm{d}x - 2k_1 \int\limits_{\mathbb{R}^3} (\partial_t^\ell \mathcal{D}^\alpha \xi) v \cdot \nabla(\partial_t^\ell \mathcal{D}^\alpha \xi) \,\mathrm{d}x \\
&\quad -2k_1\tau^2 \int\limits_{\mathbb{R}^3} v \cdot \nabla(\partial_t^\ell \mathcal{D}^\alpha v) \cdot (\partial_t^\ell \mathcal{D}^\alpha v) \,\mathrm{d}x + \frac{2}{k_1} \int\limits_{\mathbb{R}^3} (\frac{1}{\bar{\varrho}} - \frac{1}{\varrho})(\partial_t^\ell \mathcal{D}^\alpha v) \cdot \nabla(\partial_t^\ell \mathcal{D}^\alpha \xi) \,\mathrm{d}x \\
&\quad +2\tau^2 \int\limits_{\mathbb{R}^3} (\frac{\varrho}{\bar{\varrho}} - 1)(\partial_t^\ell \mathcal{D}^\alpha v) \cdot (\partial_t^\ell \mathcal{D}^\alpha v_t) \,\mathrm{d}x + \tau^2 \int\limits_{\mathbb{R}^3} \frac{\zeta_t}{\bar{\varrho}} |\partial_t^\ell \mathcal{D}^\alpha v|^2 \,\mathrm{d}x \\
&\quad -2 \int\limits_{\mathbb{R}^3} \frac{\xi}{p} (\partial_t^\ell \mathcal{D}^\alpha \xi)(\partial_t^\ell \mathcal{D}^\alpha \xi_t) \,\mathrm{d}x - \int\limits_{\mathbb{R}^3} \partial_t(\frac{\xi}{p})(\partial_t^\ell \mathcal{D}^\alpha \xi)^2 \,\mathrm{d}x \\
&\lesssim \sqrt{\epsilon}(\mathcal{E}_X[\xi](t) + \mathcal{E}[v](t)) - 2 \int\limits_{\mathbb{R}^3} \frac{\xi}{p} (\partial_t^\ell \mathcal{D}^\alpha \xi)[\partial_t^\ell \mathcal{D}^\alpha \xi_t + k_1 \gamma p \nabla \cdot (\partial_t^\ell \mathcal{D}^\alpha v)] \,\mathrm{d}x \\
&\quad + \frac{2}{k_1} \int\limits_{\mathbb{R}^3} (\frac{1}{\bar{\varrho}} - \frac{1}{\varrho})(\partial_t^\ell \mathcal{D}^\alpha v) \cdot [\nabla(\partial_t^\ell \mathcal{D}^\alpha \xi) + k_1\tau^2 \varrho(\partial_t^\ell \mathcal{D}^\alpha v_t)] \,\mathrm{d}x.
\end{aligned} \tag{3.4}$$

Apply $\partial_t^\ell \mathcal{D}^\alpha$ to $(2.1)_1$, where $0 \le \ell \le 2, \ell + |\alpha| = 4$, we get

$$\begin{aligned}
\partial_t^\ell \mathcal{D}^\alpha \xi_t + k_1 \gamma p \nabla \cdot (\partial_t^\ell \mathcal{D}^\alpha v) &= -k_1 \partial_t^\ell \mathcal{D}^\alpha (v \cdot \nabla \xi) \\
&\quad -k_1\gamma \sum_{\ell_1 + |\alpha_1| > 0} \partial_t^{\ell_1} \mathcal{D}^{\alpha_1} \xi \nabla \cdot (\partial_t^{\ell_2} \mathcal{D}^{\alpha_2} v).
\end{aligned} \tag{3.5}$$

Plug (3.5) into the following integral, we get

$$\begin{aligned}
& \int\limits_{\mathbb{R}^3} \frac{\xi}{p} (\partial_t^\ell \mathcal{D}^\alpha \xi)(\partial_t^\ell \mathcal{D}^\alpha \xi_t + k_1 \gamma p \nabla \cdot (\partial_t^\ell \mathcal{D}^\alpha v)) \,\mathrm{d}x = \int\limits_{\mathbb{R}^3} \frac{\xi}{p} (\partial_t^\ell \mathcal{D}^\alpha \xi)[R.H.S.\ of\ (3.5)] \,\mathrm{d}x \\
&\lesssim \sqrt{\epsilon}(\mathcal{E}_X[\xi](t) + \mathcal{E}[v](t)) + \frac{k_1}{2} \int\limits_{\mathbb{R}^3} |\partial_t^\ell \mathcal{D}^\alpha \xi|^2 \nabla \cdot (\frac{\xi}{p} v) \,\mathrm{d}x \\
&\lesssim \sqrt{\epsilon}(\mathcal{E}_X[\xi](t) + \mathcal{E}[v](t)).
\end{aligned} \tag{3.6}$$

Apply $\partial_t^\ell \mathcal{D}^\alpha$ to $k_1\tau^2 v_t + k_1^2 \tau^2 v \cdot \nabla v + k_1 v + \frac{1}{\varrho} \nabla \xi = 0$, where $0 \le \ell \le 2, \ell + |\alpha| = 4$, we get

$$\begin{aligned}
& \frac{1}{\varrho} \nabla(\partial_t^\ell \mathcal{D}^\alpha \xi) + k_1 \tau^2 (\partial_t^\ell \mathcal{D}^\alpha v_t) \\
&= -k_1 \partial_t^\ell \mathcal{D}^\alpha v - k_1^2 \tau^2 \partial_t^\ell \mathcal{D}^\alpha (v \cdot \nabla v) - \sum_{\ell_1 + |\alpha_1| > 0} \partial_t^{\ell_1} \mathcal{D}^{\alpha_1} (\frac{1}{\varrho}) \partial_t^{\ell_2} \mathcal{D}^{\alpha_2} \nabla \xi.
\end{aligned} \tag{3.7}$$

Plug (3.7) into the following integral, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (\tfrac{1}{\bar\varrho} - \tfrac{1}{\varrho})(\partial_t^\ell \mathcal{D}^\alpha v) \cdot [\nabla(\partial_t^\ell \mathcal{D}^\alpha \xi) + k_1 \tau^2 \varrho \partial_t^\ell \mathcal{D}^\alpha v_t] \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} (\tfrac{1}{\bar\varrho} - \tfrac{1}{\varrho})(\partial_t^\ell \mathcal{D}^\alpha v) \cdot \varrho [R.H.S.\ of\ (3.7)] \,\mathrm{d}x \\
&\lesssim \tfrac{k_1^2 \tau}{2} \int_{\mathbb{R}^3} |\partial_t^\ell \mathcal{D}^\alpha v|^2 \nabla \cdot [(\tfrac{\varrho}{\bar\varrho} - 1)\tau v] \,\mathrm{d}x + \sqrt{\epsilon}(\mathcal{E}_X[\xi](t) + \mathcal{E}[v](t)) \\
&\lesssim \sqrt{\epsilon}(\mathcal{E}_X[\xi](t) + \mathcal{E}[v](t)).
\end{aligned}
\tag{3.8}
$$

Plug (3.6) and (3.8) into (3.4), we get

$$
I_1 - \tfrac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} \tfrac{\xi}{p} (\partial_t^\ell \mathcal{D}^\alpha \xi)^2 \,\mathrm{d}x + \tau^2 \tfrac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} (\tfrac{\varrho}{\bar\varrho} - 1) |\partial_t^\ell \mathcal{D}^\alpha v|^2 \,\mathrm{d}x \lesssim \sqrt{\epsilon}(\mathcal{E}_X[\xi](t) + \mathcal{E}[v](t)).
\tag{3.9}
$$

After Summing ℓ and α, we have

$$
\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{0\le\ell\le2, 1\le\ell+|\alpha|\le4} \int_{\mathbb{R}^3} |\partial_t^\ell \mathcal{D}^\alpha \xi|^2 \,\mathrm{d}x - \sum_{0\le\ell\le2, \ell+|\alpha|=4} \int_{\mathbb{R}^3} \tfrac{\xi}{p} (\partial_t^\ell \mathcal{D}^\alpha \xi)^2 \,\mathrm{d}x \Big) \\
&+\tau^2 \tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{0\le\ell\le2, 1\le\ell+|\alpha|\le4} \int_{\mathbb{R}^3} |\partial_t^\ell \mathcal{D}^\alpha v|^2 \,\mathrm{d}x + \sum_{0\le\ell\le2, \ell+|\alpha|=4} \int_{\mathbb{R}^3} (\tfrac{\varrho}{\bar\varrho} - 1) |\partial_t \mathcal{D}^\alpha v|^2 \,\mathrm{d}x \Big) \\
&+2\Big(\sum_{0\le\ell\le2, 1\le\ell+|\alpha|\le4} \int_{\mathbb{R}^3} |\partial_t^\ell \mathcal{D}^\alpha v|^2 \,\mathrm{d}x + \sum_{0\le\ell\le2, \ell+|\alpha|=4} \int_{\mathbb{R}^3} (\tfrac{\varrho}{\bar\varrho} - 1) |\partial_t^\ell \mathcal{D}^\alpha v|^2 \,\mathrm{d}x \Big) \\
&\lesssim \sqrt{\epsilon}(\mathcal{E}_X[\xi](t) + \mathcal{E}[v](t)).
\end{aligned}
\tag{3.10}
$$

