ULTRA-CONTRACTIVITY FOR KELLER-SEGEL MODEL WITH DIFFUSION EXPONENT $m > 1 - 2/d$

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Abstract. This paper establishes the hyper-contractivity in $L^\infty(\mathbb{R}^d)$ (it's known as ultra-contractivity) for the multi-dimensional Keller-Segel systems with the diffusion exponent $m > 1 - 2/d$. The results show that for the super-critical and critical case $1-2/d < m \le 2-2/d$, if $\|U_0\|_{d(2-m)/2} < C_{d,m}$ where $C_{d,m}$ is a universal constant, then for any $t > 0$, $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)}$ is bounded and decays as t goes to infinity. For the subcritical case $m > 2 - 2/d$, the solution $u(\cdot,t) \in L^\infty(\mathbb{R}^d)$ with any initial data $U_0 \in L^1_+(\mathbb{R}^d)$ for any positive time.

1. **Introduction and main theorem.** We consider the Keller-Segel model in spatial dimension $d \ge 3$:

$$\begin{cases} u_t = \Delta u^m - \nabla \cdot (u \nabla c)\,,\ x \in \mathbb{R}^d,\ t \ge 0, \\ -\Delta c = u,\ x \in \mathbb{R}^d,\ t \ge 0, \\ u(x,0) = U_0(x) \ge 0,\ x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where the diffusion exponent m is supercritical $0 < m < 2 - 2/d$, critical $m_c := 2-2/d$, and subcritical $m > 2-2/d$ respectively. This model was proposed by Keller

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and Segel [13] to describe the biological phenomenon chemotaxis. Here $u(x,t)$ represents the bacteria density, $c(x,t)$ represents the chemical substance concentration and it is given by the fundamental solution

$$c(x,t) = c_d \int_{\mathbb{R}^d} \frac{u(y,t)}{|x-y|^{d-2}} dy\,, \tag{1.2}$$

where

$$c_d = \frac{1}{d(d-2)\alpha_d}\,, \qquad \alpha_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}\,, \tag{1.3}$$

α_d is the volume of d-dimensional unit ball. The case $m > 1$ is called slow diffusion and the case $m < 1$ is called fast diffusion [19, 20, 8].

The main characteristic of equation (1.1) is the competition between the diffusion and the nonlocal aggregation. This is well represented by the free energy for $m > 1$

$$F(u) = \frac{1}{m-1} \int_{\mathbb{R}^d} u^m(x)dx - \frac{1}{2} \int_{\mathbb{R}^d} uc(x)dx. \tag{1.4}$$

For $m = 1$, the first term of (1.4) is replaced by $\int_{\mathbb{R}^d} u \log u dx$ [16]. According to different m, the competition results in different behaviors. Taking the mass invariant scaling $u_\lambda(x) = \lambda u\left(\lambda^{1/d}x, \lambda t\right)$ into account we can observe that for the supercritical case $1 \le m < 2-2/d$, the aggregation dominates the diffusion for high density (large λ) and the density has finite-time blow-up [11, 12, 6, 17, 16, 4]. While for low density (small λ), the diffusion dominates the aggregation and the density has infinite-time spreading [17, 18, 16, 2]. On the contrary, for the subcritical case $m > 2-2/d$, the aggregation dominates the diffusion for low density and prevents spreading, while for high density, the diffusion dominates the aggregation thus blow-up is precluded [17, 18, 14].

In this paper, we mainly focus on the hyper-contractivity for the Keller-Segel model with $m \le 2-2/d$ and $m > 2-2/d$ respectively. For non-degenerate Keller-Segel equation with $m = 1, d = 2$, Blanchet, Dolbeault and Perthame [5] showed that if the initial data $\|U_0\|_1 < 8\pi$ and $U_0 \log U_0 \in L^1(\mathbb{R}^d)$, then for any $1 < q < \infty$ and any $t > 0$, there exists a continuous function $h_q(t)$ satisfying that for $t \to 0$

$$h_q(t) \to \infty$$

and

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)} \le \|U_0\|_1 h_q(t).$$

Later, in 2012, Calvez, Corrias and Ebde [7] proved the local in time hyper-contractive property for $m = 1, d \ge 3$, it reads that if $U_0 \in (L^1 \cap L^a)(\mathbb{R}^d)$, $a > d/2$ arbitrarily close to $d/2$, there exists a finite time $T_a = C(a)\left(\int_{R^d} U_0^a dx\right)^{-\frac{1}{a-d/2}}$ and a local weak solution $u \in L^\infty\left((0,T_a); (L^1 \cap L^a)(\mathbb{R}^d)\right)$ satisfying that for any $a < q < \infty$, there exists a constant C not depending on $\|U_0\|_{L^q(\mathbb{R}^d)}$ such that

$$\int_{\mathbb{R}^d} u(\cdot,t)^q dx \le C\left(1 + t^{1-q}\right), \ a.e.\ t \in (0,T_a).$$

For general m, in our previous paper [3], it is showed that for $0 < m \le 2-2/d$, if the initial data $\|U_0\|_{d(2-m)/2} < C_{d,m}$ where $C_{d,m}$ is a universal constant depending on d, m, then there exists a global weak solution. Furthermore, for $0 < m < 1-2/d$, the solution will vanish at finite time, and for $m = 1-2/d$, the $L^q(1 < q < \infty)$ norm has exponentially decay in time with the initial data in L^q norm. On the other hand, for supercritical and critical case $1-2/d < m \le 2-2/d$, the solution satisfies

$\|u(\cdot,t)\|_q \le \frac{C(d,m,\|U_0\|_1)}{t^\alpha}$ for any $t>0$ and any $1<q<\infty$, here α is a positive constant. For the subcritical case $m>2-2/d$, if the initial data $U_0\in L^1_+(\mathbb{R}^d)$, then the solution will be bounded in $L^q(\mathbb{R}^d)$ for any $1<q<\infty$.

For the hyper-contractive property in L^∞ norm (it's also known as ultra-contractivity [10]), Corrias and Perthame [9] proved the hyper-contractivity for the parabolic-parabolic Keller-Segel model ($d\ge 3$)

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla c)\,,\ x\in\mathbb{R}^d,\ t\ge 0,\\ c_t-\Delta c = u-c,\ x\in\mathbb{R}^d,\ t\ge 0,\\ u(x,0)=U_0(x)\ge 0,\ x\in\mathbb{R}^d. \end{cases} \tag{1.5}$$

The results show that if $U_0\in(L^1\cap L^a)(\mathbb{R}^d)$, $d/2<a\le d$, $\nabla c_0\in L^d(\mathbb{R}^d)$, there is a constant $C(d,a)$ such that for

$$\|U_0\|_{L^a(\mathbb{R}^d)}+\|\nabla c_0\|_{L^d(\mathbb{R}^d)}\le C(d,a),$$

the parabolic-parabolic system has a weak solution satisfying the hyper-contractivity type estimate for any $\epsilon>0$

$$\|u(\cdot,t)-G(t)*U_0\|_{L^\infty(\mathbb{R}^d)}\le Ct^{\frac12-d+\epsilon},\ t\to\infty,$$

where $G(t)=\frac{1}{(2\pi t)^{d/2}}e^{-\frac{|x|^2}{2t}}$ is the heat kernel. In this paper, we will extend the hyper-contractivity result in [3] to L^∞ norm for general m. The main results are given below

Theorem 1.1. *Let $d\ge 3$, $p=\frac{d(2-m)}{2}$ and $m>1-2/d$. Assume $U_0\in L^1_+(\mathbb{R}^d)$,*

(i) *For the supercritical case and the critical case $1-2/d<m\le 2-2/d$, denote $\eta:=C_{d,m}^{2-m}-\|U_0\|_p^{2-m}$ where $C_{d,m}$ is a universal constant given by (3.1), if $\eta>0$, then there exists a global weak solution of (1.1) satisfying that for $0<t\le 1$*

$$\|u(\cdot,t)\|_\infty \tag{1.6}$$
$$\le \max\left[1,C(\eta,\|U_0\|_1,m,d)\right]\left(\frac{1}{t^{\frac{(p+\epsilon_0-1)(3+m+dm/2)}{\epsilon_0(2m+d+2/d)}\frac{m+d+1}{m-1+2/d}}}+\frac{1}{t^{\frac{m+d+1}{m-1+2/d}}}\right)\cdot\frac{1}{t^{d/2}},$$

and for $1<t<\infty$

$$\|u(\cdot,t)\|_\infty \tag{1.7}$$
$$\le \max\left[1,C(\eta,\|U_0\|_1,m,d)\right]\left(\frac{1}{t^{\frac{(p+\epsilon_0-1)(3+m+dm/2)}{\epsilon_0(2m+d+2/d)}\frac{m+d+1}{m-1+2/d}}}+\frac{1}{t^{\frac{m+d+1}{m-1+2/d}}}\right).$$

where ϵ_0 satisfies $\frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2S_d^{-1}}-\|U_0\|_p^{2-m}=\frac{\eta}{2}$.

