EXISTENCE THEOREMS FOR A MULTIDIMENSIONAL CRYSTAL SURFACE MODEL*

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Abstract. In this paper we study the existence assertion of the initial boundary value problem for the equation $\frac{\partial u}{\partial t} = \Delta e^{-\Delta u}$. This problem arises in the mathematical description of the evolution of crystal surfaces. Our investigations reveal that the exponent in the equation can have a singular part in the sense of the Lebesgue decomposition theorem, and the exponential nonlinearity somehow "cancels" it out. The net result is that we obtain a solution u that satisfies the equation and the initial boundary conditions in the almost everywhere (a.e.) sense.

Key words. existence, nonlinear fourth-order parabolic equations, nonlinear functions of distributions, crystal surface model

AMS subject classifications. 35D30, 35Q99, 35A01

DOI. 10.1137/16M1059400

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$ and $T > 0$. Consider the initial boundary value problem

$$\frac{\partial u}{\partial t} = \Delta e^{-\Delta u} \quad \text{in } \Omega_T \equiv \Omega \times (0,T), \tag{1}$$

$$\nabla u \cdot \nu = \nabla e^{-\Delta u} \cdot \nu = 0 \quad \text{on } \Sigma_T \equiv \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x) \quad \text{on } \Omega, \tag{3}$$

where ν is the unit outward normal vector to the boundary.

Equation (1) can be used to describe the evolution of a crystal surface. In this case, u is the surface height. Crystal films have been used in many high-tech devices, and thus how to accurately predict the motion of a crystal surface has attracted a lot of attention. Several PDE models have been proposed; see [7] and [8] and the references therein. Crystal films are obtained as the continuum limit of a family of kinetic Monte Carlo models of crystal surface relaxation that includes both the solid-on-solid and discrete Gaussian models. In [8], equations of the form

$$\frac{\partial u}{\partial t} = \Delta e^{-\mathrm{div}[\sigma(\nabla u)]}, \quad \sigma(\xi) = \nabla V(\xi), \tag{4}$$

have been derived. The potential function V is a homogeneous function of degree p. In many applications, one can simply take $V(\xi) = |\xi|^p$. In [8], a number of interesting qualitative features of the large scale behavior of the equations are investigated numerically for various values of p. Our equation (1) roughly corresponds to the case where $p = 2$. The objective of this paper is to offer a rigorous mathematical analysis of this equation. It represents the first step in our plan to investigate the more complicated models. We refer the reader to [3] for a related one-dimensional model.

*Received by the editors February 1, 2016; accepted for publication (in revised form) August 2, 2016; published electronically November 1, 2016.

http://www.siam.org/journals/sima/48-6/M105940.html

Funding: The work of the first author was partially supported by KI-Net NSF RNMS grant 1107291 and NSF grant DMS 1514826.

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We can also look at our problem from another perspective. Denote an energy functional by $\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$. Equation (1) can be recast as a Cahn–Hilliard equation with curvature-dependent mobility,

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(e^{\frac{\delta \mathcal{E}}{\delta u}}\nabla \frac{\delta \mathcal{E}}{\delta u}\right), \tag{5}$$

and it possesses an energy-dissipation relation

$$\frac{d\mathcal{E}}{dt} = -\int_\Omega e^{\frac{\delta\mathcal{E}}{\delta u}}|\nabla \frac{\delta \mathcal{E}}{\delta u}|^2\,dx = -4\int_\Omega |\nabla e^{\frac{1}{2}\frac{\delta\mathcal{E}}{\delta u}}|^2\,dx.$$

The exponential nonlinearity in the curvature-dependent mobility models the asymmetric behavior of the convex and concave crystal surface in solid-on-solid interface growth. This shares some similarities with the asymmetric exclusion process [2]. Some in-depth discussions on the exponential nonlinearity and numerical simulations were conducted in [8]. A more general energy functional can be taken as $\mathcal{E}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx$, $1 \le p \le 2$.

Fourth-order nonlinear parabolic equations arise in a variety of physical settings. Two well-known examples are the thin film equation [1] and the quantum drift-diffusion model [6]. A well-known difficulty in the study of fourth-order equations is that the maximum principle is no longer valid for such equations. As a result, the techniques one often uses in the analysis of second-order equations are mostly not applicable. To gain some insight into our problem, we perform some formal analysis. That is, we assume that u is a classical solution of (1) in our subsequent calculations. To begin, we move the term on the right-hand side of (1) to the left-hand side, square both sides of the resulting equation, integrate it over Ω, note that

$$-\int_\Omega \partial_t u\,\Delta e^{-\Delta u}\,dx = -\int_\Omega \partial_t \Delta u\, e^{-\Delta u}\,dx = \frac{d}{dt}\int_\Omega e^{-\Delta u(x,t)}\,dx,$$

and thereby obtain

$$2\frac{d}{dt}\int_\Omega e^{-\Delta u(x,t)}\,dx + \int_\Omega (\partial_t u)^2\,dx + \int_\Omega \left(\Delta e^{-\Delta u}\right)^2\,dx = 0.$$

Integrating this over $(0,s)$, $0 < s \le T$, yields

$$\begin{aligned} &2\int_\Omega e^{-\Delta u(x,s)}\,dx + \int_{\Omega_s} (\partial_t u)^2\,dx\,dt \\ &+\int_{\Omega_s} \left(\Delta e^{-\Delta u}\right)^2\,dx\,dt = 2\int_\Omega e^{-\Delta u_0(x)}\,dx, \end{aligned} \tag{6}$$

where $\Omega_s = \Omega \times (0,s)$. This is our first a priori estimate.

Take the gradient of both sides of (1), take the dot product of the resulting equation with ∇u, and then integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u(x,t)|^2\,dx + 4\int_\Omega |\nabla e^{-\frac{\Delta u}{2}}|^2\,dx = 0. \tag{7}$$

Here we have used the fact that

$$\begin{aligned} \int_\Omega \nabla\Delta e^{-\Delta u}\cdot\nabla u\,dx &= \int_\Omega \nabla e^{-\Delta u}\cdot\nabla\Delta u\,dx \\ &= -\int_\Omega e^{-\Delta u}|\nabla \Delta u|^2\,dx = -4\int_\Omega |\nabla e^{-\frac{\Delta u}{2}}|^2\,dx. \end{aligned}$$

Integrate (7) with respect to t to arrive at our second a priori estimate,

$$\frac{1}{2}\int_\Omega |\nabla u(x,s)|^2\,dx + 4\int_{\Omega_s} |\nabla e^{-\frac{\Delta u}{2}}|^2\,dx\,dt = \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2\,dx. \tag{8}$$

Now we differentiate (1) with respect to t, multiply through the resulting equation by $\partial_t u$, and integrate over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (\partial_t u)^2\,dx &= \int_\Omega \partial_t \Delta e^{-\Delta u}\,\partial_t u\,dx = \int_\Omega \partial_t e^{-\Delta u}\,\partial_t \Delta u\,dx\\ &= -\int_\Omega e^{-\Delta u}\,|\partial_t \Delta u|^2\,dx = -4\int_{\Omega_s} |\partial_t e^{-\frac{\Delta u}{2}}|^2\,dx,\end{aligned}$$

from whence follows our third a priori estimate,

$$\begin{aligned}\frac{1}{2}\int_\Omega |\partial_t u(x,s)|^2\,dx + 4\int_{\Omega_s} |\partial_t e^{-\frac{\Delta u}{2}}|^2\,dx\,dt &= \frac{1}{2}\int_\Omega |\partial_t u(x,0)|^2\,dx\\ &= \frac{1}{2}\int_\Omega |\Delta e^{-\Delta u_0(x)}|^2\,dx.\end{aligned} \tag{9}$$

Set

$$\rho = e^{-\Delta u}.$$

Then we have

$$-\Delta u = \ln\rho \quad \text{on } \Omega. \tag{10}$$

Integrate the above equation over Ω to obtain

$$\int_\Omega \ln\rho\,dx = 0.$$

Keeping this and (6) in mind, we estimate

$$\begin{aligned}\int_\Omega |\ln\rho|\,dx &= \int_\Omega (\ln\rho)^+\,dx + \int_\Omega (\ln\rho)^-\,dx\\ &= 2\int_\Omega (\ln\rho)^+\,dx - \int_\Omega \ln\rho\,dx \le 2\int_\Omega \rho\,dx \le 2\int_\Omega e^{-\Delta u_0(x)}\,dx.\end{aligned} \tag{11}$$

To summarize our preceding calculations, we have the following:

(E1) $\partial_t u, |\nabla u| \in L^\infty(0,T;L^2(\Omega)),\ \Delta u \in L^\infty(0,T;L^1(\Omega))$;

(E2) $\rho \in L^\infty(0,T;L^1(\Omega)),\ \Delta\rho \in L^\infty(0,T;L^2(\Omega)),\ \sqrt{\rho} \in W^{1,2}(\Omega_T)$.

Note that by taking the Laplacian of both sides of (1) we can derive the equation

$$\frac{\partial \ln\rho}{\partial t} + \Delta^2\rho = 0.$$

Unfortunately, this equation is of little use to us. As we have seen, we need to take advantage of the specific nonlinear structure of the original equation (1) to obtain enough a priori estimates.

