GRAPH INVARIANTS FROM IDEAS IN PHYSICS AND NUMBER THEORY

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Abstract. We study free scalar field theory on a graph, which gives rise to a modified version of discrete Green's function on a graph studied in [8]. We show that this gives rise to a graph invariant, which is closely related to the 2-dimensional Weisfeiler-Lehman algorithm for graph isomorphism testing. We then consider the same theory over the integers, which leads to the consideration of certain quadratic forms over the integers as initiated in [14], associated to the graphs. The quadratic form represented by the combinatorial Laplacian respects a well-behaved *wedge sum* of graphs, and appears to capture important graph properties regarding graph embeddings into surfaces, namely the graph genus and the dual graphs.

1. Introduction

The main motivation of this paper is to try to initiate a new perspective to the study of graphs, from ideas familiar in quantum field theory. This single perspective leads naturally to both interesting considerations on the graph isomorphism problem, and unexpected relations between topological graph theory and the theory of quadratic forms over integers.

More specifically, in section 2, we study one of the simplest quantum field theories defined on a graph, namely a real free scalar field theory, with a varying mass parameter. Here we use the notion of quantum field theory in a sense similar to lattice gauge theory: we apply some of its very basic ideas, in a situation where there are only finitely many degrees of freedom. Its two-point correlation function gives us a version of the discrete Green's function. This function showed up in [8] for different purposes. In section 3, we explain that this Green's function directly gives rise to a graph invariant, which turns out to be very close to the 2-dim Weisfeiler-Lehman algorithm for graph isomorphism testing.

In section 4, we consider the same field theory path integral over the integers, which leads to the consideration of another type of graph invariants, by viewing graphs as quadratic forms over the integers, which to the authors' knowledge first appeared in [14]: let A_G be the adjacency matrix of a finite simple graph G, and M_G be its combinatorial Laplacian matrix, and u be a fixed nonnegative integer, $M_G + uI$ then defines a quadratic form over $\mathbb{Z}$. It is obvious that the isomorphism type of the quadratic form, is an invariant of the graph. It was proved in [14] that for any pair of cospectral graphs, when u is sufficiently big depending on the graph pair, the form $M_G + uI$ could be used to distinguish the pair of graphs. Note that a different idea of quadratic forms on graphs was explored in [1]. We study the case $u = 0$, and show that the situation is very different from the case of a big u, and what happens appears to be rather interesting. In particular, we first observe there is a commutative monoid structure, on the set of equivalence classes of these forms associated with graphs, given by a well-defined *wedge sum* of graphs 4.4. We hope this structure to be useful, in studying the decomposition of graphs. Next, we propose Conjecture 4.10, and Conjecture 4.12 regarding its inverse statement, which aim to characterize precisely, what are the graph properties, that are captured by the quadratic form of the Laplacian ($u = 0$): we conjecture these properties are the graph genus g, and the dual (multi-)graphs associated to embeddings of the graph into the genus g surface. We then provide extensive evidence for Conjecture 4.10, in particular we prove Lemma 4.18, and remark on what in addition needs to be done, to prove Conjecture 4.10. We give remarks on how conjectures 4.10 and 4.12 could be related with 2-dimensional quantum gravity, and how a possible further extension of them could relate topological graph theory with the very rich number theoretic theory of quadratic forms. Locally, as suggested in [14], for each prime number p, one could view the quadratic form over the p-adic integers $\mathbb{Z}_p$, and there one has the easily computable p-adic symbols, which are complete local invariants of the form. We show some computational evidence, suggesting that these p-adic symbols could distinguish the well-known graph pairs in [5] constructed by Cai, Fürer and Immerman, where 2-dim (and in some of the cases we consider, also higher dim) Weisfeiler-Lehman algorithm fails to distinguish the graphs. The combinatorial meaning of these p-adic symbols, in terms of graphs, and the classification of graphs according to these symbols, look interesting. A hint on this is described in Remark 4.25, and discussions before it.

The graph invariant in section 3 belongs to the framework of spectral graph theory: it is constructed using eigenspaces of the Laplacian matrix. As stated in [10, 11], there is a hope to discover very useful invariants from this approach. Our idea is related to the idea of graph angles that is surveyed in [11, 4]. Also, [3] is of relevance to our idea, where the authors use the eigenspace to constrain the action of the automorphism group of the graph, on the coset space of the eigenspace.

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2. Free scalar field theory on a graph

Let G be a graph with $|G| = n$ vertices, choose an arbitrary labeling of the vertices by $V = \{x_1, x_2, \ldots, x_n\}$, and let M_G denote its $n \times n$ (combinatorial) Laplacian matrix under this basis: for $i \neq j$, the i, j-th entry is equal to -1 if there is an edge between x_i and x_j, and is equal to 0 otherwise. The diagonal entries are the degrees of the vertices, so that the sum of any column of M_G is equal to 0. From the definition, the matrix M_G is symmetric. M_G represents the combinatorial Laplacian operator under the dual basis. Let λ_k, $k = 0, 1, \ldots, m$ denote the set of different eigenvalues of M_G by increasing order. For each k, let the column vectors $\phi_k^1, \ldots, \phi_k^{l_k}$ denote an orthonormal basis of the corresponding eigenspace E_k.

Consider an Euclidean real scalar field theory on the graph G: the space of fields is then the space of all real valued functions on vertices of G, which is an n-dimensional real vector space. We write the free field Lagrangian with a mass parameter $u = \text{mass}^2$ in direct analogy with the familiar Lagrangian in the continuous situation:

$$\mathfrak{L} = \sum_{e \in E} (\nabla_e \phi)^2 + u\phi^2, \tag{1}$$

where ∇_e is the graph gradient with respect to a directed edge e, $(\nabla_e \phi)^2$ is independent of the choice of the orientation of e, and E is the set of edges of G. One can consult [7] for these notations. We have the usual Green's formula

$$\int_G (\nabla_e \phi)^2 dx = \int_G \phi \Delta \phi dx, \tag{2}$$

where Δ is the Laplacian.

As the same with usual quantum field theory (QFT) on a manifold, we consider two-point correlation functions defined by

$$\langle \phi(x)\phi(y)\rangle = \frac{\int \phi(x)\phi(y)e^{-\int \phi(\Delta+u)\phi dx} D\phi}{\int e^{-\int \phi(\Delta+u)\phi dx} D\phi}. \tag{3}$$

We allow x and y to be equal, as there will be no short distance problems in our situation. This is a finite dimensional path integral of the type that is often used as toy model to introduce the Feynman rules in physics textbooks, and it is free of divergences. However, in our simple situation here, this is our path integral. We know very well how to evaluate this by undergraduate calculus with familiar result: the denominator equals the determinant of the Laplacian to the power $-\frac{1}{2}$, which cancels with a factor coming from the numerator. Up to a nonzero constant scalar, what is left, is a sum over different eigenvalues of the form

$$\mathcal{T}_u^G(x,y) \equiv \sum_{k=0}^{m} \frac{t_k(x,y)}{\lambda_k + u}, \tag{4}$$

which may be viewed as a discrete version of the Fourier transform of the D'Alembert propagator, the familiar result in usual QFT. The individual $t_k(x,y)$ for each eigenvalue may be recovered as residues near different poles of the two-point correlation function, as we vary the parameter u.

It is straightforward to check that the function $\langle \phi(x)\phi(y)\rangle$ satisfies a discrete version of the quantum equation of motion

$$L_x \langle \phi(x)\phi(y)\rangle = \delta_{x,y}, \tag{5}$$

where L_x is the Laplacian operator on coordinate x, and the delta function $\delta_{x,y}$ on a graph is given by

$$\delta_{x,y} = \begin{cases} 0, & x \neq y, \\ 1, & x = y. \end{cases}$$

Therefore, we call the two-point correlation function as a discrete Green's function. In addition, upon a choice of labeling of vertices, as we have done, Equation (5) becomes the statement that $\langle \phi(x)\phi(y)\rangle$ as a matrix, is the inverse of $M_G + uI$, where I is the identity matrix. So obviously, it determines the graph up to isomorphism.

Remark 2.1. The two-point correlation function determines the graph up to isomorphism, thus it also determines the QFT on the graph, and therefore all of its correlation functions. This can be viewed as a baby version of Wick's theorem in the graph case.

