ERROR ESTIMATE OF THE PARTICLE METHOD FOR THE b-EQUATION*

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Abstract. In this paper, we establish the optimal error estimate of the particle method for a family of nonlinear evolutionary partial differential equations, or the so-called b-equation. The b-equation, including the Camassa-Holm equation and the Degasperis-Procesi equation, has many applications in diverse scientific fields. The particle method is an approximation of the b-equation in Lagrangian representation. We also prove short-time existence, uniqueness and regularity of the Lagrangian representation of the b-equation.

Key words. Camassa-Holm equation, Degasperis-Procesi equation, Lagrangian representation, classical solution, particle method, peakon solutions, error estimate.

AMS subject classifications. 35B65, 35C08, 35D35, 65M15, 65M75.

1. Introduction. In this paper, the Cauchy problems of the following family of nonlinear evolutionary PDEs (named b-equation) in 1-dimensional case are considered. For $x \in \mathbf{R},\ t > 0$,

$$\partial_t m + \partial_x(um) + (b-1)m\partial_x u = 0, \quad m = (1 - \alpha^2\partial_{xx})u, \tag{1.1}$$

with $b > 1$ and subject to the initial condition

$$m(x, 0) = m_0(x), \quad x \in \mathbf{R}. \tag{1.2}$$

Here, functions $m(x, t)$ and $u(x, t)$ represent the momentum and velocity respectively. The velocity function $u(x, t)$ can also be expressed as a convolution of $m(x, t)$ with the kernel $G(x)$,

$$u(x, t) = G * m = \int_R G(x - y)m(y, t)dy. \tag{1.3}$$

It is well-known that the equation (1.1) has solitary wave solutions of the form $u(x, t) = aG(x - ct)$ with speed $c = -aG(0)$ which is proportional to the amplitude of the solution. The parameter b is the stretching factor. We refer to [25] for details. These kinds of evolutionary equations are established in diverse scientific fields based on different choices of parameter b and a special choice of the kernel. In this paper, we take

$$G(x) = \frac{1}{2\alpha}e^{-|x|/\alpha}, \tag{1.4}$$

where, α is the length scale of kernel and $G(x)$ is the fundamental solution for Helmholtz operator $1 - \alpha^2\partial_{xx}$, i.e., $(1 - \alpha^2\partial_{xx})G = \delta$ and δ is the Dirac δ distribution. For periodic domain, $G(x + 1) = G(x)$, the kernel is given by

$$G(x) = \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})}. \tag{1.5}$$

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When $b = 2$, the associated b-equation is the so-called Camassa-Holm (C-H) equation, which was established by Camassa and Holm to model the unidirectional propagation of waves on the free surface of a shallow layer of water ($u(x,t)$ representing the water's free surface above a flat bottom) [2]. This equation was also independently derived by Dai [17] to model the nonlinear waves in cylindrical hyper-elastic rods with $u(x,t)$ representing the radial stretch relative to a pre-stressed state. In the case of $b = 3$, the associated equation is named the Degasperis-Procesi (D-P) equation and it is used to model the propagation of nonlinear dispersive waves [18]. In higher dimensional cases, the corresponding equation is called the Euler-Poincaré equation, which appears in the mathematical model of fully nonlinear shallow water waves [4, 25]. Beyond these, this equation has many further applications in computer vision [26] and computational anatomy [27].

Mathematical analysis of the Cauchy problems for both the C-H equation and the D-P equation have been extensively studied in the literature. Those researches are mainly concentrated on the well-posedness, such as the existence and uniqueness for the local classical solution, the global weak solutions, and blow-up behavior [1, 8, 14, 32].

However, those kinds of mathematical theories cannot be directly used in the error analysis of the particle method because the particle method is an approximation of the b-equation in Lagrangian representation $X(\xi,t),\ \ p(\xi,t)$. In this paper, we will establish a mathematical theory, such as well-posedness and regularity for the Lagrangian dynamics of the b-equation as described in the following paragraph. To our knowledge, this is the first result in this direction.

Denoting the time derivative of $(X(\xi,t),p(\xi,t))$ by $(\dot{X}(\xi,t),\dot{p}(\xi,t))$ and regarding ξ (we call it Lagrangian label) as a parameter, the Lagrangian dynamics of the b-equation is given by

$$\dot{X}(\xi,t) = \int_{-L}^{L} G(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta, \tag{1.6}$$

$$\dot{p}(\xi,t) = -(b-1)p(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta, \tag{1.7}$$

with initial data $X(\xi,0) = \xi$, $p(\xi,0) = m_0(X(\xi,0)) = m_0(\xi)$. We suppose that supp $m_0 \subset [-L,L]$. The solution to the b-equation, $u(x,t)$ and $m(x,t)$, can be constructed by the Lagrangian representation $(X(\xi,t),p(\xi,t))$ in the following form

$$u(x,t) = \int_{-L}^{L} p(\eta,t)G(x - X(\eta,t))d\eta, \tag{1.8}$$

$$m(x,t) = \int_{-L}^{L} p(\eta,t)\delta(x - X(\eta,t))d\eta. \tag{1.9}$$

We prove local existence of the unique solution to this dynamical system under condition $m_0 \in L^1[-L,L]$ in Theorem 2.1. In this theorem, we also obtain a property of well-separation of trajectory: $X_\xi(\xi,t) \geq C$ for some constant $C > 0$. This property plays a crucial role in the higher order regularity analysis of the Lagrangian dynamics and error estimates of the particle method in Section 2 and 3, respectively. In Theorem 2.2, under the condition $m_0 \in C_c^2[-L,L]$, we prove that $(u(x,t),m(x,t))$, given by (1.8)-(1.9), is the classical solution to the b-equation.

Some traditional numerical methods, such as finite difference methods, finite element methods and spectral methods, have been proposed in [3, 9, 24, 31, 33] to compute the numerical solution for the C-H equation and the D-P equation [30]. It is well known that the C-H equation has solitary solutions with form $u(x,t) = ae^{-|x-ct|}$, named as peakon [15, 20, 28, 11, 12, 13, 19, 29]. A remarkable characteristic of those solutions is the discontinuity in their first derivative at peaks and those peakons are the leading driver of the time evolution. As a result, many of these conventional numerical methods require very fine grids along with adaptive techniques for a better simulation of the behavior of the peakon solution.

Based on the characteristic of peakon solution, in [3], Camassa, Huang and Lee developed a particle method to solve the C-H equation numerically. The particle method is exactly an N-peakon solution to the C-H equation, where N is the number of particles in the particle method. The evolution of peakons is given by a singular system of Ordinary Differential Equations (ODEs), which reassembles the Lagrangian dynamics (1.6)-(1.7). In their paper [3], the authors provided an elegant proof of the global existence of the solution of this ODEs by using the Lax Pair and the complete integrability of the C-H equation when the initial data $m_0 \geq 0$. They also provided a formal error estimate and a clean 2nd order accuracy check with some interesting numerical experiments. In some subsequent works, the particle method had been applied to solve the C-H equation, the Euler-Poincaré equation(2-D) [5, 6, 7, 21] and the D-P equation [19, 25] numerically. Compared with traditional numerical methods, the particle method has two main advantages: (i) it possesses low numerical diffusion which allows one to capture a variety of nonlinear waves with high resolution; (ii) it is easy and accurate to handle the peakon solutions.

To improve the efficiency and accuracy, one can use a mollifier with scale $\epsilon \geq 0$ to regularize the kernel. For $\epsilon > 0$ and $\epsilon = 0$, this method is analogous to the vortex blob method [16] and the point vortex method [23] for the incompressible Euler equations, respectively.

The main purpose of this paper is to provide an optimal error estimate of the particle methods, with or without regularization, for the b-equation. The framework of our analysis is similar to that of vortex method for the incompressible Euler equations [16, 23]. Stability and error estimate of the particle method are first established under an *a-priori assumption* (3.22) where the numerical particles remain well separated like the exact ones. Then, this *a-priori assumption* is justified by the result of error estimate. The advantage of using the *a-priori assumption* is to avoid the singularity of the kernel in estimations. We establish the optimal error estimate between the numerical solution of the particle method and the exact Lagrangian representation $(X(\xi,t), p(\xi,t))$ in Theorem 3.1 for both $\epsilon = 0$ and $\epsilon > 0$. As a by-product, we also prove the existence of the solution to the resulting singular ODEs of the particle method without using the complete integrability and positive condition on the initial data. The error estimate of $u(x,t)$ in C^1 norm is provided by Theorem 3.2 and the error estimate of $m(x,t)$ is measured in the Lipschitz distance in Theorem 3.3.

The remainder of this paper is organized as follows. In Section 2, we introduce the Lagrangian dynamics for the b-equation. We prove the local existence, uniqueness, regularity and a property of well-separation of trajectory: $X_\xi(\xi,t) \geq C > 0$ of these dynamics. In Section 3, we describe the particle method and introduce the mollifier regularization of the particle method. We also provide the error estimate of the numerical integration involving singular kernel G'. Then, we state and prove our main results about the error estimates for the particle method in terms of the Lagrangian representation $(X(\xi,t),\ p(\xi,t))$ and the classical solution $(u(x,t),\ m(x,t))$ to the b-equation. A concluding remark is given in Section 4. The appendix provides some additional details and proofs that were omitted in the main text.

2. Lagrangian dynamics of the b-equation and short-time existence of classical solution. In this section, we first introduce the Lagrangian dynamics of the b-equation. Then, we study the existence and regularity of the Lagrangian representation $(X(\xi,t),\ p(\xi,t))$ to this dynamics. A classical solution $(u(x,t),\ m(x,t))$ of (1.1) can be constructed by this Lagrangian representation. The regularity result obtained in this section will be used in our error analysis of the particle method in Section 3. The following function space will be used in this section.

For nonnegative integers $m,\ n$ and real number $T>0$, we denote $U_T=[-L,L]\times(0,T]$ and introduce the function space [22]

$$C^n_m(U_T):=\left\{u:U_T\to\mathbf{R}:\quad \partial_x^\beta u\in C(U_T),\quad |\beta|\le n;\quad \partial_t^\alpha u\in C(U_T),\quad |\alpha|\le m\right\}.$$

2.1. Lagrangian dynamics of the b-equation. We first introduce the Lagrangian dynamics of the b-equation.

$$\dot{X}(\xi,t)=u(X(\xi,t),t) \tag{2.1}$$

$$\dot{p}(\xi,t)=-(b-1)p(\xi,t)\partial_x u(X(\xi,t),t) \tag{2.2}$$

$$X(\xi,0)=\xi,\quad p(\xi,0)=m_0(\xi),\quad \text{supp}\,\{m_0\}\subset[-L,L], \tag{2.3}$$

where, $\xi\in[-L,L]$ is the Lagrangian label and $u(x,t)$ is given by (1.8).

The solution to (1.1)-(1.2) can be constructed by the Lagrangian representation $(X(\xi,t),\ p(\xi,t))$ as described in (1.8)-(1.9). It is easy to find that $u(x,t)=G*m(x,t)$. Moreover, the following lemma shows that the pair $(u(x,t),\ m(x,t))$ is indeed the classical solution to (1.1)-(1.2).

LEMMA 2.1. *Assuming that $X(\xi,t)\in C^3_1(U_\delta),\ p(\xi,t)\in C^2_1(U_\delta)$ are the solutions of (2.1)-(2.3)-(1.8) and $(u(x,t),m(x,t))$ defined by (1.8)(1.9) satisfying $u(x,t)\in C^3_1(\mathbf{R}\times(0,\delta])$ for some constant $\delta>0$, then, $(u(x,t),m(x,t))$ is the classical solutions of (1.1)-(1.2) .*

Proof. For any test function $\phi\in C_c^\infty(\mathbf{R})$, with notation $\langle\phi,\psi\rangle:=\int_{-\infty}^{+\infty}\phi(x)\psi(x)dx$, a direct computation shows that

$$\langle\phi,m\rangle=\int_{-\infty}^{+\infty}\phi(x)\int_{-L}^{L}\delta(x-X(\eta,t))p(\eta,t))d\eta dx=\int_{-L}^{L}p(\eta,t)\phi(X(\eta,t))d\eta.$$

Therefore, by using (2.1)(2.2), we have

$$\begin{aligned}\langle\phi,m_t\rangle=\frac{d}{dt}\langle\phi,m\rangle&=\int_{-L}^{L}\dot{p}(\eta,t)\phi(X(\eta,t))+p(\eta,t)\phi^{'}(X(\eta,t)))\dot{X}(\eta,t)d\eta\\&=\int_{-L}^{L}\left[-(b-1)p(\eta,t)u_x(X(\eta,t),t)\right]\phi(X(\eta,t))+p(\eta,t)(\phi^{'}u)(X(\eta,t))d\eta.\end{aligned}$$

The above equality can be rewritten as

$$\begin{aligned}\langle\phi,m_t\rangle&=\int_{-L}^{L}p(\eta,t)\int_{-\infty}^{+\infty}\delta(x-X(\eta,t))\left[-(b-1)u_x\phi+\phi^{'}u\right](x)dxd\eta\\&=\int_{-\infty}^{+\infty}\left[-(b-1)u_x\phi+\phi^{'}u\right](x)\int_{-L}^{L}p(\eta,t)\delta(x-X(\eta,t))d\eta dx\\&=\langle\phi^{'}u,m\rangle-(b-1)\langle u_x\phi,m\rangle=-\langle\phi,(um)_x\rangle-(b-1)\langle\phi,mu_x\rangle\\&=-\langle\phi,(um)_x+(b-1)mu_x\rangle.\end{aligned}$$

This means that $m_t + (um)_x + (b-1)mu_x = 0$ since that $\phi(x)$ is arbitrary. □

For further analysis, we need more properties about the kernel G. It is easy to verify that the kernel $G(x) = \frac{1}{2\alpha}e^{-|x|/\alpha}$ satisfies

- $G(x)$ is even function and $G'(x)$ is odd function and $\|G\|_{L^\infty} = \frac{1}{2\alpha}$, $\|G^{'}\|_{L^\infty} = \frac{1}{2\alpha^2}$;
- $|G(a)-G(b)| \le \frac{1}{2\alpha^2}|a-b|, \quad \forall\, a,b; \quad |G^{'}(a)-G^{'}(b)| \le \frac{1}{2\alpha^3}|a-b|, \quad ab > 0.$

In general, we assume that the kernel G satisfies the following properties:

1. $G(x)$ and $G^{'}(x)$ are bounded.
2. The following inequalities hold,

$$|G(a) - G(b)| \le K_1\, |a-b|, \qquad \forall \quad a, b. \tag{2.4}$$

$$|G^{'}(a) - G^{'}(b)| \le K_2\, |a-b|, \qquad ab > 0. \tag{2.5}$$

Before the proof of the existence of solution to (2.1)-(2.3), we state some useful priori estimates.