(ii) *For the subcritical case $m>2-2/d$, if $m=2$, we also assume $U_0\log U_0\in L^1(\mathbb{R}^d)$ and if $m>2$, we assume $U_0\in L^{m-1}(\mathbb{R}^d)$, then*

$$\|u(\cdot,t)\|_\infty\le\max\left[1,C(\|U_0\|_1,m,d)\right]\left(1+\frac{1}{t^{m+d+1}}\right)\cdot\frac{1}{t^{d/2}},\qquad 0<t\le 1,$$

$$\|u(\cdot,t)\|_\infty\le\max\left[1,C(\|U_0\|_1,m,d)\right]\left(1+\frac{1}{t^{m+d+1}}\right),\qquad 1<t<\infty.$$

Furthermore, for any $T>t_0>0$, the weak solution has the following regularities

$$u(x,t)\in L^\infty\left(t_0,T;L^1_+\cap L^\infty(\mathbb{R}^d)\right)\cap L^2\left(t_0,T;H^1(\mathbb{R}^d)\right), \tag{1.8}$$

and

$$u_t \in L^{p_2}\left(0,T;W^{-1,p_1}_{loc}(\mathbb{R}^d)\right)\cap L^2\left(t_0,T;H^{-1}(\mathbb{R}^d)\right) \text{ for some } p_1,p_2\geq 1. \tag{1.9}$$

This paper is organized as follows. In Section 2, we list some preliminary lemmas which will be used to prove the L^∞ norm. Section 3 is devoted to show the main theorem on hyper-contractive property in $L^\infty(\mathbb{R}^d)$. Finally, Section 4 considers the boundedness in $L^\infty(\mathbb{R}^d)$ uniformly in time.

2. **Preliminary.** Before proving hyper-contractive estimates, we need the following preparations, some lemmas have been proved in [3].

Lemma 2.1. *Let $1<\frac{b}{a}<\frac{2d}{a(d-2)}$ and $\frac{b}{a}<\frac{2}{a}+\frac{2}{d}$. Assume $w\in L^1_+(\mathbb{R}^d)$ and $w^{1/a}\in H^1(\mathbb{R}^d)$ with $a>0$, then*

$$\|w\|_{b/a}^{b/a}\leq C(\delta)C_0^{-\frac{1}{\delta-1}}\|w\|_1^\gamma + C_0\|\nabla w^{1/a}\|_2^2,$$

where

$$\delta=\frac{2\left(\frac{1}{a}-\frac{d-2}{2d}\right)}{\frac{b}{a}-1},\gamma=1+\frac{2b-2a}{2d-bd+2a},$$

and $C(\delta)=\delta^{-\frac{1}{\delta-1}}\frac{S_d^{-\frac{b\theta\delta'}{2}}}{\delta'}$ with $\theta=\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}-\frac{d-2}{2d}}$ and $\delta'=\frac{\delta}{\delta-1}$. C_0 is an arbitrary positive constant.

Proof. The Sobolev inequality reads as follows

$$S_d\|u\|^2_{2d/(d+2)}\leq\|\nabla u\|_2^2,\ S_d=\frac{d(d-2)}{4}2^{2/d}\pi^{1+1/d}\Gamma\left(\frac{d+1}{2}\right)^{-2/d}, \tag{2.1}$$

taking $u=w^{1/a}$ in (2.1) and the interpolation inequality with $1<b/a<\frac{2d}{a(d-2)}$ yields

$$\|w\|_{b/a}\leq\|w\|_1^{1-\theta}\|w\|^\theta_{\frac{2d}{a(d-2)}}=\|w\|_1^{1-\theta}\|w^{1/a}\|^{\theta a}_{2d/(d-2)}\leq S_d^{-\theta a/2}\|w\|_1^{1-\theta}\|\nabla w^{1/a}\|_2^{\theta a},$$

whence follows

$$\|w\|_{b/a}^{b/a}\leq C(d)\|w\|_1^{(1-\theta)b/a}\|\nabla w^{1/a}\|_2^{b\theta}, \tag{2.2}$$

where

$$\theta=\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{a}-\frac{d-2}{2d}},\ C(d)=S_d^{-b\theta/2}.$$

It is easy to verify that $b\theta<2$ if $b/a<\frac{2}{a}+\frac{2}{d}$. Therefore, by the Young inequality we have

$$\|w\|_{b/a}^{b/a}\leq C(d)^{\delta'}\frac{\beta^{-\delta'}}{\delta'}\|w\|_1^{\frac{b}{a}(1-\theta)\delta'}+\frac{\beta^\delta}{\delta}\|\nabla w^{1/a}\|_2^{b\theta\delta},$$

here $\delta'=\frac{\delta}{\delta-1}$ and $b\theta\delta=2$ such that

$$\delta=\frac{2\left(\frac{1}{a}-\frac{d-2}{2d}\right)}{b/a-1}.$$

Let $C_0=\frac{\beta^\delta}{\delta}$ and thus $\beta^{-\delta'}=(C_0\delta)^{-\frac{1}{\delta}\frac{\delta}{\delta-1}}$. We denote $C(\delta)=\delta^{-\frac{1}{\delta-1}}\frac{C(d)^{\delta'}}{\delta'}$, $\gamma=\frac{b}{a}(1-\theta)\delta'$, this concludes the proof. □

Now taking

$$a = \frac{2}{m+q-1},\ b = \frac{2q}{m+q-1},\ C_0 = \frac{2mq(q-1)}{(m+q-1)^2},\ w = u$$

in Lemma 2.1 we obtain the following lemma

Lemma 2.2. *Let $d \geq 3$, $q > 1, m > 1 - 2/d$, assume $u \in L^1_+(\mathbb{R}^d)$ and $u^{\frac{m+q-1}{2}} \in H^1(\mathbb{R}^d)$, then*

$$\left(\|u\|_q^q\right)^{1+\frac{m-1+2/d}{q-1}} \leq S_d^{-1}\|\nabla u^{(q+m-1)/2}\|_2^2 \|u\|_1^{\frac{1}{q-1}(2q/d+m-1)}. \tag{2.3}$$

and

$$\|u\|_q^q \leq \frac{2mq(q-1)}{(m+q-1)^2}\|\nabla u^{\frac{m+q-1}{2}}\|_2^2 + \left(1-\frac{\alpha_0}{2}\right)\left[S_d\frac{2mq(q-1)}{(m+q-1)^2}\frac{2}{\alpha_0}\right]^{\frac{1}{1-2/\alpha_0}}\|u\|_1^{\delta_0},$$

where $\delta_0 = 1 + \frac{2(q-1)}{dm-d+2}$, $\alpha_0 = \frac{2(q-1)}{m+q-2+2/d} < 2$ for $m > 1 - 2/d$.

Similarly letting

$$a = \frac{2}{m+q-1},\ b = \frac{2(q+1)}{m+q-1},\ C_0 = \frac{2mq}{(m+q-1)^2}$$

in Lemma 2.1 leads to

Lemma 2.3. *Let $d \geq 3$, $q > 0, m > 2 - 2/d$, assume $u \in L^1_+(\mathbb{R}^d)$ and $u^{\frac{m+q-1}{2}} \in H^1(\mathbb{R}^d)$, then*

$$\|u\|_{q+1}^{q+1} \leq \frac{2mq}{(m+q-1)^2}\|\nabla u^{\frac{m+q-1}{2}}\|_2^2 + \left(1-\frac{\alpha}{2}\right)\left[S_d\frac{2mq}{(m+q-1)^2}\frac{2}{\alpha}\right]^{\frac{1}{1-2/\alpha}}\|u\|_1^{\eta},$$

where $\eta = 1 + \frac{2q}{dm-2d+2}$, $\alpha = \frac{2q}{m+q-2+2/d} < 2$ for $m > 2 - 2/d$.

For the supercritical case $0 < m < 2 - 2/d$, choosing particular a, b in (2.2) of Lemma 2.1 and using the Young inequality one has the following lemma which will be used in the next sections.

Lemma 2.4. *Let $d \geq 3$, $0 < m \leq 2 - 2/d$, $p = \frac{d(2-m)}{2}, q \geq p$ and $u \in L^1_+(\mathbb{R}^d)$. Then*

$$\|u\|_{q+1}^{q+1} \leq S_d^{-1}\|\nabla u^{(m+q-1)/2}\|_2^2 \|u\|_p^{2-m}, \tag{2.4}$$

and for $q \geq r > p$

$$\begin{aligned}\|u\|_{q+1}^{q+1} &\leq S_d^{-\frac{\alpha}{2}}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^{\alpha}\|u\|_r^{\beta} \\ &\leq \frac{2mq}{(m+q-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 + C(q,r,d)\left(\|u\|_r^r\right)^{\delta},\end{aligned} \tag{2.5}$$

where

$$\alpha = \frac{2(q-r+1)}{q-r+1+2(r-p)/d} < 2, \qquad \beta = q+1-\frac{m+q-1}{2}\alpha,$$

$$\delta = \frac{\beta}{r(1-\alpha/2)} = 1 + \frac{1+q-r}{r-p},$$

$$C(q,r,d) = \left[\frac{2mq[q-r+1+2(r-p)/d]}{S_d^{-1}(q+m-1)^2(q-r+1)}\right]^{-\frac{d(q-r+1)}{2(r-p)}}\frac{2(r-p)}{d(q-r+1)+2(r-p)}.$$

Now we define the weak solution which we will deal with throughout this paper.