By (3.2) + (3.10), we proved that there exists a constant $C_1 > 0$ such that

$$
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_1[\xi](t) + \tau^2 \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_2[v](t) + 2\mathcal{E}_2[v](t) \le C_1 \sqrt{\epsilon}(\mathcal{E}_X[\xi](t) + \mathcal{E}[v](t)).
\tag{3.11}
$$

Thus, Lemma 3.1 is proved. □

The structure of the equations (2.1) implies $\mathcal{E}_X[\xi](t)$ can be estimates by $\mathcal{E}[v](t)$, as the following lemma stated:

Lemma 3.2. *For any fixed $T \in (0, +\infty), \tau \in [0, 1]$, if*

$$
\sup_{0\le t\le T} \mathcal{E}[\xi, \tau v, \phi, \zeta](t) \le \epsilon,
$$

where $0 < \epsilon \ll 1$, then there exists a constant $c_3 > 0$ such that for any $t \in [0, T]$,

$$
\mathcal{E}_X[\xi](t) \le c_3 \mathcal{E}[v](t).
\tag{3.12}
$$

Proof. Apply $\mathcal{D}^\alpha$ to $\nabla\xi = -k_1\tau^2\varrho v_t - k_1^2\tau^2\varrho v\cdot\nabla v - k_1\varrho v$, where $0\le|\alpha|\le 3$, we get

$$\begin{aligned}\|\mathcal{D}^\alpha\nabla\xi\|_{L^2(\mathbb{R}^3)}^2 &\lesssim \tau^4\|\mathcal{D}^\alpha(\varrho v_t)\|_{L^2(\mathbb{R}^3)}^2 + \tau^4\|\mathcal{D}^\alpha(\varrho v\cdot\nabla v)\|_{L^2(\mathbb{R}^3)}^2 + \|\mathcal{D}^\alpha(\varrho v)\|_{L^2(\mathbb{R}^3)}^2\\ &\lesssim \|\mathcal{D}^\alpha v_t\|_{L^2(\mathbb{R}^3)}^2 + \|\mathcal{D}^\alpha v\|_{L^2(\mathbb{R}^3)}^2 + \mathcal{E}[\zeta]\mathcal{E}[v](t) + \mathcal{E}[\tau v](t)\mathcal{E}[v](t)\\ &\quad +\mathcal{E}[\tau v](t)\mathcal{E}[\zeta]\mathcal{E}[v](t).\end{aligned}\tag{3.13}$$

Apply $\mathcal{D}^\alpha$ to $\xi_t = -k_1 v\cdot\nabla\xi - k_1\gamma p\nabla\cdot v$, where $0\le|\alpha|\le 3$, we get

$$\begin{aligned}\|\mathcal{D}^\alpha\xi_t\|_{L^2(\mathbb{R}^3)}^2 &\lesssim \|\mathcal{D}^\alpha(v\cdot\nabla\xi)\|_{L^2(\mathbb{R}^3)}^2 + \|\mathcal{D}^\alpha(p\nabla\cdot v)\|_{L^2(\mathbb{R}^3)}^2\\ &\lesssim \|\mathcal{D}^\alpha\nabla\cdot v\|_{L^2(\mathbb{R}^3)}^2 + \mathcal{E}_X[\xi]\mathcal{E}[v](t).\end{aligned}\tag{3.14}$$

Similar to the above estimate, apply $\partial_t\mathcal{D}^\alpha$ to ξ_t, where $0\le|\alpha|\le 2$, we get

$$\|\mathcal{D}^\alpha\xi_{tt}\|_{L^2(\mathbb{R}^3)}^2 \lesssim \|\mathcal{D}^\alpha\nabla\cdot v_t\|_{L^2(\mathbb{R}^3)}^2 + \mathcal{E}_X[\xi]\mathcal{E}[v](t).\tag{3.15}$$

By (3.13) + (3.14) + (3.15), we have

$$\begin{aligned}\mathcal{E}_X[\xi](t) &\le C_2\mathcal{E}[v](t) + C_2\mathcal{E}[\zeta]\mathcal{E}[v](t) + C_2\mathcal{E}[\tau v](t)\mathcal{E}[v](t)\\ &\quad +C_2\mathcal{E}[\tau v](t)\mathcal{E}[\zeta]\mathcal{E}[v](t) + C_2\mathcal{E}_X[\xi]\mathcal{E}[v](t)\\ &\le C_2\mathcal{E}_X[\xi]\mathcal{E}[v](t) + C_2(1+2\epsilon+\epsilon^2)\mathcal{E}[v](t),\end{aligned}\tag{3.16}$$

for some $C_2 > 0$.

Assume ϵ is so small that $C_2\mathcal{E}[v](t)\le\frac{1}{2}$. Let $c_3 = 2C_2(1+2\epsilon+\epsilon^2)$, we get $\mathcal{E}_X[\xi](t)\le c_3\mathcal{E}[v](t)$. Thus, Lemma 3.2 is proved. □

Based on the above a priori estimates, we prove not only the uniform L^∞ bound of $\mathcal{E}[\xi,\tau v](t)$, but also the uniform bound of $\int_0^T \mathcal{E}[v](s)\,\mathrm{d}s$.

Lemma 3.3. *For any fixed $T\in(0,+\infty), \tau\in[0,1]$, if*

$$\sup_{0\le t\le T}\mathcal{E}[\xi,\tau v,\phi,\zeta](t)\le\epsilon,$$

where $0<\epsilon\ll 1$, then there exists a constant $c_4>0$ such that for any $t\in[0,T]$,

$$\begin{aligned}&\mathcal{E}[\xi,\tau v](t)\le c_4\|(\xi_0,\mathcal{U}_0)\|_{H^4(\mathbb{R}^3)}^2,\\ &\int_0^T\mathcal{E}[v](s)\,\mathrm{d}s\le c_4\|(\xi_0,\mathcal{U}_0)\|_{H^4(\mathbb{R}^3)}^2.\end{aligned}\tag{3.17}$$

Proof. Since Lemmas 3.1 and 3.2 are proved, plug (3.12) into (3.1), then we get

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1[\xi](t)+\tau^2\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2[v](t)+2\mathcal{E}_2[v](t) &\le C_1(1+c_3)\sqrt{\epsilon}\mathcal{E}[v](t)\\ &\le \tfrac{C_1(1+c_3)}{c_1}\sqrt{\epsilon}\mathcal{E}_2[v](t).\end{aligned}\tag{3.18}$$

Since ϵ is sufficiently small, we assume $\epsilon \le \frac{c_1^2}{C_1^2(1+c_3)^2}$, then,

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1[\xi](t)+\tau^2\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_2[v](t)+\mathcal{E}_2[v](t)\le 0.\tag{3.19}$$

By using Lemma 2.2, it is easy to get

$$\mathcal{E}[\xi,\tau v](t)+\int_0^t\mathcal{E}[v](s)\,\mathrm{d}s\le c_4\|(\xi_0,\mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)},\tag{3.20}$$

where $c_4=\frac{C_3}{c_1}$. Thus, Lemma 3.3 is proved. $\square$

The following lemma concerns the uniform bound of $\mathcal{E}[\phi](t)$. Here, the finiteness of T plays a key role in the proof.