We are ready to present our definition of a weak solution.

Definition. *We say that a pair* (u,ρ) *is a weak solution to* (1)–(3) *if the following conditions hold:*

(D1) $\rho \in L^\infty(0,T;W^{2,2}(\Omega))$, $u \in C([0,T];L^2(\Omega))$, $\frac{\partial u}{\partial t}, |\nabla u| \in L^\infty(0,T;L^2(\Omega))$, $\Delta u \in \mathcal{M}(\overline{\Omega_T}) \cap L^2(0,T;\left(W^{1,2}(\Omega)\right)^*)$, *where* $(W^{1,2}(\Omega))^*$ *is the dual space of* $W^{1,2}(\Omega)$ *and* $\mathcal{M}(\overline{\Omega_T})$ *is the space of signed Radon measures on* $\overline{\Omega_T}$.

(D2) *The term* $-\Delta u$ *has a decomposition*

$$-\Delta u = g_a + \nu_s$$

with respect to the Lebesgue measure in the sense of the Lebesgue decomposition theorem [4, p. 42], *where* g_a *is the absolutely continuous part of* $-\Delta u$ *and* ν_s *is the singular part. That is, the support of* $\nu_s \equiv A_0$ *has Lebesgue measure* 0. *Then there holds*

$$\rho = e^{g_a} \quad \textit{almost everywhere (a.e.) on } \Omega_T. \tag{12}$$

(D3) *We have*

$$\frac{\partial u}{\partial t} = \Delta\rho \quad \textit{a.e. on } \Omega_T, \tag{13}$$

$$\nabla\rho\cdot\nu = 0 \quad \textit{a.e. on } \Sigma_T. \tag{14}$$

The initial condition (3) *is satisfied in the space* $C([0,T];L^2(\Omega))$, *while the boundary condition* $\nabla u\cdot\nu = 0$ *on* Σ_T *is satisfied in the sense*

$$\int_0^T \langle -\Delta u, \xi\rangle\, dt = \int_{\Omega_T} \nabla u\cdot\nabla\xi\, dx\, dt \quad \textit{for all } \xi \in L^2\left(0,T;W^{1,2}(\Omega)\right),$$

where $\langle\cdot,\cdot\rangle$ *is the duality pairing between* $\left(W^{1,2}(\Omega)\right)^*$ *and* $W^{1,2}(\Omega)$.

We would like to make a few remarks about the definition. The possible existence of a singular part in $-\Delta u$ is due to the fact that (1) offers little a priori control over the exponent term. To be precise, by (E1) we only have an L^1 bound on this term. This constitutes the main difficulty in our mathematical analysis of the problem. It does seem counterintuitive that we can still have (12)–(13) in spite of the fact that $\nu_s \neq 0$. We can gain some insight by examining this example. Set

$$K_j(s) = \begin{cases} 0 & \text{if } s > \frac{1}{j}, \\ -\frac{j}{2} & \text{if } \frac{1}{j} \le s \le \frac{1}{j}, \\ 0 & \text{if } s < -\frac{1}{j}, \qquad j = 1, 2, \ldots. \end{cases}$$

It is well known that $K_j \rightharpoonup \delta_0$ weakly in the space of measures, where δ_0 is the Dirac delta function concentrated at 0. However, the sequence $\{e^{K_j}\}$ is well behaved. In fact, we have $e^{K_j} \to 1$ strongly in $L^s(-1,1)$ for each $s \ge 1$. The key to our success is that we have managed to show that A_0, the support of ν_s, is roughly the set where $-\Delta u$ is negative infinity. Formally, the composite function $e^{-\Delta u}$ is 0 there. Thus this function can still behave rather well. It is the exploitation of this subtle cancellation effect that constitutes the core of our development.

Obviously, g_a here is not the negative Laplacian of u in the sense of distributions. In section 3, we will offer another perspective from which to view the function. It can be taken to be the negative Laplacian of u in a pointwise a.e. sense [9].

Physically, the singularities represent rupture defects in the surface evolution due to the asymmetric nature in the exponential curvature-dependent mobility. Rupture in the epitaxial growth models was carefully studied numerically in [8].

Our main result is the following theorem.

THEOREM 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary, and assume $u_0 \in W^{2,\infty}(\Omega) \cap W^{4,2}(\Omega)$. Then there is a weak solution to* (1)–(3)*. Furthermore, the uniqueness assertion holds for those solutions whose Laplacians have no singular parts. That is, there is only one weak solution u to* (1)–(3) *with $\nu_s = 0$.*

For each $T > 0$ there is a weak solution u on Ω_T. In this sense, our weak solution is global.

Our result is also sharp. We cannot rule out the existence of a singular part even in the case where the space dimension is 1. We can illustrate this in the following stationary solution. Let $\Omega = (-1, 1)$. Define

$$u(x) = \begin{cases} -(x+1)^2 + 1 & \text{if } -1 \le x \le 0, \\ -(x-1)^2 + 1 & \text{if } 0 < x \le 1. \end{cases}$$

An elementary calculation shows that $-\Delta u = 2 - 4\delta_0$. By our definition, u is a stationary solution of (1)–(2). More generally, given any $\nu_0 \in \left(W^{1,2}(\Omega)\right)^* \cap \mathcal{M}(\overline{\Omega})$, consider the problem

$$\begin{aligned} -\Delta u &= -\frac{\nu_0(\overline{\Omega})}{|\Omega|} + \nu_0 \quad \text{in } \Omega, \\ \nabla u \cdot \nu &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

This problem has a solution u in the space $W^{1,2}(\Omega)$. Obviously, it is also a stationary solution of (1)–(2), provided that the support of ν_0 has Lebesgue measure 0. An interesting open question arises: if the initial function is very smooth, how long does it take for $-\Delta u$ to develop singularity?

The uniqueness part of Theorem 1 is easy. We will prove it right here. Let v be another solution of (1)–(3) with the property that the singular part of $-\Delta v$ is 0. Then we have

$$\frac{\partial}{\partial t}(u - v) = \Delta e^{-\Delta u} - \Delta e^{-\Delta v}.$$

Multiply through this equation by $u - v$ and then integrate the resulting equation over Ω to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega (u-v)^2\, dx &= \int_\Omega \left(\Delta e^{-\Delta u} - \Delta e^{-\Delta v}\right)(u-v)\, dx \\ &= \langle \Delta u - \Delta v, e^{-\Delta u} - e^{-\Delta v}\rangle. \end{aligned}$$

Integrate the above equation over $(0, s)$ to yield

$$\begin{aligned} \frac{1}{2}\int_\Omega (u(x,s) - v(x,s))^2\, dx &= \int_0^s \langle \Delta u - \Delta v, e^{-\Delta u} - e^{-\Delta v}\rangle\, dt \\ &= \int_{\Omega_s} (\Delta u - \Delta v)(e^{-\Delta u} - e^{-\Delta v})\, dx\, dt \le 0, \end{aligned}$$

where $\Omega_s = \Omega \times (0, s)$. This completes the proof.

An easy way to remove the singular part in $-\Delta u$ is by adding a lower order perturbation to (1). To be precise, we consider the problem

$$\frac{\partial u}{\partial t} = \Delta e^{-\Delta u} + \delta \Delta u \quad \text{in } \Omega_T, \tag{15}$$

$$\nabla u \cdot \nu = \nabla e^{-\Delta u} \cdot \nu = 0 \quad \text{on } \Sigma_T, \tag{16}$$

$$u(x, 0) = u_0(x) \quad \text{on } \Omega, \tag{17}$$

where $\delta > 0$ is a small perturbation parameter. In this case, we will have $\Delta u \in L^2(\Omega_T)$. To see this, we use $-\Delta u$ as a test function in (15) to obtain

$$\frac{1}{2}\int_\Omega |\nabla u(x,s)|^2\,dx + 4\int_{\Omega_s} |\nabla e^{-\frac{1}{2}\Delta u}|^2\,dx\,dt + \delta\int_{\Omega_s} (\Delta u)^2\,dx\,dt = \frac{1}{2}\int_\Omega |\nabla u_0(x)|^2\,dx. \tag{18}$$

Our development will imply the existence of a unique weak solution $u = u_\delta$ to (15)–(17) and its weak* convergence to a weak solution of (1)–(3) in the space $L^\infty(0,T;W^{1,2}(\Omega))$ as $\delta \to 0$. That is, we have the following corollary.

COROLLARY 1. *Under the assumptions of Theorem* 1 *there is a unique weak solution u_δ to* (15)–(17) *with $\Delta u_\delta \in L^2(\Omega_T)$. Furthermore, $u_\delta \rightharpoonup u$ weak* in $L^\infty(0,T;W^{1,2}(\Omega))$ as $\delta \to 0$, where u is a weak solution to* (1)–(3).

A solution to (1)–(3) will be constructed as the limit of a sequence of approximate solutions. The key is to design an approximation scheme so that all the calculations in the derivation of (6)–(11) can be justified. This is accomplished in sections 2 and 3. To be more specific, in section 2 we state a few preparatory lemmas and present our approximate problem. The existence of a classical solution is established for the problem. We form a sequence of approximate solutions based upon implicit discretization in the time variable. Section 3 is devoted to the proof that the estimates (6)–(11) are all preserved for the sequence, and this is enough to justify passing to the limit.