Definition 2.5. (Weak solution) Let $U_0 \in L^1_+(\mathbb{R}^d)$ be the initial data and $T \in (0,\infty)$. c is the concentration associated with u. u is a weak solution to the system (1.1) with initial data U_0 and it satisfies:

(i) Regularity:

$$u \in L^{\max(m,2)}\left(0,T; L^1_+ \cap L^{\max\left(m,\frac{2d}{d+2}\right)}(\mathbb{R}^d)\right), \tag{2.6}$$

$$\partial_t u \in L^{p_2}\left(0,T; W^{-1,p_1}_{loc}(\mathbb{R}^d)\right) \text{ for some } p_1, p_2 \ge 1. \tag{2.7}$$

(ii) For $\forall\ \psi \in C_0^\infty(\mathbb{R}^d)$ and any $0 < t < \infty$

$$\begin{aligned}&\int_{\mathbb{R}^d} \psi u(\cdot,t)dx - \int_{\mathbb{R}^d} \psi U_0 dx = \int_0^t \int_{\mathbb{R}^d} \Delta\psi u^m dxds \\ &- \frac{c_d(d-2)}{2}\int_0^t \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{[\nabla\psi(x)-\nabla\psi(y)]\cdot(x-y)}{|x-y|^2}\frac{u(x,s)u(y,s)}{|x-y|^{d-2}}dxdyds.\end{aligned} \tag{2.8}$$

Remark 1. Notice that the regularity (2.6) is enough to make sense of each term in (2.8). By the HLS inequality [15] one has

$$\begin{aligned}&\iint_{\mathbb{R}^d\times\mathbb{R}^d} \left|\frac{[\nabla\psi(x)-\nabla\psi(y)]\cdot(x-y)}{|x-y|^2}\right| \frac{u(x,t)u(y,t)}{|x-y|^{d-2}}dxdy \\ \le& C\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{u(x,t)u(y,t)}{|x-y|^{d-2}}dxdy \\ \le& C\|u(x)\|^2_{2d/(d+2)} < \infty.\end{aligned}$$

Before showing the global existence results for $0 < m < 2-2/d$, we need the following lemma.

Lemma 2.6. *Assume $y(t) \ge 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \le \alpha - \beta y(t)^a$ for $\alpha > 0, \beta > 0$, then*

(i) *For $a > 1$, $y(t)$ has the following hyper-contractive property*

$$y(t) \le (\alpha/\beta)^{1/a} + \left[\frac{1}{\beta(a-1)t}\right]^{\frac{1}{a-1}}, \quad \text{for } t > 0. \tag{2.9}$$

Furthermore, if $y(0)$ is bounded, then

$$y(t) \le \max\left(y(0), (\alpha/\beta)^{1/a}\right). \tag{2.10}$$

(ii) *For $a = 1$, $y(t)$ decays exponentially*

$$y(t) \le \alpha/\beta + y(0)e^{-\beta t}.$$

(iii) *For $a < 1$, $\alpha = 0$, $y(t)$ has finite time extinction, that's there exists a $0 < T_{ext} \le \frac{y^{1-a}(0)}{\beta(1-a)}$ such that $y(t) = 0$ for all $t > T_{ext}$.*

Proof. The lemma was proved in [3] except (2.10), here we give the proof for (2.10). The ODE inequality can be recast as

$$y'(t) \le \beta\left[(\alpha/\beta)^{\frac{1}{a}a} - y(t)^a\right].$$

Case 1. If $y(0) \le (\alpha/\beta)^{1/a}$, then by contradiction arguments we have that for any $t > 0$

$$y(t) \le (\alpha/\beta)^{1/a}.$$

Case 2. For $y(0) > (\alpha/\beta)^{1/a}$, if $y(t) > (\alpha/\beta)^{1/a}$ for all $t > 0$, then $y'(t) < 0$ and thus $y(t) < y(0)$. Otherwise, denote t_0 as the first time such that $y(t_0) = (\alpha/\beta)^{1/a}$, then

$$\begin{aligned} &y'(t) < 0,\ 0 \le t < t_0, \\ &y(t) \le (\alpha/\beta)^{1/a},\ t > t_0. \end{aligned}$$

Collecting the two cases we obtain

$$y(t) \le \max\Big(y(0), (\alpha/\beta)^{1/a}\Big).$$

□

3. **The hyper-contractive estimates and proof of the main theorem.** In this section, we will show the hyper-contractive property for $m > 1 - 2/d$. Firstly we define a constant which is related to the initial condition for the existence results:

$$C_{d,m} := \left(\frac{4mp}{(m+p-1)^2 S_d^{-1}}\right)^{\frac{1}{2-m}},\ p = \frac{d(2-m)}{2}, \tag{3.1}$$

where S_d is given by (2.1). The following theorems give the hyper-contractive of L^q for any $1 < q < \infty$ which is proved in [3]. For the supercritical and critical cases,

Theorem 3.1 ([3]). *Let $d \ge 3$, $0 < m \le 2 - 2/d$ and $p = \frac{d(2-m)}{2}$, $\eta := C_{d,m}^{2-m} - \|U_0\|_p^{2-m}$. Assume $U_0 \in L^1_+(\mathbb{R}^d)$ and $\eta > 0$, then there exists a global weak solution u such that $\|u(\cdot,t)\|_p < C_{d,m}$ for all $t \ge 0$. Furthermore,*

- ***(i)*** *For $0 < m < 1 - 2/d$, there exists a minimal extinction time $T_{ext}(\|U_0\|_1, \eta, p)$ such that the weak solution vanishes a.e. in $\mathbb{R}^d$ for all $t \ge T_{ext}$.*
- ***(ii)*** *For $m = 1 - 2/d$, the weak solution decays exponentially*

$$\|u(\cdot,t)\|_p \le \|U_0\|_p e^{-\frac{\eta}{\|U_0\|_1^{1/(p-1)}}\frac{(p-1)}{p}t}. \tag{3.2}$$

- ***(iii)*** *For $1 - 2/d < m \le 2 - 2/d$, the weak solution satisfies mass conservation and the following hyper-contractive estimates hold true that for any $t > 0$ and $1 \le q \le p$*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^q \le C(\eta, \|U_0\|_1, q) t^{-\frac{q-1}{m-1+2/d}}, \tag{3.3}$$

and for $p < q < \infty$

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^d)}^q \le C(\eta, \|U_0\|_1, q)\left(t^{-\frac{(p+\epsilon_0-1)(1+q-p)}{(q+m+2/d-2)\epsilon_0}\frac{q-1}{m-1+2/d}} + t^{-\frac{q-1}{m-1+2/d}}\right), \tag{3.4}$$

where ϵ_0 satisfies $\frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2 S_d^{-1}} - \|U_0\|_p^{2-m} = \frac{\eta}{2}$.

Theorem 3.2 ([3]). *For $m > 2 - 2/d$, assume $U_0 \in L^1_+(\mathbb{R}^d)$. Assume also $U_0 \log U_0 \in L^1(\mathbb{R}^d)$ for $m = 2$ and $U_0 \in L^{m-1}(\mathbb{R}^d)$ for $m > 2$, then there exists a weak solution globally in time satisfying the following hyper-contractive property that for any $1 < q < \infty$*

$$\|u\|_q^q \le C(\|U_0\|_1, q, m, d) + \left[\frac{q-1}{t}\right]^{q-1}, \quad \textit{for any } t > 0. \tag{3.5}$$

Using the boundedness of $\|u\|_q$ for any $1<q<\infty$ we can prove our main result about the hyper-contractivity in L^∞ estimates.

Proof of Theorem 1.1. The global existence of a weak solution was proved in [3]. Now we will give the proof of the hyper-contractivity in $L^\infty(\mathbb{R}^d)$ for any positive time. Firstly we denote $q_k := 3^k+m+d+1$ and estimate $\int_{\mathbb{R}^d} u^{q_k}dx$.

Step 1. (The L^{q_k} estimate) Multiplying equation (1.1) with $u^{q_k-1}(q_k>1)$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^{q_k}dx = -\frac{4mq_k(q_k-1)}{(q_k+m-1)^2}\int_{\mathbb{R}^d}\left|\nabla u^{\frac{q_k+m-1}{2}}\right|^2 dx + (q_k-1)\int_{\mathbb{R}^d} u^{q_k+1}dx, \quad (3.6)$$

from Lemma 2.1 by letting

$$a=\frac{2q_{k-1}}{q_k+m-1},\ b=\frac{2(q_k+1)}{q_k+m-1},\ w=u^{a\frac{q_k+m-1}{2}}$$

we obtain

$$\int_{\mathbb{R}^d} u^{q_k+1}dx \le C(\delta_1)C_1^{-\frac{1}{\delta_1-1}}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\gamma_1} + C_1\left\|\nabla u^{\frac{q_k+m-1}{2}}\right\|_2^2, \quad (3.7)$$

where $\delta_1 = \frac{2\left(\frac{1}{a}-\frac{d-2}{2d}\right)}{\frac{b}{a}-1} = O(1)$ and $\gamma_1 = 1+\frac{2b-2a}{2d-bd+2a} \le 3$ with $m>0$, C_1 is a positive constant to be determined. It's easy to verify that $1<b/a<\frac{2d}{a(d-2)}$ and $b/a<2/a+2/d$.