Lemma 3.4. *For any fixed $T\in(0,+\infty),\tau\in[0,1]$, if*

$$\sup_{0\le t\le T}\mathcal{E}[\xi,\tau v,\phi,\zeta](t)\le\epsilon,$$

where $0<\epsilon\ll 1$, then there are constants $c_5>0, c_6>0$ such that for any $t\in[0,T]$,

$$\mathcal{E}[\phi](t)\le c_5\|\phi_0\|^2_{H^4(\mathbb{R}^3)}\exp\{c_6T\|(\xi_0,\mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)}\}.\tag{3.21}$$

Proof. Let $\partial_t^\ell\mathcal{D}^\alpha\phi\cdot\partial_t^\ell\mathcal{D}^\alpha(2.1)_3$, where $0\le\ell\le 2, 0\le\ell+|\alpha|\le 4$, we get

$$(|\partial_t^\ell\mathcal{D}^\alpha\phi|^2)_t=-v\cdot\nabla|\partial_t^\ell\mathcal{D}^\alpha\phi|^2-2\partial_t^\ell\mathcal{D}^\alpha\phi[\partial_t^\ell\mathcal{D}^\alpha,v\cdot\nabla]\phi.\tag{3.22}$$

Integrate (3.22) in $\mathbb{R}^3$, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\partial_t^\ell\mathcal{D}^\alpha\phi|^2\,\mathrm{d}x=\int_{\mathbb{R}^3}|\partial_t^\ell\mathcal{D}^\alpha\phi|^2\nabla\cdot v\,\mathrm{d}x-2\int_{\mathbb{R}^3}\partial_t^\ell\mathcal{D}^\alpha\phi[\partial_t^\ell\mathcal{D}^\alpha,v\cdot\nabla]\phi\,\mathrm{d}x:=I_2.\tag{3.23}$$

When $\ell+|\alpha|\le 4$, it is easy to check $I_2\lesssim\mathcal{E}[v](t)^{\frac12}\mathcal{E}[\phi](t)$ by using commutator estimates. Sum ℓ,α, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[\phi](t)\le C_4\mathcal{E}[v](t)^{\frac12}\mathcal{E}[\phi](t).\tag{3.24}$$

Integrate (3.24) in $(0,t)$, where $t \in [0,T]$, we obtain the uniform a priori estimate for $\mathcal{E}[\phi](t)$ which is independent of τ.

$$
\begin{aligned}
\mathcal{E}[\phi](t) &\le c_5\|\phi_0\|^2_{H^4(\mathbb{R}^3)} \exp\{\int_0^t C_4\mathcal{E}[v](s)^{\frac{1}{2}}\,\mathrm{d}s\} \\
&\le c_5\|\phi_0\|^2_{H^4(\mathbb{R}^3)} \exp\{C_4T\int_0^T \mathcal{E}[v](s)\,\mathrm{d}s\} \\
&\le c_5\|\phi_0\|^2_{H^4(\mathbb{R}^3)} \exp\{c_6T\|(\xi_0,\mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)}\},
\end{aligned}
\tag{3.25}
$$

where $c_5>0, c_6=C_4c_4>0$. Thus, Lemma 3.4 is proved. $\square$

Due to $\zeta = \varrho(\xi+\bar{p},\phi+\bar{S})-\bar{\varrho}$, we can estimate $\mathcal{E}[\zeta](t)$ in the following lemma:

Lemma 3.5. *For any fixed $T\in(0,+\infty), \tau\in[0,1]$, if*

$$
\sup_{0\le t\le T}\mathcal{E}[\xi,\tau v,\phi,\zeta](t)\le\epsilon,
$$

where $0<\epsilon\ll 1$, then there are constants $c_7>0, c_8>0$ such that for any $t\in[0,T]$,

$$
\mathcal{E}[\zeta](t)\le c_7\|\phi_0\|^2_{H^4(\mathbb{R}^3)}\exp\{c_6T\|(\xi_0,\mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)}\}+c_8\|(\xi_0,\mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)}.
\tag{3.26}
$$

Proof. Since ζ is expressed in terms of ξ and ϕ, namely,

$$
\zeta=\tfrac{1}{\sqrt[\gamma]{A}}\exp\{-\tfrac{\bar{S}}{\gamma}\}\Big[(\xi+\bar{p})^{\frac{1}{\gamma}}\exp\{-\tfrac{\phi}{\gamma}\}-\bar{p}^{\frac{1}{\gamma}}\Big],
\tag{3.27}
$$

it is easy to get the tame estimate of composed functions (see [28]):

$$
\begin{aligned}
&\mathcal{E}[\zeta](t)\le C_5\mathcal{E}[\xi](t)+C_5\mathcal{E}[\phi](t)\\
&\le c_7\|\phi_0\|^2_{H^4(\mathbb{R}^3)}\exp\{c_6T\|(\xi_0,\mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)}\}+c_8\|(\xi_0,\mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)},
\end{aligned}
\tag{3.28}
$$

where $c_7=c_5C_5>0, c_8=c_4C_5>0$. Thus, Lemma 3.5 is proved. $\square$

Lemmas 3.3, 3.4, 3.5 imply that $\mathcal{E}[\xi,\tau v,\phi,\zeta](t)$ is bounded by fixed T and the initial data, i.e. $\|(\xi_0,\mathcal{U}_0)\|_{H^4(\mathbb{R}^3)}, \|\phi_0\|_{H^4(\mathbb{R}^3)}$. Thus, as long as ϵ_0, which is the bound of the initial data and depends on T, is chosen to be small enough, the a priori assumption $\sup\limits_{0\le t\le T}\mathcal{E}[\xi,\tau v,\phi,\zeta](t)\le\epsilon$ will be valid.

4. Relaxation Limit of the Relaxing Equations

In this section, we study the relaxation limit of the relaxing equaitons (2.1).

Before we prove the relaxation limit of the relaxing equations (2.1), we need to prove the uniqueness of the relaxed equations (2.2), which is necessary for the proof of the relaxation limit (namely, Theorem 4.3).

Lemma 4.1. *Assume* $(\xi^1, \phi^1) \in C^1(\mathbb{R}^3 \times [0,T])$ *and* $(\xi^2, \phi^2) \in C^1(\mathbb{R}^3 \times [0,T])$ *are two solutions of the relaxed equations* (2.2) *with the same data* $(\lim\limits_{\tau\to 0} p_0(x,\tau) - \bar{p}, \lim\limits_{\tau\to 0} S_0(x,\tau) - \bar{S})$, *then* $\xi_1 = \xi_2,\ \phi_1 = \phi_2$.

Proof. Set

$$
\begin{aligned}
&\hat{\xi} = \xi^1 - \xi^2, \quad \hat{v} = v^1 - v^2, \quad \hat{\phi} = \phi^1 - \phi^2, \quad \hat{\zeta} = \zeta^1 - \zeta^2,\\
&\hat{\hat{\xi}} = \tfrac{\xi^1+\xi^2}{2}, \qquad \hat{\hat{v}} = \tfrac{v^1+v^2}{2}, \qquad \hat{\hat{\phi}} = \tfrac{\phi^1+\phi^2}{2}, \qquad \hat{\hat{\varrho}} = \tfrac{\varrho^1+\varrho^2}{2},
\end{aligned}
$$

then $(\hat{\xi}, \hat{v}, \hat{\phi}, \hat{\zeta})$ satisfy the following equations:

$$
\begin{cases}
\hat{\xi}_t + k_2 \nabla\cdot\hat{v} = -\gamma k_1 \hat{\hat{\xi}} \nabla\cdot\hat{v} - \gamma k_1 \hat{\xi}\nabla\cdot\hat{\hat{v}} - k_1 \hat{\hat{v}}\cdot\nabla\hat{\xi} - k_1 \hat{v}\cdot\nabla\hat{\hat{\xi}},\\
k_2\nabla\hat{\xi} + \hat{v} = \frac{1}{k_1}\left(\frac{\zeta^1}{2\varrho^1\bar{\varrho}} + \frac{\zeta^2}{2\varrho^2\bar{\varrho}}\right)\nabla\hat{\xi} + \frac{\hat{\zeta}}{k_1\varrho^1\varrho^2}\nabla\hat{\hat{\xi}},\\
\hat{\phi}_t = -k_1\hat{\hat{v}}\cdot\nabla\hat{\phi} - k_1\hat{v}\cdot\nabla\hat{\hat{\phi}},\\
(\hat{\xi},\hat{\phi})(x,0) = (0,0).
\end{cases}
\tag{4.1}
$$

Let $\hat{\xi}\cdot(4.1)_1 + \hat{v}\cdot(4.1)_2$, integrate in $\mathbb{R}^3$, note that $\int\limits_{\mathbb{R}^3} \nabla\cdot(\hat{\xi}\hat{v})\,\mathrm{d}x = 0$, $\sup\limits_{0\le t\le T} \mathcal{E}[\hat{\xi}, \tau\hat{v}, \hat{\phi}, \hat{\zeta}](t) \le 4\epsilon$, $\sup\limits_{0\le t\le T} \mathcal{E}[\hat{\hat{\xi}}, \tau\hat{\hat{v}}, \hat{\hat{\phi}}](t) \le \epsilon$, then we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} |\hat{\xi}|^2\,\mathrm{d}x + 2\int\limits_{\mathbb{R}^3} |\hat{v}|^2\,\mathrm{d}x\\
&= \int\limits_{\mathbb{R}^3} 2\gamma k_1 \hat{\hat{\xi}}\hat{v}\cdot\nabla\hat{\xi} + 2(\gamma-1)k_1\hat{\xi}\hat{v}\cdot\nabla\hat{\hat{\xi}} - (2\gamma-1)k_1\hat{\xi}\hat{\xi}\nabla\cdot\hat{\hat{v}}\\
&\quad + \frac{1}{k_1}\left(\frac{\zeta^1}{\varrho^1\bar{\varrho}} + \frac{\zeta^2}{\varrho^2\bar{\varrho}}\right)\hat{v}\cdot\nabla\hat{\xi} + \frac{2\hat{\zeta}}{k_1\varrho^1\varrho^2}\hat{v}\cdot\nabla\hat{\hat{\xi}}\,\mathrm{d}x\\
&\lesssim \sqrt{\epsilon}\big(\|\hat{v}\|_{L^2(\mathbb{R}^3)}\|\nabla\hat{\xi}\|_{L^2(\mathbb{R}^3)} + \|\hat{v}\|_{L^2(\mathbb{R}^3)}\|\hat{\xi}\|_{L^2(\mathbb{R}^3)} + \|\hat{\xi}\|^2_{L^2(\mathbb{R}^3)}\\
&\quad + \|\hat{v}\|_{L^2(\mathbb{R}^3)}\|\hat{\zeta}\|_{L^2(\mathbb{R}^3)}\big).
\end{aligned}
\tag{4.2}
$$