Furthermore, one can then study various operations on graphs, and try to see how the two-point correlation function changes accordingly. This is interesting because, theoretically, it is almost always important to understand how invariants change under important operations. On the other hand, the two-point correlation function can provide a measure on when two given graphs are considered *almost isomorphic*, which may be useful in practice. For example, if we have a large data presented as a big graph, one should expect that the data given may contain a little marginal error, and so being able to make sense of and detect *almost isomorphic* graphs looks to be a practically important problem.

For example, suppose we delete an edge (adding an edge will be just the opposite, of course) between two vertices x_1 and x_2, and get a new graph, called $\widetilde{G}$. Let us try to write down the two-point correlation function for $\widetilde{G}$ in terms of data of G and the two vertices x_1 and x_2. From the form of (3), we know that this operation may only possibly affect the term $e^{\int \phi\Delta\phi dx}$. For this term, at any vertex other than x_1 and x_2, the action of the Laplacian is unaffected by definition. At x_1, the integral $\int \phi\Delta\phi dx$ changes by $\phi(x_1)(\phi(x_1) - \phi(x_2))$, and at x_2, the integral changes by $\phi(x_2)(\phi(x_2) - \phi(x_1))$. Therefore, the two-point correlation function for $\widetilde{G}$ can be expressed as

$$\langle \phi(x)\phi(y) \rangle_{\widetilde{G}} = \frac{\int \phi(x)\phi(y) e^{\int \phi \Delta \phi dx} e^{(\phi(x_1)-\phi(x_2))^2} D\phi}{\int e^{\int \phi \Delta \phi dx} e^{(\phi(x_1)-\phi(x_2))^2} D\phi}. \tag{6}$$

Again, the above can be explicitly calculated by Gaussian integrals, and one may then compare it with the two-point correlation function of G, and analyze the difference in various situations. One elementary observation is that, roughly speaking, difference of values of eigenfunctions at vertices x_1 and x_2 contribute to the difference of two-point correlation functions. Furthermore, the two-point correlation function is more sensitive to the difference at smaller eigenvalues. This is consistent with the physics picture: smaller eigenvalues correspond to lower energy modes, and if the low energy modes for two graphs are close, then we have a sense that these two graphs are close to each other.

Remark 2.2. The individual functions $t_k(x, y)$ will change in a more complicated manner, and probably one should not expect a particularly nice formula for the change of $t_k(x, y)$ similar to (6), because e.g. even the number of distinct eigenvalues and the dimension of eigenspaces may jump, and there may be complications from cross terms. The combination $\langle \phi(x)\phi(y) \rangle$ takes into account all of these and the change of it can be presented by the simple formula above.

It looks quite possible that one may study more elaborated quantum field theories on a general graph, especially with the help of various topological and geometrical concepts for graphs that are developed for graphs recently [20, 16, 17, 19].

3. A graph invariant

Suppose we have another graph H with n vertices, and upon a choice of an arbitrary labeling of the vertices, we get another Laplacian matrix M_H. The problem of whether G and H are isomorphic graphs, amounts to the linear algebra question of whether there exists a permutation matrix P, such that $P^\top M_G P = M_H$. (Note that $P^\top = P^{-1}$.) In spectral graph theory, people study the real spectrum of M_G, as an invariant of the graph under isomorphisms, however, the spectrum itself is not sufficient for the graph isomorphism problem. Two graphs can have the same real spectrum but fail to be isomorphic, and these are called cospectral graphs. On the other hand, the eigenfunctions contain much more information than just the eigenvalues. The apparent question of dealing with the eigenfunctions or eigenspaces is that they are not preserved under graph isomorphisms, but instead, the eigenspaces also transform by permutations. So, in order to use them appropriately in the graph isomorphism problem, one needs to find suitable invariants associated with the eigenfunctions.

We denote $t_k(x, y) = \sum_{i=1}^{l_k} \phi_k^i(x)\phi_k^i(y)$, and $T(x, y) = \langle t_0(x, y) \dots t_m(x, y) \rangle$. It is obvious that the vector function $T(x, y)$ does not depend on the choice of the orthonormal basis, and it can be constructed directly from the graph Laplacian independent of the choice of a labeling of vertices, therefore it is an intrinsically defined function on $G \times G$. The set of $1 \times (m + 1)$ vectors $T(x, y)$ counting multiplicity, marked by each corresponding eigenvalue, where x, y range among all pairs of vertices of G, is therefore an invariant of the graph, which we denote by ST. This invariant is clearly polynomial time computable, and furthermore the elements of this set can be ordered in order for comparisons. In the following, we explain how this invariant arises directly from the free scalar field theory, and how it is related to the 2-dimensional Weisfeiler-Lehman algorithm.

Remark 3.1. The above method is linear algebra that can also work for suitable variations of the Laplacian matrix, for example, the normalized Laplacian. Furthermore, the discussion can actually be applied to more general situations, such as multi-graphs.

As the two-point function matrix is the inverse of $M_G + uI$, by the adjugate matrix formula of an inverse matrix, we have

$$\langle \phi(x)\phi(y) \rangle = \frac{(-1)^{x+y} A_{y,x}}{\det(M_G + uI)}, \tag{7}$$

where $A_{y,x}$ is the y, x-th cofactor of $M_G + uI$, which is a polynomial in u of integral coefficients of degree less than n. Since our discrete Green's function can be written as an integral of the heat kernel which is positive, one expects $\langle \phi(x)\phi(y) \rangle$ to be positive. In fact, one has the following stronger fact.

Lemma 3.2. All coefficients of the polynomial $(-1)^{x+y} A_{y,x}$ are positive.

Proof. This is a simple verification by induction. □

We consider the graph invariant given by the set of values (actually a set of functions of u) of the two-point correlation function, counting multiplicities. We have

$$\langle \phi(x)\phi(y) \rangle = \sum_{k=0}^{m} \sum_{i=1}^{l_k} \frac{\phi_k^i(x)\phi_k^i(y)}{\lambda_k + u}, \tag{8}$$

and by basic linear algebra, more generally,

$$(M_G + uI)^\alpha = \sum_{k=0}^{m} \sum_{i=1}^{l_k} (\phi_k^i(x)\phi_k^i(y))(\lambda_k + u)^\alpha, \tag{9}$$

for any $\alpha \in \mathbb{R}$. Note that one can take such arbitrary powers of a positive semi-definite matrix.

Therefore, if for two graphs G and H, the invariant we are considering are the same, it will mean that there exists a permutation Q of n^2 elements acting linearly on $n \times n$ matrices by permuting the corresponding elements, such that

$$Q \langle \phi(x)\phi(y) \rangle_G = \langle \phi(x)\phi(y) \rangle_H . \tag{10}$$

By the above equation combined with taking residues of (8), we have, for every k,

$$Q \left\langle \sum_{i=1}^{l_k} \phi_k^i(x)\phi_k^i(y) \right\rangle_G = \left\langle \sum_{i=1}^{l_k} \phi_k^i(x)\phi_k^i(y) \right\rangle_H . \tag{11}$$

Therefore by (9), we have

$$Q(M_G + uI)^\alpha = (M_H + uI)^\alpha, \tag{12}$$

for all $\alpha \in \mathbb{R}$.

Remark 3.3. Conversely, one convinces oneself easily that, if (12) holds, then the pair of graphs are cospectral, and our graph invariant takes the same value for the pair.

Equation (12) gives interesting identities. e.g. Taking $\alpha = 0$, one derives that Q preserves the diagonal. Taking α to be positive integers, and $u = 0$, one gets infinitely many identities with more or less clear combinatorial meaning.

On the other hand, as it is clear from the above derivation, the set of $1 \times (m+1)$ vectors $T(x, y)$ marked by eigenvalues, which we denoted by ST, as an invariant of the graph, is equivalent to the above set of values of two-point correlation functions.

It turns out that, (12) is a consequence of the 2-dimensional Weisfeiler-Lehman algorithm, as shown in Theorem 3 of [2]. It is not yet clear to us if 2-dimensional Weisfeiler-Lehman is strictly stronger than (12), and if so, to what extent. As a consequence, the strongly regular graphs of a given type, and the famous pairs of graphs constructed in [5] (which we will refer to as CFI graphs in the following) have the same ST invariants. On the other hand, it is conceivable that a variation of the physics construction (e.g. considering the set of values of $2k$ point functions) may be closely related with higher dimensional Weisfeiler-Lehman algorithm, and its variations.