LEMMA 2.2. *Assuming that $p(\xi,t) \in C_1^2(U_\delta)$ is a solution to (2.1)-(2.3) and $G^{'}$ is an odd function, then $\int_{-L}^{L} p(\eta,t)d\eta$ is independent of time variable t. Particularly, if $p_0 \ge 0$, then $\|p(\cdot,t)\|_{L^1} = const.$ $0 \le t \le \delta$.*

Proof. Using the fact that $G^{'}$ is an odd function, one has

$$\frac{d}{dt}\int_{-L}^{L} p(\eta,t)d\eta = -(b-1)\int_{-L}^{L}\int_{-L}^{L} p(\xi,t)G^{'}(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta d\xi = 0.$$

□

LEMMA 2.3. *Assume that $m_0 \in L^1[-L,L]$ and $(X(\xi,t), p(\xi,t))$ satisfying (2.1)-(2.3) and let $T_0 := \frac{2\alpha^2}{(b-1)\|m_0\|_{L^1}}$. Then, for $0 < t \le T < T_0$, the following estimate holds*

$$\|p(\cdot,t)\|_{L^1} \le \frac{2\alpha^2\|m_0\|_{L^1}}{2\alpha^2 - (b-1)T\|m_0\|_{L^1}} =: B_1.$$

Proof. According to (2.2)(2.3), we have

$$|p(\xi,t)| \le |m_0(\xi)| + (b-1)\int_0^t |p(\xi,s)| \int_{-L}^{L} |G^{'}(X(\xi,s) - X(\eta,s))p(\eta,s)|ds.$$

By integrating this inequality on $[-L, L]$, one has

$$\|p(\cdot,t)\|_{L^1} \le \|m_0\|_{L^1} + (b-1)\|G'\|_{L^\infty}\int_0^t \|p(\cdot,s)\|_{L^1}^2 ds.$$

From Gronwall's inequality and $\|p(\cdot,0)\|_{L^1} = \|m_0\|_{L^1}$, we arrive at

$$\|p(\cdot,t)\|_{L^1} \le \frac{2\alpha^2\|m_0\|_{L^1}}{2\alpha^2 - (b-1)T\|m_0\|_{L^1}}.$$

□

2.2. Local existence and uniqueness of the solution to the Lagrangian dynamics. We denote the right hand side (RHS) of (2.1) and (2.2) as functionals $f(X,p)$, $g(X,p)$, respectively. Then, (2.1)-(2.3) can be rewritten as

$$\dot{X}(\xi,t) = f(X,p) \tag{2.6}$$
$$\dot{p}(\xi,t) = g(X,p) \tag{2.7}$$
$$X(\xi,0) = \xi, \;\; p(\xi,0) = m_0(\xi). \tag{2.8}$$

In this subsection, we establish the short-time existence and uniqueness of the solution to the Lagrangian dynamics (2.6)-(2.8) in the following theorem.

THEOREM 2.1. *Assuming that $m_0(x) \in L^1[-L,L]$ with $\|m_0\|_{L^1} = M_0$, then, there exist constants $C_1 > 0$, $\delta > 0$, such that (2.6)-(2.8) has a unique solution $(X(\xi,t), p(\xi,t))$ satisfying*

$$X(\xi,t) \in C_1^0(U_\delta) \tag{2.9}$$

$$\frac{1}{C_1} \le X_\xi(\xi,t) \le C_1, \qquad (\xi,t) \in U_\delta \tag{2.10}$$

$$p(\cdot,t) \in L^1[-L,L]), \quad \textit{for} \quad 0 < t \le \delta; \quad p(\xi,\cdot) \in C[0,\delta], \quad \textit{for} \quad \xi \in [-L,L]. \tag{2.11}$$

We split the proof of this theorem into the following three items and state them in Lemma 2.4, 2.5 2.6, respectively.

(1) With notation

$$T_1 := \min\left\{\frac{1}{4(b-1)M_0\|G'\|_{L^\infty}}, \quad T_0\right\},$$

we prove that, for any given $p \in C([0,T_1], L^1[-L,L])$ satisfying $\|p\|_{C(0,T_1;L^1[-L,L])} \le 2M_0$, there exists unique solution $X_p(\xi,t) \in C_1^0(U_{t_1})$ to (2.6) for some sufficient small constant $t_1 > 0$. The subscript p represents that X_p depends on p.

(2) We prove that $\partial_\xi X_p(\xi,t)$ exists and satisfies (2.10).

(3) We prove that there exists unique $p(\xi,t)$ satisfying (2.7) and

$$p(\cdot,t) \in L^1[-L,L], \; 0 < t \le \delta; \quad p(\xi,\cdot) \in C[0,\delta], \; \xi \in [-L,L]; \quad \|p\|_{C(0,T_1;L^1[-L,L])} \le 2M_0.$$

To prove (1), for a fixed $p(\xi,t)$, the system (2.6) can be recast as,

$$\dot{X}_p(\xi,t) = \int_{-L}^{L} G(X_p(\xi,t) - X_p(\eta,t))p(\eta,t)d\eta =: F(X_p), \tag{2.12}$$
$$X(\xi,0) = \xi. \tag{2.13}$$

Denote

$$\mathcal{P} = \left\{p(\xi,t) \in C([0,T_1], L^1[-L,L]) \; : \; \max_{0\le t\le t_1} \|p(\cdot,t)\|_{L^1} \le 2M_0, \quad p(\xi,0) = m_0(\xi)\right\} \tag{2.14}$$

then, we have

LEMMA 2.4. *For any given $p(\xi,t) \in \mathcal{P}$, the system (2.12)-(2.13) has a unique solution X_p in $C_1^0(U_{t_1})$, $t_1 = \min\left\{\frac{1}{8M_0K_1}, \frac{1}{\|G\|_{L^\infty}}, \quad T_1\right\}$.*

Proof. (2.12) can be rewritten as its equivalent integral equation

$$Y(\xi,t)=\xi+\int_0^t\int_{-L}^{L}G(Y(\xi,s)-Y(\eta,s))p(\eta,s)d\eta ds=:\mathcal{T}_p(Y). \tag{2.15}$$

Here, we have simplified X_p by Y. We use the contraction mapping theorem to prove existence of a unique solution to (2.15). To this end, by defining the closed subset

$$\mathcal{M}=\left\{Y(\xi,t)\in C(U_{t_1})\ :\ \max_{0\le t\le t_1}\|Y(\cdot,t)\|_{L^\infty}\le C^{**}:=L+2M_0\right\},$$

we should show that the mapping $\mathcal{T}_p$ is a contraction on $\mathcal{M}$ for sufficient small t_1.
Step 1: It is clear that $\mathcal{T}_p(Y)(\xi,t)\in C(U_{t_1})$. By using (2.14), one has

$$\|\mathcal{T}_p(Y)(\cdot,t)\|_{L^\infty}\le L+2M_0t_1\|G\|_{L^\infty}\le C^{**}$$

Then, $\mathcal{T}_p$ maps $\mathcal{M}$ onto itself by taking $t_1\|G\|_{L^\infty}\le 1$.
Step 2. The operator $\mathcal{T}_p$ is a contraction. Actually, by using the property (2.4), we have

$$\begin{aligned}
&|(\mathcal{T}_p(Y_1)-\mathcal{T}_p(Y_2))(\xi,t)|\\
\le&\int_0^t\int_{-L}^{L}|\left[G(Y_1(\xi,s)-Y_1(\eta,s))-G(Y_2(\xi,s)-Y_2(\eta,s))\right]p(\eta,s)|d\eta ds\\
\le&K_1\int_0^t\int_{-L}^{L}\left[|Y_1(\xi,s)-Y_2(\xi,s)|+|Y_1(\eta,s)-Y_2(\eta,s)|\right]|p(\eta,s)|d\eta ds\\
\le&4M_0K_1\int_0^{t_1}\|(Y_1-Y_2)(\cdot,s)\|_{L^\infty}ds.
\end{aligned}$$

This means that

$$\|(\mathcal{T}_p(Y_1)-\mathcal{T}_p(Y_2))\|_{\mathcal{M}}\le 4M_0K_1t_1\|(Y_1-Y_2)\|_{\mathcal{M}},$$

where we have used the notation $\|Y\|_{\mathcal{M}}:=\sup_{t\in[0,t_1]}\|Y(\cdot,t)\|_{L^\infty}$. Therefore, the mapping $\mathcal{T}_p$ is contractive by setting

$$t_1=\min\left\{\frac{1}{8M_0K_1},\quad\frac{1}{\|G\|_{L^\infty}}\right\}.$$

By the contraction mapping theorem, the system (2.12)-(2.13) has a unique solution on $[0,t_1]$. It is obvious that $Y\in C_1^0(U_{t_1})$ since it is the solution to (2.12) and $F(X_p)\in C(U_{t_1})$. This ends the proof. ☐

We now turn to proving that $\partial_\xi X_p(\xi,t)$ exists and satisfies (2.10). This result plays an important role throughout the remainder of this paper. For simplicity in notation, we use $X(\xi,t)$ to denote the solution $X_p(\xi,t)$ of (2.12)-(2.13) in the following lemma.

LEMMA 2.5. *Assume that $p(\xi,t)\in L^1(U_{t_1})$ for some $t_1>0$. let $C_1=\exp(\|G'\|_{L^\infty}\int_0^{t_1}\|p(\cdot,s)\|_{L^1}ds)$, and let $X(\xi,t)$ be the solution of (2.12)-(2.13) in $(0,t_1]$. Then, $X_\xi(\xi,t)$ exists and satisfies*

$$\frac{1}{C_1}\le X_\xi(\xi,t)\le C_1,\qquad(\xi,t)\in U_{t_1}.$$

Proof. By formally differentiating (2.12) with respect to ξ, the existence of $X_\xi(\xi,t)$ is related to the following system

$$\dot{X}_\xi(\xi,t) = X_\xi(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta,$$

with initial data $X_\xi(\xi,0)=1$.

One can find that this linear system has a solution. This in turn means that the derivative of $X(\xi,t)$ with respect to ξ exists and has the following representation

$$X_\xi(\xi,t)=e^{A(\xi,t)},\ A(\xi,t)=\int_0^t\int_{-L}^{L} G^{'}(X(\xi,s)-X(\eta,s))p(\eta,s)d\eta ds.$$

A direct computation shows that

$$|A(\xi,t)|\le \int_0^t |\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta ds| \le \|G^{'}\|_{L^\infty}\int_0^{t_1}\|p(\cdot,s)\|_{L^1}ds.$$

Therefore, our assertion holds with $C_1 = \exp(\|G^{'}\|_{L^\infty}\int_0^{t_1}\|p(\cdot,s)\|_{L^1}ds)$. This ends the proof of this lemma. □

Now, we can state the last part of the proof of Theorem 2.1, the existence and uniqueness of the solution to (2.7).

LEMMA 2.6. *For initial data $m_0(\xi)\in L^1[-L,L]$, there exists $0<\delta\le t_1$, such that the following system*

$$\dot{p}(\xi,t)=-(b-1)p(\xi,t)\int_{-L}^{L} G^{'}(X_p(\xi,t)-X_p(\eta,t))p(\eta,t)d\eta, \tag{2.16}$$

subject to initial data $p(\xi,0)=m_0(\xi)$, has a unique solution $p\in\mathcal{P}(U_\delta)$. Here, $X_p(\xi,t)$ is the solution of (2.12).

Proof. The equivalent integral equation of (2.16) is given by

$$p(\xi,t)=m_0(\xi)-(b-1)\int_0^t p(\xi,s)\int_{-L}^{L} G'(X_p(\xi,s)-X_p(\eta,s))p(\eta,s)d\eta ds =: \mathcal{L}(p).$$

To prove that there exists a unique solution to (2.16), we define the closed subset

$$\mathcal{N} := \big\{p(\xi,t)\ :\quad p(\cdot,t)\in L^1[-L,L],\quad \|p(\cdot,t)\|_{L^1}\le 2M_0,\quad 0\le t\le\delta\big\}.$$

Then, we will show that $\mathcal{L}$ is a contraction mapping on $\mathcal{N}$.

(1) $\mathcal{L}$ maps $\mathcal{N}$ onto $\mathcal{N}$. By noticing that $\delta\le\frac{1}{4(b-1)M_0\|G'\|_{L^\infty}}$, one has

$$\|\mathcal{L}(p)(\cdot,t)\|_{L^1}\le\|m_0\|_{L^1}+4(b-1)M_0^2\delta\|G'\|_{L^\infty}\le 2M_0.$$

(2) $\mathcal{L}$ is a contraction. For simplicity in notation, we denote X_1,X_2 as X_{p_1},X_{p_2}, respectively. Then,

$$|(\mathcal{L}(p_1)-\mathcal{L}(p_2))(\xi,t)|\le(b-1)\int_0^t |I_1(\xi,s)|+|I_2(\xi,s)|ds$$

with notations

$$I_1(\xi,t) = p_1(\xi,t)\int_{-L}^{\xi} p_1(\eta,t)G^{'}(X_1(\xi,t)-X_1(\eta,t))d\eta$$
$$-p_2(\xi,t)\int_{-L}^{\xi} p_2(\eta,t)G^{'}(X_2(\xi,t)-X_2(\eta,t))d\eta,$$
$$I_2(\xi,t) = p_1(\xi,t)\int_{\xi}^{L} p_1(\eta,t)G^{'}(X_1(\xi,t)-X_1(\eta,t))d\eta$$
$$-p_2(\xi,t)\int_{\xi}^{L} p_2(\eta,t)G^{'}(X_2(\xi,t)-X_2(\eta,t))d\eta.$$

In above notations, we have divided the interval $[-L,L]$ into $[-L,\xi]\bigcup[\xi,L]$ to avoid integrant crossing the discontinues point of $G^{'}$ at zero. Consequently, the monotonicity of the flow mapping $X(\xi,t)$ can be used in each sub-interval. We should keep in mind that this kind of integral will always be dealt with in such a way throughout this paper. Then, the first term $I_1(\xi,t)$ can be estimated as follows.

$$|I_1(\xi,t)| \le |(p_1(\xi,t)-p_2(\xi,t))\int_{-L}^{\xi} p_1(\eta,t)G^{'}(X_1(\xi,t)-X_1(\eta,t))d\eta|$$
$$+|p_2(\xi,t)\int_{-L}^{\xi}(p_1(\eta,t)-p_2(\eta,t))G^{'}(X_1(\xi,t)-X_1(\eta,t))d\eta|$$
$$+|p_2(\xi,t)\int_{-L}^{\xi} p_2(\eta,t)\left[G^{'}(X_1(\xi,t)-X_1(\eta,t))-G^{'}(X_2(\xi,t)-X_2(\eta,t))\right]d\eta|.$$

According to (2.10), we know that $[X_1(\xi,t)-X_1(\eta,t)]\times[X_2(\xi,t)-X_2(\eta,t)] > 0$. Then, by using the property (2.5), we have

$$|I_1(\xi,t)| \le 2M_0\|G^{'}\|_{L^\infty}|p_1(\xi,t)-p_2(\xi,t)| + \|G^{'}\|_{L^\infty}|p_2(\xi,t)|\|p_1(\cdot,t)-p_2(\cdot,t)\|_{L^1}$$
$$+2K_2|p_2(\xi,t)|\|X_1(\cdot,t)-X_2(\cdot,t)\|_{L^\infty}\|p_2(\cdot,t)\|_{L^1}.$$

In the same way, we also have

$$|I_2(\xi,t)| \le 2M_0\|G^{'}\|_{L^\infty}|p_1(\xi,t)-p_2(\xi,t)| + \|G^{'}\|_{L^\infty}|p_2(\xi,t)|\|p_1(\cdot,t)-p_2(\cdot,t)\|_{L^1}$$
$$+2K_2|p_2(\xi,t)|\|X_1(\cdot,t)-X_2(\cdot,t)\|_{L^\infty}\|p_2(\cdot,t)\|_{L^1}.$$

Therefore, by choosing constant $K=(b-1)\max\left\{8M_0\|G^{'}\|_{L^\infty},\ \ 16M_0^2K_2\right\}$, we arrive at

$$\|(\mathcal{L}(p_1)-\mathcal{L}(p_2))(\cdot,t)\|_{L^1} \le K\int_0^t(\|p_1(\cdot,s)-p_2(\cdot,s)\|_{L^1}+\|(X_1-X_2)(\cdot,s)\|_{L^\infty})ds. \tag{2.17}$$

On the other hand, according to (2.15)

$$(X_1-X_2)(\xi,t)$$
$$=\int_0^t\int_{-L}^{L}\left[G(X_1(\xi,s)-X_1(\eta,s))p_1(\eta,s)-G(X_2(\xi,s)-X_2(\eta,s))p_2(\eta,s)\right]d\eta ds.$$

Hence, by taking the L^∞ norm, one has

$$\|(X_1-X_2)(\cdot,t)\|_{L^\infty} \le \int_0^t \|G\|_{L^\infty}\|p_1(\cdot,s)-p_2(\cdot,s)\|_{L^1}+4M_0K_1\|(X_1-X_2)(\cdot,s)\|_{L^\infty}ds.$$

Then, Gronwall's inequality yields

$$\|(X_1 - X_2)(\cdot,t)\|_{L^\infty} \le \delta e^{4M_0K_1\delta}\|G\|_{L^\infty}\|p_1 - p_2\|_{\mathcal{N}}, \tag{2.18}$$

with notation $\|Y\|_{\mathcal{N}} := \sup_{t\in[0,\delta]}\|Y(\cdot,t)\|_{L^1}$. Combining (2.17) and (2.18), one has

$$\|(\mathcal{L}(p_1) - \mathcal{L}(p_2))\|_{\mathcal{N}} \le K\delta\|p_1 - p_2\|_{\mathcal{N}} + K\delta^2 e^{4M_0K_1\delta}\|G\|_{L^\infty}\|p_1 - p_2\|_{\mathcal{N}},$$

As a result, for sufficient small δ, we have

$$K\delta + K\delta^2 e^{4M_0K_1\delta}\|G\|_{L^\infty} < 1,$$

the mapping $\mathcal{L}$ is a contraction. The argument is closed here by the contraction mapping theorem. □

Remark 2.1. *In the proof of the existence and uniqueness of the Lagrangian dynamics (2.6)-(2.8), we only use the general assumptions on the kernel G. As a by-product, We know that the solution $X(\xi,t)$ satisfies $\frac{1}{C_1} \le X_\xi \le C_1$. This means that the particles will not cross over at any time $t < \delta$.*

2.3. Regularity results. In this subsection, we improve the regularity for $(X(\xi,t), p(\xi,t))$ under the condition $m_0 \in C_c^1[-L,L]$ in Lemma 2.9, and then under condition $m_0 \in C_c^2[-L,L]$ in Lemma 2.10.