2. Approximate problems. Before we present our approximate problems, we state a few preparatory lemmas.

LEMMA 1. *Let Ω be a bounded domain in $\mathbb{R}^N$.*

(i) *If Ω is convex, then*

$$\int_\Omega (\Delta u)^2\,dx \ge \int_\Omega |\nabla^2 u|^2\,dx \quad \text{for all } u \in W^{2,2}(\Omega) \text{ with } \nabla u\cdot\nu = 0 \text{ on } \partial\Omega. \tag{19}$$

(ii) *If $\partial\Omega$ is C^2, then there is a positive constant c depending only on N, Ω, and the smoothness of the boundary such that*

$$\int_\Omega (\Delta u)^2 dx + \int_\Omega |\nabla u|^2 dx \ge c\int_\Omega |\nabla^2 u|^2 dx \tag{20}$$

for all $u \in W^{2,2}(\Omega)$ with $\nabla u \cdot \nu = 0$ on $\partial\Omega$.

We refer the reader to [10] for some background information on this lemma.

Our existence theorem is based on the following fixed point theorem, which is often called the Leray–Schauder theorem [5, p. 280].

LEMMA 2. *Let B be a map from a Banach space $\mathcal{B}$ into itself. Assume that*

(H1) *B is continuous;*
(H2) *the images of bounded sets of B are precompact; and*
(H3) *there exists a constant c such that*

$$\|z\|_{\mathcal{B}} \le c$$

for all $z \in \mathcal{B}$ and $\sigma \in [0,1]$ satisfying $z = \sigma B(z)$.

Then B has a fixed point.

Relevant interpolation inequalities for Sobolev spaces are listed in the following lemma.

Lemma 3. *Let Ω be a bounded domain in $\mathbb{R}^N$. Denote by $\|\cdot\|_p$ the norm in the space $L^p(\Omega)$. Then we have the following:*

1. $\|f\|_q \le \varepsilon\|f\|_r + \varepsilon^\sigma\|f\|_p$, *where* $\varepsilon > 0$, $p \le q < r$, *and* $\sigma = (\frac{1}{p} - \frac{1}{q})/(\frac{1}{q} - \frac{1}{r})$;
2. *for each* $\varepsilon > 0$ *and each* $p \in [2, 2^*)$, *where* $2^* = \frac{2N}{N-2}$ *if* $N > 2$ *and any number bigger than 2 if* $N = 2$, *there is a positive number* $c = c(\varepsilon, p)$ *such that*

$$\|f\|_p \le \varepsilon\|\nabla f\|_2 + c\|f\|_1 \quad \text{for all } f \in W^{1,2}(\Omega), \tag{21}$$

$$\|\nabla g\|_p \le \varepsilon\|\nabla^2 g\|_2 + c\|g\|_1 \quad \text{for all } g \in W^{2,2}(\Omega). \tag{22}$$

Finally, we collect a few frequently used elementary inequalities in the following lemma.

Lemma 4. *For* $x, y \in \mathbb{R}^N$, $s, t \in \mathbb{R}$, *and* $a, b \in \mathbb{R}^+$, *we have the following:*

3. $x \cdot (x - y) \ge \frac{1}{2}(|x|^2 - |y|^2)$;
4. *if f is an increasing function on $\mathbb{R}$ and F an antiderivative of f, then*

$$f(s)(s - t) \ge F(s) - F(t).$$

In particular, there hold the inequalities

$$e^s(s - t) \ge e^s - e^t \quad \text{and} \tag{23}$$

$$s(e^s - e^t) \ge e^s s - e^t t - (s - t); \tag{24}$$

5. *we have*

$$(e^s - e^t)(s - t) \ge 2\left(e^{\frac{s}{2}} - e^{\frac{t}{2}}\right)^2; \tag{25}$$

6. *there hold*

$$(a + b)^\alpha \le a^\alpha + b^\alpha \quad \text{if } 0 < \alpha \le 1,$$
$$(a + b)^\alpha \le 2^{\alpha - 1}(a^\alpha + b^\alpha) \quad \text{if } \alpha > 1,$$
$$ab \le \varepsilon a^p + \frac{1}{\varepsilon^{q/p}} b^q \text{ if } \varepsilon > 0,\ p,\ q > 1 \text{ with } \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

Obviously, only (24) and (25) deserve some attention. To see (24), we calculate

$$\begin{aligned} s(e^s - e^t) &= \ln e^s(e^s - e^t) \\ &\ge \int_{e^t}^{e^s} \ln\tau\, d\tau = \tau\ln\tau - \tau\,\Big|_{e^t}^{e^s} \\ &= e^s s - e^t t - (s - t). \end{aligned}$$

The proof of (25) is also rather elementary. Without loss of generality, we assume $s > t$. The mean value theorem asserts that there is an $\eta \in (\frac{t}{2}, \frac{s}{2})$ such that

$$e^{\frac{s}{2}} - e^{\frac{t}{2}} = e^\eta \frac{s - t}{2} \le (e^{\frac{s}{2}} + e^{\frac{t}{2}})\frac{s - t}{2}.$$

Multiplying through this inequality by $e^{\frac{s}{2}} - e^{\frac{t}{2}}$ yields the desired result.

Now we are ready to introduce our approximate problems. Let

$$\tau > 0, \quad v \in C^\alpha(\overline{\Omega}), \quad \alpha \in (0, 1), \tag{26}$$

be given. Consider the problem

$$-\Delta e^{\psi} + \tau\psi = -\frac{u - v}{\tau} \quad \text{in } \Omega, \tag{27}$$

$$-\Delta u + \tau u = \psi \quad \text{in } \Omega, \tag{28}$$

$$\nabla u \cdot \nu = \nabla e^{\psi} \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{29}$$

This is our approximating problem. Basically, we have transformed a fourth-order equation into a system of two second-order equations, and furthermore, we have discretized the time derivative, thereby turning a parabolic problem into an elliptic one. The tricky part here is that we have kept the exponential nonlinearity in (27). If we had taken the exponential term to be our new unknown function, which seemed to be a very natural choice at first glance, we would have had to show that the new unknown was positive, which was essentially equivalent to trying to find a nonnegative solution to the equation $-\Delta\varphi + \tau\varphi = f$ when we had no information on how f changed its sign. Obviously, there is no hope of accomplishing that. It is also interesting that we just need to add the two low-order terms $\tau\psi, \tau u$ in order to "regularize" the problem.

PROPOSITION 1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with C^2 boundary, and assume that* (26) *holds. Then there is a classical solution to* (27)*–*(29)*.*

Proof. The existence assertion will be established via the Leray–Schauder theorem. For this purpose, we define an operator B from $L^{\infty}(\Omega)$ into itself as follows: for each $g \in L^{\infty}(\Omega)$ we say $B(g) = \psi$ if ψ is the unique solution of the linear boundary value problem

$$-\text{div}\,(e^{g}\nabla\psi) + \tau\psi = -\frac{u - v}{\tau} \quad \text{in } \Omega, \tag{30}$$

$$\nabla\psi \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{31}$$

where u solves the linear problem

$$-\Delta u + \tau u = g \quad \text{in } \Omega, \tag{32}$$

$$\nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{33}$$

Observe that since $g \in L^{\infty}(\Omega)$, (30)–(31) is uniformly elliptic. According to the classical regularity theory for linear elliptic equations, we can conclude that the two preceding problems both have a unique solution in the space $C^{\beta}(\overline{\Omega})$ for some $\beta \in (0, 1)$ [5, Chap. 8]. We can infer from this that B is well defined and continuous and maps bounded sets into precompact sets. It remains to show that there is a positive number c such that

$$\|\psi\|_{\infty} \le c \tag{34}$$

for all $\psi \in L^{\infty}(\Omega)$ and $\sigma \in [0, 1]$ satisfying

$$\psi = \sigma B(\psi).$$

This equation is equivalent to the boundary value problem

$$-\Delta e^{\psi} + \tau\psi = -\sigma\frac{u - v}{\tau} \quad \text{in } \Omega, \tag{35}$$

$$-\Delta u + \tau u = \psi \quad \text{in } \Omega, \tag{36}$$

$$\nabla u \cdot \nu = \nabla e^{\psi} \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{37}$$

The inequality (34) will be a consequence of the following two claims.

CLAIM 1. $\|\psi\|_p \le \frac{1}{\tau^2}\|u-v\|_p$ *for each* $p>2$. *In particular, we have*

$$\|\psi\|_\infty \le \frac{1}{\tau^2}\|u-v\|_\infty. \tag{38}$$

Let $p>2$ be given. Then the function $|\psi|^{p-2}\psi$ lies in $W^{1,2}(\Omega)$ and $\nabla\left(|\psi|^{p-2}\psi\right) = (p-1)|\psi|^{p-2}\nabla\psi$. Multiply through (35) by this function and integrate the resulting equation over Ω to obtain

$$\begin{aligned}(p-1)\int_\Omega e^\psi|\psi|^{p-2}|\nabla\psi|^2\,dx + \tau\int_\Omega|\psi|^p\,dx &= -\frac{\sigma}{\tau}\int_\Omega(u-v)|\psi|^{p-2}\psi\,dx\\ &\le \frac{1}{\tau}\int_\Omega|u-v||\psi|^{p-1}\,dx\\ &\le \frac{1}{\tau}\|u-v\|_p\|\psi\|_p^{p-1}.\end{aligned}$$

Dropping the first integral in the above inequality yields the claim.