4. Quadratic form invariants for graphs

If we consider the free scalar field theory as in section 2 over $\mathbb{Z}$ on a connected graph, namely, we take u to be a nonnegative integer, and allow ϕ to take only integral values, then the partition function $\int e^{-\int \phi(\Delta+u)\phi dx} D\phi$ becomes a theta function of the lattice $M_G + uI$. This leads us to consider these quadratic forms associated to the graphs, and will be the content of this section.

Following [14], let A_G be the adjacency matrix of a graph G, and M_G be its combinatorial Laplacian matrix, and $f \in \mathbb{Z}[x]$ a polynomial with integral coefficients. As a permutation matrix lies in $\mathrm{GL}(n, \mathbb{Z})$, the isomorphism class of the quadratic form over $\mathbb{Z}$, represented by $f(M_G)$ (or $f(A_G)$), is an invariant of the graph. Suppose G and H are cospectral graphs with respect to M_G, it was shown in [14] (Theorem 4.1) that for big enough positive integer u which possibly depending on the graph pair, the form $M_G + uI$ always distinguishes the graphs in the pair. In terminology of quantum field theory, we may understand this result as saying that, when the mass u is big enough, the form $M_G + uI$ determines the graph. We now study what happens when $u = 0$. For ease of terminology, for the remainder of this section, let us call the quadratic form over $\mathbb{Z}$, represented by the combinatorial Laplacian of a graph G, as the quadratic form of G.

Let G_1, G_2 be two graphs, and x be a vertex of G_1, and $y_1, \dots, y_k$ be a set of vertices of G_2, we define a new graph G to be the disjoint union of G_1 and G_2, together with an edge between each pair of vertices x, y_i, $i = 1, \dots, k$. We have the following

Lemma 4.1. The isomorphism class of the quadratic form represented by the combinatorial Laplacian of G, is independent of the choice of x.

Proof. Let n_1 and n_2 denote the vertex numbers of G_1 and G_2, respectively. After a possible vertex permutation, the Laplacian matrix M_G of G is formed by two diagonal blocks $M_{G_1} + kE^{n_1,n_1}_{n_1,n_1}$ and $M_{G_2} + \sum_{i=1}^k E^{n_2,n_2}_{i,i}$, together with additional -1 at positions $(n_1, n_1+1), \dots, (n_1, n_1+k)$ and $(n_1+1, n_1), \dots, (n_1+k, n_1)$, where $E^{s,t}_{i,j}$ denote the $s \times t$ matrix with entry 1 at position (i, j), and entry 0 everywhere else. M_{G_1}, M_{G_2} denote the Laplacian matrices of G_1 and G_2 respectively. We do the operation of adding the last n_2 rows to row n_1, and then adding the last n_2 columns to column n_1, the resulting matrix N_G is formed by the block M_{G_1}, and the block $M_{G_2} + \sum_{i=1}^k E^{n_2,n_2}_{i,i}$.

We have that $M_G \cong N_G$ as quadratic forms, and the isomorphism class of N_G is obviously independent of the choice of x. Therefore, the isomorphism class of M_G is independent of the choice of x. □

Next, we define the wedge sum graph $G_1 \vee_{x,y} G_2$ of G_1 and G_2 with respect to the vertex x of G_1, and the vertex y of G_2, to be the graph formed by the quotient of the disjoint union of G_1 and G_2, by identifying x and y. We have the following

Corollary 4.2. The isomorphism class of the quadratic form of $G_1 \vee_{x,y} G_2$, in independent of the choices of x and y.

Proof. Let us denote the set of neighbors of y in G_2, by $\{y_1, \dots, y_k\}$. Observe that $G_1 \vee_{x,y} G_2$, is the same graph as that described before lemma 4.1, with respect to G_1, the subgraph of G_2 formed by deleting the vertex y and all edges connecting with it, x and $y_1, \dots, y_k$. Therefore, by lemma 4.1, the isomorphism class of the quadratic form in question is independent of the choice of x. By symmetry, it is also independent of y. □

Definition 4.3. We impose an equivalence relation among graphs, by declaring two graphs to be equivalent, if and only if the quadratic forms represented by their Laplacian matrices are equivalent as quadratic forms, and call the set of such equivalence classes G_f.

Theorem 4.4. The above wedge sum of graphs is well defined on the equivalence classes, and it gives rise to a structure of commutative monoid on the equivalence classes.

Remark 4.5. As a result, any two trees of the same vertex number are in the same equivalence class, as they can be formed via step-by-step wedge sums of two-vertex trees. The reader is invited to check this by hand.

Proof. First, the equivalence class of $G_1 \vee_{x,y} G_2$ is independent of the choices of x, y by corollary 4.2. Denote as before, $n_1 = |G_1|$, and $n_2 = |G_2|$.

Next, we view $G_1 \vee_{x,y} G_2$ as in the proof of corollary 4.2. Then by doing the row and column operations as in the proof of lemma 4.1, the quadratic form of $G_1 \vee_{x,y} G_2$ can be represented by a block diagonal matrix, where one block is the combinatorial Laplacian M_1 of G_1, and the other block is of dimension $n_2 - 1$, and is determined by G_2 and Y. Next, suppose G_1 and H_1 are two graphs in the same equivalence class, and x' be any vertex of H_1. The quadratic form of $H_1 \vee_{x',y} G_2$ can also be represented in the same way, where one block is the combinatorial Laplacian M_1' of H_1, and the other block is determined by G_2 and y, identical to that of $G_1 \vee_{x,y} G_2$. Since we have $M_1 \cong M_1'$ as quadratic forms, there exists $T \in \mathrm{GL}(n_1, \mathbb{Z})$, such that $T^\top M_1 T = M_1'$, therefore the forms of $G_1 \vee_{x,y} G_2$ and $H_1 \vee_{x',y} G_2$ are equivalent, via the block diagonal matrix $\mathrm{diag}(T, I_{n_2-1})$. This proves our wedge sum of graphs is well defined on the equivalence classes.

Next, corollary 4.2 implies that on the equivalence classes, the sum is associative and commutative. Furthermore, the one-vertex graph P affords the identity element, and G_f becomes a commutative monoid. □

Definition 4.6. [18] A connected graph G is called a *block*, if and only if G cannot be written as a sum $G_1 \vee_{x,y} G_2$ of two strictly smaller graphs.

Obviously the blocks generate the submonoid of G_f consisting of connected graphs. We have the following lemma.

Lemma 4.7. Any connected graph has a unique decomposition into a wedge sum of blocks.

Proof. First, an easy induction shows such a decomposition always exists.

Next, to prove uniqueness, we consider the set of gluing vertices in the graph G: namely, the set of vertices where the graph can be separated as a wedge sum of two components, glued along the vertex. If there is no such vertex, then by definition the graph is simple, and there is nothing to prove. Otherwise, pick any such vertex v, then G is separated into components, glued along v, which we call components of G, with respect to v. Obviously, any simple graph in the decomposition of G, which is obviously connected, must appear as a subgraph of components of G, with respect to v. Suppose there is another graph G' isomorphic to G as a graph, which is also written as a wedge sum of simple graphs. Then under this graph isomorphism, v must be mapped to a gluing vertex v' of G'. Next, the components of G with respect to v must map bijectively to isomorphic components of G' with respect to v'. Again, any simple graph in the decomposition of G', must appear as a subgraph of components of G' with respect to v'. By induction, each such component has a unique decomposition into simple graphs, thus the decomposition of G' must be the same as that of G. □

Remark 4.8. Therefore, to understand the quadratic form of a connected graph, it suffices to understand the forms of its simple components, and we do not need to worry about how the simple components glue.

To go further, we propose the following

Conjecture 4.9. Let G_1 and G_2 be two connected planar graphs, and suppose there exist embeddings of the two graphs into the 2-sphere, such that the (geometric) dual graphs with respect to the embeddings are isomorphic as multi-graphs, then the quadratic forms over $\mathbb{Z}$, represented by the combinatorial Laplacian of G_1 and G_2 are equivalent.