Plug $\nabla\hat{\xi} = -k_1\hat{\hat{\varrho}}\hat{v} - k_1\hat{\zeta}\hat{\hat{v}}$ into (4.2), apply Young's inequality, then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} |\hat{\xi}|^2\,\mathrm{d}x + 2\int\limits_{\mathbb{R}^3} |\hat{v}|^2\,\mathrm{d}x \leq \|\hat{v}\|^2_{L^2(\mathbb{R}^3)} + C\sqrt{\epsilon}(\|\hat{\xi}\|^2_{L^2(\mathbb{R}^3)} + \|\hat{\zeta}\|^2_{L^2(\mathbb{R}^3)}). \tag{4.3}$$

Let $\hat{\phi}\cdot(4.1)_3$, we get $(|\hat{\phi}|^2)_t = -2k_1\hat{\phi}\hat{\hat{v}}\cdot\nabla\hat{\phi} - 2k_1\hat{\phi}\hat{v}\cdot\nabla\hat{\hat{\phi}}$. Integrate in $\mathbb{R}^3$ and apply Young's inequality, then we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} |\hat{\phi}|^2\,\mathrm{d}x &= \int\limits_{\mathbb{R}^3} k_1|\hat{\phi}|^2\nabla\cdot\hat{\hat{v}} - 2k_1\hat{\phi}\hat{v}\cdot\nabla\hat{\hat{\phi}}\,\mathrm{d}x \\ &\leq \|\hat{v}\|^2_{L^2(\mathbb{R}^3)} + C\sqrt{\epsilon}\|\hat{\phi}\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{4.4}$$

Sum (4.3) and (4.4), note that $\|\hat{\zeta}\|_{L^2(\mathbb{R}^3)} \lesssim \|\hat{\xi}\|_{L^2(\mathbb{R}^3)} + \|\hat{\phi}\|_{L^2(\mathbb{R}^3)}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int\limits_{\mathbb{R}^3} |\hat{\xi}|^2\,\mathrm{d}x + \int\limits_{\mathbb{R}^3} |\hat{\phi}|^2\,\mathrm{d}x\Big) \leq C_6\sqrt{\epsilon}(\|\hat{\xi}\|^2_{L^2(\mathbb{R}^3)} + \|\hat{\phi}\|^2_{L^2(\mathbb{R}^3)}), \tag{4.5}$$

for some constant $C_6 > 0$. Then

$$\|\hat{\xi}\|^2_{L^2(\mathbb{R}^3)} + \|\hat{\phi}\|^2_{L^2(\mathbb{R}^3)} \leq (\|\hat{\xi}|_{t=0}\|^2_{L^2(\mathbb{R}^3)} + \|\hat{\phi}_{t=0}\|^2_{L^2(\mathbb{R}^3)})\exp\{C_6\sqrt{\epsilon}T\} = 0. \tag{4.6}$$

Since $\xi_1, \phi_1, \xi_2, \phi_2 \in C^1(\mathbb{R}^3\times[0,T])$, we have $\xi_1 = \xi_2$, $\phi_1 = \phi_2$. Thus, Lemma 4.1 is proved. $\square$

Remark 4.2. *(i) In the compactness argument of the following Theorem* 4.3*, the solutions of the relaxing equations* (2.1) *have enough uniform regularities to guarantee the limits* $\xi, \phi \in C^1(\mathbb{R}^3\times[0,T])$.

(ii) For the relaxed system (2.2)*,* ζ, v *are functions of* ξ, ϕ*, thus* $\zeta \in C^1(\mathbb{R}^3\times[0,T])$, $v \in C^0(\mathbb{R}^3\times[0,T])$.

In the finite time interval $[0,T]$, the bounds of $\mathcal{E}[\xi](t), \mathcal{E}[\tau v](t), \mathcal{E}[\phi](t), \mathcal{E}[\zeta](t)$ and $\int\limits_0^T \mathcal{E}[v](s)\,\mathrm{d}s$ are uniform with respect to τ, thus we have enough compactness to pass to the limits in the relaxing equations (2.1). The following theorem states the relaxation limit of the relaxing Cauchy problem (2.1).

Theorem 4.3. *Suppose the conditions are the same with those of Theorem* 1.1*, then the problem* (2.1) *admits a unique solution* (ξ, v, ϕ, ζ) *in* $[0,T]$ *satisfying*

$$\sup_{0\leq t\leq T}\mathcal{E}[\xi, \tau v, \phi, \zeta](t) + \int\limits_0^T \mathcal{E}[v](t)\,\mathrm{d}t \leq C(T, \epsilon_0), \tag{4.7}$$

such that as $\tau \to 0$*,*

$$
\begin{aligned}
&(\xi, \phi, \zeta) \to (\tilde{\xi}, \tilde{\phi}, \tilde{\zeta}) \ \textit{in}\ C([0,T], C^{2+\mu_1}(\mathcal{K}) \cap W^{3,\mu_2}(\mathcal{K})),\\
&v \rightharpoonup \tilde{v} \quad \textit{in} \bigcap_{0\le \ell \le 2} H^{\ell}([0,T], H^{4-\ell}(\mathbb{R}^3)),
\end{aligned}
\tag{4.8}
$$

where $\mu_1 \in [0, \frac{1}{2}), \mu_2 \in [2, 6)$*,* $\mathcal{K}$ *denotes any compact subset of* $\mathbb{R}^3$*,* $(\tilde{\xi}, \tilde{v}, \tilde{\phi}, \tilde{\zeta})$ *is the unique classical solution to the relaxed equations* (2.2)*.*

Proof. By Lemmas 3.3, 3.4, 3.5, we have the uniform bound (4.7) and the bound of the solution of (1.2), i.e., $(\hat{p}, \hat{u}, \hat{S}, \hat{\varrho})(x, t')$ satisfies

$$
\begin{aligned}
&\sup_{0\le t' \le T/\tau} \mathcal{E}[\hat{p}-\bar{p}, \hat{u}, \hat{S}-\bar{S}, \hat{\varrho}-\bar{\varrho}](t') + \tfrac{1}{\tau}\int_0^{T/\tau} \mathcal{E}[\hat{u}](t')\,\mathrm{d}t'\\
&= \sup_{0\le t\le T} \mathcal{E}[\xi, k_1\tau v, \phi, \zeta](t) + k_1^2 \int_0^T \mathcal{E}[v](t)\,\mathrm{d}t\\
&\lesssim \|(\xi_0, \mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)} + \|\phi_0\|^2_{H^4(\mathbb{R}^3)} \exp\{CT\|(\xi_0, \mathcal{U}_0)\|^2_{H^4(\mathbb{R}^3)}\}.
\end{aligned}
\tag{4.9}
$$

When the initial data are sufficiently small, we have the global existence of classical solutions to non-isentropic Euler equations with damping (1.2) (see [20]), then we get classical solutions to (1.2) in the time interval $[0, T/\tau]$ for any $\tau > 0$, and then we can construct the unique classical solution of the relaxing equations (1.1) in $[0, T]$ via the time rescaling (1.3).

By Aubin's Lemma (see [29]), we get the compact embedding:

$$
L^\infty([0,T], H^4(\mathbb{R}^3)) \cap H^1([0,T], H^3(\mathbb{R}^3)) \hookrightarrow\hookrightarrow C([0,T], C^{2+\mu_1}(\mathcal{K}) \cap W^{3,\mu_2}(\mathcal{K})),
\tag{4.10}
$$

where $\mu_1 \in [0, \frac{1}{2}), \mu_2 \in [2, 6)$, $\mathcal{K}$ is any compact subset of $\mathbb{R}^3$. Apply (4.10) to (ξ, ϕ, ζ), (4.7) implies $\xi, \phi, \zeta \in \bigcap_{0\le\ell\le 2} H^{\ell}([0,T], H^{4-\ell}(\mathbb{R}^3))$, then we have a subsequence which satisfies (4.8). In fact, (4.8) is the whole sequence convergence, due to the uniqueness of the relaxed equations (2.2) proved in Lemma 4.1.