We first provide the following technical lemma about the continuity with respect to ξ, t for the integral involving G'.

Lemma 2.7. *Assuming that $X(\xi,t) \in C(U_{\delta_1})$ satisfying (2.10) and g satisfies $\|g(\cdot,t)\|_{L^\infty}$, $\|\partial_t g(\cdot,t)\|_{L^1} \le M$, $0 < t \le \delta_1$ for some $0 < \delta_1 \le t_1$, denoting*

$$A(\xi,t) := \int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))g(\eta,t)d\eta, \tag{2.19}$$

then, $A(\xi,t) \in C(U_{\delta_1})$.

Proof. For any $\xi_1 < \xi_2$, t_1, t_2, according to the definition of $A(\xi,t)$, one has

$$\begin{aligned} A(\xi_1,t_1) - A(\xi_2,t_2) = &\int_{-L}^{L} G'(X(\xi_1,t_1) - X(\eta,t_1))g(\eta,t_1)d\eta \\ &- \int_{-L}^{L} G'(X(\xi_2,t_2) - X(\eta,t_2))g(\eta,t_2)d\eta. \end{aligned}$$

To avoid variables crossing the discontinues point of G' at zero, we split the integral range into the following three parts.

$$\begin{aligned} &|A(\xi_1,t_1) - A(\xi_2,t_2)| \\ \le &\int_{-L}^{L} |G'(X(\xi_1,t_1) - X(\eta,t_1))||g(\eta,t_1) - g(\eta,t_2)|d\eta \\ &+ \int_{-L}^{\xi_1} |G'(X(\xi_1,t_1) - X(\eta,t_1)) - G'(X(\xi_2,t_2) - X(\eta,t_2))||g(\eta,t_2)|d\eta \\ &+ \int_{\xi_1}^{\xi_2} |G'(X(\xi_1,t_1) - X(\eta,t_1)) - G'(X(\xi_2,t_2) - X(\eta,t_2))||g(\eta,t_2)|d\eta \\ &+ \int_{\xi_2}^{L} |G'(X(\xi_1,t_1) - X(\eta,t_1)) - G'(X(\xi_2,t_2) - X(\eta,t_2))||g(\eta,t_2)|d\eta. \end{aligned}$$

Then, by using monotonicity of $X(\xi,t)$ (2.10) and (2.5), we have

$$|A(\xi_1,t_1)-A(\xi_2,t_2)| \le M\|G^{'}\|_{L^\infty}|t_1-t_2| + 2LMK_2|X(\xi_1,t_1)-X(\xi_2,t_2)| + 2M\|G^{'}\|_{L^\infty}|\xi_1-\xi_2| + MK_2\int_{-L}^{L}|X(\eta,t_1)-X(\eta,t_2)|d\eta.$$

Hence, $A(\xi,t)\in C(U_{\delta_1})$ since $X(\xi,t)\in C(U_{\delta_1})$ and $\|A(\cdot,t)\|_{L^\infty}\le 2LM\|G^{'}\|_{L^\infty}$. □

With this lemma at hand, we now state the regularity results under the condition $m_0(\xi)\in C[-L,L]$.

LEMMA 2.8. *Assuming that the initial data $m_0(\xi)\in C[-L,L]$ and $(X(\xi,t),p(\xi,t))$ is a solution to (2.6)-(2.8) satisfying (2.10) and $\|p(\cdot,t)\|_{L^1}\le 2M_0,\ 0\le t\le\delta$, then, we have*

$$\partial_t p(\xi,t)\in C(U_\delta);\quad X(\xi,t)\in C_1^1(U_\delta). \tag{2.20}$$

PROOF: Taking the L^1 norm to (2.2), one has

$$\|\partial_t p(\cdot,t)\|_{L^1}\le (b-1)\|G^{'}\|_{L^\infty}\|p(\cdot,t)\|_{L^1}^2\le 4(b-1)M_0^2\|G^{'}\|_{L^\infty}.$$

We recast (2.2) as

$$\dot p(\xi,t)=p(\xi,t)Q(\xi,t). \tag{2.21}$$

with $Q(\xi,t):=-(b-1)\int_{-L}^{L}G^{'}(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta$. Then,

$$p(\xi,t)=m_0(\xi)e^{\int_0^t Q(\xi,s)ds}, \tag{2.22}$$

and the following estimate holds

$$\|Q(\cdot,s)\|_{L^\infty}\le 2M_0(b-1)\|G^{'}\|_{L^\infty}.$$

According to (2.22), we also have

$$\|p(\cdot,t)\|_{L^\infty}\le \|m_0\|_{L^\infty}e^{2M_0(b-1)\delta\|G^{'}\|_{L^\infty}}.$$

Therefore, $Q(\xi,t)\in C(U_\delta)$ by Lemma 2.7 and $p(\xi,t)\in C(U_\delta)$ according to (2.22) and $m_0(\xi)\in C[-L,L]$. Then, from (2.21), we know that $\partial_t p(\xi,t)\in C(U_\delta)$. On the other hand, as it is shown in Lemma 2.5 that

$$X_\xi(\xi,t)=e^{A(\xi,t)},\, A(\xi,t)=\int_0^t\int_{-L}^{L}G^{'}(X(\xi,s)-X(\eta,s))p(\eta,s)d\eta ds.$$

Therefore, by using Lemma 2.7 again, we know that $A(\xi,t)$ belongs to $C(U_\delta)$. This means that $X(\xi,t)\in C_1^1(U_\delta)$. □

In summary, under the condition $m_0\in C[-L,L]$, we arrive at, with constant C_1 defined in Lemma 2.5,

$$X(\xi,t)\in C_1^1(U_\delta);\quad \frac{1}{C_1}\le X_\xi(\xi,t)\le C_1,\quad (\xi,t)\in U_\delta;\quad p(\xi,t)\in C_1^0(U_\delta). \tag{2.23}$$

In the next Lemma, we improve the regularity results when the initial condition is given by $m_0 \in C_c^1[-L, L]$.

Lemma 2.9. *Assume that $m_0 \in C_c^1[-L, L]$ and let $(X(\xi,t),\ p(\xi,t))$ be the solution of (2.6)-(2.8) satisfying (2.10) and $\|p(\cdot,t)\|_{L^1} \le 2M_0, \quad 0 \le t \le \delta$. Then, for $0 \le t \le \delta$,*

$$p(\xi,t) \in C_1^1(U_\delta), \quad X(\xi,t) \in C_1^2(U_\delta)$$

and

$$p(\pm L,t) = p_\xi(\pm L,t) = 0;$$

$$\max\{\|p(\cdot,t)\|_{C^0}, \quad \|X(\cdot,t)\|_{C^1}\} \le C(\delta, \|m_0\|_{C^0}, \|G\|_{L^\infty}, \|G'\|_{L^\infty}, b).$$

Proof. Step 1. For any $\xi \in [-L, L]$, one has, from (2.7)(2.8)

$$\dot{p}(\xi,t) = -(b-1)p(\xi,t)\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta =: p(\xi,t)Q(\xi,t).$$

By taking $\xi = \pm L$, we have

$$\frac{dp(\pm L,t)}{dt} = p(\pm L,t)Q(\pm L,t)$$

with initial data $p(\pm L, 0) = m_0(\pm L) = 0$. Therefore, $p(\pm L,t) = 0$ for $0 < t \le \delta$.
Step 2. We estimate the bounds of $\|p(\cdot,t)\|_{C^0}, \quad \|X(\cdot,t)\|_{C^1}$ for $0 < t \le \delta$. According to (2.6)-(2.8),

$$X(\xi,t) = \xi + \int_0^t \int_{-L}^{L} G(X(\xi,s) - X(\eta,s))p(\eta,s)d\eta ds.$$

Therefore,

$$\|X(\cdot,t)\|_{L^\infty} \le L + 2M_0 t\|G\|_{L^\infty} \le L + 2M_0\delta\|G\|_{L^\infty}.$$

On the other hand,

$$p(\xi,t) = m_0(\xi) - (b-1)\int_0^t p(\xi,s)\int_{-L}^{L} G'(X(\xi,t) - X(\eta,s))p(\eta,s)d\eta ds.$$

Hence, by taking L^∞ norm, one has

$$\|p(\cdot,t)\|_{L^\infty} \le \|m_0\|_{L^\infty} + 2M_0(b-1)\|G'\|_{L^\infty}\int_0^t \|p(\cdot,s)\|_{L^\infty}ds.$$

Then, Gronwall's inequality yields

$$\|p(\cdot,t)\|_{L^\infty} \le \|m_0\|_{L^\infty}\left\{1 + 2(b-1)\delta M_0\|G'\|_{L^\infty}\exp\left[2(b-1)\delta M_0\|G'\|_{L^\infty}\right]\right\}.$$

We now turn to estimate $\|X_\xi(\cdot,t)\|_{L^\infty}$. A direct computation shows that

$$X_\xi(\xi,t) = 1 + \int_0^t X_\xi(\xi,s)\int_{-L}^{L} G'(X(\xi,s) - X(\eta,s))p(\eta,s)d\eta ds. \tag{2.24}$$

Then,

$$\|X_\xi(\cdot,t)\|_{L^\infty} \le 1 + 2M_0\|G'\|_{L^\infty}\int_0^t \|X_\xi(\cdot,s)\|_{L^\infty}ds.$$

Therefore, by recalling Gronwall's inequality, one has

$$\|X_\xi(\xi,t)\|_{L^\infty} \le 1 + 2\delta M_0\|G'\|_{L^\infty}\exp(2\delta M_0\|G'\|_{L^\infty}).$$

Step 3. We show that $X_{\xi\xi}(\xi,t) \in C(U_{\delta^*}),\ p_\xi(\cdot,t) \in C(U_{\delta^*}),\ 0 < \delta^* \le \delta$. Taking time derivative to (2.24), one has

$$\dot{X}_\xi(\xi,t) = X_\xi(\xi,t)\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta.$$

Then, differentiating it with respect to ξ, one has

$$\begin{aligned}\dot{X_{\xi\xi}}(\xi,t) &= X_{\xi\xi}(\xi,t)\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta \\ &\quad + X_\xi(\xi,t)\partial_\xi\left\{\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta\right\}.\end{aligned} \tag{2.25}$$

We notice that

$$\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta = -\int_{-L}^{L}\frac{p}{X_\eta}(\eta,t)\partial_\eta G(X(\xi,t) - X(\eta,t))d\eta.$$

Then, by using integration by parts and $p(\pm L,t) = 0$, we have

$$\begin{aligned}\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta &= \int_{-L}^{L} G(X(\xi,t) - X(\eta,t))\partial_\eta\frac{p}{X_\eta}(\eta,t)d\eta \\ &= \int_{-L}^{L} G(X(\xi,t) - X(\eta,t))\frac{X_\eta p_\eta - pX_{\eta\eta}}{X_\eta^2}(\eta,t)d\eta.\end{aligned} \tag{2.26}$$

Therefore, by substituting (2.26) into (2.25), the second order derivative of X satisfies

$$\begin{aligned}\dot{X}_{\xi\xi}(\xi,t) \;=\; & X_{\xi\xi}(\xi,t)\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta \\ & + X_\xi^2(\xi,t)\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))\frac{X_\eta p_\eta - pX_{\eta\eta}}{X_\eta^2}(\eta,t)d\eta \\ =:\; & F_1(\xi,t).\end{aligned} \tag{2.27}$$

On the other hand, taking $\xi-$ derivative to the following equation

$$\dot{p}(\xi,t) = -(b-1)p(\xi,t)\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta,$$

one has

$$\begin{aligned}\dot{p}_\xi(\xi,t) = &-(b-1)p_\xi(\xi,t)\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta \\ &-(b-1)p(\xi,t)\partial_\xi\int_{-L}^{L} G'(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta.\end{aligned}$$

The second term can be dealt with as follows.

$$\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta = \int_{-L}^{L}\frac{-1}{X_\eta(\eta,t)}\partial_\eta G(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta.$$

Then, by using integration by parts and $p(\pm L,t)=0$ again, we have

$$\begin{aligned}&\partial_\xi \int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta\\ &= X_\xi(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{p_\eta X_\eta - pX_{\eta\eta}}{X_\eta^2}(\eta,t)d\eta.\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{p}_\xi(\xi,t) \;=\; &-(b-1)p_\xi(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta \\ &-(b-1)p(\xi,t)X_\xi(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{p_\eta X_\eta - pX_{\eta\eta}}{X_\eta^2}(\eta,t)d\eta\\ =:&\; F_2(\xi,t).\end{aligned} \tag{2.28}$$

Combining (2.27) with (2.28), we obtain the following system for $(X_{\xi\xi}(\xi,t),p_\xi(\xi,t))$

$$\dot{X}_{\xi\xi}(\xi,t) = F_1(\xi,t) \tag{2.29}$$

$$\dot{p}_\xi(\xi,t) = F_2(\xi,t) \tag{2.30}$$

$$X_{\xi\xi}(\xi,t) = 0, \quad p_\xi(\xi,0) = m_0^{'}(\xi). \tag{2.31}$$

The denominators in $F_1(\xi,t),\ F_2(\xi,t)$ are not vanishing thanks to the fact that $X_\eta > \frac{1}{C_1} > 0$. The existence of the unique solution to this system can be proven by using the contraction mapping theorem as stated in the following claim. The detailed proof is given in Appendix.

Claim 2.1. *Assuming that $m_0 \in C_c^1[-L,L]$ and $(p(\xi,t),\ X(\xi,t))$ is the solution to (2.6)-(2.8) satisfying*

$$p(\xi,t)\in C_1^0(U_\delta); \quad X(\xi,t)\in C_1^1(U_\delta); \quad \frac{1}{C_1}\le X_\xi(\xi,t)\le C_1, \quad (\xi,t)\in U_\delta$$

then, there exists $0<\delta^*\le\delta$ such that the system (2.29)-(2.31) has a unique solution $X_{\xi\xi}(\xi,t)\in C(U_{\delta^*}),\ p_\xi(\xi,t)\in C(U_{\delta^*})$.

According to this Claim, we know that $p_\xi(\xi,t)\in C(U_{\delta^*}),\ X_{\xi\xi}(\xi,t)\in C(U_{\delta^*})$ for sufficient small $0<\delta^*\le\delta$. With a little abuse of notation, we till use δ as δ^* in the following analysis. Combining with (2.23), we have $p(\xi,t)\in C_1^1(U_\delta),\ X(\xi,t)\in C_1^2(U_\delta)$.