Furthermore, one could try to remove the planar graph restriction: in general, any graph can be embedding into a higher genus surface, and the minimal genus of such surfaces is called the genus of the graph. Given any such embedding, the (geometric) dual (multi-)graph can obviously be defined in the same way, as that in the planar case. The above conjecture could be extended to the following

Conjecture 4.10. Let G_1 and G_2 be two connected graphs of the same genus g, and further suppose that G_1 and G_2 have embeddings into the genus g surface Σ_g, such that the corresponding dual graphs are isomorphic as multi-graphs, then the quadratic forms over $\mathbb{Z}$, represented by the combinatorial Laplacian of G_1 and G_2 are equivalent.

Remark 4.11. Note that, if one drops the condition that G_1 and G_2 having the same genus, the theorem would become false: e.g. take the 3-path and the 3-cycle, both can be embedded into a torus, such that the dual graph consists of a single vertex, and 3 self-edges. However the quadratic forms of the 3-path and the 3-cycle are not equivalent.

One could try to think about an inverse statement. The condition that two graphs have the same quadratic forms is an equivalence relation, so the condition on graph embedding into surface must also be formulated as an equivalence relation, and we propose the following

Conjecture 4.12. Let G_1 and G_2 be two connected graphs. Suppose the quadratic forms over $\mathbb{Z}$, represented by the combinatorial Laplacian of the two graphs are equivalent, then they have the same genus g, and there exist a finite series of graphs $G_1 = H_1, H_2, \ldots, H_s = G_2$, such that each pair H_i, H_{i+1}, $i = 1, \ldots, s-1$ satisfies the assumptions in Conjecture 4.10.

Remark 4.13. The above conjectures link the geometry and combinatorics of graphs, with the arithmetic of quadratic forms, which a priori are totally different subjects.

We first exhibit some examples whose dual graphs with respect to certain embeddings are isomorphic. Figure 1 shows a pair of planar graphs (in color blue) constructed by *gluing* two 3-cycles with a 4-cycle, embedded into a sphere. The dual graph is shown in red. Figure 2 shows embeddings into a torus of a pair of genus 1 graphs constructed by gluing two 4-cycles with a complete graph on 5 vertices.

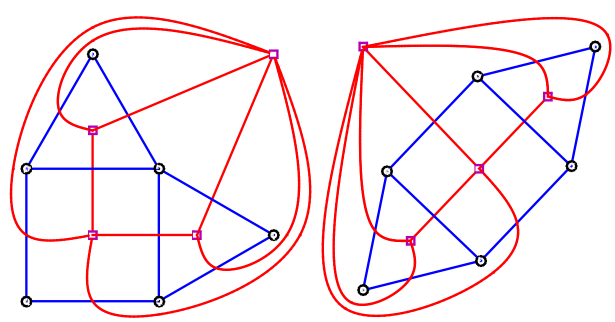

FIGURE 1. A pair of planar graphs embedded into the sphere, with isomorphic dual graphs

In Figure 2, note that the pair of graphs with isomorphic dual graphs is constructed by gluing two 4-cycles to K_5 in a way, such that those cycles lie in the same region of the torus, i.e. connected components of the complement of the embedded image of K_5. In the same way, one can construct examples of graph pairs of higher genus, with embeddings into surface of the corresponding genus, for which the dual graphs are isomorphic, by gluing several cycles to a complete graph K_n, or to other types of graphs, in this specific way.

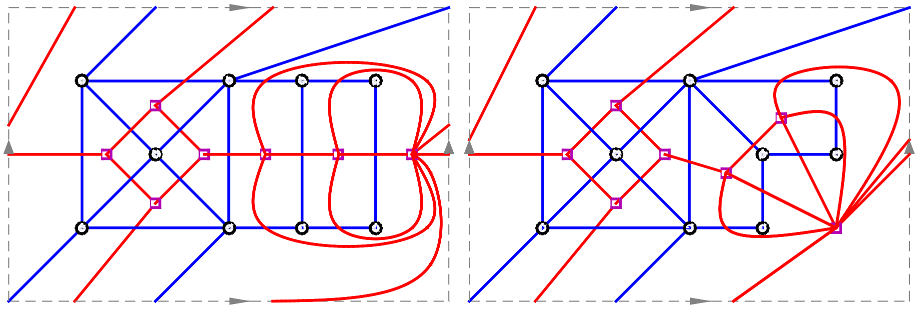

FIGURE 2. A pair of genus one graphs embedded into the torus, with isomorphic dual graphs

Remark 4.14. Conjecture 4.10 has some resemblence with the discrete approach to 2-dimensional quantum gravity [15]. In the latter, one takes a genus g surface and considers *polygonulations* of the surface, which are graphs minimally embedded in the surface, and they are dual to Feyman (multi)-graphs of the matrix model of 2d gravity. Thus, in the path integral, if instead of integrating over geometries of the surface, one takes a different approach of summing over graphs, then for two graphs satisfying the conditions of Conjecture 4.10, it is reasonable to expect that they should be identified in the path integral and be counted only once. If the inverse of Conjecture 4.10 is true, then the sum can be rephrased as summing over isomorphism classes of the corresponding quadratic forms.

Remark 4.15. If one allows arbitrary integral weights on edges and vertices, then one can realize any finite dimensional quadratic form over $\mathbb{Z}$ by a unique weighted graph, via its combinatorial Laplacian. It would be extremely interesting, if the conjectures 4.10 and 4.12 could be generalized to such weighted graphs, as then we would be able to translate the condition of two arbitrary forms being equivalent, to a graph theorecitc condition. Therefore in particular, it would be very interesting to see how e.g. the rich theory of binary forms translate into graph theoretic statements.

For this sake, one idea is to find a physics theory, which should supposedly be dual to the free scalar field theory over $\mathbb{Z}$ on graphs, that captures the topological graph theoretic conditions in Conjecture 4.10. 2d quantum gravity might not be a good candidate for this purpose, as it does not care much about properties of individual graphs. It is clear that the free scalar field theory can easily make sense in more general settings.

Let us next present some consistency checks of the conjectures. First, suppose a (connected) graph pair G_1, G_2 of genus g satisfies the assumptions of Conjecture 4.10, then from [24], Thm 6-11, and the references therein, the embeddings of G_1 and G_2 into the genus g surface Σ_g are the so-called *2-cell embeddings*, meaning that each region is a 2-disk, so in particular, Euler's formula applies, and we have $V - E + F = g$ for both G_1 and G_2, where V and E are vertex and edge numbers for the graphs, respectively, and F is the number of regions. As E and F are equal to E and V of that of the dual graphs, we deduce that G_1 and G_2 have the same number of vertices, which is a necessary condition for the conjecture to be true.

Furthermore, take any wedge sum $H_1 \vee_{x,y} H_2$ of two given connected graphs H_1 and H_2, we know from corollary 4.2 that, the quadratic form of the wedge sum, is independent of the choice of vertices x, y. Therefore, if the inverse statement is true, the genus of the wedge sum should be independent of this choice of vertices as well. According to Thm 6-18 of [24] and the references therein, the genus of such a wedge sum is the sum of the genera of H_1 and H_2, which is indeed independent of the choice of x, y.

Remark 4.16. In addition, intuitively, for pairs of connected planar graphs H_1 and H_2, take any two wedge sums of them, and take two embeddings of them into the sphere, such that the restrictions of the embeddings to H_1 and H_2 are the same, then informally, the embeddings are connected by a finite sequence of *deformations* of the graph, and *reflections* with respect to certain vertex that transforms between neighboring regions, so they should satisfy the conclusion of the inverse statement 4.12. One also has a similar intuition for the higher genus case.

Note that in figure 1, the second graph results from the first graph by *moving* the triangle on the right, one step clockwise. We observe the following lemma, which explicitly establishes the equivalence of quadratic forms of a graph pair, resulting from such a situation:

Lemma 4.17. Given a graph G_1 with a degree 3 vertex v_1, adjacent with v_2 and v_3, and a degree 2 vertex v_4, adjacent with v_3 and v_1, let us define a second graph G_2, by deleting the edge $[v_4, v_3]$ from G_1, and adding the edge $[v_4, v_2]$. Then the combinatorial Laplacians M_{G_1} and M_{G_2} are equivalent as quadratic forms over $\mathbb{Z}$.