Substituting (3.7) into (3.6) we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^{q_k}dx \le& \left(C_1(q_k-1)-\frac{4mq_k(q_k-1)}{(q_k+m-1)^2}\right)\int_{\mathbb{R}^d}\left|\nabla u^{\frac{q_k+m-1}{2}}\right|^2 dx\\ &+C(\delta_1)(q_k-1)C_1^{-\frac{1}{\delta_1-1}}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\gamma_1}. \quad (3.8)\end{aligned}$$

We can see that for $k\to\infty$,

$$\frac{4mq_k(q_k-1)}{(q_k+m-1)^2}\to 4m,$$

therefore, in order to control the term $\int_{\mathbb{R}^d}\left|\nabla u^{\frac{q_k+m-1}{2}}\right|^2 dx$ in (3.8), since $q_k>m+1$, we can choose $C_1=\frac{m}{2(q_k-1)}$, $C_2=m/2$ such that

$$C_1(q_k-1)-\frac{4mq_k(q_k-1)}{(q_k+m-1)^2}\le -C_2, \quad (3.9)$$

this follows

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} u^{q_k}dx \le& -C_2\int_{\mathbb{R}^d}\left|\nabla u^{\frac{q_k+m-1}{2}}\right|^2 dx\\ &+C(\delta_1)(q_k-1)C_1^{-\frac{1}{\delta_1-1}}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\gamma_1}. \quad (3.10)\end{aligned}$$

On the other hand, from (2.2) of Lemma 2.1 letting

$$a=\frac{2q_{k-1}}{q_k+m-1},\ b=\frac{2q_k}{q_k+m-1}$$

one has

$$\left(\|u\|_{q_k}^{q_k}\right)^{1+\frac{m-1+2q_{k-1}/d}{q_k-q_{k-1}}} \le S_d^{-1}\|\nabla u^{(q_k+m-1)/2}\|_2^2\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\frac{1}{q_k-q_{k-1}}(2q_k/d+m-1)},$$

substituting it into (3.10) follows that for any $t > t_0$ with fixed $t_0 > 0$

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^{q_k}dx \le -\frac{C_2}{S_d^{-1}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\frac{1}{q_k-q_{k-1}}(2q_k/d+m-1)}}\left(\|u\|_{q_k}^{q_k}\right)^{1+\frac{m-1+2q_{k-1}/d}{q_k-q_{k-1}}} + C(\delta_1)q_k^{\frac{1}{1-1/\delta_1}}\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\gamma_1},$$

where for $m > 0$

$$\gamma_1 = 1 + \frac{2q_k - 2q_{k-1} + 2}{dm - 2d + 2q_{k-1}} < 3,\ \delta_1 = 1 + \frac{m-2+2q_{k-1}/d}{q_k - q_{k-1} + 1} \ge 1 + 1/d.$$

Since $q_k > 1$, thus for any $t > t_0 > 0$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^{q_k}dx \le -\frac{C(m,d)}{\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\frac{1}{q_k-q_{k-1}}(2q_k/d+m-1)}}\left(\|u\|_{q_k}^{q_k}\right)^{1+\frac{m-1+2q_{k-1}/d}{q_k-q_{k-1}}} + C(\delta_1)q_k^{d+1}\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\gamma_1},$$

Step 2. (Iterative procedures and hyper-contractive estimates) By applying Lemma 2.6, letting $y_k(t) = \int_{\mathbb{R}^d} u^{q_k}dx$ and taking

$$a = 1 + \frac{m-1+2q_{k-1}/d}{q_k - q_{k-1}} \ge 1 + 1/d, \quad \text{if } m > 0,$$

$$\beta(t_0) = \frac{C(m,d)}{\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\frac{1}{q_k-q_{k-1}}(2q_k/d+m-1)}},$$

$$\alpha(t_0) = C(\delta_1)q_k^{d+1}\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\gamma_1},$$

in the ODE inequality (2.9), then

$$y_k(t) \le [\alpha(t_0)/\beta(t_0)]^{1/a} + \left[\frac{1}{\beta(t_0)(a-1)(t-t_0)}\right]^{1/(a-1)},\ t > t_0, \tag{3.11}$$

plugging $a, \alpha(t_0), \beta(t_0)$ into (3.11) yields that for any $t > t_0 > 0$

$$\begin{aligned} y_k(t) \le & C(m,d)q_k^{\frac{d+1}{a}}\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\left(\gamma_1+\frac{2q_k/d+m-1}{q_k-q_{k-1}}\right)\frac{1}{a}} \\ & + \left[\frac{C(m,d)\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{\frac{m-1+2q_k/d}{q_k-q_{k-1}}}}{(a-1)(t-t_0)}\right]^{\frac{1}{a-1}} \\ \le & C(m,d)q_k^{d+1}\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{A} + \left[\frac{C(m,d)\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)}{(t-t_0)^{1/\eta_0}}\right]^{B} \\ \le & \max[1, C(m,d)][2(m+d+1)3^k]^{d+1}\cdot \\ & \left(\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)^{A} + \left[\frac{\sup_{t_0<t<\infty}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}dx\right)}{(t-t_0)^{1/\eta_0}}\right]^{B}\right), \end{aligned} \tag{3.12}$$

where we have used $a-1\geq 1/d$ and for $m>0$

$$\eta_0=\frac{2q_k/d+m-1}{q_k-q_{k-1}}\geq\frac{d}{3},$$
$$A=\frac{\gamma_1+\eta_0}{a}=\frac{1+\frac{2q_k-2q_{k-1}+2}{dm-2d+2q_{k-1}}+\frac{2q_k/d+m-1}{q_k-q_{k-1}}}{1+\frac{2q_{k-1}/d+m-1}{q_k-q_{k-1}}}\leq 3,$$
$$B=\frac{\eta_0}{a-1}=\frac{2q_k/d+m-1}{2q_{k-1}/d+m-1}\leq 3,$$

denote $C_0=\max[1,C(m,d)][2(m+d+1)]^{d+1}$, from (3.12) one has that for any $t_0<t<\infty$

$$\begin{aligned} y_k(t)&\leq C_0 3^{(d+1)k}\left[\sup_{t_0<t<\infty}y_{k-1}^A(t)+\left(\frac{\sup_{t_0<t<\infty}y_{k-1}(t)}{(t-t_0)^{1/\eta_0}}\right)^B\right]\\ &\leq 2C_0 3^{(d+1)k}\max\left\{1,\sup_{t_0<t<\infty}y_{k-1}^3(t),\left(\frac{\sup_{t_0<t<\infty}y_{k-1}(t)}{(t-t_0)^{1/\eta_0}}\right)^3\right\}. \end{aligned}\tag{3.13}$$

Next we will analyze the inequality (3.13).

If $0<t\leq 1$, take $0<(t-t_0)^{1/\eta_0}<1$, then $\frac{1}{\eta_0}\leq\frac{d}{3}$ gives rise to

$$\begin{aligned} y_k(t)&\leq 2C_0 3^{(d+1)k}\max\left\{1,\left(\frac{\sup_{t_0<t<\infty}y_{k-1}(t)}{(t-t_0)^{1/\eta_0}}\right)^3\right\}\\ &\leq\frac{2C_0}{(t-t_0)^d}3^{(d+1)k}\max\left(1,\sup_{t_0<t<\infty}y_{k-1}^3(t)\right), \end{aligned}$$

then after some iterative procedures for any fixed t,t_0, we have

$$y_k(t)\leq\left(\frac{2C_0}{(t-t_0)^d}\right)^{\frac{3^k-1}{2}}3^{(d+1)\left(\frac{3^{k+1}}{4}-\frac{k}{2}-\frac{3}{4}\right)}\max\left(\sup_{t_0<t<\infty}y_0^{3^k}(t),1\right).\tag{3.14}$$

Recalling $q_k=3^k+m+d+1$, taking the power $\frac{1}{q_k}$ to both sides of (3.14) we conclude that for $t_0<t\leq 1$

$$\|u(\cdot,t)\|_\infty\leq\frac{\sqrt{2C_0}}{(t-t_0)^{d/2}}3^{3(d+1)/4}\max\left(\sup_{t_0<t<\infty}\|u(t)\|_{m+d+2}^{m+d+2},1\right),\tag{3.15}$$

take $t_0=t/2$ we have

$$\|u(\cdot,t)\|_\infty\leq\frac{C(d,m)}{t^{d/2}}3^{3(d+1)/4}\max\left(\sup_{t/2<s<\infty}\|u(s)\|_{m+d+2}^{m+d+2},1\right),\ 0<t\leq 1.\tag{3.16}$$

Similarly, if $1<t<\infty$, taking $t-t_0>1/2$ in (3.13) we have

$$y_k(t)\leq C_1 3^{(d+1)k}\max\{1,\sup_{t_0<t<\infty}y_{k-1}^3(t)\},$$

where $C_1=2C_0 2^{1/\eta_0}$, this follows

$$y_k(t)\leq C_1^{\frac{3^k-1}{2}}3^{(d+1)\left(\frac{3^{k+1}}{4}-\frac{k}{2}-\frac{3}{4}\right)}\max\left(\sup_{t_0<t<\infty}y_0^{3^k}(t),1\right),\tag{3.17}$$

taking the power $\frac{1}{q_k}$ to both sides of (3.17) we conclude that for $1/2 < t - t_0 < \infty$

$$\|u(\cdot,t)\|_\infty \le \sqrt{C_1} 3^{3(d+1)/4} \max\left(\sup_{t_0<t<\infty} \|u(t)\|_{m+d+2}^{m+d+2}, 1 \right), \tag{3.18}$$

taking $t_0 = t/2$ in (3.18) follows

$$\|u(\cdot,t)\|_\infty \le \sqrt{C_1} 3^{3(d+1)/4} \max\left(\sup_{t_0<t<\infty} \|u(t)\|_{m+d+2}^{m+d+2}, 1 \right), \ 1 < t < \infty, \tag{3.19}$$