CLAIM 2. $\|\psi\|_2 \le \frac{1}{\tau^2}\|v\|_2$.

Upon using ψ as a test function in (35), we obtain

$$\int_\Omega e^\psi|\nabla\psi|^2\,dx + \tau\int_\Omega\psi^2\,dx = -\frac{\sigma}{\tau}\int_\Omega u\psi\,dx + \frac{\sigma}{\tau}\int_\Omega v\psi\,dx. \tag{39}$$

By using u as a test function in (36), we can derive

$$\int_\Omega u\psi\,dx = \int_\Omega|\nabla u|^2\,dx + \tau\int_\Omega u^2\,dx \ge 0.$$

Keeping this in mind, we deduce from (39) that

$$\tau\int_\Omega\psi^2\,dx \le \frac{\sigma}{\tau}\int_\Omega v\psi\,dx \le \frac{1}{\tau}\int_\Omega|v||\psi|\,dx \le \frac{1}{\tau}\|v\|_2\|\psi\|_2.$$

The claim follows.

Equipped with these two claims, we can employ a bootstrap argument to obtain (34). If $N\le 3$, then Claim 2 and the classical L^∞ estimate [5, p. 189] already imply (34). Thus, from here on, we will assume that

$$N \ge 4.$$

Square both sides of (36) and integrate to obtain

$$\int_\Omega(\Delta u)^2dx + \tau^2\int_\Omega u^2dx + 2\tau\int_\Omega|\nabla u|^2dx = \int_\Omega\psi^2dx.$$

This together with (ii) of Lemma 1 and Claim 2 implies that

$$\|u\|_{W^{2,2}(\Omega)} \le c. \tag{40}$$

If this is enough to yield that

$$\|u\|_q \le c \ \text{ for some } q > \tfrac{N}{2}, \tag{41}$$

then due to Claim 1 we also have that $\|\psi\|_q \le c$. Applying the classical L^∞ estimate to (36), we arrive at

$$\|u\|_\infty \le c\|u\|_2 + c\|\psi\|_q \le c. \tag{42}$$

This combined with (38) gives the desired result. If (40) is not enough for (41) to hold, by the Sobolev embedding theorem we must have that $N \ge 8$ and

$$\|u\|_{\frac{2N}{N-4}} \le c.$$

In view of Claim 1 and the classical regularity theory for linear elliptic equations, we have

$$\|u\|_{W^{2,\frac{2N}{N-4}}(\Omega)} \le c.$$

Does this imply (41)? If it does, we are done. If not, repeat the above argument. Obviously, we can reach (41) in a finite number of steps. Since ψ is bounded, we can conclude from the classical regularity theory that (ψ, u) is a classical solution. The proof is complete. ∎

3. Proof of Theorem 1. The proof of Theorem 1 will be divided into several propositions. To begin, we present our approximation scheme. This is based on Proposition 1. Then we proceed to derive estimates similar to (6) and (8)–(11) for our approximate problems. These estimates are shown to be sufficient to justify passing to the limit.

Let $T > 0$ be given. For each $j \in \{1, 2, \dots, \}$ we divide the time interval $[0, T]$ into j equal subintervals. Set

$$\tau = \frac{T}{j}.$$

We discretize (1)–(3) as follows. For $k = 1, \dots, j$, we recursively solve the system

$$\frac{u_k - u_{k-1}}{\tau} - \Delta e^{\psi_k} + \tau\psi_k = 0 \quad \text{in } \Omega, \tag{43}$$

$$-\Delta u_k + \tau u_k = \psi_k \quad \text{in } \Omega, \tag{44}$$

$$\nabla e^{\psi_k} \cdot \nu = \nabla u_k = 0 \quad \text{on } \partial\Omega. \tag{45}$$

Introduce the functions

$$\tilde{u}_j(x,t) = \frac{t - t_{k-1}}{\tau} u_k(x) + \left(1 - \frac{t - t_{k-1}}{\tau}\right) u_{k-1}(x),\ x \in \Omega,\ t \in (t_{k-1}, t_k], \tag{46}$$

$$\bar{u}_j(x,t) = u_k(x), \quad x \in \Omega, \quad t \in (t_{k-1}, t_k], \tag{47}$$

$$\bar{\psi}_j(x,t) = \psi_k(x), \quad x \in \Omega, \quad t \in (t_{k-1}, t_k], \tag{48}$$

where $t_k = k\tau$. We can rewrite (43)–(45) as

$$\frac{\partial \tilde{u}_j}{\partial t} - \Delta e^{\bar{\psi}_j} + \tau\bar{\psi}_j = 0 \quad \text{in } \Omega_T, \tag{49}$$

$$-\Delta \bar{u}_j + \tau \bar{u}_j = \bar{\psi}_j \quad \text{in } \Omega_T. \tag{50}$$

We proceed to derive a priori estimates for the sequence of approximate solutions $\{\tilde{u}_j, \bar{u}_j, \bar{\psi}_j\}$. The discretized version of (6) is the following proposition.

PROPOSITION 2. *We have*

$$2\max_{0\le t\le T}\int_\Omega e^{\bar\psi_j(x,t)}\,dx+\int_{\Omega_T}\left(\Delta e^{\bar\psi_j}\right)^2\,dxdt+\int_{\Omega_T}\left(\frac{\partial\tilde u_j}{\partial t}\right)^2\,dxdt$$
$$+\tau^2\int_{\Omega_T}\bar\psi_j{}^2\,dxdt+2\tau\int_{\Omega_T}e^{\bar\psi_j}|\nabla\bar\psi_j|^2\,dxdt+2\tau\int_{\Omega_T}|\nabla e^{\bar\psi_j}|^2\,dxdt$$
$$+2\tau^2\int_{\{\bar\psi_j>0\}}\bar\psi_je^{\bar\psi_j}\,dxdt+\max_{0\le t\le T}\left(\tau\int_\Omega|\nabla\bar u_j|^2\,dx+\tau^2\int_\Omega\bar u_j{}^2\,dx\right)$$
$$\le 6\int_\Omega e^{\psi_0}\,dx+3\tau\int_\Omega|\nabla u_0|^2\,dx+3\tau^2\int_\Omega u_0^2\,dx+6\tau^2e^{-1}|\Omega_T|,$$

where

(51) $$\psi_0=-\Delta u_0+\tau u_0.$$

Proof. For each $k\in\{1,2,\dots,j\}$, we square both sides of (43), integrate the resulting equation over Ω, and thereby obtain

$$\int_\Omega\left(\left(\Delta e^{\psi_k}\right)^2+\tau^2\psi_k{}^2+\left(\frac{u_k-u_{k-1}}{\tau}\right)^2\right)dx$$
(52) $$-2\tau\int_\Omega\psi_k\,\Delta e^{\psi_k}\,dx-\frac{2}{\tau}\int_\Omega\Delta e^{\psi_k}(u_k-u_{k-1})\,dx+2\int_\Omega\psi_k(u_k-u_{k-1})\,dx=0.$$

Now we begin to evaluate the last three integrals on the left-hand side of the above equation. First, we easily have

(53) $$-2\tau\int_\Omega\psi_k\,\Delta e^{\psi_k}\,dx=2\tau\int_\Omega\nabla\psi_k\cdot\nabla e^{\psi_k}\,dx=2\tau\int_\Omega e^{\psi_k}|\nabla\psi_k|^2\,dx.$$

After calculating the integral $-\frac{2}{\tau}\int_\Omega\Delta e^{\psi_k}(u_k-u_{k-1})\,dx$, we multiply through (43) by e^{ψ_k} and then integrate over Ω to yield

(54) $$-\int_\Omega e^{\psi_k}(u_k-u_{k-1})\,dx=\tau\int_\Omega|\nabla e^{\psi_k}|^2\,dx+\tau^2\int_\Omega\psi_ke^{\psi_k}\,dx.$$

In view of (44), we obtain

(55) $$-\Delta(u_k-u_{k-1})+\tau(u_k-u_{k-1})=\psi_k-\psi_{k-1}.$$

Keeping (55), (23), and (54) in mind, we estimate

$$-\frac{2}{\tau}\int_\Omega\Delta e^{\psi_k}(u_k-u_{k-1})\,dx=-\frac{2}{\tau}\int_\Omega e^{\psi_k}\Delta(u_k-u_{k-1})\,dx$$
$$=\frac{2}{\tau}\int_\Omega e^{\psi_k}(\psi_k-\psi_{k-1})\,dx-2\int_\Omega e^{\psi_k}(u_k-u_{k-1})\,dx$$
$$\ge\frac{2}{\tau}\int_\Omega\left(e^{\psi_k}-e^{\psi_{k-1}}\right)\,dx$$
(56) $$+2\tau\int_\Omega|\nabla e^{\psi_k}|^2\,dx+2\tau^2\int_\Omega\psi_k\,e^{\psi_k}\,dx.$$