Proof. This follows directly from the following Lemma 4.18 by taking $G = G_1 - \{v_1, v_4\}$ and $H = \{v_1, v_2, v_3, v_4\}$. □

Lemma 4.18. Given two graphs G and H, two pairs of vertices $(v_G, w_G) \in G \times G$, $(v_H, w_H) \in H \times H$ such that one of the following holds:

(i) $v_G \sim w_G$ and $v_H \sim w_H$;
(ii) $v_G \not\sim w_G$ and $v_H \not\sim w_H$.

We define the wedge sum graph $G \vee^{(w_G, w_H)}_{(v_G, v_H)} H$ of G and H with respect to the vertices pairs $(v_G, w_G) \in G \times G$, $(v_H, w_H) \in H \times H$ to be a graph formed by identifying v_G with v_H and w_G with w_H, respectively. The quadratic forms $G \vee^{(w_G, w_H)}_{(v_G, v_H)} H$ and $G \vee^{(w_G, v_H)}_{(v_G, w_H)} H$ are globally equivalent.

Proof. For convenience, let the labels v_G, w_G, v_H, and w_H be 1, 2, 1, and 2, respectively. The Laplacian matrix M_G is of the form

$$M_G = \left[\begin{array}{cc|c} d_1^G & s & \mathbf{g}_1^\top \\ s & d_2^G & \mathbf{g}_2^\top \\ \hline \mathbf{g}_1 & \mathbf{g}_2 & \widetilde{M}_G \end{array}\right],$$

where $d_1^G = \deg(v_G)$, $d_2^G = \deg(w_G)$, s is either 0 or -1, and $\mathbf{g}_1 = (g_1^{(i)})$, $\mathbf{g}_2 = (g_2^{(i)})$, $\widetilde{M}_G$ are the remaining parts of the matrix M_G. Similarly, the Laplacian matrix M_H is of the form

$$M_H = \left[\begin{array}{cc|c} d_1^H & s & \mathbf{h}_1^\top \\ s & d_2^H & \mathbf{h}_2^\top \\ \hline \mathbf{h}_1 & \mathbf{h}_2 & \widetilde{M}_H \end{array}\right],$$

where $d_1^H = \deg(v_H)$, $d_2^H = \deg(w_H)$, and $\mathbf{h}_1 = (h_1^{(i)})$, $\mathbf{h}_2 = (h_2^{(i)})$, $\widetilde{M}_H$ are the remaining parts of the matrix M_H. Then the Laplacian matrix $M_{G \vee^{(w_G, w_H)}_{(v_G, v_H)} H}$ is of the form

$$M_{G \vee^{(w_G, w_H)}_{(v_G, v_H)} H} = \left[\begin{array}{cc|c|c} d_1^G + d_1^H + s & s & \mathbf{g}_1^\top & \mathbf{h}_1^\top \\ s & d_2^G + d_2^H + s & \mathbf{g}_2^\top & \mathbf{h}_2^\top \\ \hline \mathbf{g}_1 & \mathbf{g}_2 & \widetilde{M}_G & O^\top \\ \hline \mathbf{h}_1 & \mathbf{h}_2 & O & \widetilde{M}_H \end{array}\right],$$

where O is a zero matrix. Note that the row sum of M_H is zero. We have the equalities

$$\sum_i h_k^{(i)} = -d_k^H - s, \; k = 1, 2, \tag{13}$$

and

$$\sum_i \widetilde{M}_H(\cdot, i) = -\mathbf{h}_1 - \mathbf{h}_2. \tag{14}$$

One verifies by direct computation using (13) and (14),

$$TM_{G\vee^{(w_G,w_H)}_{(v_G,v_H)}H}T^\top = \left[\begin{array}{cc|c|c} d_1^G + d_2^H + s & s & \mathbf{g}_1^\top & \mathbf{h}_2^\top \\ s & d_2^G + d_1^H + s & \mathbf{g}_2^\top & \mathbf{h}_1^\top \\ \hline \mathbf{g}_1 & \mathbf{g}_2 & \widetilde{M}_G & O^\top \\ \hline \mathbf{h}_2 & \mathbf{h}_1 & O & \widetilde{M}_H \end{array}\right] = G \vee^{(w_G,v_H)}_{(v_G,w_H)} H,$$

where

$$T = \left(\prod_{k=|G|+1}^{|G|+|H|} E_{-R_k \to R_k}\right)\left(\prod_{k=|G|+1}^{|G|+|H|} E_{R_2+R_k \to R_2}\right)\left(\prod_{k=|G|+1}^{|G|+|H|} E_{R_1+R_k \to R_1}\right).$$

□

Remark 4.19. It is clear from the above proof that, the lemma holds in the more general situation, where one of G and H is a multi-graph, or even more generally, a graph with arbitrary integral weights on edges and vertices.

Lemma 4.18 is a step towards a proof of Conjecture 4.10. Next, we give the following definitions.

Definition 4.20. A pair of vertices (v, w) on a graph G is said to be a *two-cut* on G if $G - \{v, w\}$ is disconnected.

To slightly reformulate Lemma 4.18 in a more apparent way, we define the flip operation on G with respect to a two-cut (v, w).

Definition 4.21. Given a graph G and a two-cut (v, w) on G. Suppose $G = G_1 \vee^{(w,v)}_{(v,w)} G_2$, where G_1 and G_2 are two subgraphs of G. We define the *flip* operation on G with respect to the two-cut (v, w) by

$$\mathrm{flip}_{(v,w)}G = G_1 \vee^{(w,v)}_{(v,w)} G_2.$$

Remark 4.22. It is clear that the flip operation preserves dual graphs of planar graphs with respect to any given embedding.

We may define an equivalence relation $\overset{\text{flip}}{\sim}$ as following.

Definition 4.23. Let G and H be two connected graphs. The graph G is said to be flip-equivalent to the graph H, denoted by $G \overset{\text{flip}}{\sim} H$, if there exists finitely many two-cuts (v_i, w_i) on G, $i = 1, \ldots, n$, such that $\mathrm{flip}_{(v_n,w_n)} \circ \cdots \circ \mathrm{flip}_{(v_1,w_1)}G = H$.

The following theorem follows directly from Lemma 4.18.

Theorem 4.24. Let G and H be two connected planar graphs such that $G \overset{\text{flip}}{\sim} H$. Then the quadratic forms over $\mathbb{Z}$, represented by the combinatorial Laplacian of G and H are equivalent.

All examples of graph pairs satisfying the conditions of Conjecture 4.10 that are known to the authors to date, are related by the above flips. On the other hand, given any genus g graph G embedded into the surface Σ_g, the embedding determines an embedding of its dual (multi-)graph, and conversely, the graph itself is determined by this embedding of its dual graph D into Σ_g. Therefore, if one could show that all possible embeddings of D into Σ_g give rise to graphs which are related by flips, then Conjecture 4.10 would follow. For enumeration of graph embeddings into surface, there is a very useful result that provides a one-to-one correspondence of possible embeddings, with a very simple algebraic object called *graph rotations*. See e.g. section 6-6, [24]. It looks hopeful that one may try to use this result to investigate all possible embeddings of the dual graph into Σ_g.

We next briefly describe some experimental results regarding these quadratic forms. Recall that to compare two quadratic forms over $\mathbb{Z}$, one can first compare the forms over the p-adic integers $\mathbb{Z}_p$ for each prime p, where the so-called p-adic symbols are complete local invariants, and are very easy to compute. If the two forms are equivalent over $\mathbb{Z}_p$ for each such p, then they are said to be in the same genus. If the forms are positive definite, and have large dimension, and large discriminant, a genus often contains a huge number of forms, as predicted by the Smith-Minkowski-Siegel mass formula [23]. On the other hand, if the forms are indefinite, and the dimension is at least 3, then there is a so-called *spinor genus* that refines the genus, and is easily computable, and is a complete invariant that determines the form over $\mathbb{Z}$ [9].

For consideration of graph isomorphism testing, the CFI graph pairs is a famous construction, where any fixed dimensional Weisfeiler-Lehman algorithm, thus also our ST invariant from the two-point function, fails. We next exhibit some experimental results suggesting that $2-$ adic symbols could distinguish the CFI pairs.