Step 3. (Boundedness and decay in L^∞ norm for supercritical, critical cases) For $1 - 2/d < m \le 2 - 2/d$, by virtue of **(iii)** of Theorem 3.1, due to $m + d + 2 > \frac{d(2-m)}{2} = p$ we have that for any $0 < t < \infty$

$$\begin{aligned}
\|u(t)\|_{m+d+2}^{m+d+2} \le & C(\eta, \|U_0\|_1, m, d) \cdot \\
& \left(\frac{1}{t^{\frac{(p+\epsilon_0-1)(1+m+d+2-p)}{\epsilon_0(m+d+2+m+2/d-2)} \frac{m+d+2-1}{m-1+2/d}}} + \frac{1}{t^{\frac{m+d+2-1}{(m+d+2)(m-1+2/d)}}} \right) \\
= & C(\eta, \|U_0\|_1, m, d) \left(\frac{1}{t^{\frac{(p+\epsilon_0-1)(3+m+dm/2)}{\epsilon_0(2m+d+2/d)} \frac{m+d+1}{m-1+2/d}}} + \frac{1}{t^{\frac{m+d+1}{m-1+2/d}}} \right),
\end{aligned}$$

where $\eta = C_{d,m} - \|U_0\|_p$ and ϵ_0 satisfies $\frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2 S_d^{-1}} = \eta/2$, then (3.16) and (3.19) follow the boundedness of the solution that for $0 < t \le 1$

$$\begin{aligned}
\|u(\cdot,t)\|_\infty \le & \max\left[1, C(\eta, \|U_0\|_1, m, d)\right] \cdot \\
& \left(\frac{1}{t^{\frac{(p+\epsilon_0-1)(3+m+dm/2)}{\epsilon_0(2m+d+2/d)} \frac{m+d+1}{m-1+2/d}}} + \frac{1}{t^{\frac{m+d+1}{m-1+2/d}}} \right) \frac{1}{t^{d/2}},
\end{aligned}$$

and for $1 < t < \infty$

$$\|u(\cdot,t)\|_\infty \le \max\left[1, C(\eta, \|U_0\|_1, m, d)\right] \left(\frac{1}{t^{\frac{(p+\epsilon_0-1)(3+m+dm/2)}{\epsilon_0(2m+d+2/d)} \frac{m+d+1}{m-1+2/d}}} + \frac{1}{t^{\frac{m+d+1}{m-1+2/d}}} \right).$$

Step 4. (Boundedness in L^∞ norm for subcritical case) For $m > 2 - 2/d$, by Theorem 3.2, we have for any $1 < q < \infty$,

$$\|u\|_q^q \le C(\|U_0\|_1, q, m, d) + \left[\frac{q-1}{t} \right]^{q-1}, \quad \text{for any } t > 0. \tag{3.20}$$

Similar to (3.16) and (3.18) we obtain

$$\|u(\cdot,t)\|_\infty \le \max\left[1, C(\|U_0\|_1, m, d)\right] \left(1 + \frac{1}{t^{m+d+2-1}} \right) \cdot \frac{1}{t^{d/2}}, \qquad 0 < t \le 1,$$

$$\|u(\cdot,t)\|_\infty \le \max\left[1, C(\|U_0\|_1, m, d)\right] \left(1 + \frac{1}{t^{m+d+2-1}} \right), \qquad 1 < t < \infty.$$

Step 5. (Time regularity) Previously we have the following basic estimates that for any $T > 0$

$$\|u\|_{L^\infty\left(0,T; L^1_+ \cap L^p(\mathbb{R}^d)\right)} \le C, \tag{3.21}$$

$$\left\| \nabla u^{\frac{m+r-1}{2}} \right\|_{L^2(0,T;L^2(\mathbb{R}^d))} \le C, \ 1 < r \le p, \tag{3.22}$$

$$\|u\|_{L^{p+1}(0,T;L^{p+1}(\mathbb{R}^d))} \le C. \tag{3.23}$$

After some computations we obtain the time regularity

$$\|u_t\|_{L^{\min\left(2,\frac{2(p+1)}{4-m}\right)}\left(0,T;W_{loc}^{-1,\frac{2p}{p+2}}(\mathbb{R}^d)\right)} \le C. \tag{3.24}$$

On the other hand, using the L^∞ bound for any $t>0$, it's easy to verity that for any $T>t_0>0$

$$\|u\|_{L^\infty\left(t_0,T;L^1_+\cap L^\infty(\mathbb{R}^d)\right)} \le C,$$
$$\left\|\nabla u^{\frac{m+q-1}{2}}\right\|_{L^2(t_0,T;L^2(\mathbb{R}^d))} \le C, \quad \text{for any } 1<q<\infty,$$

here we can choose $\frac{m+q-1}{2} \ge 1$ such that the solution satisfies the following gradient estimates

$$\|\nabla u\|_{L^2(t_0,T;L^2(\mathbb{R}^d))} \le C,$$

then some computations by using the above regularities verify the regularities (1.8) and (1.9). This ends the proof. □

Using Theorem 2.17 in [3] for the local existence results directly leads to the following corollary.

Corollary 1. *For $0<m<2-2/d$, if $U_0 \in L^1_+\cap L^q(\mathbb{R}^d)$ for some $m \le q<\infty$ and $p<q<\infty$, then there exists a finite time T_q depending on $\|U_0\|_q$ and a local weak solution $u(x,t)$ such that*

$$\|u(\cdot,t)\|_{L^\infty} \le \frac{C(\|U_0\|_q,q)}{t^\alpha}, \ 0<t<T_q/2,$$

where α is a positive constant.

Notice that the local existence results in Theorem 2.17 of [3] also hold true for $m>0$.

4. **The uniform estimates in $L^\infty(\mathbb{R}^d)$.** In this section, we will show that if $U_0 \in L^1_+\cap L^\infty(\mathbb{R}^d)$, then the solution is bounded in $L^\infty(\mathbb{R}^d)$ uniformly in time instead of hyper-contractivity in Section 3. Firstly, we will give the proof for the boundedness in $L^q(\mathbb{R}^d)(1<q<\infty)$ uniformly in time in the following proposition.

Proposition 1. *Let $d\ge 3$,*

1. *For $0<m\le 2-2/d$ and $p=\frac{d(2-m)}{2}\ge 1$, $\eta := C_{d,m}^{2-m} - \|U_0\|_p^{2-m}$, if $U_0\in L^1_+\cap L^q(\mathbb{R}^d)$ for some $1<q<\infty$ and $\eta>0$, then there exists a global weak solution u such that*

$$\|u\|_q^q \le C(\|U_0\|_1,q)\|U_0\|_p^{\frac{p(q-1)}{p-1}}, \ 1<q\le p,$$
$$\|u\|_q^q \le \|U_0\|_q^q + C(\|U_0\|_1,q)\left(\|U_0\|_q^q\right)^{\frac{p+\epsilon_0-1}{\epsilon_0}\frac{q-p+1}{q+m-2+2/d}}, \ p<q<\infty, \tag{4.1}$$

where ϵ_0 satisfies

$$\frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2 S_d^{-1}} - \|U_0\|_p^{2-m} = \frac{\eta}{2}. \tag{4.2}$$

2. For $m > 2 - 2/d$, if $U_0 \in L^1_+ \cap L^q(\mathbb{R}^d)$ for some $1 < q < \infty$, then

$$\|u\|_q^q \le \|U_0\|_q^q + \left(1 - \frac{\alpha_0}{2}\right)\left[S_d \frac{2mq(q-1)}{(m+q-1)^2}\frac{2}{\alpha_0}\right]^{\frac{1}{1-2/\alpha_0}} \|U_0\|_1^{1+\frac{2(q-1)}{dm-d+2}}$$
$$+ (q-1)\left(1 - \frac{\alpha}{2}\right)\left[S_d \frac{2mq}{(m+q-1)^2}\frac{2}{\alpha}\right]^{\frac{1}{1-2/\alpha}} \|U_0\|_1^{1+\frac{2q}{dm-2d+2}}, \tag{4.3}$$

where $\alpha = \frac{2q}{m+q-2+2/d}$, $\alpha_0 = \frac{2(q-1)}{m+q-2+2/d}$.

Proof. **Step 1.** (Uniform L^p estimates for $0 < m < 2 - 2/d$) Firstly it's obtained by multiplying the equation (1.1) with pu^{p-1} leads to

$$\frac{d}{dt}\int u^p dx + \frac{4mp(p-1)}{(m+p-1)^2}\int \left|\nabla u^{(m+p-1)/2}\right|^2 dx$$
$$=(p-1)\int u^{p+1} dx \le (p-1)S_d^{-1}\|\nabla u^{(m+p-1)/2}\|_2^2 \|u\|_p^{2-m}, \tag{4.4}$$

where the last inequality (4.4) follows from (2.4) with $q = p$. Hence one has

$$\frac{d}{dt}\int u^p dx + S_d^{-1}(p-1)\left(C_{d,m}^{2-m} - \|u\|_p^{2-m}\right)\int \left|\nabla u^{(m+p-1)/2}\right|^2 dx \le 0. \tag{4.5}$$

Since $\|U_0\|_p < C_{d,m}$, so the following estimate holds true for any $t > 0$

$$\|u(\cdot,t)\|_p < \|U_0\|_p < C_{d,m}, \tag{4.6}$$

Step 2. (Finite time extinction for $0 < m < 1 - 2/d$) It follows from (2.3) by using $\|u\|_1 \le \|U_0\|_1$ that

$$\frac{\left(\|u\|_p^p\right)^{1+\frac{m-1+2/d}{p-1}}}{S_d^{-1}\|U_0\|_1^{\frac{1}{p-1}}} \le \|\nabla u^{\frac{p+m-1}{2}}\|_2^2. \tag{4.7}$$

Substituting (4.7) into (4.4) arrives at

$$\frac{d}{dt}\int u^p dx + \frac{(p-1)\eta}{\|U_0\|_1^{\frac{1}{p-1}}}\left(\int u^p dx\right)^{\delta} \le 0, \tag{4.8}$$

where $\delta = 1 + \frac{m-1+2/d}{p-1} < 1$ for $m < 1 - 2/d$. Hence in view of Lemma 2.6 **(iii)**, there exists a finite time $0 < T_{ext} \le \frac{(\|U_0\|_p^p)^{1-\delta}}{C_p(1-\delta)}$ with $0 < \delta = 1 + \frac{m-1+2/d}{p-1} < 1$ such that $\|u(\cdot,t)\|_p$ will vanish a.e. in $\mathbb{R}^d$ for all $t > T_{ext}$, thus the solution will extinct at finite time.