Next, we have enough compactness to pass to the limits in the relaxing equations (2.1), then it is standard to prove that the strong limits of (ξ, ϕ, ζ), namely $(\tilde{\xi}, \tilde{\phi}, \tilde{\zeta})$, and the weak limit of v, namely $\tilde{\tilde{v}}$, satisfy the relaxed equations

(2.2) in the distributional sense:

$$
\begin{cases}
\int\limits_0^t \int\limits_{\mathbb{R}^3} \partial_t \varphi_1 \tilde{\xi} + \gamma k_1 \tilde{\tilde{v}} \cdot (\tilde{\xi} \nabla \varphi_1 + \varphi_1 \nabla \tilde{\xi}) + \gamma k_1 \bar{p} \tilde{\tilde{v}} \cdot \nabla \varphi_1 - k_1 \varphi_1 \tilde{\tilde{v}} \cdot \nabla \tilde{\xi} \, \mathrm{d}x \mathrm{d}s \\
\qquad\qquad\qquad\qquad\qquad\qquad = - \int\limits_{\mathbb{R}^3} \lim\limits_{\tau \to 0} \xi_0(x, \tau) \varphi_1(\cdot, 0) \, \mathrm{d}x, \\
\int\limits_0^t \int\limits_{\mathbb{R}^3} \tilde{\tilde{v}} \cdot \vec{\varphi}_2 + \frac{1}{k_1 \bar{\varrho}} \nabla \tilde{\xi} \cdot \vec{\varphi}_2 \, \mathrm{d}x \mathrm{d}s = 0, \\
\int\limits_0^t \int\limits_{\mathbb{R}^3} \tilde{\phi} \partial_t \varphi_3 - k_1 \tilde{\tilde{v}} \cdot \nabla \tilde{\phi} \varphi_3 \, \mathrm{d}x \mathrm{d}s \ = - \int\limits_{\mathbb{R}^3} \lim\limits_{\tau \to 0} \phi_0(x, \tau) \varphi_3(\cdot, 0) \, \mathrm{d}x,
\end{cases}
\tag{4.11}
$$

where $\varphi_1, \vec{\varphi}_2, \varphi_3 \in C_0^\infty(\mathbb{R}^3 \times [0, T))$. The proof is very standard, so the details of passing to the limits are omitted here. Moreover, $\tilde{p} = \bar{p} + \tilde{\xi} = A(\tilde{\zeta} + \bar{\varrho})^\gamma \exp\{\bar{S} + \tilde{\phi}\} = A \tilde{\varrho}^\gamma e^{\tilde{S}}$ holds pointwisely due to the strong convergence of (ξ, ϕ, ζ). Therefore, $(\tilde{\xi}, \tilde{\tilde{v}}, \tilde{\phi}, \tilde{\zeta})$ is a distributional solution to the relaxed equations (2.2).

Since C^1-regularity is the local in spacetime property, it is easy to know that $(\tilde{\xi}, \tilde{\phi}) \in C^1(\mathbb{R}^3 \times [0, T])$. By Lemma 4.1, $(\tilde{\xi}, \tilde{\phi})$ is the unique classical solution to the relaxed equations (2.2). We define $\tilde{v} = -\frac{1}{k_1 \bar{\varrho}} \nabla \tilde{\xi}$, then $(\tilde{\xi}, \tilde{v}, \tilde{\phi}, \tilde{\zeta})$ satisfy (2.2) in the classical sense.

However, $\tilde{v}$ and $\tilde{\tilde{v}}$ satisfy $(4.11)_2$, they differ up to at most zero measure set (a subset of $\mathbb{R}^3 \times \{t = 0\}$ for the ill-prepared data and the empty set for the well-prepared data). So $\tilde{v}$ is also the weak limit of v, and $\tilde{\tilde{v}}$ in (4.11) can be replaced by $\tilde{v}$. Thus, Theorem 4.3 is proved. □

5. Strong Convergence Rates of the Pressure, Entropy and Density

In this section, we estimate the strong convergence rates of the pressure, entropy and density, for which there is no initial layer.

Letting $\tau = 0$ in the proofs of a priori estimates for the relaxing equations (2.1), we can similarly get the regularities for the relaxed equations (2.2):

$$
\begin{aligned}
&\partial_t^\ell \tilde{\xi}, \ \partial_t^\ell \tilde{\phi}, \ \partial_t^\ell \tilde{\zeta} \in L^\infty([0, T], H^{4-\ell}(\mathbb{R}^3)), 0 \le \ell \le 2, \\
&\tilde{v} \in \bigcap_{0 \le \ell \le 2} H^\ell([0, T], H^{4-\ell}(\mathbb{R}^3)).
\end{aligned}
\tag{5.1}
$$

Note that we do not have $\partial_t^\ell \tilde{v} \in L^\infty([0, T], H^{4-\ell}(\mathbb{R}^3)), 0 \le \ell \le 2$. Thus, the solution to (2.2) has enough regularities to estimate the convergence rates.

In order to estimate the convergence rates of the pressure, entropy and density, we need to estimate the $L^\infty([0,T],H^2(\mathbb{R}^3))\cap H^1([0,T],H^1(\mathbb{R}^3))$ norm of the differences between the variables of the relaxing equations and the variables of the relaxed equations. The results are stated in the following theorem:

Theorem 5.1. *Let (ξ,v,ϕ,ζ) and $(\tilde{\xi},\tilde{v},\tilde{\phi},\tilde{\zeta})$ are the solutions obtained in Theorem 4.3. Assume $\|(p_0(x,\tau)-\lim\limits_{\tau\to0}p_0(x,\tau),S_0(x,\tau)-\lim\limits_{\tau\to0}S_0(x,\tau))\|_{H^3(\mathbb{R}^3)}\le O(\tau^{\alpha_1})$, then as $\tau\to0$,*

$$\|(\xi-\tilde{\xi},\phi-\tilde{\phi},\zeta-\tilde{\zeta})\|_{C([0,T],C^{1+\mu_1}(\mathbb{R}^3)\cap W^{2,\mu_2}(\mathbb{R}^3))}\le O(\tau^{\min\{1,\alpha_1\}}),\tag{5.2}$$

where $\mu_1\in[0,\frac{1}{2}],\mu_2\in[2,6]$.

Proof. (ξ,v,ϕ,ζ) is the unique classical solution of (2.1), and $(\tilde{\xi},\tilde{v},\tilde{\phi},\tilde{\zeta})$ is the unique classical solution of (2.2). Denote $\tilde{\rho}=\bar{\varrho}+\tilde{\zeta}$, set

$$\begin{aligned}&\hat{\xi}=\xi-\tilde{\xi},\quad \hat{v}=v-\tilde{v},\quad \hat{\phi}=\phi-\tilde{\phi},\quad \hat{\zeta}=\varrho-\tilde{\varrho},\\ &\hat{\hat{\xi}}=\tfrac{\xi+\tilde{\xi}}{2},\qquad \hat{\hat{v}}=\tfrac{v+\tilde{v}}{2},\qquad \hat{\hat{\phi}}=\tfrac{\phi+\tilde{\phi}}{2},\qquad \hat{\hat{\varrho}}=\tfrac{\varrho+\tilde{\rho}}{2},\end{aligned}$$

then $(\hat{\xi},\hat{v},\hat{\phi},\hat{\zeta})$ satisfy the following equations:

$$\begin{cases}\hat{\xi}_t+k_2\nabla\cdot\hat{v}=-\gamma k_1\hat{\hat{\xi}}\nabla\cdot\hat{v}-\gamma k_1\hat{\xi}\nabla\cdot\hat{\hat{v}}-k_1\hat{\hat{v}}\cdot\nabla\hat{\xi}-k_1\hat{v}\cdot\nabla\hat{\hat{\xi}},\\ k_2\nabla\hat{\xi}+\hat{v}=\frac{1}{k_1}(\frac{\zeta}{2\varrho\bar{\varrho}}+\frac{\tilde{\zeta}}{2\tilde{\rho}\bar{\varrho}})\nabla\hat{\xi}+\frac{\hat{\zeta}}{k_1\varrho\tilde{\rho}}\nabla\hat{\hat{\xi}}-\tau^2v_t-k_1\tau^2v\cdot\nabla v,\\ \hat{\phi}_t=-k_1\hat{\hat{v}}\cdot\nabla\hat{\phi}-k_1\hat{v}\cdot\nabla\hat{\hat{\phi}},\\ (\hat{\xi},\hat{\phi})(x,0)=(p_0(x,\tau)-\lim\limits_{\tau\to0}p_0(x,\tau),S_0(x,\tau)-\lim\limits_{\tau\to0}S_0(x,\tau)).\end{cases}\tag{5.3}$$