To construct CFI regular graphs (We actually tested also non-regular CFI graphs, which behave in a similar way.), first we use GENREG [21] to generate the set of all k-regular graphs of n vertices $\mathrm{reg}(n,k)$. For each graph in $\mathrm{reg}(n,k)$, we construct the corresponding CFI pair. For convenience, we denote the set of all the CFI pairs with respect to $\mathrm{reg}(n,k)$ by $\mathrm{CFI}(\mathrm{reg}(n,k))$.

We use Sage to compute the p-adic symbols of the quadratic form A_G for CFI graphs of vertex number less than or equal to 100, for which $\det A_G \neq 0$. The result is written in Table 3 and Table 4. According to the result, all the tested CFI pairs can be distinguished by the 2-adic symbol of the quadratic form A_G. There are some recent related studies on this in the literature, see e.g. [13] and [12]. On the other hand, for odd primes p, the p-adic symbols of each CFI pair are identical.

We can try to read off the combinatorial meaning of some information contained in these p-adic symbols, directly from definitions. e.g., apparently, there is a bijection from the kernel of A_G mod 2, as a vector space over the finite field $\mathbb{F}_2$, to the set of subsets of the vertex set of the graph, such that every vertex of the graph is connected with an even number of vertices in the subset, whereas the dimension of the kernel of A_G mod 2, is the simplest piece of information contained in the 2-adic symbol of A_G.

Remark 4.25. One can see from Table 1 and Table 2 that, the 4-element set $\mathrm{srg}(28,12,6,4)$ is split into a subset of 3 graphs, and a subset of a single graph, by the 2-adic symbol for A_G. One can check that, the one graph that is singled out in either way, is exactly the line graph of the complete graph K_8. So the other 3-element subset is the set of the so-called *Chang graphs* [6].

Remark 4.26. The computer experimental results on p-adic symbols of the quadratic form for the combinatorial Laplacian of strongly regular graphs are written in Table 5 and Table 6. The classification of $\mathrm{srg}(25,12,5,6)$ according to the p-adic symbols of the combinatorial Laplacian, distinguishes the only graph of 25 vertices with a transitive group, which is known as the Paley graph of order 25 [22]. Again, this classification distinguishes the Chang graphs among $\mathrm{srg}(28,12,6,4)$.

Remark 4.27. The (global) indefinite forms A_G and A_H cannot distinguish strongly regular graphs in general, as one readily checks that their determinants are in general not big enough for the spinor genus to contain more than one class in its genus: see e.g. Corollary 22 on page 395 of [9].

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Appendix A. Tables for graphs and quadratic forms

Graph Type	#Graphs	Eigenvalues$^{\text{Multiplicities}}$	m	#Quadratic Forms
srg(16, 6, 2, 2)	2	$(-2)^9, 2^6, 6^1$	2	2
			-2	1
			-6	2
srg(25, 12, 5, 6)	15	$(-3)^{12}, 2^{12}, 12^1$	0	15
			2	15
			3	4 $[10, 3, 1, 1]$
			-2	4 $[10, 3, 1, 1]$
			-12	15
srg(26, 10, 3, 4)	10	$(-3)^{12}, 2^{13}, 10^1$	0	10
			2	10
			3	3 $[7, 2, 1]$
			-2	4 $[5, 2, 2, 1]$
			-10	10
srg(28, 12, 6, 4)	4	$(-2)^{20}, 4^7, 12^1$	0	4
			2	2 $[3, 1]$
			-4	4
			-12	4
srg(29, 14, 6, 7)	41	$(\frac{-1\pm\sqrt{29}}{2})^{14}, 14^1$	0	41
			2	41
			-14	41
srg(35, 18, 9, 9)	3854	$(-3)^{20}, 3^{14}, 18^1$	2	3854
srg(36, 14, 4, 6)	180	$(-4)^{14}, 2^{21}, 14^1$	0	180
			2	180
			4	155
			-2	9 $[66, 44, 43, 11, 9, 2, 2, 2, 1]$
			-14	180
srg(40, 12, 2, 4)	28	$(-4)^{15}, 2^{24}, 12^1$	0	28
			2	28
			4	28
			-2	6 $[13, 8, 3, 2, 1, 1]$
			-12	28
srg(45, 12, 3, 3)	78	$(-3)^{24}, 3^{20}, 12^1$	2	78
			3	21
			-3	76
			-12	78
srg(50, 21, 8, 9)	18	$(-4)^{24}, 3^{25}, 21^1$	0	18
			2	18
			4	17
			-3	5 $[10, 4, 2, 1, 1]$
			-21	18
srg(64, 18, 2, 6)	167	$(-6)^{18}, 2^{45}, 18^1$	0	167
			6	167
			-2	4 $[156, 9, 1, 1]$
			-18	167

Table 1. The number of distinct quadratic forms $(A_G + mI)^2$ with respect to srg(n, k, λ, μ).

Graph Type	p	p-Adic Symbols	#Graphs
srg(16, 6, 2, 2)	2 3	[0, 6, 7, 0, 0], [1, 4, 1, 0, 0], [2, 6, 3, 1, 4] [0, 15, 1], [1, 1, −1]	1
	2 3	[0, 6, 3, 0, 0], [1, 4, 3, 1, 2], [2, 6, 5, 1, 6] [0, 15, 1], [1, 1, −1]	1
srg(25, 12, 5, 6)	2 3	[0, 12, 1, 0, 0], [1, 12, 1, 0, 0], [2, 1, 3, 1, 3] [0, 12, 1], [1, 13, 1]	10
	2 3	[0, 12, 5, 0, 0], [1, 12, 5, 0, 0], [2, 1, 3, 1, 3] [0, 12, −1], [1, 13, −1]	5
srg(26, 10, 3, 4)	2 3 5	[0, 12, 1, 0, 0], [1, 14, 5, 1, 6] [0, 14, −1], [1, 12, 1] [0, 25, −1], [1, 1, −1]	5
	2 3 5	[0, 12, 5, 0, 0], [1, 14, 1, 1, 6] [0, 14, 1], [1, 12, −1] [0, 25, −1], [1, 1, −1]	2
	2 3 5	[0, 12, 5, 0, 0], [1, 14, 1, 1, 2] [0, 14, −1], [1, 12, 1] [0, 25, −1], [1, 1, −1]	2
	2 3 5	[0, 12, 1, 0, 0], [1, 14, 5, 1, 2] [0, 14, 1], [1, 12, −1] [0, 25, −1], [1, 1, −1]	1
srg(28, 12, 6, 4)	2 3	[0, 8, 5, 0, 0], [1, 12, 5, 0, 0], [3, 8, 3, 1, 6] [0, 27, 1], [1, 1, 1]	3
	2 3	[0, 6, 3, 0, 0], [1, 15, 7, 1, 5], [3, 7, 7, 1, 1] [0, 27, 1], [1, 1, 1]	1
srg(29, 14, 6, 7)	2 7	[0, 28, 1, 0, 0], [1, 1, 7, 1, 7] [0, 14, −1], [1, 15, −1]	41
srg(35, 18, 9, 9)	2 3	[0, 34, 7, 0, 0], [1, 1, 7, 1, 7] [0, 13, −1], [1, 8, −1], [2, 14, −1]	3816
	2 3	[0, 34, 7, 0, 0], [1, 1, 7, 1, 7] [0, 13, 1], [1, 8, −1], [2, 14, 1]	37
	2 3	[0, 34, 7, 0, 0], [1, 1, 7, 1, 7] [0, 11, −1], [1, 12, −1], [2, 12, −1]	1
srg(36, 14, 4, 6)	2 7	[0, 14, 7, 0, 0], [1, 8, 5, 0, 0], [3, 14, 5, 1, 6] [0, 35, 1], [1, 1, 1]	109
	2 7	[0, 14, 3, 0, 0], [1, 8, 1, 0, 0], [3, 14, 5, 1, 2] [0, 35, 1], [1, 1, 1]	48
	2 7	[0, 12, 1, 0, 0], [1, 10, 3, 0, 0], [2, 2, 7, 0, 0], [3, 12, 3, 1, 6] [0, 35, 1], [1, 1, 1]	19
	2 7	[0, 12, 1, 0, 0], [1, 10, 3, 0, 0], [2, 2, 3, 0, 0], [3, 12, 7, 1, 2] [0, 35, 1], [1, 1, 1]	1
	2 7	[0, 10, 7, 0, 0], [1, 12, 5, 0, 0], [2, 4, 1, 0, 0], [3, 10, 5, 1, 6] [0, 35, 1], [1, 1, 1]	2
	2 7	[0, 8, 1, 0, 0], [1, 14, 3, 0, 0], [2, 6, 7, 0, 0], [3, 8, 3, 1, 6] [0, 35, 1], [1, 1, 1]	1

TABLE 2. The p-adic symbols for quadratic form A_G with respect to $\mathrm{srg}(n, k, \lambda, \mu)$ by using Sage.