Step 4. Taking $\xi=\pm L$ in (2.28), $p_\xi(\pm L,t)$ satisfies

$$\dot{p}_\xi(\pm L,t) = -(b-1)p_\xi(\pm L,t)\int_{-L}^{L} G^{'}(X(\pm L,t)-X(\eta,t))p(\eta,t)d\eta \tag{2.32}$$

subject to initial data $p_\xi(\pm L,0)=m_0^{'}(\pm L)=0$. Therefore, $p_\xi(\pm L,t)=0,\ 0\le t\le\delta$. The proof is finished here. ◻

Finally, we have the following higher order regularity results if $m_0 \in C_c^2[-L, L]$.

LEMMA 2.10. *Assuming that $m_0 \in C_c^2[-L, L]$, $(X(\xi, t), p(\xi, t))$ is a solution to (2.6)-(2.8) satisfying $p(\pm L, t) = p_\xi(\pm L, t) = 0$, $\|p(\cdot, t)\|_{L^1} \leq 2M_0$, $0 \leq t \leq \delta$ and (2.10), then, the following regularity results hold*

$$X(\xi, t) \in C_1^3(U_\delta); \qquad p(\xi, t) \in C_1^2(U_\delta).$$

There also exists constant $\tilde{C}_1 := C(\delta, \|m_0\|_{C^2}, \|G\|_{L^\infty}, \|G^{'}\|_{L^\infty}, b)$, such that

$$\|p(\cdot, t)\|_{C^2}, \quad \|X(\cdot, t)\|_{C^3} \leq \tilde{C}_1, \quad \forall \ \ 0 \leq t \leq \delta$$

Proof. We have proven that $p(\xi, t) \in C_1^1(U_{\delta^*})$, $X(\xi, t) \in C_1^2(U_{\delta^*})$ in Lemma 2.9.

Step 1. We estimate $\|X_{\xi\xi}(\cdot, t)\|_{L^\infty}, \|p_\xi(\cdot, t)\|_{L^\infty}$. According to (2.29), a direct computation yields

$$\begin{aligned}
&\|\dot{X}_{\xi\xi}(\cdot, t)\|_{L^\infty} \\
&\leq 2M_0\|G^{'}\|_{L^\infty}\|X_{\xi\xi}(\cdot, t)\|_{L^\infty} + C_1^4\|G^{'}\|_{L^\infty}\left[2C_1 L\|p_\xi(\cdot, t)\|_{L^\infty} + 2M_0\|X_{\xi\xi}(\cdot, t)\|_{L^\infty}\right]
\end{aligned}$$

where, C_1 is defined in (2.10). According to (2.30), with C being a generic constant, one has

$$\|\dot{p}_\xi(\cdot, t)\|_{L^\infty} \leq C(\|p_\xi(\cdot, t)\|_{L^\infty} + \|X_{\xi\xi}(\cdot, t)\|_{L^\infty}).$$

Then, by using Gronwall's inequality, we have

$$\|X_{\xi\xi}(\cdot, t)\|_{L^\infty} + \|p_\xi(\cdot, t)\|_{L^\infty} \leq C(\delta, \|m_0\|_{C^1}, \|G\|_{L^\infty}, \|G^{'}\|_{L^\infty}, b).$$

Step 2. We prove that there exist $p_{\xi\xi}(\xi, t), X_{\xi\xi\xi}(\xi, t) \in C(U_\delta)$. The following system for those two unknowns can be derived in the same way as the derivation of (2.29)-(2.31) by noticing the fact that $p_\xi(\pm L, t) = p_\xi(\pm L, t) = 0$, $0 \leq t \leq \delta$.

$$\begin{aligned}
\dot{X}_{\xi\xi\xi}(\xi, t) = {} & X_{\xi\xi\xi}(\xi, t)\int_{-L}^{L} D(\xi, \eta, t)X_\xi(\xi, t)p(\eta, t)d\eta + X_{\xi\xi}^2(\xi, t)\int_{-L}^{L} D(\xi, \eta, t)p(\eta, t)d\eta \\
& + X_{\xi\xi}X_\xi^2\int_{-L}^{L} D(\xi, \eta, t)S(\eta, t)d\eta. \qquad (2.33)
\end{aligned}$$

$$\begin{aligned}
\dot{p}_{\xi\xi}(\xi, t) = {} & -(b-1)p_{\xi\xi}(\xi, t)\int_{-L}^{L} D(\xi, \eta, t)p(\eta, t)d\eta \qquad (2.34) \\
& -(b-1)\left[2p_\xi(\xi, t)X_\xi(\xi, t) + p(\xi, t)X_{\xi\xi}(\xi, t)\right]\int_{-L}^{L} D(\xi, \eta, t)S(\eta, t)d\eta \\
& -(b-1)p(\xi, t)X_\xi^2(\xi, t)\int_{-L}^{L} D(\xi, \eta, t)\frac{S_\eta X_\eta - SX_{\eta\eta}}{X_\eta^2}(\eta, t)d\eta \\
X_{\xi\xi\xi}(\xi, 0) = {} & 0, \quad p_{\xi\xi}(\xi, 0) = m_0^{''}(\xi), \qquad (2.35)
\end{aligned}$$

where,

$$S(\eta, t) := \frac{p_\eta X_\eta - pX_{\eta\eta}}{X_\eta^2}(\eta, t), \quad D(\xi, \eta, t) := G^{'}(X(\xi, t) - X(\eta, t)).$$

The proof of the existence of $p_{\xi\xi}(\xi,t)\in C(U_\delta)$ and $X_{\xi\xi\xi}(\xi,t)\in C(U_\delta)$ to (2.33)-(2.35) is also similar to the proof of Claim 2.1. This proves that $X(\xi,t)\in C_1^3(U_\delta),\ p(\xi,t)\in C_1^2(U_\delta)$.

Step 3. Finally, we bound $\|X_{\xi\xi\xi}(\cdot,t)\|_{L^\infty}, \|p_{\xi\xi}(\cdot,t)\|_{L^\infty}$. According to (2.33)(2.34), we have

$$\|\dot{p}_{\xi\xi}(\cdot,t)\|_{L^\infty}\le C(\|p_{\xi\xi}(\cdot,t)\|_{L^\infty}+\|X_{\xi\xi\xi}(\cdot,t)\|_{L^\infty})+C_2.$$

$$\|\dot{X}_{\xi\xi\xi}(\cdot,t)\|_{L^\infty}\le C(\|p_{\xi\xi}(\cdot,t)\|_{L^\infty}+\|X_{\xi\xi\xi}(\cdot,t)\|_{L^\infty})+C_2.$$

Where, $C,\ C_2$ are generic constants. Therefore, by recalling Gronwall's inequality, we have

$$\|X_{\xi\xi\xi}(\cdot,t)\|_{L^\infty}+\|p_{\xi\xi}(\cdot,t)\|_{L^\infty}\le C(\delta,\|m_0\|_{C^2},\|G\|_{L^\infty},\|G^{'}\|_{L^\infty},b)=:\tilde{C}_1.$$

The proof is finished. □

With these regularity results at hand, we can obtain the regularity result for $u(x,t)$. To this end, we first state a lemma.

LEMMA 2.11. *Assuming that* $g(\xi,t)\in C(U_\delta), \max_{0\le t\le\delta}\|g(\cdot,t)\|_{L^\infty}\le M,\ X(\xi,t)\in C_1^1(U_\delta)$ *satisfying (2.10) and letting*

$$B(x,t):=\int_{-L}^{L}G^{'}(x-X(\eta,t))g(\eta,t)d\eta, \tag{2.36}$$

then, $B(x,t)\in C(\mathbf{R}\times(0,\delta])$.

Proof. Recalling $A(\xi,t)$ given by (2.19), we formally extend the defined domain of $A(\xi,t)$ from $\xi\in[-L,L]$ to $\mathbf{R}$. For a given $(x,t)\ \in\mathbf{R}\times(0,\delta]$, there exists $\xi\in\mathbf{R}$ such that $X(\xi,t)=x$, then

$$B(x,t)=\int_{-L}^{L}G^{'}(X(\xi,t)-X(\eta,t))g(\eta,t)d\eta=A(\xi,t).$$

In other words, $A(\xi,t)$ is the Lagrangian description of $B(x,t)$. There exist three ranges for $\xi\in\mathbf{R}$.

Range 1: $\xi\in(-L,L)$. We have proven that $A(\xi,t)$ is continuous with respect to (ξ,t) in Lemma 2.7 by noticing that the condition $\|\partial_t g(\cdot,t)\|_{L^1}\le M$ can be replaced by $g(\xi,t)\in C(U_\delta)$. On the other hand, we know that $X(\xi,t)\in C_1^1(U_\delta)$ satisfying $\frac{1}{C_1}\le X_\xi(\xi,t)\le C_1$. Therefore, $B(x,t)$ is continuous.

Range 2: $X(\pm L,t)=x$. We only need to prove it when $X(-L,t)=x$. The proof of the other case is the same. It is easy to find that, for any small $\epsilon>0$, there are two cases for $(x-x_1)^2+(t-t_1)^2<\epsilon$. The first case is that $X(\xi_1,t_1)=x_1,\ \xi_1\in(-L,L)$. The other case is that $X(\xi_1,t_1)=x_1,\ \xi_1\in\mathbf{R}\setminus[-L,L]$. The first case can be proven in the similar way with range 1. The second case is similar to the following situation.

Range 3: $\xi\in\mathbf{R}\setminus[-L,L]$. For any (x_1,t_1) be sufficiently closed to (x,t), we notice that

$$\begin{aligned}&|B(x_1,t_1)-B(x,t)|\\ &\le\|G^{'}\|_{L^\infty}\int_{-L}^{L}|g(\eta,t_1)-g(\eta,t)|d\eta+M\int_{-L}^{L}|G^{'}(x_1-X(\eta,t_1))-G^{'}(x-X(\eta,t))|d\eta\\ &\le\|G^{'}\|_{L^\infty}\int_{-L}^{L}|g(\eta,t_1)-g(\eta,t)|d\eta+MK_2\int_{-L}^{L}\left[|x_1-x|+|X(\eta,t_1)-X(\eta,t)|\right]d\eta.\end{aligned}$$

Therefore, according to $g(\xi,t)\in C(U_\delta)$ and $X(\xi,t)\in C_1^1(U_\delta)$, $B(x,t)$ is continuous at (x,t). It is easy to verify that $\|B(\cdot,t)\|_{L^\infty(\mathbf{R})}\le 2LM\|G'\|_{L^\infty}$. The proof of this lemma is completed. □

Now, we can state our main results about the regularity of $u(x,t)$.

THEOREM 2.2. *Assuming that $m_0\in C_c^2[-L,L]$, then there exists $\delta>0$, such that the Lagrangian dynamic (2.1)-(2.3) has a unique solution $X(\xi,t)\in C_1^3(U_\delta)$, $p(\xi,t)\in C_1^2(U_\delta)$ satisfying $p(\pm L,t)=p_\xi(\pm L,t)=0,\ 0\le t\le\delta$ and (2.10). Furthermore, $u(x,t)$ given by (1.8) belongs to $C_1^3(\mathbf{R}\times(0,\delta])$ and it is a classical solution to (1.1)-(1.2).*

Proof. According to Theorem 2.1 and Lemma 2.10, we know that the Lagrangian dynamic (2.1)-(2.3) has a unique solution $X(\xi,t)\in C_1^3(U_\delta)$, $p(\xi,t)\in C_1^2(U_\delta)$ satisfying $p(\pm L,t)=p_\xi(\pm L,t)=0,\ 0\le t\le\delta$ and (2.10). On the other hand, from (1.8), one has

$$\partial_t u(x,t)=\int_{-L}^{L}G(x-X(\eta,t))p_t(\eta,t)d\eta-\int_{-L}^{L}X_t(\eta,t)G'(x-X(\eta,t))p(\eta,t)d\eta.$$

According to Lemma 2.7, Lemma 2.10, Lemma 2.11 and $X(\xi,t)\in C_1^3(U_\delta)$, $p(\xi,t)\in C_1^2(U_\delta)$, we know that $\partial_t u(x,t)\in C(\mathbf{R}\times(0,\delta])$.

We now turn to analyzing the derivatives with respect to the spatial variable x. Actually, by using integration by parts and $p(\pm L,t)=0$, we have

$$\begin{aligned}u_x(x,t)=\int_{-L}^{L}p(\eta,t)\partial_x G(x-X(\eta,t))d\eta&=-\int_{-L}^{L}\frac{p(\eta,t)}{X_\eta(\eta,t)}\partial_\eta G(x-X(\eta,t))d\eta\\&=\int_{-L}^{L}\partial_\eta(\frac{p(\eta,t)}{X_\eta(\eta,t)})G(x-X(\eta,t))d\eta.\end{aligned}$$

In the same way as above, noticing that $p(\pm L,t)=p_\xi(\pm L,t)=0$, we have

$$\begin{aligned}u_{xx}(x,t)&=\int_{-L}^{L}\partial_\eta[\frac{1}{X_\eta(\eta,t)}\partial_\eta(\frac{p(\eta,t)}{X_\eta(\eta,t)})]G(x-X(\eta,t))d\eta,\\u_{xxx}(x,t)&=\int_{-L}^{L}\partial_\eta[\frac{1}{X_\eta(\eta,t)}\partial_\eta(\frac{p(\eta,t)}{X_\eta(\eta,t)})]G'(x-X(\eta,t))d\eta.\end{aligned}$$

It is clear that $u_x(x,t)$, $u_{xx}(x,t)\in C(\mathbf{R}\times(0,\delta])$. For $u_{xxx}(x,t)$, we have $\partial_\eta[\frac{1}{X_\eta(\cdot,t)}\partial_\eta(\frac{p(\cdot,t)}{X_\eta(\cdot,t)})]\in C_1^0(U_\delta)$ and $\|\partial_\eta[\frac{1}{X_\eta(\cdot,t)}\partial_\eta(\frac{p(\cdot,t)}{X_\eta(\cdot,t)})]\|_{L^\infty}$ is bounded according to Lemma 2.10 and the property (2.10). Therefore, by using Lemma 2.11 again, we have

$$u_x(x,t),\ u_{xx}(x,t),\ u_{xxx}(x,t)\in C(\mathbf{R}\times(0,\delta]).$$

Then, Lemma 2.1 tells us that $u(x,t)$ is the classical solution to (1.1)-(1.2). The proof is completed. □

With these regularity results at hand, we provide the error estimate of the particle method for the b-equation in the following section.

3. Error estimate of the particle method for the b-equation. We have proven that, under the initial condition $m_0 \in C_c^2[-L, L]$, there exists $\delta > 0$ such that the Lagrangian dynamics (2.1)-(2.3) has a unique solution $X(\xi, t) \in C_1^3(U_\delta),\ p(\xi, t) \in C_1^2(U_\delta)$ satisfying $p(\pm L, t) = p_\xi(\pm L, t) = 0,\ 0 \le t \le \delta$ and (2.10). Furthermore, (1.1)-(1.2) has classical solution $u(x, t) \in C_1^3(\mathbf{R} \times (0, \delta])$. In this section, we let $T_{\max} > 0$ be the largest time of the existence of classical solution to the b-equation. In other words, $T_{\max}$ satisfies

$$\|X(\cdot, t)\|_{C^3} + \|p(\cdot, t)\|_{C^2} < +\infty, 0 \le t < T_{\max},$$

$$\limsup_{t \to T_{\max}} (\|X(\cdot, t)\|_{C^3} + \|p(\cdot, t)\|_{C^2}) = +\infty.$$

For fixed $T,\ 0 < T < T_{\max}$, let the constants $C_T,\ M_T,\ C_1$ be

$$C_T := \max_{0 \le t \le T} (\|X(\cdot, t)\|_{C^3} + \|p(\cdot, t)\|_{C^2}) \tag{3.1}$$

$$M_T := 2L \exp\left[(b-1)\|G'\|_{L^\infty} \int_0^T \|p(\cdot, t)\|_{L^1} dt\right] C_T \tag{3.2}$$

$$C_1 := \exp(\|G'\|_{L^\infty} \int_0^T \|p(\cdot, t)\|_{L^1} dt) \tag{3.3}$$

We also assume that $X(\xi, t)$ satisfies the following property

$$\frac{1}{C_1} \le X_\xi(\xi, t) \le C_1, \tag{3.4}$$

as it was proven for $X(\xi, t)$ in Lemma 2.5.