The last integral in (52) can be estimated as follows:

$$\begin{aligned} 2\int_\Omega \psi_k(u_k-u_{k-1})\,dx &= 2\int_\Omega(-\Delta u_k+\tau u_k)(u_k-u_{k-1})\,dx \\ &= 2\int_\Omega \nabla u_k\nabla(u_k-u_{k-1})\,dx + 2\tau\int_\Omega u_k(u_k-u_{k-1})\,dx \\ &\geq \int_\Omega\left(|\nabla u_k|^2-|\nabla u_{k-1}|^2\right)dx+\tau\int_\Omega\left({u_k}^2-{u_{k-1}}^2\right)dx. \end{aligned} \tag{57}$$

Collecting (53), (56), and (57) in (52) gives

$$\begin{aligned} &\int_\Omega\left(\left(\Delta e^{\psi_k}\right)^2+\tau^2{\psi_k}^2+\left(\frac{u_k-u_{k-1}}{\tau}\right)^2\right)dx \\ &+2\tau\int_\Omega e^{\psi_k}|\nabla\psi_k|^2\,dx+\frac{2}{\tau}\int_\Omega\left(e^{\psi_k}-e^{\psi_{k-1}}\right)dx+2\tau\int_\Omega|\nabla e^{\psi_k}|^2\,dx \\ &+2\tau^2\int_\Omega \psi_k\, e^{\psi_k}\,dx+\int_\Omega\left(|\nabla u_k|^2-|\nabla u_{k-1}|^2\right)dx \\ &+\tau\int_\Omega\left({u_k}^2-{u_{k-1}}^2\right)dx\leq 0. \end{aligned} \tag{58}$$

Multiplying through the inequality by τ and summing up the resulting one over k, we obtain that for each $s\in\{\tau,2\tau,\dots,j\tau\}$ there holds

$$\begin{aligned} &2\int_\Omega e^{\bar\psi_j(x,s)}\,dx+\int_{\Omega_s}\left(\Delta e^{\bar\psi_j}\right)^2dxdt+\int_{\Omega_s}\left(\frac{\partial\tilde u_j}{\partial t}\right)^2dxdt \\ &+\tau^2\int_{\Omega_s}{\bar\psi_j}^2\,dxdt+2\tau\int_{\Omega_s}e^{\bar\psi_j}|\nabla\bar\psi_j|^2\,dxdt+2\tau\int_{\Omega_s}|\nabla e^{\bar\psi_j}|^2\,dxdt \\ &+2\tau^2\int_{\{\bar\psi_j>0\}}\bar\psi_j\, e^{\bar\psi_j}\,dxdt+\max_{0\leq t\leq T}\left(\tau\int_\Omega|\nabla\bar u_j|^2\,dx+\tau^2\int_\Omega{\bar u_j}^2\,dx\right) \\ &\leq 2\int_\Omega e^{\psi_0}\,dx+\tau\int_\Omega|\nabla u_0|^2\,dx+\tau^2\int_\Omega u_0^2\,dx-2\tau^2\int_{\{\bar\psi_j\leq 0\}}\bar\psi_j e^{\bar\psi_j}\,dxdt. \end{aligned}$$

It is not difficult to see that the above inequality is valid for each $s\in(0,T]$. Also, the function $\theta(s)=se^s\geq -e^{-1}$ for $s\in(-\infty,0)$, and thus the desired result follows. ◻

Proposition 3. *There holds*

$$\begin{aligned} &\frac12\max_{0\leq t\leq T}\int_\Omega|\nabla\bar u_j|^2\,dx+\int_{\Omega_T}e^{\bar\psi_j}|\nabla\bar\psi_j|^2\,dxdt+\frac{\tau}{2}\max_{0\leq t\leq T}\int_\Omega{\bar u_j}^2\,dx \\ &\quad+\tau\int_{\Omega_T}\left(\Delta\bar u_j\right)^2dx+\tau^2\int_{\Omega_T}|\nabla\bar u_j|^2\,dx \\ &\leq\frac32\int_\Omega|\nabla u_0|^2\,dx+\frac{3\tau}{2}\int_{\Omega_T}u_0^2\,dx. \end{aligned} \tag{59}$$

Obviously, this proposition is the discretized version of (8).

Proof. Take the gradient of both sides of (43), then take the dot product of the resulting equation with ∇u_k, and integrate over Ω to obtain

$$\int_\Omega\frac{\nabla u_k-\nabla u_{k-1}}{\tau}\cdot\nabla u_k\,dx-\int_\Omega\nabla\left(\Delta e^{\psi_k}\right)\cdot\nabla u_k\,dx+\tau\int_\Omega\nabla\psi_k\cdot\nabla u_k\,dx=0. \tag{60}$$

The second integral in the preceding equation is computed as follows:

$$
\begin{aligned}
-\int_\Omega \nabla\left(\Delta e^{\psi_k}\right)\cdot\nabla u_k\,dx &= \int_\Omega \Delta e^{\psi_k}\,\Delta u_k\,dx\\
&= \int_\Omega \Delta e^{\psi_k}\left(-\psi_k+\tau u_k\right)dx\\
&= \int_\Omega e^{\psi_k}|\nabla e^{\psi_k}|^2\,dx+\tau\int_\Omega \Delta e^{\psi_k}\,u_k\,dx. \qquad (61)
\end{aligned}
$$

To estimate the last integral in the above expression, we multiply through (43) by u_k and then integrate to obtain

$$
\tau\int_\Omega \Delta e^{\psi_k}\,u_k\,dx=\int_\Omega (u_k-u_{k-1})\,u_k\,dx+\tau^2\int_\Omega \psi_k\,u_k\,dx\geq\frac{1}{2}\int_\Omega\left({u_k}^2-{u_{k-1}}^2\right)\,dx. \qquad (62)
$$

The last step is due to the fact that (44) asserts that

$$
\int_\Omega \psi_k\,u_k\,dx=\int_\Omega|\nabla u_k|^2\,dx+\tau\int_\Omega {u_k}^2\,dx\geq 0.
$$

Combining (61) with (62) yields

$$
-\int_\Omega \nabla\left(\Delta e^{\psi_k}\right)\cdot\nabla u_k\,dx\geq\int_\Omega e^{\psi_k}|\nabla e^{\psi_k}|^2\,dx+\frac{1}{2}\int_\Omega\left({u_k}^2-{u_{k-1}}^2\right)\,dx. \qquad (63)
$$

Calculating the last integral in (60), we have

$$
\begin{aligned}
\tau\int_\Omega \nabla\psi_k\cdot\nabla u_k\,dx &= -\tau\int_\Omega \psi_k\,\Delta u_k\,dx\\
&= -\tau\int_\Omega\left(-\Delta u_k+\tau u_k\right)\Delta u_k\,dx\\
&= \tau\int_\Omega\left(\Delta u_k\right)^2\,dx+\tau^2\int_\Omega|\nabla u_k|^2\,dx. \qquad (64)
\end{aligned}
$$

Substituting (63) and (64) into (60) yields

$$
\begin{aligned}
&\frac{1}{2\tau}\int_\Omega\left(|\nabla u_k|^2-|\nabla u_{k-1}|^2\right)dx+\int_\Omega e^{\psi_k}|\nabla e^{\psi_k}|^2dx\\
&+\frac{1}{2}\int_\Omega\left({u_k}^2-{u_{k-1}}^2\right)dx+\tau\int_\Omega\left(\Delta u_k\right)^2dx+\tau^2\int_\Omega|\nabla u_k|^2dx\leq 0. \qquad (65)
\end{aligned}
$$

Then the proposition follows from multiplying through the above inequality by τ and summing up the resulting one over k. ∎

To obtain the discretized version of (9), we need the following result.

PROPOSITION 4. *We have*

$$
\int_\Omega\left(\frac{u_1-u_0}{\tau}\right)^2\,dx\leq c. \qquad (66)
$$

Here and in what follows, the letter c denotes a positive number independent of τ.

Proof. We easily infer from (58) that

$$\int_\Omega \left(\frac{u_1-u_0}{\tau}\right)^2 dx + \frac{2}{\tau}\int_\Omega e^{\psi_1}(\psi_1-\psi_0)\,dx$$
$$+2\tau^2\int_\Omega \psi_1\, e^{\psi_1}\,dx \le \int_\Omega |\nabla u_0|^2\,dx + \tau\int_\Omega u_0^2\,dx. \tag{67}$$

Thus we only need to make sure that the second integral in the above inequality does not cause any problems. For this purpose, we estimate that

$$\begin{aligned}\frac{2}{\tau}\int_\Omega e^{\psi_1}(\psi_1-\psi_0)\,dx &= \frac{2}{\tau}\int_\Omega (e^{\psi_1}-e^{\psi_0})(\psi_1-\psi_0)\,dx\\ &\quad+\frac{2}{\tau}\int_\Omega e^{\psi_0}(\psi_1-\psi_0)\,dx\\ &\ge \frac{2}{\tau}\int_\Omega e^{\psi_0}(\psi_1-\psi_0)\,dx\\ &= \frac{2}{\tau}\int_\Omega e^{\psi_0}(-\Delta(u_1-u_0)+\tau(u_1-u_0))\,dx\\ &= -\frac{2}{\tau}\int_\Omega \Delta e^{\psi_0}(u_1-u_0)\,dx + 2\int_\Omega e^{\psi_0}(u_1-u_0)\,dx\\ &\ge -\frac{1}{4}\int_\Omega \left(\frac{u_1-u_0}{\tau}\right)^2 dx - 4\int_\Omega \left(\Delta e^{\psi_0}\right)^2 dx\\ &\quad-\frac{1}{4}\int_\Omega \left(\frac{u_1-u_0}{\tau}\right)^2 dx - 4\tau^2\int_\Omega \left(e^{\psi_0}\right)^2 dx,\end{aligned}$$

from whence the proposition follows. ∎

Now we are ready to present the next proposition.