Graph Type	#Vertices	p	p-Adic Symbols
CFI(reg(4,3))	40	2	$[0, 30, 7, 0, 0], [1, 4, 1, 0, 0], [2, 6, 3, 1, 4]$
		3	$[0, 39, -1], [1, 1, 1]$
		2	$[0, 30, 7, 0, 0], [1, 4, 1, 1, 0], [2, 6, 3, 1, 4]$
		3	$[0, 39, -1], [1, 1, 1]$
CFI(reg(8,3))	80	2	$[0, 62, 7, 0, 0], [1, 6, 7, 0, 0], [2, 10, 7, 0, 0], [3, 2, 7, 0, 0]$
		3	$[0, 79, -1], [1, 1, -1]$
		5	$[0, 79, 1], [1, 1, -1]$
		2	$[0, 62, 7, 0, 0], [1, 6, 7, 1, 0], [2, 10, 7, 0, 0], [3, 2, 7, 0, 0]$
		3	$[0, 79, -1], [1, 1, -1]$
		5	$[0, 79, 1], [1, 1, -1]$
CFI(reg(8,3))	80	2	$[0, 62, 7, 0, 0], [1, 8, 1, 0, 0], [2, 6, 7, 0, 0], [3, 4, 1, 0, 0]$
		3	$[0, 78, -1], [1, 2, -1]$
		2	$[0, 62, 7, 0, 0], [1, 8, 1, 1, 0], [2, 6, 7, 0, 0], [3, 4, 1, 0, 0]$
		3	$[0, 78, -1], [1, 2, -1]$
CFI(reg(8,3))	80	2	$[0, 60, 1, 0, 0], [1, 8, 1, 0, 0], [2, 12, 1, 0, 0]$
		3	$[0, 78, -1], [1, 2, -1]$
		2	$[0, 62, 7, 0, 0], [1, 4, 1, 0, 0], [2, 14, 7, 0, 0]$
		3	$[0, 78, -1], [1, 2, -1]$
CFI(reg(8,3))	80	2	$[0, 62, 7, 0, 0], [1, 8, 1, 0, 0], [2, 6, 7, 0, 0], [3, 4, 5, 0, 0]$
		3	$[0, 79, 1], [1, 1, -1]$
		2	$[0, 62, 7, 0, 0], [1, 8, 1, 1, 0], [2, 6, 7, 0, 0], [3, 4, 5, 0, 0]$
		3	$[0, 79, 1], [1, 1, -1]$

TABLE 3. The p-adic symbols for quadratic form A_G with respect to CFI(reg(n, 3)) by using Sage.

Graph Type	#Vertices	p	p-Adic Symbols
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,10,3,1,0],[2,6,7,1,0],[3,6,7,0,0]$
		3	$[0,99,-1],[1,1,1]$
		2	$[0,78,7,0,0],[1,10,3,0,0],[2,6,7,1,0],[3,6,7,0,0]$
		3	$[0,99,-1],[1,1,1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,12,1,1,0],[2,2,7,0,0],[3,8,5,0,0]$
		3	$[0,99,-1],[1,1,1]$
		2	$[0,78,7,0,0],[1,12,1,0,0],[2,2,7,0,0],[3,8,5,0,0]$
		3	$[0,99,-1],[1,1,1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,12,5,1,0],[2,2,7,0,0],[3,8,1,0,0]$
		3	$[0,99,1],[1,1,1]$
		7	$[0,99,1],[1,1,-1]$
		2	$[0,78,7,0,0],[1,12,5,0,0],[2,2,7,0,0],[3,8,1,0,0]$
		3	$[0,99,1],[1,1,1]$
		7	$[0,99,1],[1,1,-1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,8,1,1,0],[2,10,7,0,0],[4,4,3,1,2]$
		3	$[0,99,1],[1,1,1]$
		2	$[0,78,7,0,0],[1,8,1,0,0],[2,10,7,0,0],[4,4,3,1,2]$
		3	$[0,99,1],[1,1,1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,10,7,1,0],[2,6,3,1,4],[3,6,7,0,0]$
		3	$[0,99,-1],[1,1,1]$
		2	$[0,78,7,0,0],[1,10,7,0,0],[2,6,3,1,4],[3,6,7,0,0]$
		3	$[0,99,-1],[1,1,1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,12,1,1,0],[2,2,7,0,0],[3,6,3,0,0],[4,2,7,0,0]$
		3	$[0,99,-1],[1,1,1]$
		2	$[0,78,7,0,0],[1,12,1,0,0],[2,2,7,0,0],[3,6,3,0,0],[4,2,7,0,0]$
		3	$[0,99,-1],[1,1,1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,12,1,1,0],[2,2,7,0,0],[3,6,3,0,0],[4,2,1,1,6]$
		3	$[0,99,1],[1,1,1]$
		2	$[0,78,7,0,0],[1,12,1,0,0],[2,2,7,0,0],[3,6,3,0,0],[4,2,1,1,6]$
		3	$[0,99,1],[1,1,1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,12,5,1,0],[2,2,7,0,0],[3,8,1,0,0]$
		3	$[0,99,-1],[1,1,1]$
		2	$[0,78,7,0,0],[1,12,5,0,0],[2,2,7,0,0],[3,8,1,0,0]$
		3	$[0,99,-1],[1,1,1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,12,5,1,0],[2,2,7,0,0],[3,8,5,0,0]$
		3	$[0,98,-1],[1,2,-1]$
		2	$[0,78,7,0,0],[1,12,5,0,0],[2,2,7,0,0],[3,8,5,0,0]$
		3	$[0,98,-1],[1,2,-1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,12,1,0,0],[2,2,7,0,0],[3,8,1,0,0]$
		3	$[0,98,-1],[1,2,-1]$
		2	$[0,76,1,0,0],[1,16,1,0,0],[3,8,1,0,0]$
		3	$[0,98,-1],[1,2,-1]$
CFI(reg(10,3))	100	2	$[0,78,7,0,0],[1,8,1,1,0],[2,10,7,0,0],[4,4,5,0,0]$
		3	$[0,99,-1],[1,1,1]$
		2	$[0,78,7,0,0],[1,8,1,0,0],[2,10,7,0,0],[4,4,5,0,0]$
		3	$[0,99,-1],[1,1,1]$

TABLE 4. The p-adic symbols for quadratic form A_G with respect to CFI(reg$(10,3)$) by using Sage.