In this section, we use the uniform grid $\{\xi_i\}_{i=1}^N \subset [-L, L]$ with mesh size $h := \frac{2L}{N}$ given by

$$\xi_i = -L + (i - \frac{1}{2})h,\ i = 1, 2, \cdots,\ N. \tag{3.5}$$

The solution of the particle method, denoted by $\{X_i(t),\ p_i(t)\}_{i=1}^N$, will be used to approximate the Lagrangian dynamics of the b-equation. We provide error estimates between the numerical solution of the particle method and the exact Lagrangian representation, i.e. $X(\xi_i, t) - X_i(t),\ p(\xi_i, t) - p_i(t)$ in Theorem 3.1. Consequently, the error estimates between the exact solution $u(x, t),\ m(x, t)$ and the numerical solutions $u_{\epsilon,h}(x, t),\ m_{\epsilon,h}(x, t)$ are also obtained in Theorem 3.2, 3.3, respectively. In this section, the ℓ^1 norm for a vector $Y = (y_1, y_2, \cdots, y_N)$ is used. This norm is defined by

$$\|Y\|_h = h \sum_{i=1}^N |y_i|.$$

We divide this section into four subsections. The first subsection is to introduce the particle method and some estimates for exact particle trajectories. The particle method with $\epsilon = 0$ is known as the point vortex method in computational fluid dynamics. In the case of $\epsilon > 0$, it is known as the vortex blob method. We will provide error estimates for both $\epsilon = 0,\ \epsilon > 0$ at the same time in this section. In the second subsection, we give the error estimate of the numerical integration involved $G,\ G^{'}$. Then, in the third subsection, the error estimate for Lagrangian representation $X(\xi, t),\ p(\xi, t)$ of the particle method is provided. The last subsection is devoted to the error analysis of $u(x, t),\ m(x, t)$ of the particle method.

3.1. The particle method . In this subsection, we first provide some estimates on exact particle trajectories, i.e. $\{X(\xi_i, t),\ p(\xi_i, t)\}_{i=1}^{N}$, which are the exact solutions of the following ODEs

$$\frac{dX(\xi_i, t)}{dt} = \int_{-L}^{L} G(X(\xi_i, t) - X(\eta, t))p(\eta, t)d\eta \tag{3.6}$$

$$\frac{dp(\xi_i, t)}{dt} = -(b-1)p(\xi_i, t)\int_{-L}^{L} G^{'}(X(\xi_i, t) - X(\eta, t))p(\eta, t)d\eta, \tag{3.7}$$

with initial data $X(\xi_i, 0) = \xi_i,\ p(\xi_i, 0) = m_0(\xi_i)$.

Denoting

$$\begin{aligned} P(t) &= [p(\xi_1, t), p(\xi_2, t), \cdots, p(\xi_N, t)]\,, \\ m_0 &= [(m_0(\xi_1), m_0(\xi_1), \cdots, m_0(\xi_N))]\,, \end{aligned} \tag{3.8}$$

then, we have

LEMMA 3.1. *Assume that $X(\xi, t) \in C_1^3(U_T), p(\xi, t) \in C_1^2(U_T)$ is the solution to (2.1)-(2.3) satisfying (3.4) in $[0, T]$ and the initial mesh is an uniform mesh with size h, then, for any $0 \le t \le T$, the following estimates hold.*

$$\frac{1}{C_1}h \le X(\xi_{i+1}, t) - X(\xi_i, t) \le C_1 h,\ i = 1, 2, \cdots N-1, \tag{3.9}$$

$$\|P(t)\|_h \le M_T,$$

where, the constant C_1 is defined in (3.3) and M_T is defined in (3.2).

Proof. A direct computation shows that, for $0 < t \le T$

$$X(\xi_{i+1}, t) - X(\xi_i, t) = X_\xi(\tau_i, t)h,\ \tau_i \in (\xi_i, \xi_{i+1}).$$

Therefore, (3.9) holds by using (3.4).

For the second estimate, according to (3.7), we have by directly computation that

$$|p(\xi_i, t)| \le |m_0(\xi_i)| + (b-1)\int_0^t |p(\xi_i, s)|\|G^{'}\|_{L^\infty}\|p(\cdot, s)\|_{L^1}ds.$$

Then, Gronwall's inequality leads to

$$\begin{aligned} \|P(t)\|_h &\le \|m_0\|_h \exp((b-1)\|G^{'}\|_{L^\infty}\int_0^T \|p(\cdot, s)\|_{L^1}ds) \\ &\le 2L\exp((b-1)\|G^{'}\|_{L^\infty}\int_0^T \|p(\cdot, s)\|_{L^1}ds)C_T = M_T. \end{aligned}$$

The proof is finished. □

Now we introduce the mollifier and the particle method for the b-equation.

DIFINITION 3.1. *(1) Define the mollifier* $\rho(x) = f(|x|) \geq 0, \rho(x) \in C^k(\mathbf{R}),\ k \geq 2$ *satisfying*

$$\int_{\mathbf{R}} \rho(x)dx = 1, \quad supp\{\rho\} \subset \{x \in \mathbf{R}| \quad |x| < 1\} \tag{3.10}$$

(2) For each $\epsilon > 0$, *set*

$$\rho_\epsilon(x) := \frac{1}{\epsilon}\rho(\frac{x}{\epsilon}).$$

LEMMA 3.2. *Assuming that* $G = \frac{1}{2\alpha}e^{-\frac{|x|}{\alpha}}$, *then the mollified kernel* $G_\epsilon = G * \rho_\epsilon(x)$ *satisfies:*

(1) $|G_\epsilon(a) - G_\epsilon(b)| \leq \frac{1}{2\alpha^2}|a-b|,\ \forall a, b \in \mathbf{R}$ *and* $\epsilon \geq 0$; $G_\epsilon^{'}(0) = 0$;
(2) $|G^{'}(a) - G_\epsilon^{'}(a)| \leq \frac{1}{2\alpha^3}\epsilon$ *holds for* $0 \leq \epsilon < |a|$;
(3) $|G_\epsilon^{'}(a) - G_\epsilon^{'}(b)| \leq \frac{1}{2\alpha^3}|a-b|$, $ab > 0$ *holds for* $0 \leq \epsilon < \min\{|a|, |b|\}$.

Proof. (1) A direct computation shows that

$$G_\epsilon(a) - G_\epsilon(b) = G_\epsilon^{'}(\tau)(a-b).$$

According to the fact that $\|G_\epsilon^{'}\|_{L^\infty} \leq \|G^{'}\|_{L^\infty} = \frac{1}{2\alpha^2}$, we confirm the assertion. Since that $G^{'}(0) = 0$, the second statement is obvious.
(2) We only need to prove this inequality for the case $0 < \epsilon < a$. By directly estimating, one has

$$|G^{'}(a) - G_\epsilon^{'}(a)| \leq \int_{-\epsilon}^{\epsilon} |G^{'}(a) - G^{'}(a-y)|\rho_\epsilon(y)dy.$$

Then, the property (2.5) of $G^{'}$ leads to

$$|G^{'}(a) - G_\epsilon^{'}(a)| \leq \frac{1}{2\alpha^3}\int_{-\epsilon}^{\epsilon} |y|\rho_\epsilon(y)dy \leq \frac{1}{2\alpha^3}\epsilon.$$

(3) We only need to consider the case $\epsilon > 0$. A direct estimate shows that

$$\begin{aligned}
|G_\epsilon^{'}(a) - G_\epsilon^{'}(b)| &\leq \int_{-\infty}^{+\infty} \rho_\epsilon(y)|G^{'}(a-y) - G^{'}(b-y)|dy \\
&= \int_{-\epsilon}^{+\epsilon} \rho_\epsilon(y)|\frac{1}{\alpha}\text{sgn}(y-a)G(a-y) - \frac{1}{\alpha}\text{sgn}(y-b)G(b-y)|dy \\
&= \int_{-\epsilon}^{+\epsilon} \rho_\epsilon(y)\frac{1}{\alpha}|G(a-y) - G(b-y)|dy \leq \frac{1}{2\alpha^3}|a-b|.
\end{aligned}$$

This ends the proof. □

With $\{\xi_i\}_{i=1}^N$ and G_ϵ at hand, we introduce the particle method for the b-equation as follows. For $i = 1, \cdots, N$, the particle method for the b-equation reads

$$\frac{dX_i(t)}{dt} = h\sum_{j=1}^{N} G_\epsilon(X_i(t) - X_j(t))p_j(t) \tag{3.11}$$

$$\frac{dp_i(t)}{dt} = -h(b-1)p_i(t)\sum_{j=1}^{N} G_\epsilon^{'}(X_i(t) - X_j(t))p_j(t) \tag{3.12}$$

$$X_i(0) = \xi_i,\ p_i(0) = m_0(\xi_i). \tag{3.13}$$

Here, $X_i(t), p_i(t)$ represent the location of the ith particle and its weight, and N denote the total number of particles. Once $X_i(t), p_i(t)$ are determined, we can obtain $(u_h^\epsilon(x,t),\ m_h^\epsilon(x,t))$, the numerical approximation of $(u(x,t),\ m(x,t))$, by

$$u_h^\epsilon(x,t) = \sum_{j=1}^{N} p_j(t)G_\epsilon(x - X_j(t)) \tag{3.14}$$

$$m_h^\epsilon(x,t) = \sum_{j=1}^{N} p_j(t)\rho_\epsilon(x - X_j(t)). \tag{3.15}$$

3.2. Numerical integration involving $G,\ G^{'}$. In this subsection, we estimate the numerical integration involving the kernel G and $G^{'}$.

Proposition 3.1. *If $X(\xi,t) \in C_1^3(U_T), p(\xi,t) \in C_1^2(U_T)$ satisfy (3.4) in $[0,T]$, G is given by (1.4) and C_T is given by (3.1), ξ_j is defined by (3.5) , then, there exists constant C_0, for any $0 \le t \le T,\ x \in \mathbf{R},\ X(\xi,t) = x$*

$$(1)\quad \left|\int_{-L}^{L} G(x - X(\eta,t))p(\eta,t)d\eta - h\sum_{j=1}^{N} G(x - X(\xi_j,t))p(\xi_j,t)\right| \le C_0h^2. \tag{3.16}$$

Furthermore, if $\xi \in (-\infty,-L]\bigcup\{\xi_1,\cdots,\xi_N\}\bigcup[L,+\infty)$, then

$$(2)\quad \left|\int_{-L}^{L} G^{'}(x - X(\eta,t))p(\eta,t)d\eta - h\sum_{j=1}^{N} G^{'}(x - X(\xi_j,t))p(\xi_j,t)\right| \le C_0h^2. \tag{3.17}$$

For $\xi \in (-L,L)\setminus\{\xi_1,\cdots,\xi_N\}$, (3.17) is reduced to C_0h. Here, the constant C_0 only depends on $C_T,\ \alpha,\ b$.

Proof. We first prove (3.16). For any fixed t and $x \in \mathbf{R}$, there exists ξ such that $X(\xi,t) = x$. There are three cases for the range of ξ.

Case 1. $X(\xi,t) = x,\ \xi \in (-\infty,-L)$. In this case, $X(\xi,t) - X(\eta,t) < 0$, as a result

$$G(x - X(\eta,t)) = \frac{1}{2\alpha}\exp(\frac{x - X(\eta,t)}{\alpha}),\ G^{'}(x - X(\eta,t)) = \frac{1}{2\alpha^2}\exp(\frac{x - X(\eta,t)}{\alpha})$$

the integrant in those two integral belong to $C^2[-L,L]$. The estimate holds by standard arguments on numerical integration.

Case 2. $X(\xi,t) = x,\ \xi \in [L,+\infty)$. It is similar with case 1.

Case 3. $X(\xi,t)=x,\ \xi\in[\xi_i-\frac{h}{2},\xi_i+\frac{h}{2})$ for some $1\le i\le N$.

$$\begin{aligned}I_h&=\int_{-L}^{L}G(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta-h\sum_j G(X(\xi,t)-X(\xi_j,t))p(\xi_j,t)\\&=\sum_{j<i}\int_{\xi_j-\frac{h}{2}}^{\xi_j+\frac{h}{2}}G(X(\xi,t)-X(\eta,t))p(\eta,t)-G(X(\xi,t)-X(\xi_j,t))p(\xi_j,t)d\eta\\&\quad+\sum_{j>i}\int_{\xi_j-\frac{h}{2}}^{\xi_j+\frac{h}{2}}G(X(\xi,t)-X(\eta,t))p(\eta,t)-G(X(\xi,t)-X(\xi_j,t))p(\xi_j,t)d\eta\\&\quad+\int_{\xi_i-\frac{h}{2}}^{\xi_i+\frac{h}{2}}G(X(\xi,t)-X(\eta,t))p(\eta,t)-G(X(\xi,t)-X(\xi_i,t))p(\xi_i,t)d\eta.\end{aligned}$$

Let $I_h=:I_1+I_2+I_3$, then we turn to estimating $I_1,\ I_2,\ I_3$. First, one has

$$\begin{aligned}|I_1|+|I_2|\le&\sum_{j<i}\|G(X(\xi,t)-X(\eta,t))p(\eta,t)\|_{C^2[\xi_j-\frac{h}{2},\xi_j+\frac{h}{2}]}h^3\\&+\sum_{j>i}\|G(X(\xi,t)-X(\eta,t))p(\eta,t)\|_{C^2[\xi_j-\frac{h}{2},\xi_j+\frac{h}{2}]}h^3\end{aligned}$$

up to a constant factor by a standard argument of numerical integration. The term I_3 can be estimated as follows. By means of the property (2.4), we have, with C being a generic constant,

$$\begin{aligned}|I_3|\le&\int_{\xi_i-\frac{h}{2}}^{\xi_i+\frac{h}{2}}|G(X(\xi,t)-X(\eta,t))-G(X(\xi,t)-X(\xi_i,t))||p(\eta,t)|d\eta\\&+\int_{\xi_i-\frac{h}{2}}^{\xi_i+\frac{h}{2}}|G(X(\xi,t)-X(\xi_i,t))||p(\eta,t)-p(\xi_i,t)|d\eta\\\le&K_1\int_{\xi_i-\frac{h}{2}}^{\xi_i+\frac{h}{2}}|X(\eta,t)-X(\xi_i,t)||p(\eta,t)|+\|G\|_{L^\infty}\|p(\cdot,t)\|_{C^1}|\eta-\xi_i|d\eta\\\le&C\int_{\xi_i-\frac{h}{2}}^{\xi_i+\frac{h}{2}}\{\|X(\cdot,t)\|_{C^1}\|p(\cdot,t)\|_{C^0}+\|G\|_{L^\infty}\|p(\cdot,t)\|_{C^1}\}\frac{h}{2}d\eta\\=&C\{\|X(\cdot,t)\|_{C^1}\|p(\cdot,t)\|_{C^0}+\|G\|_{L^\infty}\|p(\cdot,t)\|_{C^1}\}\frac{h^2}{2}.\end{aligned}$$

A direct computation shows that, for each integral interval $I_j=(\xi_j-\frac{h}{2},\xi_j+\frac{h}{2})$, $j\neq i$

$$\begin{aligned}\|G(X(\xi,t)-X(\eta,t))p(\eta,t)\|_{C^2(I_j)}\le&C(\|x_\xi(\cdot,t)\|_{L^\infty}^2\|p(\cdot,t)\|_{L^\infty}+\|x_{\xi\xi}(\cdot,t)\|_{L^\infty}^2\|p(\cdot,t)\|_{L^\infty}\\&+\|x_\xi(\cdot,t)\|_{L^\infty}^2\|p_\xi(\cdot,t)\|_{L^\infty}+\|p_{\xi\xi}(\cdot,t)\|_{L^\infty}).\end{aligned}$$

Therefore, we arrive at

$$|I_h|\le|I_1|+|I_2|+|I_3|\le\sum_{j\neq i}C_{I_j}h^3+C_{I_i}h^2$$

with constants C_{I_j} only depending on $C_T,\ \alpha,\ b$. Hence, the proof of (3.16) is completed here.