Step 3. (Uniform L^{r_0} estimate with $r_0 := p + \epsilon_0$ for ϵ_0 small enough for $1 - 2/d \le m < 2 - 2/d$) Using (2.4) with $q = r_0$ deduces

$$\frac{d}{dt}\int u^{r_0} dx + \frac{4mr_0(r_0-1)}{(r_0+m-1)^2}\int \left|\nabla u^{(m+r_0-1)/2}\right|^2 dx$$
$$=(r_0-1)\int u^{r_0+1} dx$$
$$\le (r_0-1)S_d^{-1}\|\nabla u^{(r_0+m-1)/2}\|_2^2 \|u\|_p^{2-m}$$
$$\le (r_0-1)S_d^{-1}\|\nabla u^{(r_0+m-1)/2}\|_2^2 \|U_0\|_p^{2-m}. \tag{4.9}$$

The last inequality is derived from (4.6). If we choose ϵ_0 such that

$$\frac{\eta}{2} := \frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2 S_d^{-1}} - \|U_0\|_p^{2-m} < \eta, \tag{4.10}$$

then one has

$$\frac{d}{dt}\int u^{r_0}dx + S_d^{-1}(r_0-1)\frac{\eta}{2}\int \left|\nabla u^{(m+r_0-1)/2}\right|^2 dx \le 0, \tag{4.11}$$

then we obtain the uniform estimates for $\|u\|_{r_0}$

$$\|u(\cdot,t)\|_{r_0} \le \|U_0\|_{r_0}. \tag{4.12}$$

Step 4. (Uniform L^q estimates for $q > r_0$ with $U_0 \in L^q(\mathbb{R}^d)$ and $1-2/d \le m < 2-2/d$) For $q > r_0$, taking $r = r_0$ in (2.5) and using (4.12) one has

$$\begin{aligned}
&\frac{d}{dt}\|u\|_q^q + \frac{4qm(q-1)}{(q+m-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 \\
=&(q-1)\int u^{q+1}dx \\
\le&\frac{2mq(q-1)}{(m+q-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 + C(q,r_0,d)\left(\|u\|_{r_0}^{r_0}\right)^{\delta}, \\
\le&\frac{2mq(q-1)}{(m+q-1)^2}\left\|\nabla u^{\frac{q+m-1}{2}}\right\|_2^2 + C(q,r_0,d)\left(\|U_0\|_{r_0}^{r_0}\right)^{\delta},
\end{aligned} \tag{4.13}$$

where $\delta = 1 + \frac{1+q-r_0}{r_0-p}$.

Collecting (2.3) and (4.13) yields

$$\begin{aligned}
\frac{d}{dt}\|u\|_q^q \le& -\frac{2mq(q-1)}{S_d^{-1}(m+q-1)^2\|U_0\|_1^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}}\left(\|u\|_q^q\right)^{1+\frac{m-1+2/d}{q-1}} \\
&+ C(q,r_0,d)\left(\|U_0\|_{r_0}^{r_0}\right)^{\delta}.
\end{aligned} \tag{4.14}$$

From Lemma 2.6 by letting

$$y(t) = \|u\|_q^q,\ \ \alpha = C(q,r_0,d)\left(\|U_0\|_{r_0}^{r_0}\right)^{\delta},\ \ \beta = \frac{2mq(q-1)}{S_d^{-1}(m+q-1)^2\|U_0\|_1^{\frac{1}{q-1}\left(1+\frac{2(q-p)}{d}\right)}},$$

Case 1. **($1-2/d < m < 2-2/d$)** $a = 1 + \frac{m-1+2/d}{q-1} > 1$, by (2.10) of Lemma 2.6 we have

$$\begin{aligned}
\|u\|_q^q \le& \max\left(\|U_0\|_q^q, C(\|U_0\|_1,q)(\|U_0\|_{r_0}^{r_0})^{\frac{\delta}{a}}\right) \\
\le& \max\left(\|U_0\|_q^q, C(\|U_0\|_1,q)\left(\|U_0\|_q^q\right)^{\frac{r_0-1}{r_0-p}\frac{q-p+1}{q+m-2+2/d}}\right),
\end{aligned}$$

where we have used the interpolation inequality in the last inequality for $1 < p < r_0 < q$.

Case 2. **($m = 1-2/d$)** $a = 1$, from Lemma 2.6 one has

$$y(t) \le \alpha/\beta + y(0), \tag{4.15}$$

$$\|u(\cdot,t)\|_q^q \le \|U_0\|_q^q + C(\|U_0\|_1,q)\left(\|U_0\|_q^q\right)^{\frac{r_0-1}{q-1}\frac{q-p+1}{r_0-p}}. \tag{4.16}$$

Thus we conclude that for $m \ge 1-2/d$

$$\|u\|_q^q \le \|U_0\|_q^q + C(\|U_0\|_1,q)\left(\|U_0\|_q^q\right)^{\frac{r_0-1}{r_0-p}\frac{q-p+1}{q+m-2+2/d}}.$$

Step 5. (Uniform L^q estimates with $U_0 \in L^q(\mathbb{R}^d)$ and $m > 2-2/d$) Taking Lemma 2.3 into account we have the following estimates

$$
\begin{aligned}
&\frac{d}{dt}\|u\|_q^q + \frac{4mq}{(m+q-1)^2}\|\nabla u^{\frac{m+q-1}{2}}\|_2^2 = (q-1)\|u\|_{q+1}^{q+1} \\
\le& \frac{2mq(q-1)}{(m+q-1)^2}\|\nabla u^{\frac{m+q-1}{2}}\|_2^2 \\
&+ (q-1)\left(1-\frac{\alpha}{2}\right)\left[S_d\frac{2mq}{(m+q-1)^2}\frac{2}{\alpha}\right]^{\frac{1}{1-2/\alpha}}\|u\|_1^{1+\frac{2q}{dm-2d+2}}.
\end{aligned}
$$

Here $\alpha = \frac{2q}{m+q-2+2/d}$, combining Lemma 2.2 leads to

$$
\begin{aligned}
\frac{d}{dt}\|u\|_q^q \le& -\|u\|_q^q + \left(1-\frac{\alpha_0}{2}\right)\left[S_d\frac{2mq(q-1)}{(m+q-1)^2}\frac{2}{\alpha_0}\right]^{\frac{1}{1-2/\alpha_0}}\|u\|_1^{1+\frac{2(q-1)}{dm-d+2}} \\
&+ (q-1)\left(1-\frac{\alpha}{2}\right)\left[S_d\frac{2mq}{(m+q-1)^2}\frac{2}{\alpha}\right]^{\frac{1}{1-2/\alpha}}\|u\|_1^{1+\frac{2q}{dm-2d+2}},
\end{aligned}
$$

where $\alpha_0 = \frac{2(q-1)}{m+q-2+2/d}$. By Gronwall's inequality we obtain the conclusion.

As to the regularity process and global existence, we can refer to [3] for precise results. Thus ends the proof. □

The following lemma is proved by the spirit of [1] which will be used to estimate the boundedness in $L^\infty(\mathbb{R}^d)$.

Lemma 4.1. *Assume $y_k(t) \ge 0,\ k = 0,1,2,...$ are C^1 functions for $t > 0$ satisfying*

$$
y_k'(t) \le -y_k + a_k\left(y_{k-1}^{\gamma_1}(t) + y_{k-1}^{\gamma_2}(t)\right), \tag{4.17}
$$

where $a_k = \bar{a}3^{rk} > 1$ with $\bar{a}, r$ are positive bounded constants and $0 < \gamma_2 < \gamma_1 \le 3$. Assume also that there exists a bounded constant $K \ge 1$ such that $y_k(0) \le K^{3^k}$, then

$$
y_k(t) \le (2\bar{a})^{\frac{3^k-1}{2}} 3^{r\left(\frac{3^{k+1}}{4}-\frac{k}{2}-\frac{3}{4}\right)} \max\left\{\sup_{t\ge0} y_0^{3^k}(t), K^{3^k}\right\}. \tag{4.18}
$$

Proof. Multiplying e^t to both sides of (4.17) yields

$$
\begin{aligned}
\left(e^t y_k(t)\right)' \le a_k e^t\left(y_{k-1}^{\gamma_1}(t) + y_{k-1}^{\gamma_2}(t)\right) \le 2a_k e^t \max\{1, \sup_{t\ge0} y_{k-1}^3(t)\}, \\
y_k(t) \le (1-e^{-t})2a_k \max\left\{1, \sup_{t\ge0} y_{k-1}^3(t)\right\} + e^{-t}y_k(0) \qquad (4.19) \\
\le 2a_k \max\left\{1, \sup_{t\ge0} y_{k-1}^3(t), y_k(0)\right\} \\
\le 2a_k \max\left\{1, \sup_{t\ge0} y_{k-1}^3(t), K^{3^k}\right\} = 2a_k \max\left\{\sup_{t\ge0} y_{k-1}^3(t), K^{3^k}\right\}.
\end{aligned}
$$

Then from (4.19) after some iterative steps we have

$$
\begin{aligned}
y_k(t) &\le 2a_k(2a_{k-1})^3(2a_{k-2})^{3^2}(2a_{k-3})^{3^3}\cdots(2a_1)^{3^{k-1}}\max\left\{\sup_{t\ge 0} y_0^{3^k}(t), K^{3^k}\right\}\\
&= (2\bar{a})^{1+3+3^2+3^3+\cdots 3^{k-1}} 3^{r\left(k+3(k-1)+3^2(k-2)+\cdots+3^{k-1}\right)}\max\left\{\sup_{t\ge 0} y_0^{3^k}(t), K^{3^k}\right\}\\
&= (2\bar{a})^{\frac{3^k-1}{2}} 3^{r\left(\frac{3^{k+1}}{4}-\frac{k}{2}-\frac{3}{4}\right)}\max\left\{\sup_{t\ge 0} y_0^{3^k}(t), K^{3^k}\right\}.
\end{aligned}
$$

□

Now we are in a position to prove the L^∞ bound.