Graph Type	p	p-Adic Symbols	#Graphs
srg(16, 6, 2, 2)	2	$[0, 6, 7, 0, 0], [3, 5, 7, 1, 7], [5, 4, 1, 0, 0]$	1
	2	$[0, 6, 7, 0, 0], [1, 1, 1, 1, 1], [3, 2, 7, 0, 0], [4, 2, 1, 1, 6], [5, 4, 1, 0, 0]$	1
srg(25, 12, 5, 6)	2 3 5	$[0, 12, 1, 0, 0], [1, 12, 1, 0, 0]$ $[0, 12, 1], [1, 12, 1]$ $[0, 12, -1], [1, 2, 1], [2, 10, -1]$	10
	2 3 5	$[0, 12, 5, 0, 0], [1, 12, 5, 0, 0]$ $[0, 12, -1], [1, 12, -1]$ $[0, 11, 1], [1, 4, 1], [2, 9, 1]$	4
	2 3 5	$[0, 12, 5, 0, 0], [1, 12, 5, 0, 0]$ $[0, 12, -1], [1, 12, -1]$ $[0, 9, 1], [1, 8, 1], [2, 7, 1]$	1
srg(26, 10, 3, 4)	2 13	$[0, 12, 5, 0, 0], [2, 1, 1, 1, 1], [3, 12, 1, 0, 0]$ $[0, 14, -1], [1, 11, -1]$	5
	2 13	$[0, 12, 1, 0, 0], [2, 1, 1, 1, 1], [3, 12, 5, 0, 0]$ $[0, 14, 1], [1, 11, 1]$	2
	2 13	$[0, 12, 5, 0, 0], [2, 1, 5, 1, 5], [3, 12, 5, 0, 0]$ $[0, 14, -1], [1, 11, -1]$	2
	2 13	$[0, 12, 1, 0, 0], [2, 1, 5, 1, 5], [3, 12, 1, 0, 0]$ $[0, 14, 1], [1, 11, 1]$	1
srg(28, 12, 6, 4)	2 7	$[0, 8, 5, 0, 0], [1, 12, 5, 0, 0], [3, 1, 5, 1, 5], [4, 6, 3, 0, 0]$ $[0, 8, 1], [1, 19, 1]$	3
	2 7	$[0, 6, 3, 0, 0], [1, 15, 3, 1, 1], [4, 6, 7, 0, 0]$ $[0, 8, 1], [1, 19, 1]$	1
srg(29, 14, 6, 7)	2 7 29	$[0, 28, 5, 0, 0]$ $[0, 14, -1], [1, 14, -1]$ $[0, 15, 1], [1, 13, 1]$	41
srg(36, 14, 4, 6)	2 3	$[0, 14, 3, 0, 0], [2, 9, 1, 1, 5], [3, 12, 1, 0, 0]$ $[0, 14, -1], [1, 7, -1], [2, 2, 1], [3, 12, 1]$	73
	2 3	$[0, 14, 7, 0, 0], [2, 9, 5, 1, 5], [3, 12, 1, 0, 0]$ $[0, 14, 1], [1, 7, 1], [2, 2, -1], [3, 12, -1]$	39
	2 3	$[0, 14, 3, 0, 0], [2, 9, 5, 1, 1], [3, 12, 5, 0, 0]$ $[0, 14, -1], [1, 7, -1], [2, 2, 1], [3, 12, 1]$	36
	2 3	$[0, 12, 5, 0, 0], [1, 2, 7, 0, 0], [2, 11, 3, 1, 1], [3, 10, 3, 0, 0]$ $[0, 14, -1], [1, 7, -1], [2, 2, 1], [3, 12, 1]$	12
	2 3	$[0, 12, 5, 0, 0], [1, 2, 7, 0, 0], [2, 11, 7, 1, 5], [3, 10, 7, 0, 0]$ $[0, 14, -1], [1, 7, -1], [2, 2, 1], [3, 12, 1]$	8
	2 3	$[0, 14, 7, 0, 0], [2, 9, 1, 1, 1], [3, 12, 5, 0, 0]$ $[0, 14, 1], [1, 7, 1], [2, 2, -1], [3, 12, -1]$	5
	2 3	$[0, 14, 7, 0, 0], [2, 9, 5, 1, 5], [3, 12, 1, 0, 0]$ $[0, 13, -1], [1, 9, -1], [2, 1, -1], [3, 12, -1]$	2
	2 3	$[0, 14, 7, 0, 0], [2, 9, 5, 1, 5], [3, 12, 1, 0, 0]$ $[0, 12, -1], [1, 9, -1], [2, 4, 1], [3, 10, 1]$	2
	2 3	$[0, 10, 3, 0, 0], [1, 4, 1, 0, 0], [2, 13, 5, 1, 1], [3, 8, 5, 0, 0]$ $[0, 14, -1], [1, 7, -1], [2, 2, 1], [3, 12, 1]$	2
	2 3	$[0, 8, 5, 0, 0], [1, 6, 7, 0, 0], [2, 15, 3, 1, 1], [3, 6, 3, 0, 0]$ $[0, 14, -1], [1, 7, -1], [2, 2, 1], [3, 12, 1]$	1

TABLE 5. The p-adic symbols for quadratic form of combinatorial Laplacian with respect to $\mathrm{srg}(n, k, \lambda, \mu)$ by using Sage.

Graph Type	p	p-Adic Symbols	#Graphs
srg(40, 12, 2, 4)	2	$[0,16,5,0,0],[1,8,1,0,0],[3,1,3,1,3],[5,14,3,0,0]$	17
	5	$[0,16,-1],[1,23,1]$	
	2	$[0,14,7,0,0],[1,10,3,0,0],[3,1,7,1,7],[4,2,7,0,0],[5,12,1,0,0]$	5
	5	$[0,16,-1],[1,23,1]$	
	2	$[0,14,7,0,0],[1,10,3,0,0],[3,1,3,1,3],[4,2,3,0,0],[5,12,1,0,0]$	3
	5	$[0,16,-1],[1,23,1]$	
	2	$[0,12,1,0,0],[1,12,5,0,0],[3,1,3,1,3],[4,4,5,0,0],[5,10,7,0,0]$	1
	5	$[0,16,-1],[1,23,1]$	
	2	$[0,12,1,0,0],[1,12,5,0,0],[3,1,7,1,7],[4,4,1,0,0],[5,10,7,0,0]$	1
	5	$[0,16,-1],[1,23,1]$	
	2	$[0,10,7,0,0],[1,14,3,0,0],[3,1,7,1,7],[4,6,7,0,0],[5,8,1,0,0]$	1
	5	$[0,16,-1],[1,23,1]$	
srg(45, 12, 3, 3)	2	$[0,44,5,0,0]$	19
	3	$[0,20,1],[1,4,-1],[2,2,-1],[3,18,-1]$	
	5	$[0,21,-1],[1,23,-1]$	
	2	$[0,44,5,0,0]$	18
	3	$[0,19,-1],[1,6,1],[2,1,-1],[3,18,-1]$	
	5	$[0,21,-1],[1,23,-1]$	
	2	$[0,44,5,0,0]$	13
	3	$[0,20,-1],[1,4,1],[2,2,1],[3,18,1]$	
	5	$[0,21,-1],[1,23,-1]$	
	2	$[0,44,5,0,0]$	8
	3	$[0,19,1],[1,6,-1],[2,1,1],[3,18,1]$	
	5	$[0,21,-1],[1,23,-1]$	
	2	$[0,44,5,0,0]$	6
	3	$[0,18,1],[1,6,-1],[2,4,-1],[3,16,-1]$	
	5	$[0,21,-1],[1,23,-1]$	
	2	$[0,44,5,0,0]$	6
	3	$[0,17,1],[1,8,-1],[2,3,1],[3,16,1]$	
	5	$[0,21,-1],[1,23,-1]$	
	2	$[0,44,5,0,0]$	3
	3	$[0,18,-1],[1,6,1],[2,4,1],[3,16,1]$	
	5	$[0,21,-1],[1,23,-1]$	
	2	$[0,44,5,0,0]$	3
	3	$[0,17,-1],[1,8,1],[2,3,-1],[3,16,-1]$	
	5	$[0,21,-1],[1,23,-1]$	
	2	$[0,44,5,0,0]$	2
	3	$[0,15,-1],[1,10,1],[2,5,-1],[3,14,-1]$	
	5	$[0,21,-1],[1,23,-1]$	

TABLE 6. The p-adic symbols for quadratic form of combinatorial Laplacian with respect to $\mathrm{srg}(n,k,\lambda,\mu)$ by using Sage.

Appendix B. Codes

B.1. Sage Code for Computing p-Adic Symbols of Quadratic Forms.

```
def pAdic(A, dim):
    A = matrix(ZZ, dim, A);
    Q = QuadraticForm(ZZ, A);
    rslt = Q.CS_genus_symbol_list();
    return rslt;
```

B.2. Sage Code for Checking Local Equivalence of Quadratic Forms.

```
def CheckLocalIso(A1, A2, dim):
    A1 = matrix(ZZ, dim, A1);
    A2 = matrix(ZZ, dim, A2);
    Q1 = QuadraticForm(ZZ, A1);
    Q2 = QuadraticForm(ZZ, A2);
    rslt = Q1.is_locally_equivalent_to(Q2, check_primes_only=True);
    return rslt;
```

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