For (3.17), we only need estimate the case $\xi = \xi_i$ and the case $\xi \in [\xi_i - \frac{h}{2}, \xi_i + \frac{h}{2})$, $\xi \neq \xi_i$. For $\xi \in (-\infty, -L] \bigcup [L, +\infty)$, (3.17) can be proved in the same way as that of (3.16). We omit it here.

Step 1. $\xi = \xi_i$. For simplicity in notation, we assume that $\xi = \xi_i = 0$. Then, based on the analysis of (3.16) and the fact that $G^{'}(0) = 0$, we need to prove that

$$|\int_{-\frac{h}{2}}^{\frac{h}{2}} G^{'}(X(0,t) - X(\eta,t))p(\eta,t)d\eta| \leq Ch^2$$

with some constant C independent of h. Actually,

$$\begin{aligned} I &= \int_{-\frac{h}{2}}^{\frac{h}{2}} G^{'}(X(0,t) - X(\eta,t))p(\eta,t)d\eta \\ &= \int_{-\frac{h}{2}}^{0} G^{'}(X(0,t) - X(\eta,t))p(\eta,t)d\eta + \int_{0}^{\frac{h}{2}} G^{'}(X(0,t) - X(\eta,t))p(\eta,t)d\eta. \end{aligned}$$

Using the fact that $G^{'}(x) = -\mathrm{sgn}(x)\frac{1}{\alpha}G(x)$, we have

$$\begin{aligned} I &= \frac{1}{\alpha}\int_{0}^{\frac{h}{2}} G(X(0,t) - X(\eta,t))p(\eta,t)d\eta - \frac{1}{\alpha}\int_{-\frac{h}{2}}^{0} G(X(0,t) - X(\eta,t))p(\eta,t)d\eta \\ &= \frac{1}{\alpha}\int_{0}^{\frac{h}{2}} [G(X(0,t) - X(\eta,t))p(\eta,t) - G(X(0,t) - X(-\eta,t))p(-\eta,t)]d\eta \\ &= \frac{1}{\alpha}\int_{0}^{\frac{h}{2}} [G(X(0,t) - X(\eta,t)) - G(X(0,t) - X(-\eta,t))]p(\eta,t) \\ &\quad + \frac{1}{\alpha}\int_{0}^{\frac{h}{2}} G(X(0,t) - X(-\eta,t))[p(\eta,t) - p(-\eta,t)]d\eta. \end{aligned}$$

Therefore,

$$\begin{aligned} |I| &\leq \frac{1}{\alpha}\int_{0}^{\frac{h}{2}} \|p(\cdot,t)\|_{L^\infty}|X(\eta,t) - X(-\eta,t)| + \|G\|_{L^\infty}|p(\eta,t) - p(-\eta,t)|d\eta \\ &\leq \frac{2}{\alpha}\int_{0}^{\frac{h}{2}} [\|p(\cdot,t)\|_{L^\infty}\|X(\cdot,t)\|_{C^1} + \|G\|_{L^\infty}\|p(\cdot,t)\|_{C^1}]\eta d\eta \leq Ch^2. \end{aligned}$$

Step 2. For $\xi \in [\xi_i - \frac{h}{2}, \xi_i + \frac{h}{2})$, $\xi \neq \xi_i$, it is obvious that

$$|\int_{\xi_i - \frac{h}{2}}^{\xi_i + \frac{h}{2}} G^{'}(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta - hG^{'}(X(\xi,t) - X(0,t))p(0,t)| \leq Ch.$$

The proof is completed. ◻

With these preparations at hand, we can provide the error estimate of the particle method for the b-equation. The next subsection is devoted to the error estimate of the particle method in terms of Lagrangian representation $X(\xi,t)$, $p(\xi,t)$.

3.3. Error estimate for X, p. Based on the regularity results $X(\xi,t) \in C_1^3(U_T)$, $p(\xi,t) \in C_1^2(U_T)$, $u(x,t) \in C_1^3(\mathbf{R} \times [0,T])$ under the condition that

$m_0 \in C_c^2[-L, L]$ and the properties of $G,\ G_\epsilon$ and the flow mapping $X(\xi, t)$ satisfying (2.10), we now can establish our results about error analysis.

Analogous to (3.8), we define

$$P_h(t) = (p_1(t), p_2(t), \cdots, p_N(t))$$

and recall that the ℓ^1 norm $\|P(t)\|_h = h\sum_{i=1}^N |p(\xi_i, t)|,\ \|P_h(t)\|_h = h\sum_{i=1}^N |p_i(t)|$. The following constants will be used in this section.

$$B := \max\left\{\frac{1}{2\alpha}, 2(M_T+1), \frac{(M_T+1)}{2\alpha^2}, C_0\right\} \qquad (3.18)$$
$$+(b-1)\max\left\{\frac{(M_T+1)}{\alpha^2}, C_0, \frac{(M_T+1)^2}{\alpha^3}\right\}$$

$$B_0 := \frac{1}{4C_1}\min\left\{\frac{1}{BTe^{BT}},\ 1\right\}; \quad h_0 := \min\left\{\frac{1}{8C_1BTe^{BT}}, \frac{1}{2}\right\}.$$

Where, the constant C_0 is stated in Proposition 3.1, M_T is defined in (3.2) and C_1 is defined in (3.3), respectively. The motivation to define the constant $B_0,\ h_0$ is that we should be able to justify the *a-priori assumption* in the following error analysis. Then, we have the following theorem for error estimates of X, p.

THEOREM 3.1. *If $X(\xi, t) \in C_1^3(U_T),\ p(\xi, t) \in C_1^2(U_T)$ satisfying (3.4) is the solution to (2.1)-(2.3), the constant $h_0,\ B_0$ is defined in (3.18) and C_1 is defined in (3.3), then, for any $0 < h < h_0,\ 0 \le\ \epsilon < B_0 h,\ 0 < t \le T$, we have*
(i) The ODEs (3.11)-(3.13) has a unique solution $\{(X_j(t), p_j(t)\}_{j=1}^N$ satisfying

$$h\sum_{i=1}^N |p_i(t)| \le M_T + 1; \quad \min_{1\le i\le N-1}(X_{i+1}(t) - X_i(t)) \ge \frac{h}{4C_1},\ 0 < t \le T \qquad (3.19)$$

(ii) The following error estimate holds

$$\max_{1\le i\le N} |X(\xi_i, t) - X_i(t)| + h\sum_{i=1}^N |p(\xi_i, t) - p_i(t)| \le BTe^{BT}(\epsilon + h^2),\ 0 < t \le T \quad (3.20)$$

where, B is defined in (3.18).

Proof. (i) The RHS of (3.11)-(3.13) is Lipschitz continuous as long as (3.19) holds. As a result, the ODEs (3.11)-(3.13) has a unique solution on $(0, T]$. The proof of (3.19) is included in the following proof of (ii).

(ii) We denote $\varphi_i(t) = X(\xi_i, t) - X_i(t),\ \psi_i(t) = p(\xi_i, t) - p_i(t),\ \|\varphi(t)\|_\infty = \max_{1\le i\le N} |\varphi_i(t)|,\ \ \|\psi(t)\| = h\sum_{i=1}^N |\psi_i(t)|$ in the following analysis. The proof is divided into three parts. We estimate $\varphi_i(t),\ \psi_i(t)$ in Step 1 and Step 2, respectively. Some *a-priori-assumptions* will be used in these estimations. The estimate (3.20) is also proven in Step 2. Step 3 is devoted to justifying these *a-priori-assumptions* by using (3.20).

Step 1. We split $\varphi_i(t)$ into two parts as follows.

$$\varphi_i(t) = \int_0^t \{\int_{-L}^L G(X(\xi_i, s) - X(\eta, s))p(\eta, s)d\eta - h\sum_{j=1}^N G_\epsilon(X_i(s) - X_j(s))p_j(s)\}ds$$
$$=: \int_0^t I_1(s) + I_2(s)ds$$

where,

$$I_1(t) = \int_{-L}^{L} G(X(\xi_i,t) - X(\eta,t))p(\eta,t)d\eta - h\sum_{j=1}^{N} G_\epsilon(X(\xi_i,t) - X(\xi_j,t))p(\xi_j,t),$$

$$I_2(t) = h\sum_{j=1}^{N} G_\epsilon(X(\xi_i,t) - X(\xi_j,t))p(\xi_j,t) - h\sum_{j=1}^{N} G_\epsilon(X_i(t) - X_j(t))p_j(t).$$

We point out that $I_1(t)$ is the consistency error and $I_2(t)$ is the stability error of this numerical method. According to Proposition 3.1 and Lemma 3.2, we have

$$\begin{aligned}|I_1(t)| \le & \left|\int_{-L}^{L} G(X(\xi_i,t) - X(\eta,t))p(\eta,t)d\eta - h\sum_{j=1}^{N} G(X(\xi_i,t) - X(\xi_j,t))p(\xi_j,t)\right| \\ & + \left|h\sum_{j=1}^{N} G(X(\xi_i,t) - X(\xi_j,t))p(\xi_j,t) - h\sum_{j=1}^{N} G_\epsilon(X(\xi_i,t) - X(\xi_j,t))p(\xi_j,t)\right| \\ & \le \|P(t)\|_h\|G - G_\epsilon\|_{L^\infty} + C_0h^2,\end{aligned}$$

and

$$\begin{aligned}|I_2(t)| \le & \left|h\sum_{j=1}^{N} G_\epsilon(X(\xi_i,t) - X(\xi_j,t))p(\xi_j,t) - h\sum_{j=1}^{N} G_\epsilon(X(\xi_i,t) - X(\xi_j,t))p_j(t)\right| \\ & + \left|h\sum_{j=1}^{N} G_\epsilon(X(\xi_i,t) - X(\xi_j,t))p_j(t) - h\sum_{j=1}^{N} G_\epsilon(X_i(t) - X_j(t))p_j(t)\right| \\ & \le h\|G_\epsilon\|_{L^\infty}\sum_{j=1}^{N} |p(\xi_j,t) - p_j(t)| + 2\|\varphi\|_\infty h\sum_{j=1}^{N} |p_j(t)|.\end{aligned}$$

Therefore,

$$|I_1(t)| + |I_2(t)| \le \|P\|_h\|G - G_\epsilon\|_{L^\infty} + \|G_\epsilon\|_{L^\infty}\|\psi\| + 2\|P_h\|_h\|\varphi\|_\infty + C_0h^2. \quad (3.21)$$

We assume for a while that $\|P_h(t)\|_h$ satisfies, which will be justified in Step 3.

$$\textit{a-priori assumption:} \ \|P_h(t)\|_h \le M_T + 1, \ 0 < t \le T. \quad (3.22)$$

Using (3.21) and the *a-priori assumption* (3.22), we obtain the following estimates

$$|\varphi_i| \le \int_0^t \{\frac{1}{2\alpha^2}(M_T+1)\epsilon + \frac{1}{2\alpha}\|\psi\| + 2(M_T+1)\|\varphi\|_\infty + C_0h^2\}ds.$$

Consequently, we have

$$\|\varphi\|_\infty \le \int_0^t (\frac{1}{2\alpha}\|\psi\| + 2(M_T+1)\|\varphi\|_\infty)ds + \frac{M_T+1}{2\alpha^2}\epsilon t + C_0h^2t. \quad (3.23)$$

Step 2. We now turn to analyzing $\psi_i(t) = p(\xi_i,t) - p_i(t)$. We also split ψ_i as follows.

$$\psi_i(t) = -(b-1)\int_0^t M_1(s) + M_2(s)ds$$

with

$$M_1(t) := p(\xi_i,t)\left\{\int_{-L}^{L} G'(X(\xi_i,t)-X(\eta,t))p(\eta,t)d\eta - h\sum_{j=1}^{N} G'_\epsilon(X(\xi_i,t)-X(\xi_j,t))p(\xi_j,t)\right\}$$

$$M_2(t) := p(\xi_i,t)h\sum_{j=1}^{N} G'_\epsilon(X(\xi_i,t)-X(\xi_j,t))p(\xi_j,t) - p_i(t)h\sum_{j=1}^{N} G'_\epsilon(X_i(t)-X_j(t))p_j(t).$$

$M_1(t), M_2(t)$ are also referred as the consistency error and stability error respectively. A direct estimate shows that

$$\begin{aligned}|M_1(t)| &\le |p(\xi_i,t)\int_{-L}^{L} G'(X(\xi_i,t)-X(\eta,t))p(\eta,t)d\eta - h\sum_{j=1}^{N} G'(X(\xi_i,t)-X(\xi_j,t))p(\xi_j,t)| \\ &\quad + |p(\xi_i,t)|h\sum_{j=1}^{N}|G'(X(\xi_i,t)-X(\xi_j,t)) - G'_\epsilon(X(\xi_i,t)-X(\xi_j,t))||p(\xi_j,t)| \\ &\le \|P\|_h|p(\xi_i,t)|\frac{1}{2\alpha^3}\epsilon + C_0h^2.\end{aligned}$$

We assume for a while that $\{X_i(t)\}_{i=1}^N$ remain well separated in the sense that they satisfy, like the property of the exact particles described in Lemma 3.1.

$$\textit{a-priori assumption:} \quad \min_{1\le i\le N}(X_{i+1}(t)-X_i(t)) \ge \frac{h}{4C_1}, \ 0 < t \le T. \tag{3.24}$$

We point out that (3.24) implies $\min_{i\neq j}|(X_i(t)-X_j(t))| \ge \frac{h}{4C_1}, \ 0 < t \le T$. Then, by noticing that $G'(0) = G'_\epsilon(0) = 0$, one has

$$\begin{aligned}|M_2(t)| &\le h|p(\xi_i,t)-p_i(t)|\sum_{j=1,j\neq i}^{N}|G'_\epsilon(X(\xi_i,t)-X(\xi_j,t))p(\xi_j,t)| \\ &\quad + h|p_i(t)|\sum_{j=1,j\neq i}^{N}|G'_\epsilon(X(\xi_i,t)-X(\xi_j,t))p(\xi_j,t) - G'_\epsilon(X_i(t)-X_j(t))p(\xi_j,t)| \\ &\quad + h|p_i(t)|\sum_{j=1,j\neq i}^{N}|G'_\epsilon(X_i(t)-X_j(t))(p(\xi_j,t)-p_j(t))|.\end{aligned}$$

The second term, denoted by I_2, can be estimated as follows.

$$\begin{aligned}I_2 &\le h|p_i(t)|\sum_{j<i}^{N}|G'_\epsilon(X(\xi_i,t)-X(\xi_j,t)) - G'_\epsilon(X_i(t)-X_j(t))||p(\xi_j,t)| \\ &\quad + h|p_i(t)|\sum_{j>i}^{N}|G'_\epsilon(X(\xi_i,t)-X(\xi_j,t)) - G'_\epsilon(X_i(t)-X_j(t))||p(\xi_j,t)|.\end{aligned}$$