Proposition 5. *We have*

$$\begin{aligned}&\frac{1}{2}\max_{0\le t\le T}\int_\Omega \left(\frac{\partial \tilde{u}_j}{\partial t}\right)^2 dx + \int_{\Omega_T}\left(\frac{\partial \tilde{\sigma}_j}{\partial t}\right)^2 dxdt\\ &\quad\frac{\tau}{2}\int_{\Omega_T}|\nabla e^{\bar{\psi}_j}|^2\,dxdt + \tau^2\max_{0\le t\le T}\int_\Omega \bar{\psi}_j\, e^{\bar{\psi}_j}\,dx\\ &\quad\tau^2\int_{\Omega_T}\left|\frac{\partial}{\partial t}\nabla\tilde{u}_j\right|^2 dxdt + \tau^3\int_{\Omega_T}\left|\frac{\partial \tilde{u}_j}{\partial t}\right|^2 dxdt\\ &\le 3\tau^2\max_{0\le t\le T}\int_\Omega e^{\bar{\psi}_j}\,dx + \frac{3}{2}\int_\Omega\left(\frac{u_1-u_0}{\tau}\right)^2 dx\\ &\quad+\frac{3}{2}\int_\Omega |\nabla e^{\psi_0}|^2\,dx + 3\tau^2\int_\Omega \psi_0\, e^{\psi_0}\,dx \le c,\end{aligned}$$

where

$$\tilde{\sigma}_j(x,t) = \frac{t-t_{k-1}}{\tau}\, e^{\frac{1}{2}\psi_k(x)} + \left(1-\frac{t-t_{k-1}}{\tau}\right)e^{\frac{1}{2}\psi_{k-1}(x)}, \quad x\in\Omega,\ t\in(t_{k-1},t_k].$$

Proof. For $k=2,3,\dots,j$, we derive from (43) that

$$\frac{1}{\tau}\left(\frac{u_k-u_{k-1}}{\tau}-\frac{u_{k-1}-u_{k-2}}{\tau}\right)-\Delta\left(\frac{e^{\psi_k}-e^{\psi_{k-1}}}{\tau}\right)+\tau\frac{\psi_k-\psi_{k-1}}{\tau}=0 \quad \text{in } \Omega. \tag{68}$$

Multiply through this equation by $\frac{u_k-u_{k-1}}{\tau}$ and integrate the resulting equation over Ω to obtain

$$\frac{1}{\tau}\int_\Omega\left(\frac{u_k-u_{k-1}}{\tau}-\frac{u_{k-1}-u_{k-2}}{\tau}\right)\frac{u_k-u_{k-1}}{\tau}\,dx$$
$$-\int_\Omega\Delta\left(\frac{e^{\psi_k}-e^{\psi_{k-1}}}{\tau}\right)\frac{u_k-u_{k-1}}{\tau}\,dx+\int_\Omega\frac{\psi_k-\psi_{k-1}}{\tau}(u_k-u_{k-1})\,dx=0. \tag{69}$$

The second integral on the left-hand side of the above equation can be evaluated as follows:

$$\begin{aligned}-\int_\Omega\Delta\left(\frac{e^{\psi_k}-e^{\psi_{k-1}}}{\tau}\right)\frac{u_k-u_{k-1}}{\tau}\,dx&=-\int_\Omega\left(\frac{e^{\psi_k}-e^{\psi_{k-1}}}{\tau}\right)\Delta\left(\frac{u_k-u_{k-1}}{\tau}\right)dx\\&=\int_\Omega\left(\frac{e^{\psi_k}-e^{\psi_{k-1}}}{\tau}\right)\left(\frac{\psi_k-\psi_{k-1}}{\tau}\right)dx\\&\quad-\int_\Omega\left(\frac{e^{\psi_k}-e^{\psi_{k-1}}}{\tau}\right)(u_k-u_{k-1})\,dx.\end{aligned} \tag{70}$$

By virtue of (25), we have

$$\left(\frac{e^{\psi_k}-e^{\psi_{k-1}}}{\tau}\right)\left(\frac{\psi_k-\psi_{k-1}}{\tau}\right)dx\geq 2\left(\frac{e^{\frac{1}{2}\psi_k}-e^{\frac{1}{2}\psi_{k-1}}}{\tau}\right)^2. \tag{71}$$

To derive the estimate on the last integral in (70), we multiply through (43) by $e^{\psi_k}-e^{\psi_{k-1}}$ and obtain

$$-\int_\Omega\frac{u_k-u_{k-1}}{\tau}(e^{\psi_k}-e^{\psi_{k-1}})\,dx=\int_\Omega\nabla e^{\psi_k}\nabla(e^{\psi_k}-e^{\psi_{k-1}})\,dx \tag{72}$$
$$\begin{aligned}&+\tau\int_\Omega\psi_k(e^{\psi_k}-e^{\psi_{k-1}})\,dx\\&\geq\frac{1}{2}\int_\Omega\left(|\nabla e^{\psi_k}|^2-|\nabla e^{\psi_{k-1}}|^2\right)dx\\&+\tau\int_\Omega\left(e^{\psi_k}\psi_k-e^{\psi_{k-1}}\psi_{k-1}\right)dx-\tau\int_\Omega\left(e^{\psi_k}-e^{\psi_{k-1}}\right)dx.\end{aligned} \tag{73}$$

The last step is due to (24). The third integral on the left-hand side of (69) can be evaluated as follows:

$$\begin{aligned}\int_\Omega(\psi_k-\psi_{k-1})(u_k-u_{k-1})\,dx&=\int_\Omega\left(-\Delta(u_k-u_{k-1})+\tau(u_k-u_{k-1})\right)(u_k-u_{k-1})\,dx\\&=\int_\Omega|\nabla(u_k-u_{k-1})|^2\,dx+\tau\int_\Omega|u_k-u_{k-1}|^2\,dx.\end{aligned} \tag{74}$$

Using (70)–(74) in (69) yields

$$\begin{aligned}&\frac{1}{2\tau}\int_\Omega\left(\left(\frac{u_k-u_{k-1}}{\tau}\right)^2-\left(\frac{u_{k-1}-u_{k-2}}{\tau}\right)^2\right)dx+2\int_\Omega\left(\frac{e^{\frac{1}{2}\psi_k}-e^{\frac{1}{2}\psi_{k-1}}}{\tau}\right)^2dx\\&+\frac{1}{2}\int_\Omega\left(|\nabla e^{\psi_k}|^2-|\nabla e^{\psi_{k-1}}|^2\right)dx+\tau\int_\Omega\left(e^{\psi_k}\psi_k-e^{\psi_{k-1}}\psi_{k-1}\right)dx\\&-\tau\int_\Omega\left(e^{\psi_k}-e^{\psi_{k-1}}\right)dx+\int_\Omega|\nabla(u_k-u_{k-1})|^2\,dx+\tau\int_\Omega|u_k-u_{k-1}|^2\,dx\leq 0.\end{aligned} \tag{75}$$

Multiply through the inequality by τ, sum the resulting inequality over k, and thereby establish the desired result. □

An immediate consequence of Proposition 5 is that

$$\max_{0\le t\le T}\int_\Omega \left(\Delta e^{\bar{\psi}_j}\right)^2 dx \le c. \tag{76}$$

To see this, we recall from (49) that

$$\frac{\partial \tilde{u}_j}{\partial t} = \Delta e^{\bar{\psi}_j} - \tau\bar{\psi}_j \quad \text{in } \Omega_T. \tag{77}$$

Square both sides of this equation and integrate the resulting equation over Ω to obtain

$$\begin{aligned}\int_\Omega \left(\frac{\partial \tilde{u}_j}{\partial t}\right)^2 dx &= \int_\Omega \left(\Delta e^{\bar{\psi}_j}\right)^2 dx + \tau^2\int_\Omega \left(\bar{\psi}_j\right) dx - 2\tau\int_\Omega \bar{\psi}_j\, \Delta e^{\bar{\psi}_j}\, dx\\ &= \int_\Omega \left(\Delta e^{\bar{\psi}_j}\right)^2 dx + \tau^2\int_\Omega \left(\bar{\psi}_j\right) dx + 2\tau\int_\Omega e^{\bar{\psi}_j}|\nabla\bar{\psi}_j|^2 dx.\end{aligned}$$

This gives (76).