Let $(5.3)_1\cdot\hat{\xi}+(5.3)_2\cdot\hat{v}$, integrate in $\mathbb{R}^3$, note that $\int\limits_{\mathbb{R}^3}\nabla\cdot(\hat{\xi}\hat{v})\,\mathrm{d}x=0$, $\sup\limits_{0\le t\le T}\mathcal{E}[\hat{\xi},\tau\hat{v},\hat{\phi},\hat{\zeta}](t)\le4\epsilon$, $\sup\limits_{0\le t\le T}\mathcal{E}[\hat{\hat{\xi}},\tau\hat{\hat{v}},\hat{\hat{\phi}}](t)\le\epsilon$, then we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\hat{\xi}|^2\,\mathrm{d}x+2\int\limits_{\mathbb{R}^3}|\hat{v}|^2\,\mathrm{d}x\\ &=\int\limits_{\mathbb{R}^3}2\gamma k_1\hat{\hat{\xi}}\hat{v}\cdot\nabla\hat{\xi}+(2\gamma-2)k_1\hat{\xi}\hat{v}\cdot\nabla\hat{\hat{\xi}}-(2\gamma k_1-1)\hat{\xi}\hat{\xi}\nabla\cdot\hat{\hat{v}}\\ &\quad+\frac{1}{k_1}(\frac{\zeta}{\varrho\bar{\varrho}}+\frac{\tilde{\zeta}}{\tilde{\rho}\bar{\varrho}})\nabla\hat{\xi}\cdot\hat{v}+\frac{2\hat{\zeta}}{k_1\varrho\tilde{\rho}}\nabla\hat{\hat{\xi}}\cdot\hat{v}-2\tau^2v_t\cdot\hat{v}-2k_1\tau^2v\cdot\nabla v\cdot\hat{v}\,\mathrm{d}x.\end{aligned}\tag{5.4}$$

Plug $\nabla\hat{\xi} = -k_1\hat{\hat{\varrho}}\hat{v} - k_1\hat{\zeta}\hat{v} - \tau^2 v_t - k_1\tau^2 v\cdot\nabla v$ into (5.4), apply Young's inequality, then we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\hat{\xi}|^2\,\mathrm{d}x + 2\int\limits_{\mathbb{R}^3}|\hat{v}|^2\,\mathrm{d}x \le \|\hat{v}\|^2_{L^2(\mathbb{R}^3)} + C\sqrt{\epsilon}(\|\hat{\xi}\|^2_{L^2(\mathbb{R}^3)} + \|\hat{\zeta}\|^2_{L^2(\mathbb{R}^3)})\\ +C\tau^2(1+\sqrt{\epsilon})\mathcal{E}[v](t).\end{aligned} \tag{5.5}$$

Let $\hat{\phi}\cdot(5.3)_3$, integrate in $\mathbb{R}^3$, apply Young's inequality, then we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3}|\hat{\phi}|^2\,\mathrm{d}x &= \int\limits_{\mathbb{R}^3}k_1|\hat{\phi}|^2\nabla\cdot\hat{v} - 2k_1\hat{\phi}\hat{v}\cdot\nabla\hat{\hat{\phi}}\,\mathrm{d}x\\ &\le \|\hat{v}\|^2_{L^2(\mathbb{R}^3)} + C\sqrt{\epsilon}\|\hat{\phi}\|^2_{L^2(\mathbb{R}^3)}.\end{aligned} \tag{5.6}$$

Sum (5.5) and (5.6), note that $\|\hat{\zeta}\|_{L^2(\mathbb{R}^3)} \lesssim \|\hat{\xi}\|_{L^2(\mathbb{R}^3)} + \|\hat{\phi}\|_{L^2(\mathbb{R}^3)}$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\big(\int\limits_{\mathbb{R}^3}|\hat{\xi}|^2\,\mathrm{d}x + \int\limits_{\mathbb{R}^3}|\hat{\phi}|^2\,\mathrm{d}x\big)\\ &\le C_7\sqrt{\epsilon}(\|\hat{\xi}\|^2_{L^2(\mathbb{R}^3)} + \|\hat{\phi}\|^2_{L^2(\mathbb{R}^3)}) + \tau^2C_7(1+\sqrt{\epsilon})\mathcal{E}[v](t),\end{aligned} \tag{5.7}$$

for some constant $C_7 > 0$. Then

$$\begin{aligned}\|\hat{\xi}\|^2_{L^2(\mathbb{R}^3)} + \|\hat{\phi}\|^2_{L^2(\mathbb{R}^3)} &\le (\|\hat{\xi}|_{t=0}\|^2_{L^2(\mathbb{R}^3)} + \|\hat{\phi}_{t=0}\|^2_{L^2(\mathbb{R}^3)})\exp\{C_7\sqrt{\epsilon}t\}\\ &\quad+\tau^2C_7(1+\sqrt{\epsilon})\int\limits_0^t\exp\{C_7\sqrt{\epsilon}(t-s)\}\mathcal{E}[v](s)\,\mathrm{d}s\\ &\lesssim \tau^2 + \tau^{2\alpha_1}.\end{aligned} \tag{5.8}$$

In order to obtain the higher order a priori estimates, we need to define another energy functional:

$$\mathcal{F}[\xi](t) := \sum_{0\le\ell\le1, 1\le\ell+|\alpha|\le3}\|\partial_t^\ell\mathcal{D}^\alpha\xi(t)\|^2_{L^2(\mathbb{R}^3)}.$$

Let $\partial_t^\ell\mathcal{D}^\alpha\hat{\xi}\cdot\partial_t^\ell\mathcal{D}^\alpha(5.3)_1 + \partial_t^\ell\mathcal{D}^\alpha\hat{v}\cdot\partial_t^\ell\mathcal{D}^\alpha(5.3)_2$, where $0\le\ell\le1, 1\le\ell+|\alpha|\le3$, integrate in $\mathbb{R}^3$, plug $\nabla\hat{\xi} = -k_1\hat{\hat{\varrho}}\hat{v} - k_1\hat{\zeta}\hat{\hat{v}} - \tau^2 v_t - k_1\tau^2 v\cdot\nabla v$ into the right hand of the equation, apply Young's inequality, sum ℓ and α, then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\hat{\xi}](t) + 2\mathcal{F}[\hat{v}](t) \le \mathcal{F}[\hat{v}](t) + C\sqrt{\epsilon}(\mathcal{F}[\hat{\xi}](t) + \mathcal{F}[\hat{\zeta}](t)) + C\tau^2(1+\sqrt{\epsilon})\mathcal{E}[v](t). \tag{5.9}$$

Let $\partial_t^\ell\mathcal{D}^\alpha\hat{\phi}\cdot\partial_t^\ell\mathcal{D}^\alpha(5.3)_3$, integrate in $\mathbb{R}^3$, apply Young's inequality, sum ℓ and α, then we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[\hat{\phi}](t) \le \mathcal{F}[\hat{v}](t) + C\sqrt{\epsilon}\mathcal{F}[\hat{\phi}](t). \tag{5.10}$$

Sum (5.9) and (5.10), note that $\mathcal{F}[\hat{\zeta}](t) \lesssim \mathcal{F}[\hat{\xi}](t) + \mathcal{F}[\hat{\phi}](t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{F}[\hat{\xi}](t) + \mathcal{F}[\hat{\phi}](t)\big) \leq C_8\sqrt{\epsilon}\big(\mathcal{F}[\hat{\xi}](t) + \mathcal{F}[\hat{\phi}](t)\big) + \tau^2 C_8(1+\sqrt{\epsilon})\mathcal{E}[v](t), \tag{5.11}$$

for some constant $C_8 > 0$. Then

$$\begin{aligned}\mathcal{F}[\hat{\xi}](t) + \mathcal{F}[\hat{\phi}](t) &\leq \mathcal{F}[\hat{\xi}](0) + \mathcal{F}[\hat{\phi}](0)\exp\{C_8\sqrt{\epsilon}t\} \\ &\qquad + \tau^2 C_8(1+\sqrt{\epsilon})\int_0^t \exp\{C_8\sqrt{\epsilon}(t-s)\}\mathcal{E}[v](s)\,\mathrm{d}s \\ &\lesssim \tau^2 + \tau^{2\alpha_1}.\end{aligned} \tag{5.12}$$

Due to the estimates (5.8), (5.12), the fixed $T \in (0, +\infty)$ and the following embedding:

$$L^\infty([0,T], H^2(\mathbb{R}^3)) \cap H^1([0,T], H^1(\mathbb{R}^3)) \hookrightarrow C([0,T], C^{0+\mu_1}(\mathbb{R}^3) \cap W^{1,\mu_2}(\mathbb{R}^3)),$$

we have $\|(\xi - \tilde{\xi}, \phi - \tilde{\phi}, \zeta - \tilde{\zeta})\|_{C([0,T], C^{1+\mu_1}(\mathbb{R}^3)\cap W^{2,\mu_2}(\mathbb{R}^3))} \leq O(\tau^{\min\{1,\alpha_1\}})$.