Theorem 4.2. *Let $d \ge 3$, $m > 0$. Assume $U_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$. For $0 < m < 2-2/d$, we also assume $\|U_0\|_p < C_{d,m}$. Then there exists a weak solution of (1.1) such that for any $t > 0$*

$$
\|u\|_{L^\infty} \le C(m, d, K_0),
$$

where $K_0 = \max\{1, \|U_0\|_1, \|U_0\|_\infty\}$. Furthermore, if $\nabla U_0^m \in L^2(\mathbb{R}^d)$, then for any $T > 0$, the weak solution has the following regularities

$$
u(x,t) \in L^\infty\left(0,T; L^1_+ \cap L^\infty(\mathbb{R}^d)\right) \cap L^2\left(0,T; H^1(\mathbb{R}^d)\right),
$$

and

$$
u_t \in L^2\left(0,T; H^{-1}(\mathbb{R}^d)\right), \tag{4.20}
$$

$$
\nabla u^m \in L^\infty\left(0,T; L^2(\mathbb{R}^d)\right), \tag{4.21}
$$

$$
\left(u^{\frac{m+1}{2}}\right)_t \in L^2\left(0,T; L^2(\mathbb{R}^d)\right). \tag{4.22}
$$

Proof of Theorem 4.2. The global existence of the weak solution has been proved in Theorem 2.17 of [3] with $U_0 \in L^1_+ \cap L^\infty(\mathbb{R}^d)$. Now we will focus on the boundedness in $L^\infty(\mathbb{R}^d)$ uniformly in time. Firstly we denote $q_k = 3^k + m + d + 1$ and estimate $\int_{\mathbb{R}^d} u^{q_k}\,dx$.

Step 1. (The L^{q_k} estimate) Similar to the proof from (3.6) to (3.10) of Theorem 1.1, we also obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d} u^{q_k}\,dx \le &- C_2\int_{\mathbb{R}^d}\left|\nabla u^{\frac{q_k+m-1}{2}}\right|^2 dx\\
&+ C(\delta_1)(q_k-1)C_1^{-\frac{1}{\delta_1-1}}\left(\int_{\mathbb{R}^d} u^{q_{k-1}}\,dx\right)^{\gamma_1}.
\end{aligned}\tag{4.23}
$$

Here $C_2 = m/2$, $C_1 = \frac{m}{2(q_k-1)}$ and

$$
\gamma_1 = 1 + \frac{2\tilde{b}-2\tilde{a}}{2d-\tilde{b}d+2\tilde{a}} \le 3,\ \ \delta_1 = \frac{2\left(\frac{1}{\tilde{a}}-\frac{d-2}{2d}\right)}{\frac{\tilde{b}}{\tilde{a}}-1} = O(1),
$$

where

$$
\tilde{a} = \frac{2q_{k-1}}{q_k+m-1},\ \ \tilde{b} = \frac{2(q_k+1)}{q_k+m-1}.
$$

Moreover, taking

$$
a = \frac{2q_{k-1}}{q_k+m},\ \ b = \frac{2q_k}{q_k+m},\ \ w = u^{a\frac{q_k+m-1}{2}}
$$

in Lemma 2.1 we have

$$\int_{\mathbb{R}^d} u^{q_k} dx \le C(\delta_2) C_2^{-\frac{1}{\delta_2 - 1}} \left(\int_{\mathbb{R}^d} u^{q_k - 1} dx \right)^{\gamma_2} + C_2 \left\| \nabla u^{\frac{q_k + m - 1}{2}} \right\|_2^2, \tag{4.24}$$

where $\delta_2 = \frac{2\left(\frac{1}{a} - \frac{d-2}{2d}\right)}{\frac{b}{a} - 1} = O(1)$ and $\gamma_2 = 1 + \frac{2b - 2a}{2d - bd + 2a} \le 3$ if $m > 0$.

Plugging (4.24) into (4.23) one has

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^d} u^{q_k} dx \\ \le & - \int_{\mathbb{R}^d} u^{q_k} dx + C(\delta_1)(q_k - 1) C_1^{-\frac{1}{\delta_1 - 1}} \left(\int_{\mathbb{R}^d} u^{q_k - 1} dx \right)^{\gamma_1} \\ & + C(\delta_2) C_2^{-\frac{1}{\delta_2 - 1}} \left(\int_{\mathbb{R}^d} u^{q_k - 1} dx \right)^{\gamma_2} \\ = & - \int_{\mathbb{R}^d} u^{q_k} dx + C(\delta_1, m)(q_k - 1)^{\frac{1}{1 - \frac{1}{\delta_1}}} \left(\int_{\mathbb{R}^d} u^{q_k - 1} dx \right)^{\gamma_1} \\ & + C(\delta_2, m) \left(\int_{\mathbb{R}^d} u^{q_k - 1} dx \right)^{\gamma_2} \\ \le & - \int_{\mathbb{R}^d} u^{q_k} dx \\ & + \max\left[1, C(\delta_1, m), C(\delta_2, m)\right] q_k^{\frac{1}{1 - 1/\delta_1}} \left\{ \left(\int_{\mathbb{R}^d} u^{q_k - 1} dx \right)^{\gamma_1} + \left(\int_{\mathbb{R}^d} u^{q_k - 1} dx \right)^{\gamma_2} \right\}, \end{aligned} \tag{4.25}$$

where $\gamma_2 < \gamma_1 \le 3$ with $m > 0$.

Step 2. $\big($Uniform estimates of $L^\infty(\mathbb{R}^d)$ $\big)$ Let $K_0 = \max\left(1, \|U_0\|_1, \|U_0\|_\infty\right)$ and $K = K_0^{\frac{q_k}{3^k}} \ge 1$, then

$$y_k(0) = \|U_0\|_{q_k}^{q_k} \le \left[\max\left(\|U_0\|_1, \|U_0\|_\infty\right)\right]^{q_k} \le K_0^{q_k} = K^{3^k}. \tag{4.26}$$

Take

$$y_k(t) = \int_{\mathbb{R}^d} u^{q_k} dx, \;\; r = \frac{1}{1 - 1/\delta_1},$$

$$\bar{a} = \max\left[1, C(\delta_1, m), C(\delta_2, m)\right](m + d + 1)^r = O(1),$$

then (4.25) can be recast as

$$y_k'(t) \le -y_k(t) + \bar{a} 3^{rk} \left(y_{k-1}^{\gamma_1}(t) + y_{k-1}^{\gamma_2}(t) \right). \tag{4.27}$$

Combining (4.26) and (4.27), by Lemma 4.1 we obtain

$$\int_{\mathbb{R}^d} u^{q_k} dx \le (2\bar{a})^{\frac{3^k - 1}{2}} 3^{r\left(\frac{3^{k+1}}{4} - \frac{k}{2} - \frac{3}{4}\right)} \max\left\{ \sup_{t \ge 0} y_0^{3^k}(t), K^{3^k} \right\}. \tag{4.28}$$

Recalling $q_k = 3^k + m + d + 1$ and taking the power $\frac{1}{q_k}$ to both sides of (4.28), then the boundedness of the solution u is obtained by passing to the limit $k \to \infty$

$$\|u(t)\|_{L^\infty} \le \sqrt{2\bar{a}} 3^{3r/4} \max\left(\sup_{t \ge 0} y_0(t), K_0 \right). \tag{4.29}$$

Now we shall divide it into two cases $m > 2 - 2/d$ and $0 < m < 2 - 2/d$ to estimate $y_0(t)$.