We set

$$\bar{\rho}_j = e^{\bar{\psi}_j}. \tag{78}$$

Then we have

$$-\Delta\bar{u}_j + \tau\bar{u}_j = \bar{\psi}_j = \ln\bar{\rho}_j \quad \text{in } \Omega. \tag{79}$$

The key question to our development is how to justify passing to the limit in the above equation. The difficulty lies in the fact that we do not have enough control over the sequence $\{\ln\bar{\rho}_j\}$. In fact, we only have the following L^1 bound for the sequence.

PROPOSITION 6. *The sequence* $\{\ln\bar{\rho}_j\}$ *is bounded in* $L^\infty(0,T;L^1(\Omega))$.

Proof. We integrate (79) over Ω to obtain

$$\tau\int_\Omega \bar{u}_j\, dx = \int_\Omega \ln\bar{\rho}_j\, dx.$$

On account of Proposition 2, we have

$$\max_{0\le t\le T}\left|\int_\Omega \ln\bar{\rho}_j\, dx\right| \le c.$$

Applying Proposition 2 again, we deduce

$$\begin{aligned}\int_\Omega |\ln\bar{\rho}_j|\, dx &= \int_\Omega \left[(\ln\bar{\rho}_j)^+ + (\ln\bar{\rho}_j)^-\right] dx\\ &= 2\int_{\{\bar{\rho}_j>1\}} \ln\bar{\rho}_j\, dx - \int_\Omega \ln\bar{\rho}_j\, dx\\ &\le 2\int_\Omega \bar{\rho}_j\, dx + c \le c.\end{aligned}$$

The proposition follows. □

What saves the game is, of course, our results in Propositions 2, 3, and 5, which enable us to establish the following proposition.

PROPOSITION 7. *We have that*
(C1) *the sequence* $\{\bar{u}_j\}$ *is bounded in* $L^\infty(0,T;W^{1,2}(\Omega))$*;*
(C2) *the sequence* $\{\bar{\rho}_j\}$ *is bounded in* $L^\infty(0,T;W^{2,2}(\Omega))$*; and*
(C3) *the sequence* $\{\bar{\rho}_j\}$ *is precompact in* $L^2(0,T;W^{1,2}(\Omega))$*.*

Proof. In view of Proposition 3, we just need to show that

$$\max_{0\le t\le T}\int_\Omega |\bar{u}_j|^2 dx \le c. \tag{80}$$

To this end, we estimate

$$\begin{aligned}\tilde{u}_j{}^2(x,s) - u_0^2(x) &= \int_0^s \frac{\partial}{\partial t}\tilde{u}_j{}^2(x,t)\,dt = 2\int_0^s \tilde{u}_j(x,t)\frac{\partial}{\partial t}\tilde{u}_j(x,t)\,dt\\ &\le \int_0^s \tilde{u}_j{}^2(x,t)\,dt + \int_0^s\left(\frac{\partial}{\partial t}\tilde{u}_j(x,t)\right)^2 dt.\end{aligned}$$

Integrate this inequality over Ω and then apply Gronwall's inequalty to obtain

$$\max_{0\le t\le T}\int_\Omega |\tilde{u}_j|^2\,dx \le c. \tag{81}$$

For $t\in(t_{k-1},t_k]$, we have

$$\begin{aligned}\tilde{u}_j(x,t) - \bar{u}_j(x,t) &= \frac{t-t_k}{\tau}(u_k - u_{k-1})\\ &= (t-t_k)\frac{\partial}{\partial t}\tilde{u}_j(x,t).\end{aligned}$$

Consequently, we derive, with the aid of Proposition 3, that

$$\int_\Omega (\tilde{u}_j(x,t) - \bar{u}_j(x,t))^2\,dx \le \tau^2\int_\Omega\left(\frac{\partial}{\partial t}\tilde{u}_j(x,t)\right)^2 dx \le c\tau^2,\quad t\in(t_{k-1},t_k].$$

This implies (C1).

To prove (C2), we first conclude from (ii) of Lemma 1 that

$$\int_\Omega |\nabla^2\bar{\rho}_j|^2\,dx \le c\int_\Omega (\Delta\bar{\rho}_j)^2\,dx + c\int_\Omega |\nabla\bar{\rho}_j|^2\,dx.$$

In view of (22), we deduce that for each $\varepsilon>0$ there holds

$$\begin{aligned}\int_\Omega |\nabla\bar{\rho}_j|^2 dx &\le \varepsilon\int_\Omega |\nabla^2\bar{\rho}_j|^2 dx + c(\varepsilon)\left(\int_\Omega \bar{\rho}_j dx\right)^2\\ &\le c\varepsilon\left(\int_\Omega (\Delta\bar{\rho}_j)^2\,dx + \int_\Omega |\nabla\bar{\rho}_j|^2 dx\right) + c(\varepsilon).\end{aligned}$$

By choosing ε sufficiently small, we obtain

$$\int_\Omega |\nabla\bar{\rho}_j|^2 dx \le c. \tag{82}$$

We can easily conclude from (21) that $\{\bar{\rho}_j\}$ is also bounded in $L^\infty(0,T;L^2(\Omega))$. Furthermore, according to the Sobolev embedding theorem, we have that

(R1) $\{\bar{\rho}_j\}$ is bounded in $L^\infty(\Omega_T)$ if $N \leq 3$;
(R2) $\{\bar{\rho}_j\}$ is bounded in $L^\infty(0,T;L^p(\Omega))$ for each $p > 1$ if $N = 4$; and
(R3) $\{\bar{\rho}_j\}$ is bounded in $L^\infty(0,T;L^{\frac{2N}{N-4}}(\Omega))$ if $N > 4$.

To establish (C3), we first claim that $\{\tilde{\sigma}_j\}$ is bounded in $W^{1,2}(\Omega_T)$. To see this, we calculate, keeping in mind Proposition 3 and the fact that $\int_\Omega e^{\psi_k}|\nabla\psi_k|^2dx = 4\int_\Omega |\nabla e^{\frac{1}{2}\psi_k}|^2dx$, that

$$\begin{aligned}
\int_{\Omega_T} |\nabla\tilde{\sigma}_j|^2\,dxdt &= \sum_{k=1}^{j}\int_{t_{k-1}}^{t_k}\int_\Omega \left|\frac{t-t_k}{\tau}\nabla e^{\frac{1}{2}\psi_k} + \left(1-\frac{t-t_k}{\tau}\right)\nabla e^{\frac{1}{2}\psi_{k-1}}\right|^2 dxdt \\
&\leq \sum_{k=1}^{j}\int_{t_{k-1}}^{t_k}\left[\frac{t-t_k}{\tau}\int_\Omega \left|\nabla e^{\frac{1}{2}\psi_k}\right|^2 dx + \left(1-\frac{t-t_k}{\tau}\right)\int_\Omega \left|\nabla e^{\frac{1}{2}\psi_{k-1}}\right|^2 dx\right] dt \\
&= \sum_{k=1}^{j}\tau\left[\int_\Omega \left|\nabla e^{\frac{1}{2}\psi_k}\right|^2 dx + \int_\Omega \left|\nabla e^{\frac{1}{2}\psi_{k-1}}\right|^2 dx\right] \\
&\leq c\left[2\int_{\Omega_T}\left|\nabla e^{\frac{1}{2}\bar{\psi}_j}\right|^2 dxdt + \tau\int_\Omega \left|\nabla e^{\frac{1}{2}\psi_0}\right|^2 dx\right] \leq c.
\end{aligned}$$

This and Proposition 5 yield the claim.

Now we can conclude from the claim that $\{\tilde{\sigma}_j\}$ is precompact in $L^2(\Omega_T)$. Note that

$$\begin{aligned}
\int_{\Omega_T} |\tilde{\sigma}_j - \sqrt{\bar{\rho}_j}|^2 dxdt &= \sum_{k=1}^{j}\int_{t_{k-1}}^{t_k}(t-t_k)^2\int_\Omega\left(\frac{e^{\frac{1}{2}\psi_k} - e^{\frac{1}{2}\psi_{k-1}}}{\tau}\right)^2 dxdt \\
&= \sum_{k=1}^{j}\tau^3\int_\Omega\left(\frac{\partial}{\partial t}\tilde{\sigma}_j\right)^2 dx \\
&= \tau^2\int_{\Omega_T}\left(\frac{\partial}{\partial t}\tilde{\sigma}_j\right)^2 dxdt \leq c\tau^2. \qquad (83)
\end{aligned}$$

Subsequently, we also have that $\{\sqrt{\bar{\rho}_j}\}$ is precompact in $L^2(\Omega_T)$. As a result, we can select a subsequence of $\{\sqrt{\bar{\rho}_j}\}$, still denoted by $\{\sqrt{\bar{\rho}_j}\}$, such that

$$\sqrt{\bar{\rho}_j} \text{ converges a.e. on } \Omega_T. \qquad (84)$$

Thus $\bar{\rho}_j = \left(\sqrt{\bar{\rho}_j}\right)^2$ also converges a.e. on Ω_T. This together with (R1)–(R3) implies that $\{\bar{\rho}_j\}$ is precompact in $L^2(\Omega_T)$.