Thus, Theorem 5.1 is proved. □

However, we can not prove the strong convergence of $\hat{\xi}_t, \hat{\phi}_t, \hat{\zeta}_t$ in time interval $[0,T]$ by using the equations (5.3), because $\hat{v} = v - \tilde{v}$ in (5.3) behaves badly in the initial layer for the ill-prepared data.

6. Initial Layer and Strong Convergence of the Velocity

In this section, we prove the strong convergence of the velocity outside an initial layer for the ill-prepared data, then we prove the strong convergence of the velocity in the time interval $[0,T]$ for the well-prepared data. The convergence rate of the velocity is also estimated. Finally, we prove the thickness of the initial layer is $O(\tau^2)$.

The results about initial layer, strong convergence of the velocity and its convergence rate are stated in the following theorem:

Theorem 6.1. *Let (ξ, v, ϕ, ζ) and $(\tilde{\xi}, \tilde{v}, \tilde{\phi}, \tilde{\zeta})$ be the solutions obtained in Theorem 4.3. For the ill-prepared data, i.e.,* $\lim\limits_{\tau\to 0}\left|v_0(x,\tau) + \frac{1}{k_1\varrho_0(x,\tau)}\nabla\xi_0(x,\tau)\right|_\infty \neq 0$, *there exists an initial layer* $[0,t^*]$ *with* $t^*=C\tau^{2-\delta}$ *for* v*, where* $C>0$*,* $0<\delta<2$*, such that as* $\tau\to 0$*,* $|v(x,t^*)-\tilde{v}(x,0)|_\infty\to 0$ *and*

$$\|v-\tilde{v}\|_{C([t^*,T],\ C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))}\le O(\tau^{\min\{1,\alpha_1\}}),\mu_1\in[0,\tfrac{1}{2}],\mu_2\in[2,6].$$

If $\delta=0$*, for any constant* $C\in(0,+\infty)$*,* $v(x,C\tau^2)$ *does not converge to* $\tilde{v}(x,0)$*.*

For the well-prepared data, i.e., $\lim\limits_{\tau\to 0}\left\|v_0(x,\tau)+\frac{1}{k_1\varrho_0(x,\tau)}\nabla\xi_0(x,\tau)\right\|_{H^2(\mathbb{R}^3)}=0$*, assuming* $\left\|v_0(x,\tau)+\frac{1}{k_1\varrho_0(x,\tau)}\nabla\xi_0(x,\tau)\right\|_{H^2(\mathbb{R}^3)}\le O(\tau^{\alpha_2})$*, as* $\tau\to 0$*,*

$$\|v-\tilde{v}\|_{C([0,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))}\le O(\tau^{\min\{1,\alpha_1,\alpha_2\}}),\mu_1\in[0,\tfrac{1}{2}],\mu_2\in[2,6].$$

Proof. Let $\mathcal{D}^\alpha\eta\cdot\mathcal{D}^\alpha(2.7)$, where $|\alpha|\le 2$, we have

$$\begin{aligned}(|\mathcal{D}^\alpha\eta|^2)_t+\tfrac{2}{\tau^2}|\mathcal{D}^\alpha\eta|^2&=2\mathcal{D}^\alpha\eta\cdot\mathcal{D}^\alpha[-\tfrac{1}{\varrho}(\nabla v)\nabla\xi-\tfrac{\gamma-1}{\varrho}\nabla\xi\nabla\cdot v-\tfrac{\gamma p}{\varrho}\nabla(\nabla\cdot v)]\\&\quad-2k_1\textstyle\sum\mathcal{D}^{\alpha_1}v\cdot\nabla\mathcal{D}^{\alpha_2}\eta\cdot\mathcal{D}^\alpha\eta.\end{aligned}\tag{6.1}$$

After integrating (6.1) in $\mathbb{R}^3$, we have

$$\begin{aligned}&\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x+\tfrac{2}{\tau^2}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x\\
&=2\int_{\mathbb{R}^3}\mathcal{D}^\alpha\eta\cdot\mathcal{D}^\alpha[-\tfrac{1}{\varrho}(\nabla v)\nabla\xi-\tfrac{\gamma-1}{\varrho}\nabla\xi\nabla\cdot v-\tfrac{\gamma p}{\varrho}\nabla(\nabla\cdot v)]\,\mathrm{d}x\\
&\quad-k_1\int_{\mathbb{R}^3}v\cdot\nabla|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x-2k_1\sum_{\alpha_1>0}\int_{\mathbb{R}^3}\mathcal{D}^{\alpha_1}v\cdot\nabla\mathcal{D}^{\alpha_2}\eta\cdot\mathcal{D}^\alpha\eta\,\mathrm{d}x\\
&\le\tfrac{1}{2\tau^2}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x+2\tau^2\|\mathcal{D}^\alpha[-\tfrac{1}{\varrho}(\nabla v)\nabla\xi-\tfrac{\gamma-1}{\varrho}\nabla\xi\nabla\cdot v-\tfrac{\gamma p}{\varrho}\nabla(\nabla\cdot v)]\|^2_{L^2(\mathbb{R}^3)}\\
&\quad+k_1\int_{\mathbb{R}^3}\nabla\cdot v|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x-2k_1\sum_{\alpha_1>0}\int_{\mathbb{R}^3}\mathcal{D}^{\alpha_1}v\cdot\nabla\mathcal{D}^{\alpha_2}\eta\cdot\mathcal{D}^\alpha\eta\,\mathrm{d}x\\
&\le\tfrac{1}{2\tau^2}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x+\gamma\|\mathcal{D}^\alpha(\tfrac{1}{\varrho})\|^6_{L^6(\mathbb{R}^3)}+\|\tau\mathcal{D}^\alpha(\nabla v)\|^6_{L^6(\mathbb{R}^3)}+\gamma\|\mathcal{D}^\alpha\nabla\xi\|^6_{L^6(\mathbb{R}^3)}\\
&\quad+(\gamma-1)\|\tau\mathcal{D}^\alpha(\nabla\cdot v)\|^6_{L^6(\mathbb{R}^3)}]+C|\tau\nabla(\nabla\cdot v)|_\infty\|\nabla(\tfrac{p}{\varrho})\|^2_{H^1(\mathbb{R}^3)}\\
&\quad+C|\tfrac{p}{\varrho}|_\infty\|\tau\mathcal{D}^\alpha\nabla(\nabla\cdot v)\|^2_{L^2(\mathbb{R}^3)}+\tfrac{1}{4\tau^2}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x\cdot 4k_1\tau|\tau\nabla\cdot v|_\infty\\
&\quad+2k_1\sum_{\alpha_1>0}4\tau|\tau\mathcal{D}^{\alpha_1}v|_\infty\tfrac{1}{4\tau^2}\int_{\mathbb{R}^3}|\nabla\mathcal{D}^{\alpha_2}\eta|^2\,\mathrm{d}x\\
&\le\tfrac{1}{\tau^2}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x+C_9\mathcal{E}[\xi,\tau v,\phi,\zeta](t),\end{aligned}\tag{6.2}$$

where $0 < \tau < \min\{1, \tau_0\}$ is small enough such that

$$4k_1\tau|\tau\nabla\cdot v|_\infty + 8k_1\sum_{\alpha_1>0}\tau|\tau\mathcal{D}^{\alpha_1}v|_\infty \le C\tau\mathcal{E}[\tau v](t)^{\frac{1}{2}} \le C\tau_0\sqrt{\epsilon} \le 1.$$

Thus, it follows from (6.2) that

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x + \tfrac{1}{\tau^2}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x \le C_9\mathcal{E}[\xi,\tau v,\phi,\zeta](t),\\
&\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\exp\{\tfrac{t}{\tau^2}\}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x\big) \le C_9\exp\{\tfrac{t}{\tau^2}\}\mathcal{E}[\xi,\tau v,\phi,\zeta](t).
\end{aligned}\tag{6.3}$$

After integrating from 0 to t, we get

$$\begin{aligned}
\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x &\le \exp\{-\tfrac{t}{\tau^2}\}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta_0|^2\,\mathrm{d}x + C_9\int_0^t\exp\{-\tfrac{t-s}{\tau^2}\}\mathcal{E}[\xi,\tau v,\phi,\zeta](s)\,\mathrm{d}s\\
&\le \exp\{-\tfrac{t}{\tau^2}\}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta_0|^2\,\mathrm{d}x + C_9\epsilon\int_0^t\exp\{-\tfrac{t-s}{\tau^2}\}\,\mathrm{d}s\\
&\le \exp\{-\tfrac{t}{\tau^2}\}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta_0|^2\,\mathrm{d}x + C_9\epsilon\tau^2.
\end{aligned}\tag{6.4}$$

When $t \ge t^* = C\tau^{2-\delta}$, $\exp\{-\frac{t}{\tau^2}\} \le \exp\{-\frac{t^*}{\tau^2}\} = \exp\{-\frac{C}{\tau^\delta}\} \lesssim \tau^2$, we have

$$\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta(t)|^2\,\mathrm{d}x \le \exp\{-\tfrac{t^*}{\tau^2}\}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta_0|^2\,\mathrm{d}x + C_9\epsilon\tau^2 \lesssim \tau^2. \tag{6.5}$$