Then, by using Lemma 3.1 and the *a-priori assumption* (3.24), we know that

$$(X(\xi_i,t)-X(\xi_j,t))(X_i(t)-X_j(t)) > 0, \ for \ \ i\neq j,$$

and $0 \le \epsilon < B_0h \le \min_{i\neq j}\{|X(\xi_i,t)-X(\xi_j,t)|, |X_i(t)-X_j(t)|\}$. Therefore, by using Lemma 3.2 for the case $\epsilon > 0$ and (2.5) for the case $\epsilon = 0$, we have

$$I_2 \le h|p_i(t)|\sum_{j=1,j\neq i}^{N}\frac{1}{2\alpha^3}(|X(\xi_j,t)-X_j(t)| + |X(\xi_i,t)-X_i(t)|)|p(\xi_j,t)|.$$

We will justify this *a-priori assumption* (3.24) in Step 3. Therefore

$$|M_2(t)| \le \|P\|_h \frac{1}{2\alpha^2}|\psi_i| + \|P\|_h|p_i|\frac{1}{\alpha^3}\|\varphi\|_\infty + |p_i|\frac{1}{2\alpha^2}\|\psi\|.$$

Hence,

$$\begin{aligned}|p(\xi_i,t)-p_i(t)| \le (b-1)\int_0^t \{&\frac{\|P\|_h}{2\alpha^2}|\psi_i| + p(\xi_i,t)|\frac{\|P\|_h}{2\alpha^3}\epsilon + |p_i|\frac{\|P\|_h}{\alpha^3}\|\varphi\|_\infty + |p_i|\frac{1}{2\alpha^2}\|\psi\|\}ds\\ &+C_0(b-1)h^2t.\end{aligned}$$

Then, by summing it from $i=1$ to N, we obtain

$$\begin{aligned}\|\psi\| &\le (b-1)\int_0^t \{\|P\|_h\frac{1}{2\alpha^2}\|\psi\| + \|P\|_h\|P_h\|_h\frac{1}{\alpha^3}\|\varphi\|_\infty + \|P_h\|_h\frac{1}{2\alpha^2}\|\psi\| + \|P\|_h^2\frac{1}{2\alpha^3}\epsilon\}ds\\ &\quad +C_0(b-1)h^2t\\ &\le (b-1)\int_0^t\{\frac{M_T+1}{\alpha^2}\|\psi\| + \frac{(M_T+1)^2}{\alpha^3}\|\varphi\|_\infty + \frac{(M_T+1)^2}{2\alpha^3}\epsilon\}ds + C_0(b-1)h^2t\\ &=: \int_0^t C_2\|\psi\| + C_3\|\varphi\|_\infty ds + C_4\epsilon t + C_5h^2t,\end{aligned}$$

where the constants $C_2=(b-1)\frac{M_T+1}{\alpha^2}$, $C_3=(b-1)\frac{(M_T+1)^2}{\alpha^3}$, $C_4 := (b-1)\frac{(M_T+1)^2}{2\alpha^3}$, $C_5 := C_0(b-1)$. Let $C_6 := \max\{C_2, C_3, C_4, C_5\}$. Hence, we obtain

$$\|\psi\| \le C_6(\int_0^t \|\psi\| + \|\varphi\|_\infty ds + \epsilon t + h^2t). \tag{3.25}$$

The estimate (3.23) can be rewritten as

$$\|\varphi\|_\infty \le C_7(\int_0^t \|\psi\| + \|\varphi\|_\infty ds + \epsilon t + h^2t), \tag{3.26}$$

by setting $C_7 := \max\{\frac{1}{2\alpha},\ 2(M_T+1),\ \frac{M_T+1}{2\alpha^2},\ C_0\}$. Denoting $L(t) = \|\varphi\|_\infty + \|\psi\|$, $B := C_6 + C_7$, we have the following inequality,

$$L(t) \le B\int_0^t L(s) + \epsilon + h^2 ds.$$

Then, Gronwall's inequality yields

$$L(t) \le Bte^{Bt}(\epsilon + h^2) \le BTe^{BT}(\epsilon+h^2).$$

Step 3. Finally, we justify the *a-priori assumptions* (3.22) and (3.24).

$$\begin{aligned}\|\tilde{P}(\cdot,t)\|_h &\le \|P(\cdot,t)\|_h + h\sum_{i=1}^N |p_i(t) - p(\xi_i,t)| = \|P(\cdot,t)\|_h + \|\psi(\cdot,t)\|\\ &\le M_T + BTe^{BT}(\epsilon+h^2) \le M_T + 2BTe^{BT}h_0 \le M_T+1.\end{aligned}$$

This implies that (3.22) is actually satisfied. On the other hand,

$$\begin{aligned}X_{j+1}(t) - X_j(t) &\ge X(\xi_{j+1},t) - X(\xi_j,t) - |X(\xi_{j+1},t) - X_{j+1}(t)| - |X(\xi_j,t) - X_j(t)|\\ &\ge \frac{h}{C_1} - 2\|\varphi\|_\infty\\ &\ge \frac{h}{C_1} - 2BTe^{BT}(\epsilon+h^2) \ge \frac{h}{2C_1} - 2BTe^{BT}h^2 \ge \frac{h}{4C_1},\end{aligned}$$

since that $h \le h_0$. This in turn implies that (3.24) is actually satisfied uniformly in N for $t \in [0, T)$. The proof is completed. □

For simplicity in notations, we rewrite the result in Theorem 3.1 as the following form:

$$\|\varphi\|_\infty + \|\psi\| \le \tilde{C}(\epsilon + h^2). \tag{3.27}$$

3.4. Error estimate for $u(x,t)$, $m(x,t)$. With the error estimate for $X(\xi,t), p(\xi,t)$ at hand, according to (1.8)(1.9) and (3.14)(3.15), we obtain the estimate for $u(x,t) - u_h^\epsilon(x,t)$ and $m(x,t) - m_h^\epsilon(x,t)$ in Theorem 3.2 and 3.3, respectively.

THEOREM 3.2. *Assuming that $X(\xi,t) \in C_1^3(U_T)$, $p(\xi,t) \in C_1^2(U_T)$ is the solution to (2.1)-(2.3) satisfying (3.4), $u(x,t) \in C_1^3(\mathbf{R} \times [0,T])$ defined by (1.8) is the classical solution to (1.1)-(1.2) and $u_h^\epsilon(x,t)$ is the numerical solution given by (3.14) of the particle method, then, there exists constant $\tilde{\tilde{C}}$ independent of h and ϵ, such that, for any $0 < h < h_0$, $0 \le \epsilon < B_0 h$, $0 < t \le T$, with constants h_0, B_0 defined in (3.18), we have the following estimate.*

$$\|u(\cdot,t) - u_h^\epsilon(\cdot,t)\|_{C^0} \le \tilde{\tilde{C}}(\epsilon + h^2), \quad \|u_x(\cdot,t) - u_{h,x}^\epsilon(\cdot,t)\|_{C^0} \le \tilde{\tilde{C}}(\epsilon + h).$$

Proof. For any fixed t, and for each $x \in \mathbf{R}$, there exists ξ, such that $X(\xi,t) = x$

$$\begin{aligned}
&|u(x,t) - u_h^\epsilon(x,t)| \\
&= |\int_{-L}^{L} G(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta - h\sum_{j=1}^{N} p_j G_\epsilon(X(\xi,t) - X_j(t))| \\
&\le h|\sum_{j=1}^{N} p(\xi_j,t)G(X(\xi,t) - X(\xi_j,t)) - \sum_{j=1}^{N} p_j(t)G(X(\xi,t) - X(\xi_j,t))| \\
&\quad + h|\sum_{j=1}^{N} p_j(t)G(X(\xi,t) - X(\xi_j,t)) - \sum_{j=1}^{N} p_j(t)G(X(\xi,t) - X_j(t))| \\
&\quad + h|\sum_{j=1}^{N} p_j(t)G(X(\xi,t) - X_j(t)) - \sum_{j=1}^{N} p_j(t)G_\epsilon(X(\xi,t) - X_j(t))| + C_0 h^2 \\
&\le \frac{1}{2\alpha}\|\psi\| + (M_T + 1)\|\varphi\|_\infty + \frac{M_T + 1}{2\alpha^2}\epsilon + C_0 h^2 \le \tilde{\tilde{C}}(\epsilon + h^2).
\end{aligned}$$

In above estimations, we have used (3.27), Proposition 3.1 and Lemma 3.2.

Now we turn to estimating $|u_x(x,t) - u^{\epsilon}_{h,x}(x,t)| =: I$. In the same way as above,

$$
\begin{aligned}
I \le & \left| \int_{-L}^{L} G^{'}(X(\xi,t) - X(\eta,t))p(\eta,t)d\eta - h\sum_{j=1}^{N} p(\xi_j,t)G^{'}(X(\xi,t) - X(\xi_j,t))\right| \\
& + h\left|\sum_{j=1}^{N} p(\xi_j,t)G^{'}(X(\xi,t) - X(\xi_j,t)) - p_j(t)G^{'}(X(\xi,t) - X(\xi_j,t))\right| \\
& + h\left|\sum_{j=1}^{N} p_j(t)G^{'}(X(\xi,t) - X(\xi_j,t)) - p_j(t)G^{'}(X(\xi,t) - X_j(t))\right| \\
& + h\left|\sum_{j=1}^{N} p_j(t)G^{'}(X(\xi,t) - X_j(t)) - p_j(t)G^{'}_{\epsilon}(X(\xi,t) - X_j(t))\right| \\
=: & \, I_1 + I_2 + I_3 + I_4.
\end{aligned}
$$

For the first term I_1, according to Proposition 3.1, we have $I_1 \le C_{I_1}h$. The second term I_2 is bounded by $\|G^{'}\|_{L^\infty}\|\psi\|$. For the third term I_3, we have the following two cases.

Case 1. $\xi \in (-\infty, -L)\bigcup[L, +\infty)$, according to the second property of the kernel G (2.5), we have

$$I_3 \le \|\varphi\|_\infty \|P_h\|_h \le (M_T + 1)\|\varphi\|_\infty.$$

Case 2. $\xi \in [\xi_i - \frac{h}{2}, \xi_i - \frac{h}{2})$, then

$$
\begin{aligned}
I_3 &= h\left|\sum_{j\ne i}^{N} p_j G^{'}(X(\xi,t) - X(\xi_j,t)) - p_j G^{'}(X(\xi,t) - X_j(t))\right| + C_{I_3}h \\
&\le (M_T + 1)\|\varphi\|_\infty + C_{I_3}h
\end{aligned}
$$

by (2.5). For the last term I_4, we have

$$I_4 \le h\left|\sum_{j\ne i}^{N} p_j G^{'}(X(\xi,t) - X_j(t)) - p_j G^{'}_{\epsilon}(X(\xi,t) - X_j(t))\right| + C_{I_4}h.$$

Then, by using Lemma 3.2, we have

$$I_4 \le (M_T + 1)\epsilon + C_{I_4}h.$$

Summarizing these four terms, we arrive at $|u_x(x,t) - u^{\epsilon}_{h,x}(x,t)| \le \tilde{\tilde{C}}(\epsilon + h)$. The proof is complete. □

To estimate $m(x,t) - m_h(x,t)$, we notice that $m(x,t)$ consists of δ distribution. Therefore, we use the Lipschitz distance to measure it.

Difinition 3.2. *For any measure μ, ν, the Lipschitz distance between μ and ν is defined by*

$$Dist(\mu,\nu) := \sup_{\|\phi\|_{L^\infty}\le 1;\ Lip(\phi)\le 1} \langle \phi(x), \mu(dx) - \nu(dx)\rangle$$

with notation

$$Lip(\phi) = \sup_{x \neq y} |\frac{\phi(x) - \phi(y)}{x - y}|.$$

Then, we can estimate $m - m_h^\epsilon$ by means of the Lipschitz distance. We have

THEOREM 3.3. *Assuming that $X(\xi, t) \in C_1^3(U_T)$, $p(\xi, t) \in C_1^2(U_T)$ satisfying (3.4) is the unique solution to (2.1)-(2.3), $m(x, t)$ is defined by (1.9) and $m_h^\epsilon(x, t)$ is a numerical solution given by (3.15) of the particle method, then, for any $0 < h < h_0$, $0 \leq \epsilon < B_0 h$, $0 < t \leq T$, with constants h_0, B_0 defined in (3.18), the following estimate holds:*

$$Dist(m, m_h^\epsilon) \leq C(\epsilon + h)$$

with generic constant C independent of h and ϵ.

Proof. For any test function ϕ satisfying $\|\phi\|_{L^\infty} \leq 1$ and $Lip(\phi) \leq 1$, direct computation shows that

$$\begin{aligned}
&\langle \phi(x), m(dx) - m_h^\epsilon(dx) \rangle \\
&= \int_{-\infty}^{+\infty} \phi(x) \int_{-L}^{L} \delta(x - X(\eta, t)) p(\eta, t) d\eta dx - \int_{-\infty}^{+\infty} \phi(x) h \sum_{j=1}^{N} \rho_\epsilon(x - X_j) p_j dx \\
&= \int_{-L}^{L} \phi(X(\eta, t)) p(\eta, t) d\eta - h \sum_{j=1}^{N} \phi * \rho_\epsilon(X_j) p_j =: I_1 + I_2 + I_3
\end{aligned}$$

where, we have denoted by

$$\begin{aligned}
I_1 &= \int_{-L}^{L} \phi(X(\eta, t)) p(\eta, t) d\eta - h \sum_{j=1}^{N} \phi(X(\xi_j, t)) p(\xi_j, t) \\
I_2 &= -h \sum_{j=1}^{N} (\phi * \rho_\epsilon(X_j) - \phi(X_j)) p_j \\
I_3 &= h \sum_{j=1}^{N} \phi(X(\xi_j, t)) p(\xi_j, t) - \phi(X_j) p_j.
\end{aligned}$$

Then, we estimate I_1, I_2, I_3 one after another.

$$\begin{aligned}
|I_1| &\leq \sum_{j=1}^{N} \int_{\xi_j - \frac{h}{2}}^{\xi_j + \frac{h}{2}} |\phi(X(\eta, t) p(\eta, t) - \phi(X(\xi_j, t)) p(\xi_j, t)| d\eta \\
&\leq \sum_{j=1}^{N} \int_{\xi_j - \frac{h}{2}}^{\xi_j + \frac{h}{2}} |(\phi(X(\eta, t)) - \phi(X(\xi_j, t))) p(\eta, t)| + |\phi(X(\xi_j, t))(p(\eta, t) - p(\xi_j, t))| d\eta \\
&\leq \sum_{j=1}^{N} \int_{\xi_j - \frac{h}{2}}^{\xi_j + \frac{h}{2}} Lip(\phi) |X(\eta, t)) - X(\xi_j, t)| |p(\eta, t)| d\eta \\
&\quad + \|\phi\|_{L^\infty} \sum_{j=1}^{N} \int_{\xi_j - \frac{h}{2}}^{\xi_j + \frac{h}{2}} |p(\eta, t) - p(\xi_j, t)| d\eta \\
&\leq Lip(\phi) \|p(\cdot, t)\|_{L^1} \|X_\eta(\cdot, t)\|_{L^\infty} h + C \|p_\eta(\cdot, t)\|_{L^\infty} \|\phi\|_{L^\infty} h \leq Ch.
\end{aligned}$$

It is obvious that

$$|I_2| \le C\epsilon.$$

For the last term I_3, we have

$$\begin{aligned} |I_3| &\le h \sum_{j=1}^{N} |\phi(X(\xi_j,t)) - \phi(X_j)||p(\xi_j,t)| + |\phi(X_j)||p(\xi_j,t) - p_j(t)| \\ &\le C\ Lip(\phi)\|\varphi\|_\infty + \|\phi\|_{L^\infty}\|\psi\| \le C(\epsilon + h^2) \end{aligned}$$

by using the estimate (3.27). Summarizing the estimates above, we finish the proof. ◻

REMARK 3.1. *In the case of supp $\{m_0\}$ is unbounded, the error analysis for this situation can be done in the same way due to the finite speed of propagation of the b-equation.*

REMARK 3.2. *In [3], Camassa , Huang and Lee gave a formal error analysis for the particle method to solve the C-H equation with initial condition $m_0 > 0$, which coincides with a special choice in our analysis by setting $b = 2,\ \epsilon = 0,\ m_0 > 0$.*

4. Conclusions. In this paper, we gave a self-contained error analysis of the particle method for the b-equation (1.1)-(1.2). We first established the mathematical theory of the Lagrangian dynamics for the b-equation with a special choice of the convolution kernel G and under a suitable class of initial data. The existence, uniqueness and regularity of the Lagrangian representation $(X(\xi,t),\ p(\xi,t))$ and the solution $u(x,t)$ were established in this aper. We also proved that $X(\xi,t)$ satisfies a special property of well-separation of trajectory (2.10) and $(u(x,t),\ m(x,t))$ is the classical solution of the b-equation. In the error analysis, we used the method of *a-priori assumptions*, originated in the analysis for the the vortex methods, to avoid the singularity of the kernel in estimations. Then, we obtained optimal error estimates of the particle method for both the Lagrangian representation and the classical solution of the b-equation.