To complete the proof of (C3), we compute

$$\int_{\Omega_T}|\nabla(\bar{\rho}_j - \bar{\rho}_i)|^2\,dxdt = \int_{\Omega_T}(\Delta e^{\bar{\psi}_j} - \Delta e^{\bar{\psi}_i})(\bar{\rho}_j - \bar{\rho}_i)\,dxdt \leq c\left(\int_{\Omega_T}(\bar{\rho}_j - \bar{\rho}_i)^2\,dxdt\right)^{\frac{1}{2}} \qquad (85)$$

for each i, j. This immediately yields (C3). The proof is finished. □

We are ready to prove Theorem 1.

Proof. Passing to subsequences if necessary, we may assume that

$$\bar{u}_j \rightharpoonup u \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)), \qquad (86)$$
$$\bar{\rho}_j \to \rho \text{ strongly in } L^2(0,T;W^{1,2}(\Omega)) \text{ and a.e. on } \Omega_T. \qquad (87)$$

With the aid of Fatou's lemma, we deduce from Proposition 6 that

$$\int_{\Omega_T} |\ln \rho|\, dxdt \leq \liminf_{j\to\infty} \int_{\Omega_T} |\ln \bar{\rho}_j|\, dxdt \leq c.$$

Therefore, the set

$$A_0 = \{(x,t) \in \Omega_T : \rho(x,t) = 0\}$$

has Lebesgue measure 0. This combined with (87) asserts that

$$\ln \bar{\rho}_j \to \ln \rho \quad \text{a.e. on } \Omega_T. \tag{88}$$

Obviously, we have from (86) that

$$\tau \bar{u}_j \to 0 \text{ strongly in } L^2(0,T;W^{1,2}(\Omega)), \text{ and thus a.e on } \Omega_T \tag{89}$$

(passing to a subsequence if need be). Recall (79) to obtain

$$-\Delta \bar{u}_j \to \ln \rho \quad \text{a.e. on } \Omega_T. \tag{90}$$

On the other hand, we conclude from Propositions 3 and 6 that the sequence $\{-\Delta \bar{u}_j\}$ is bounded in both $L^1(\Omega_T)$ and $L^2(0,T;\left(W^{1,2}(\Omega)\right)^*)$. Hence we have

$$-\Delta \bar{u}_j \rightharpoonup -\Delta u \equiv \mu \quad \text{weakly in both } \mathcal{M}(\overline{\Omega_T}) \text{ and } L^2\left(0,T;\left(W^{1,2}(\Omega)\right)^*\right). \tag{91}$$

The key issue is the following: do we have

$$-\Delta u = \mu = \ln \rho?$$

The following proposition addresses this issue.

Proposition 8. *The restriction of μ to the set $\overline{\Omega_T} \setminus A_0$ is a function. This function is exactly $\ln \rho$. That is, the Lebesgue decomposition of μ with respect to the Lebesgue measure is $\ln \rho + \nu_s$, where ν_s is a measure supported in A_0, and we have*

$$\rho = e^{\mu} \quad \text{on the set } \overline{\Omega_T} \setminus A_0. \tag{92}$$

That is, $\ln \rho$ is the function g_a in the definition of a weak solution.

Proof. Keep in mind that since $\mu \in L^2(0,T;(W^{1,2}(\Omega))^*)$, each function in $L^2\left(0,T;W^{1,2}(\Omega)\right)$ is μ-measurable, and thus it is well defined except on a set of μ measure 0. Furthermore, $\int_0^T \langle \mu, v\rangle\, dt = \int_{\Omega_T} v\, d\mu$ for each $v \in L^2(0,T;W^{1,2}(\Omega))$. For $\varepsilon > 0$ let θ_ε be a smooth function on $\mathbb{R}$ having the properties

$$\theta_\varepsilon(s) = \begin{cases} 1 & \text{if } s \geq 2\varepsilon, \\ 0 & \text{if } s \leq \varepsilon, \end{cases} \qquad \text{and}$$

$$0 \leq \theta_\varepsilon \leq 1 \quad \text{on } \mathbb{R}.$$

Then it is easy to verify that we still have

$$\theta_\varepsilon(\bar{\rho}_j) \to \theta_\varepsilon(\rho) \quad \text{strongly in } L^2(0,T;W^{1,2}(\Omega)). \tag{93}$$

Pick a function ξ from $C^\infty(\overline{\Omega_T})$. Multiply through (79) by $\xi\, \theta_\varepsilon(\bar{\rho}_j)$ and integrate the resulting equation over Ω to obtain

$$-\int_{\Omega_T} \Delta \bar{u}_j \theta_\varepsilon(\bar{\rho}_j)\, \xi\, dxdt + \tau \int_{\Omega_T} \bar{u}_j\, \theta_\varepsilon(\bar{\rho}_j)\, \xi\, dxdt = \int_{\Omega_T} \ln \bar{\rho}_j\, \theta_\varepsilon(\bar{\rho}_j)\, \xi\, dxdt. \tag{94}$$

For each fixed ε we can infer from Proposition 2 that the sequence $\{\ln\bar{\rho}_j\,\theta_\varepsilon(\bar{\rho}_j)\}$ is bounded in $L^p(\Omega_T)$ for any $p>1$. This, along with (87), gives

$$\int_{\Omega_T}\ln\bar{\rho}_j\,\theta_\varepsilon(\bar{\rho}_j)\,\xi\,dxdt\to\int_{\Omega_T}\theta_\varepsilon(\rho)\ln\rho\,\xi\,dxdt.$$

Observe from (93) and (91) that

$$-\int_{\Omega_T}\Delta\bar{u}_j\,\theta_\varepsilon(\bar{\rho}_j)\,\xi\,dxdt=\int_0^T\langle-\Delta\bar{u}_j,\theta_\varepsilon(\bar{\rho}_j)\,\xi\rangle\,dt\to\int_{\Omega_T}\theta_\varepsilon(\rho)\,\xi\,d\mu. \tag{95}$$

Taking $j\to\infty$ in (94) yields

$$\int_{\Omega_T}\theta_\varepsilon(\rho)\,\xi\,d\mu=\int_{\Omega_T}\theta_\varepsilon(\rho)\ln\rho\,\xi\,dxdt. \tag{96}$$

Obviously, $\rho\in L^2\left(0,T;W^{1,2}(\Omega)\right)$, and thus it is well defined except on a set of μ measure 0. We can easily conclude from the definition of θ_ε that $\{\theta_\varepsilon(\rho)\}$ converges everywhere on the set where ρ is defined as $\varepsilon\to0$. With the aid of the dominated convergence theorem, we can take $\varepsilon\to0$ in (96) to obtain

$$\int_{\Omega_T\setminus A_0}\xi\,d\mu=\int_{\Omega_T\setminus A_0}\ln\rho\,\xi\,dxdt.$$

This is true for every $\xi\in C^\infty(\overline{\Omega_T})$, which means

$$\mu=\ln\rho\quad\text{on }\Omega_T\setminus A_0. \tag{97}$$

The proof is complete. □

With Proposition 8, we have concluded the proof of Theorem 1. □

Finally, we present another angle to look at the function g_a. Note that

$$\begin{aligned}-\int_{\Omega_T}\Delta\bar{u}_j\,\theta_\varepsilon(\bar{\rho}_j)\,\xi\,dxdt&=\int_{\Omega_T}(\nabla\bar{u}_j\cdot\nabla\theta_\varepsilon(\bar{\rho}_j))\,\xi\,dxdt+\int_{\Omega_T}(\nabla\bar{u}_j\cdot\nabla\xi)\,\theta_\varepsilon(\bar{\rho}_j)\,dxdt\\&\to\int_{\Omega_T}(\nabla u\cdot\nabla\theta_\varepsilon(\rho))\,\xi\,dxdt+\int_{\Omega_T}(\nabla u\cdot\nabla\xi)\,\theta_\varepsilon(\rho)\,dxdt.\end{aligned}$$

It immediately follows that

$$\int_{\Omega_T}(\nabla u\cdot\nabla\theta_\varepsilon(\rho))\,\xi\,dxdt+\int_{\Omega_T}(\nabla u\cdot\nabla\xi)\,\theta_\varepsilon(\rho)\,dxdt=\int_{\Omega_T}\theta_\varepsilon(\rho)\ln\rho\,\xi\,dxdt$$

for each test function ξ. That is,

$$\theta_\varepsilon(\rho)\,\ln\rho=\nabla u\cdot\nabla\theta_\varepsilon(\rho)-\operatorname{div}\left(\nabla u\,\theta_\varepsilon(\rho)\right)\quad\text{in }\Omega_T.$$

Restricting the equation to the set $\{(x,t)\in\Omega_T:\rho(x,t)>2\varepsilon\}$ yields

$$\ln\rho=-\Delta u\quad\text{on }\{(x,t)\in\Omega_T:\rho(x,t)>2\varepsilon\}.$$

This is true for each $\varepsilon>0$. We have

$$\ln\rho=-\Delta u\quad\text{on }\Omega_T\setminus A_0.$$

That is to say that we can evaluate the function $-\Delta u(x,t)$ for a.e. $(x,t)\in\Omega_T$. This idea was first used in [9]. It shows a direct connection between $\ln\rho$ and the Laplacian of $-u$.

Acknowledgments. This work was completed while the second author was visiting Duke University. He would like to express his gratitude for the hospitality of the host institution and the financial support from the KI-Net for his visit.

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