Case 1. **($m > 2 - 2/d$)** Thanks to Proposition 1, taking $q = m + d + 2$ in (4.3) and using the interpolation inequality by $U_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ we have

$$\|u(t)\|_{m+d+2}^{m+d+2} \leq \|U_0\|_{m+d+2}^{m+d+2} + C(m, d, \|U_0\|_1) \leq K_0^{m+d+2} + C(m, d, \|U_0\|_1),$$

where $K_0 = \max\{1, \|U_0\|_1, \|U_0\|_\infty\}$. Hence from (4.29) one has

$$\begin{aligned}\|u(t)\|_{L^\infty} &\leq \sqrt{2\bar{a}}3^{3r/4} \max\left(\sup_{t\geq 0} y_0(t), K_0\right) \\ &\leq \sqrt{2\bar{a}}3^{3r/4} \max\left(\|u(t)\|_{m+d+2}^{m+d+2}, K_0\right) \\ &\leq \sqrt{2\bar{a}}3^{3r/4}\left(K_0^{m+d+2} + C(m, d, \|U_0\|_1)\right).\end{aligned}$$

Case 2. **($0 < m \leq 2-2/d$)** For $0 < m \leq 2-2/d$, it's easy to verify $m+d+2 > p$, therefore by (4.1) of Proposition 1 we have

$$\begin{aligned}\|u\|_{m+d+2}^{m+d+2} \leq& C(\|U_0\|_1, m, d)\left(\|U_0\|_{m+d+2}^{m+d+2}\right)^{\frac{p+\epsilon_0-1}{\epsilon_0}\frac{m+d+2-p+1}{m+d+2+m-2+2/d}} \\ &+ \|U_0\|_{m+d+2}^{m+d+2}. \end{aligned} \tag{4.30}$$

Thus from (4.29) one has

$$\begin{aligned}\|u(t)\|_{L^\infty} &\leq \sqrt{2\bar{a}}3^{3r/4}\max\left(\|u(t)\|_{m+d+2}^{m+d+2}, K_0\right) \\ &\leq \sqrt{2\bar{a}}3^{3r/4}\left(C(\|U_0\|_1, m, d)\left(K_0^{m+d+2}\right)^{\frac{p+\epsilon_0-1}{\epsilon_0}\frac{m+3+dm/2}{2m+d+2/d}} + K_0^{m+d+2}\right),\end{aligned}$$

where ϵ_0 satisfies

$$\frac{4m(p+\epsilon_0)}{(p+\epsilon_0+m-1)^2 S_d^{-1}} - \|U_0\|_p^{2-m} = \frac{\eta}{2}. \tag{4.31}$$

Step 3. (Time regularity for $m > 1 - 2/d$) It directly follows from $u(x,t) \in L^\infty\left(0, T; L_+^1 \cap L^\infty(\mathbb{R}^d)\right)$ that

$$\begin{aligned}\|\nabla u\|_{L^2(0,T;L^2(\mathbb{R}^d))} &\leq C, \\ \|u\nabla c\|_{L^\infty(0,T;L^\infty(\mathbb{R}^d))} &\leq C, \\ \|\nabla u^m\|_{L^2(0,T;L^2(\mathbb{R}^d))} &\leq C,\end{aligned}$$

then some computations can derive the time regularities (4.20). Furthermore, Multiplying $\frac{\partial u^m}{\partial t}$ to both sides of (1.1) we obtain

$$\begin{aligned}&\frac{4m}{(m+1)^2}\int_{\mathbb{R}^d}\left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla u^m|^2 dx \\ =& -m\int_{\mathbb{R}^d}\nabla u\cdot\nabla c u^{m-1}u_t dx + m\int_{\mathbb{R}^d}u^{m+1}u_t dx \\ =& -\frac{2m}{m+1}\int_{\mathbb{R}^d}u^{\frac{m-1}{2}}\left(u^{\frac{m+1}{2}}\right)_t\nabla u\cdot\nabla c dx + m\int_{\mathbb{R}^d}u^{m+1}u_t dx \\ \leq& \frac{2m}{(m+1)^2}\int_{\mathbb{R}^d}\left|\left(u^{\frac{m+1}{2}}\right)_t\right|^2 dx + C(m)\int_{\mathbb{R}^d}\left|u^{\frac{m-1}{2}}\nabla u\cdot\nabla c\right|^2 dx + C(m)\int_{\mathbb{R}^d}u^{m+3}dx.\end{aligned}$$

Hence for any $t > 0$, from $\int_{\mathbb{R}^d} u^{m+3} dx \le C(\|U_0\|_{m+3}, d, m)$ one has

$$\begin{aligned}&\frac{2m}{(m+1)^2}\int_0^t\int_{\mathbb{R}^d}\left|\left(u^{\frac{m+1}{2}}\right)_s\right|^2 dxds+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u(t)^m|^2\,dx \qquad (4.32)\\ \le&\frac{1}{2}\int_{\mathbb{R}^d}|\nabla U_0^m|^2\,dx+C(m)\int_0^t\int_{\mathbb{R}^d}\left|\nabla u^{\frac{m+1}{2}}\cdot\nabla c\right|^2 dxds+C(\|U_0\|_{m+3},d,m)\\ \le&\frac{1}{2}\int_{\mathbb{R}^d}|\nabla U_0^m|^2dx+C(m)\|\nabla c\|^2_{L^\infty(0,\ t;L^\infty(\mathbb{R}^d))}\int_0^t\int_{\mathbb{R}^d}\left|\nabla u^{\frac{m+1}{2}}\right|^2 dxds\\ &+C(\|U_0\|_{m+3},d,m).\end{aligned}$$

It follows from the Young inequality that

$$\begin{aligned}\|\nabla c\|_{L^\infty(\mathbb{R}^d)}&=C(d)\left\|u(x)*\frac{1}{|x|^{d-1}}\right\|_{L^\infty(\mathbb{R}^d)}\\ &=C(d)\left\|\int_{0<|x-y|\le 1}\frac{u(y)}{|x-y|^{d-1}}dy+\int_{|x-y|>1}\frac{u(y)}{|x-y|^{d-1}}dy\right\|_{L^\infty(\mathbb{R}^d)}\\ &\le C(d)\left(\|u(y)\|_{L^\infty(\mathbb{R}^d)}\left\|\frac{1}{|x|^{d-1}}\right\|_{L^1(0<|x|\le 1)}+\|u\|_{L^1(\mathbb{R}^d)}\right)\\ &\le C(d)\left(\|u\|_{L^\infty(\mathbb{R}^d)}+\|u\|_{L^1(\mathbb{R}^d)}\right), \qquad (4.33)\end{aligned}$$

and the initial data $U_0 \in L^2(\mathbb{R}^d)$ leads to

$$\int_0^t\int_{\mathbb{R}^d}\left|\nabla u^{\frac{m+1}{2}}\right|^2 dxds \le C(\|U_0\|_2, d, m). \qquad (4.34)$$

Plugging (4.33) and (4.34) into (4.32) we obtain the time regularities (4.21) and (4.22). Thus completes the proof. □

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REFERENCES

[1] N. D. Alikakos, L^p bounds of solutions of reaction-diffusion equations, *Comm. Partial Differential Equations,* **4** (1979), 827–868.

[2] J. Bedrossian, Intermediate asymptotics for critical and supercritical aggregation equations and Patlak-Keller-Segel models, *Comm. Math. Sci.,* **9** (2011), 1143–1161.

[3] S. Bian and J.-G. Liu, Dynamic and steady states for multi-dimensional Keller-Segel model with diffusion exponent $m > 0$, *Comm Math Phy.,* **323** (2013), 1017–1070.

[4] A. Blanchet, J. A. Carrillo and N. Masmoudi, Infinite time aggregation for the critical Patlak-Keller-Segel model in $\mathbb{R}^2$, *Comm. Pure Appl. Math.,* **61** (2008), 1449–1481.

[5] A. Blanchet, J. Dolbeault and B. Perthame, Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions, *Eletron. J. Differ. Equ.,* 2006, 32 pp. (electronic).

[6] M. P. Brenner, P. Constantin, L. P. Kadanoff, A. Schenkel and S. C. Venkataramani, Diffusion, attraction and collapse, *Nonlinearity,* **12** (1999), 1071–1098.

[7] V. Calvez, L. Corrias and M. A. Ebde, Blow-up, concentration phenomenon and global existence for the Keller-Segel model in high dimension, *Comm. Part. Diff. Eq.,* **37** (2012), 561–584.

[8] E. A. Carlen, J. A. Carrillo and M. Loss, Hardy-Littlewood-Sobolev inequalities via fast diffusion flows, *Proc. Nat. Acad. USA,* **107** (2010), 19696–19701.

[9] L. Corrias and B. Perthame, Critical space for the parabolic-parabolic Keller-Segel model in $\mathbb{R}^d$, *C. R. Acad. Sc. Paris, Ser. I,* **342** (2006), 745–750.

[10] M. Del Pino, J. Dolbeault and I. Gentil, Nonlinear diffusions, hypercontractivity and the optimal L^p-Euclidean logarithmic Sobolev inequality, *J. Math. Anal. Appl.,* **293** (2004), 375–388.

[11] M. Herrero, E. Medina and J. L. Velázquez, Finite-time aggregation into a single point in a reaction-diffusion system, *Nonlinearity,* **10** (1997), 1739–1754.

[12] M. Herrero, E. Medina and J. L. Velázquez, Self-similar blow-up for a reaction-diffusion system, *J. Comp. Appl. Math.,* **97** (1998), 99–119.

[13] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. theor. Biol.,* **26** (1970), 399–415.

[14] I. Kim and Y. Yao, The Patlak-Keller-Segel model and its variations: properties of solutions via maximum principle, *SIAM J. Math. Anal.,* **44** (2012), 568–602.

[15] E. H. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics. V. 14, American Mathematical Society Providence, Rhode Island, 2nd edition, 2001.

[16] B. Perthame, *Transport Equations in Biology*, Birkhaeuser Verlag, Basel-Boston-Berlin, 2007.

[17] Y. Sugiyama, Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate keller-segel systems, *Diff. Int. Eqns.,* **19** (2006), 841–876.

[18] Y. Sugiyama and H. Kunii, Global existence and decay properties for a degenerate keller-segel model with a power factor in drift term, *J. Diff. Eqns.,* **227** (2006), 333–364.

[19] J. L. Vázquez, *Smoothing and Decay Estimates for Nonlinear Diffusion Equations. Equations of Porous Medium Type*, Oxford Lecture Ser. Math. Appl., vol. 33, 2006.

[20] J. L. Vázquez, *The Porous Medium Equation: Mathematical Theory*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

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