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5. Appendix: Proof of Claim 2.1.

We first recall Claim 2.1 here.

Claim 2.1 Assuming that $m_0 \in C_c^1[-L,L]$ and $(p(\xi,t),\ X(\xi,t))$ is the solution to (2.6)-(2.8) satisfying

$$p(\xi,t) \in C_1^0(U_\delta); \quad X(\xi,t) \in C_1^1(U_\delta); \quad 0 < \frac{1}{C_1} \le X_\xi(\xi,t) \le C_1, \quad (\xi,t) \in U_\delta$$

then, there exists $0 < \delta^* \le \delta$ such that the following system

$$\dot{X}_{\xi\xi}(\xi,t) = X_{\xi\xi}(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))X_\xi(\xi,t)p(\eta,t)d\eta \qquad (5.1)$$

$$+X_\xi^2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{X_\eta p_\eta - pX_{\eta\eta}}{X_\eta^2}d\eta$$

$$\dot{p}_\xi(\xi,t) = -(b-1)p_\xi(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta \qquad (5.2)$$

$$-(b-1)p(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))X_\xi(\xi,t)\frac{p_\eta(\eta,t)X_\eta - p(\eta,t)X_{\eta\eta}}{X_\eta^2}d\eta$$

$$X_{\xi\xi}(\xi,t) = 0, \qquad p_\xi(\xi,0) = m_0^{'}(\xi) \qquad (5.3)$$

has a unique solution $X_{\xi\xi}(\xi,t) \in C(U_{\delta^*}),\ p_\xi(\xi,t) \in C(U_{\delta^*})$.

Proof. Step 1. We use the notations $Y(\xi,t) = X_{\xi\xi}(\xi,t), Q(\xi,t) = p_\xi(\xi,t)$ and

$$A_1(\xi,t) = \int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))X_\xi(\xi,t)p(\eta,t)d\eta;\ A_2(\xi,t) = X_\xi^2(\xi,t)$$

$$B_1(\xi,t) = -(b-1)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))p(\eta,t)d\eta;\ B_2(\xi,t) = -(b-1)p(\xi,t)X_\xi(\xi,t).$$

Then, the system (5.1)-(5.3) can be recast as

$$\dot{Y}(\xi,t) = A_1(\xi,t)Y(\xi,t) + A_2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{Q(\eta,t)}{X_\eta(\eta,t)}d\eta \qquad (5.4)$$

$$-A_2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{p(\eta,t)}{X_\eta^2(\eta,t)}Y(\eta,t)d\eta$$

$$\dot{Q}(\xi,t) = B_1(\xi,t)Q(\xi,t) + B_2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{Q(\eta,t)}{X_\eta(\eta,t)}d\eta \qquad (5.5)$$

$$-B_2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{p(\eta,t)}{X_\eta^2(\eta,t)}Y(\eta,t)d\eta$$

$$Y(\xi,t) = 0, \qquad Q(\xi,0) = m_0^{'}(\xi) \qquad (5.6)$$

As it was proven in Lemma 2.8, we have that

$$\max\{\|p(\cdot,t)\|_{C^0},\ \ \|X(\cdot,t)\|_{C^1}\} \le C(\delta,\|m_0\|_{C^0},\alpha,b) =: C,\ 0 \le t \le \delta. \qquad (5.7)$$

According to Lemma 2.7, we know that all of $A_1(\xi,t),\ A_2(\xi,t),\ B_1(\xi,t),\ B_2(\xi,t)$ belong to $C(U_\delta)$.

Step 2. From (5.7), we know that, for $0 \le t \le \delta$

$$\|A_1(\cdot,t)\|_{L^\infty} \le 2L\|G^{'}\|_{L^\infty}C^2, \qquad \|A_2(\cdot,t)\|_{L^\infty} \le C^2.$$

Defining

$$\mathcal{Q} = \left\{Q(\xi,t) \in C(U_\delta) : \quad \max_{0\le t\le\delta}\|Q(\cdot,t)\|_{L^\infty} \le 2M_1, \quad Q(\xi,0) = m_0^{'}(\xi)\right\} \qquad (5.8)$$

with $M_1 = \|m_0\|_{C^1}$, we first consider the system (5.4) for a given $Q \in \mathcal{Q}$ by means of the contraction mapping theorem. To this end, we denote by

$$C_2 = \max\left\{2L\|G^{'}\|_{L^\infty}C^2;\ \ 4LC^2C_1\|G^{'}\|_{L^\infty};\ \ 2LC^3C_1^2\|G^{'}\|_{L^\infty}\right\},$$

and introduce the closed subset

$$\mathcal{Y} := \left\{Y(\xi,t) \in C(U_{t_1}):\quad \max_{0\le t\le t_1}\|Y(\cdot,t)\|_{L^\infty} \le M_1\right\}, \tag{5.9}$$

with $t_1 = \min\{\delta;\ \frac{1}{4C_2}\}$. We need to prove the existence of the unique solution to the following system in $\mathcal{Y}$ on $[0,t_1]$.

$$\dot{Y}(\xi,t) = A_1(\xi,t)Y(\xi,t) + A_2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{Q(\eta,t)}{X_\eta(\eta,t)}d\eta \tag{5.10}$$
$$-A_2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t)-X(\eta,t))\frac{p(\eta,t)}{X_\eta^2(\eta,t)}Y(\eta,t)d\eta$$

with initial data $Y(\xi,0)=0$.

The equivalent integral form of (5.10) reads

$$\begin{aligned} &Y(\xi,t) \qquad\qquad (5.11)\\ &= \int_0^t A_1(\xi,s)Y(\xi,s)ds + \int_0^t A_2(\xi,s)\int_{-L}^{L} G^{'}(X(\xi,s)-X(\eta,s))\frac{Q(\eta,s)}{X_\eta(\eta,s)}d\eta ds\\ &\quad -\int_0^t A_2(\xi,s)\int_{-L}^{L} G^{'}(X(\xi,s)-X(\eta,s))\frac{p(\eta,s)}{X_\eta^2(\eta,s)}Y(\eta,s)d\eta ds\\ &:= \mathcal{L}_Q(Y)(\xi,t). \end{aligned}$$

By using Lemma 2.7, we know that $\mathcal{L}_Q(Y)(\xi,t) \in C(U_t)$ and

$$\begin{aligned} &\|\mathcal{L}_Q(Y)(\cdot,t)\|_{L^\infty} \qquad\qquad (5.12)\\ &\le 2L\|G^{'}\|_{L^\infty}C^2M_1t + 4LC^2C_1\|G^{'}\|_{L^\infty}M_1t + 2LC^3C_1^2\|G^{'}\|_{L^\infty}M_1t. \end{aligned}$$

Then, by taking $3C_2t \le 1$, one has

$$\|\mathcal{L}_Q(Y)(\cdot,t)\|_{L^\infty} \le 3C_2tM_1 \le M_1.$$

This means that the operator $\mathcal{L}_Q$ maps $\mathcal{Y}$ onto itself. On the other hand,

$$\begin{aligned} &|(\mathcal{L}_Q(Y_1)-\mathcal{L}_Q(Y_2))(\xi,t)|\\ &\le \int_0^t |A_1(\xi,s)||Y_1(\xi,s)-Y_2(\xi,s)|ds + \int_0^t |A_2(\xi,s)|\int_{-L}^{L}|G^{'}(X(\xi,s)\\ &\quad -X(\eta,s))\frac{p(\eta,s)}{X_\eta^2(\eta,s)}||Y_1(\eta,s)-Y_2(\eta,s)|d\eta ds. \end{aligned}$$

Therefore,

$$\sup_{s\in[0,t]}\|(\mathcal{L}_Q(Y_1)-\mathcal{L}_Q(Y_2))(\cdot,s)\|_{L^\infty} \le 2C_2t\sup_{s\in[0,t]}\|(Y_1-Y_2)(\cdot,s)\|_{L^\infty}.$$

Then ,$\mathcal{L}_Q$ is a contraction mapping by choosing $t < \frac{1}{4C_2}$. Therefore, the contraction mapping theorem tells that (5.10) has unique solution in $C(U_{t_1})$.
Step 3. We need prove that there exists $0 < \delta^* \le t_1$ such that the following system has a unique solution in $\mathcal{Q}$.

$$\dot{Q}(\xi,t) = B_1(\xi,t)Q(\xi,t) + B_2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t) - X(\eta,t))\frac{Q(\eta,t)}{X_\eta(\eta,t)}d\eta \quad (5.13)$$
$$-B_2(\xi,t)\int_{-L}^{L} G^{'}(X(\xi,t) - X(\eta,t))\frac{p(\eta,t)}{X_\eta^2(\eta,t)}Y_Q(\eta,t)d\eta$$
$$Q(\xi,0) = m_0^{'}(\xi) \quad (5.14)$$

where $Y_Q(\xi,t)$ is the solution of (5.10), which depends on $Q(\xi,t)$. In the same way, the equivalent integral form reads

$$Q(\xi,t) = m_0^{'}(\xi) + \int_0^t B_1(\xi,s)Q(\xi,s)ds \quad (5.15)$$
$$+\int_0^t B_2(\xi,s)\int_{-L}^{L} G^{'}(X(\xi,s) - X(\eta,s))\frac{Q(\eta,s)}{X_\eta(\eta,s)}d\eta ds$$
$$-\int_0^t B_2(\xi,s)\int_{-L}^{L} G^{'}(X(\xi,s) - X(\eta,s))\frac{p(\eta,s)}{X_\eta^2(\eta,s)}Y_Q(\eta,s)d\eta := \mathcal{T}(Q)(\xi,t).$$

To prove the existence and uniqueness of solution, we recall the closed subset

$$\mathcal{Q} = \left\{Q(\xi,t) \in C(U_{\delta^*}): \quad \|Q(\cdot,t)\|_{L^\infty} \le 2M_1, \quad 0 \le t \le \delta^*, \quad Q(\xi,0) = m_0^{'}(\xi)\right\}. \quad (5.16)$$

First, we show that the operator $\mathcal{T}$ maps $\mathcal{Q}$ onto itself. Actually, by using Lemma2.7 again and the estimate

$$\|\mathcal{T}(Q)(\cdot,t)\|_{L^\infty} \le M_1 + M_1\int_0^t \|B_1(\cdot,s)\|_{L^\infty}ds + 2LM_1C_1\|G^{'}\|_{L^\infty}\int_0^t \|B_2(\cdot,s)\|_{L^\infty}da$$
$$+2LCC_1^2M_1\int_0^t \|B_2(\cdot,s)\|_{L^\infty}ds.$$

According to (5.7), we know that

$$\|B_1(\cdot,t)\|_{L^\infty} \le 2(b-1)LC\|G^{'}\|_{L^\infty}; \quad \|B_2(\cdot,t)\|_{L^\infty} \le (b-1)C^2, \ 0 \le t \le t_1.$$

Hence

$$\|\mathcal{T}(Q)(\cdot,t)\|_{L^\infty} \le M_1 + 2(b-1)LM_1C\|G^{'}\|_{L^\infty}\delta^* + 4(b-1)L^2CC_1M_1\|G^{'}\|_{L^\infty}^2\delta^*$$
$$+2(b-1)LC^3C_1^2M_1\delta^* \le 2M_1$$

by taking $\left[2(b-1)LC\|G^{'}\|_{L^\infty} + 4(b-1)L^2CC_1\|G^{'}\|_{L^\infty}^2 + 2(b-1)LC^3C_1^2\right]\delta^* := C_3\delta^* \le 1$.
Second, we show that the operator $\mathcal{T}$ is a contraction. A direct computation shows

that

$$
\begin{aligned}
&(\mathcal{T}(Q_1) - \mathcal{T}(Q_2))(\xi, t) \\
= &\int_0^t B_1(\xi, s)(Q_1(\xi, s) - Q_2(\xi, s))ds \\
&+ \int_0^t B_2(\xi, s) \int_{-L}^{L} G^{'}(X(\xi, s) - X(\eta, s)) \frac{Q_1(\eta, s) - Q_2(\eta, s)}{X_\eta(\eta, s)} d\eta ds \\
&- \int_0^t B_2(\xi, s) \int_{-L}^{L} G^{'}(X(\xi, s) - X(\eta, s)) \frac{p(\eta, s)}{X_\eta^2(\eta, s)} (Y_1(\eta, s) - Y_2(\eta, s)) d\eta ds
\end{aligned}
$$

with Y_1, Y_2 representing Y_{Q_1}, Y_{Q_2} respectively. Therefore

$$
\begin{aligned}
\|\mathcal{T}(Q_1 - Q_2)(\cdot, t)\|_{L^\infty} \le &\int_0^t \|B_1(\cdot, s)\|_{L^\infty} \|(Q_1 - Q_2)(\cdot, s)\|_{L^\infty} ds \\
&+ 2LC_1 \|G^{'}\|_\infty \int_0^t \|B_2(\cdot, s)\|_\infty \|(Q_1 - Q_2)(\cdot, s)\|_{L^\infty} ds \\
&+ 2LCC_1^2 \|G^{'}\|_{L^\infty} \int_0^t \|B_2(\cdot, s)\|_{L^\infty} \|(Y_1 - Y_2)(\cdot, s)\|_{L^\infty} ds.
\end{aligned}
$$

According to (5.11), one has

$$
\|(Y_1 - Y_2)(\cdot, t)\|_{L^\infty} \le C_2 \int_0^t \|(Y_1 - Y_2)(\cdot, s)\|_{L^\infty} + \sup_{s \in [0,t]} \|(Q_1 - Q_2)(\cdot, s)\|_{L^\infty} ds.
$$

By using Gronwall's inequality, we have

$$
\|(Y_1 - Y_2)(\cdot, t)\|_{L^\infty} \le C_2 \delta^* e^{C_2 \delta^*} \sup_{s \in [0,t]} \|(Q_1 - Q_2)(\cdot, s)\|_{L^\infty}.
$$

Therefore

$$
\begin{aligned}
&\sup_{s \in [0,t]} \|\mathcal{T}(Q_1 - Q_2)(\cdot, s)\|_{L^\infty} \\
\le &\big\{ 2(b-1)LC\|G^{'}\|_{L^\infty} \delta^* + 2L(b-1)C^2 C_1 \|G^{'}\|_{L^\infty} \delta^* \\
&+ 2(b-1)LC^3 C_1^2 \|G^{'}\|_{L^\infty} C_2 \delta^{*2} e^{C_2 \delta^*} \big\} \sup_{s \in [0,t]} \|(Q_1 - Q_2)(\cdot, s)\|_{L^\infty}.
\end{aligned}
$$

Then, by taking t satisfying

$$
\begin{aligned}
2(b-1)LC\|G^{'}\|_{L^\infty} \delta^* &+ 2L(b-1)C^2 C_1 \|G^{'}\|_{L^\infty} \delta^* \\
&+ 2(b-1)LC^3 C_1^2 \|G^{'}\|_{L^\infty} C_2 \delta^{*2} e^{C_2 \delta^*} < \frac{1}{2}
\end{aligned}
$$

the operator $\mathcal{T}$ is a contraction mapping. The proof is completed. ∎