Sum α, then we get

$$\|\eta\|^2_{L^\infty([t^*,T],H^2(\mathbb{R}^3))} \lesssim \sum_{|\alpha|\le 2}\sup_{t\in[t^*,T]}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x \lesssim \tau^2. \tag{6.6}$$

It follows from (2.8) that for any $t \in [0,T]$,

$$\begin{aligned}
\int_0^T\|\eta_t\|^2_{H^1(\mathbb{R}^3)}\,\mathrm{d}s &\lesssim \tau^4\int_0^T\|v_{tt}\|^2_{H^1(\mathbb{R}^3)}\,\mathrm{d}s + \tau^2|\tau v_t|^2_\infty\int_0^T\|\nabla v\|^2_{H^1(\mathbb{R}^3)}\,\mathrm{d}s\\
&\quad +\tau^2|\tau v|^2_\infty\int_0^T\|\nabla v_t\|^2_{H^1(\mathbb{R}^3)}\,\mathrm{d}s\\
&\lesssim \tau^4\int_0^T\|v_{tt}\|^2_{H^1(\mathbb{R}^3)}\,\mathrm{d}s + \tau^2\mathcal{E}[\tau v](t)\int_0^T\|\nabla v\|^2_{H^1(\mathbb{R}^3)}\,\mathrm{d}s\\
&\quad +\tau^2\mathcal{E}[\tau v](t)\int_0^T\|\nabla v_t\|^2_{H^1(\mathbb{R}^3)}\,\mathrm{d}s \lesssim \tau^2.
\end{aligned}\tag{6.7}$$

By using the following embedding:

$$L^\infty([t^*,T],H^2(\mathbb{R}^3))\cap H^1([t^*,T],H^1(\mathbb{R}^3)) \hookrightarrow C([t^*,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3)),$$

we have

$$\|\eta\|_{C([t^*,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))} \lesssim \tau. \tag{6.8}$$

Since $-\frac{1}{k_1\varrho}\nabla\xi \to -\frac{1}{k_1\tilde{\varrho}}\nabla\tilde{\xi} = \tilde{v}$ *in* $C([0,T],C^{1+\mu_1}(\mathbb{R}^3)\cap W^{2,\mu_2}(\mathbb{R}^3))$, we get the following convergence rate of the velocity outside the initial layer:

$$\begin{aligned} &\|v-\tilde{v}\|_{C([t^*,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))} \le \|\eta\|_{C([t^*,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))} \\ &\quad +\|\tfrac{1}{k_1\varrho}\nabla\xi - \tfrac{1}{k_1\tilde{\varrho}}\nabla\tilde{\xi}\|_{C([0,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))} \lesssim \tau^{\min\{1,\alpha_1\}}. \end{aligned} \tag{6.9}$$

Especially, we know the asymptotic behavior of the velocity near $t=0$, i.e., $v(x,t^*)$ converges to $\tilde{v}(x,0)$ pointwisely, as $\tau\to 0$.

While, for the initial data are well-prepared, i.e., $\|\eta(x,0)\|_{H^2(\mathbb{R}^2)} = O(\tau^{\alpha_2})$, it follows from (6.4) that

$$\begin{aligned} \int_{\mathbb{R}^3} |\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x &\le \exp\{-\tfrac{t}{\tau^2}\}\int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta_0|^2\,\mathrm{d}x + C_9\epsilon\tau^2 \\ &\le \int_{\mathbb{R}^3}|\mathcal{D}^\alpha\eta_0|^2\,\mathrm{d}x + C_9\epsilon\tau^2 \lesssim \tau^{2\alpha_2}+\tau^2. \end{aligned} \tag{6.10}$$

Sum α, we have that in $[0,T]$,

$$\|\eta\|^2_{L^\infty([0,T],H^2(\mathbb{R}^3))} \lesssim \sum_{|\alpha|\le 2}\int_{\mathbb{R}^3} |\mathcal{D}^\alpha\eta|^2\,\mathrm{d}x \lesssim \tau^{2\alpha_2}+\tau^2. \tag{6.11}$$

By (6.7) and (6.11), we have

$$\|\eta\|_{C([0,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))} \lesssim \tau^{\min\{1,\alpha_2\}}. \tag{6.12}$$

So we get the strong convergence of v in $[0,T]$ for the well-prepared data:

$$v\to\tilde{v} \ in\ C([0,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3)). \tag{6.13}$$

Similar to (6.9), we have $\|v-\tilde{v}\|_{C([0,T],C^{0+\mu_1}(\mathbb{R}^3)\cap W^{1,\mu_2}(\mathbb{R}^3))} \lesssim \tau^{\min\{1,\alpha_1,\alpha_2\}}$. Thus, Theorem 6.1 is proved. □

In Theorem 6.1, we show that $v\to\tilde{v}$ outside the initial layer $[0,t^*]$ when $t^*=C\tau^{2-\delta}$, $0<\delta<2$. While the following theorem shows that the thickness of the initial layer is $O(\tau^2)$. It suffices to prove that for any constant $C\in(0,+\infty)$, $\|\eta(x,C\tau^2)\|_{L^2(\mathbb{R}^3)}$ is away from zero, then $v(x,C\tau^2)\nrightarrow\tilde{v}(x,0)$ as $\tau\to 0$.

Theorem 6.2. *Assume the ill-prepared data satisfy* $\lim\limits_{\tau\to 0}\|\eta|_{t=0}\|_{L^2(\mathbb{R}^3)} = B > 0$*, then for any constant* $C \in (0, +\infty)$*, there exists* $\tau_1 > 0$ *such that* $\|\eta(x, C\tau^2)\|_{L^2(\mathbb{R}^3)}$ *is away from zero when* $\tau \in [0, \tau_1]$*.*

Proof. Due to the continuity of $\eta|_{t=0}$, we can simply assume $\|\eta|_{t=0}\|_{L^2(\mathbb{R}^3)} \geq \frac{B}{2}$ for $\tau \in [0, \tau_1]$. Set $z = \frac{t}{\tau^2}$, then $v(x,t) = \frac{V(x,z)}{\tau^2}$. Let $\hat{\eta}(x,z) = \eta(x,t)$, we get the equation of $\hat{\eta}$:

$$\partial_z \hat{\eta} + k_1 V \cdot \nabla\hat{\eta} + \hat{\eta} = -\tfrac{1}{\varrho}(\nabla V)\nabla\xi - \tfrac{\gamma-1}{\varrho}\nabla\xi\nabla\cdot V - \tfrac{\gamma p}{\varrho}\nabla(\nabla\cdot V). \tag{6.14}$$

Let (6.14) $\cdot\,\hat{\eta}$, integrate in $\mathbb{R}^3$, apply Young's inequality, then we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\int\limits_{\mathbb{R}^3}|\hat{\eta}|^2\,\mathrm{d}x + 2\int\limits_{\mathbb{R}^3}|\hat{\eta}|^2\,\mathrm{d}x \geq -\int\limits_{\mathbb{R}^3}|\hat{\eta}|^2\,\mathrm{d}x - \tau^2 C_{10}\mathcal{E}[\xi, \frac{V(x,z)}{\tau}, \phi, \zeta](t), \tag{6.15}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}z}\big(\exp\{3z\}\int\limits_{\mathbb{R}^3}|\hat{\eta}|^2\,\mathrm{d}x\big) \geq -\tau^2 C_{10}\epsilon\exp\{3z\}. \tag{6.16}$$

After integrating from 0 to z, we get

$$\begin{aligned}\|\hat{\eta}(x,z)\|^2_{L^2(\mathbb{R}^3)} &\geq \exp\{-3z\}\|\hat{\eta}|_{z=0}\|^2_{L^2(\mathbb{R}^3)} - \tau^2 C_{10}\epsilon\int\limits_0^z \exp\{3(s-z)\}\,\mathrm{d}s \\ &\geq \tfrac{B^2}{4}\exp\{-3z\} - \tfrac{\tau^2 C_{10}\epsilon}{3}(1-\exp\{-3z\}) \\ &\geq \tfrac{B^2}{8}\exp\{-3z\} \neq 0,\end{aligned} \tag{6.17}$$

where $z \in (0, +\infty)$ and we choose $\tau_1 = \sqrt{\frac{3B^2}{8C_{10}\epsilon(\exp\{3z\}-1)}} > 0$.

Then for any constant $C > 0$, $\|\eta(x, C\tau^2)\|_{L^2(\mathbb{R}^3)} = \|\hat{\eta}(x, C)\|_{L^2(\mathbb{R}^3)}$ is away from zero when $\tau \in [0, \tau_1]$. Thus, Theorem 6.2 is proved. □

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