A SIMPLE PROOF OF THE CUCKER-SMALE FLOCKING DYNAMICS AND MEAN-FIELD LIMIT*

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Abstract. We present a simple proof on the formation of flocking to the Cucker-Smale system based on the explicit construction of a Lyapunov functional. Our results also provide a unified condition on the initial states in which the exponential convergence to flocking state will occur. For large particle systems, we give a rigorous justification for the mean-field limit from the many particle Cucker-Smale system to the Vlasov equation with flocking dissipation as the number of particles goes to infinity.

Key words. Flocking, swarming, emergence, self-driven particles system, autonomous agents, Vlasov equation, Lyapunov functional, measure valued solution, Kantorovich-Rubinstein distance.

AMS subject classifications. Primary 92C17; secondary 82C22, 82C40.

1. Introduction

Collective self-driven synchronized motion of autonomous agents appears in many applications ranging from animal herding to the emergence of common languages in primitive societies [1, 9, 10, 11, 12, 14, 16, 18, 19, 20, 2, 3]. The word *flocking* in this paper refers to general phenomena where autonomous agents reach a consensus based on limited environmental information and simple rules. In the seminal work of Cucker and Smale [2, 3], they postulated a model for the flocking of birds, and verified the convergence to a consensus (the same velocity) depending on the spatial decay of the communication rate between autonomous agents.

In this paper, we present a simple and complete analysis of the flocking to the Cucker-Smale system (in short C-S system) using the explicit Lyapunov functional approach. In particular we improve the flocking estimates of the C-S system for regular and algebraically decaying communication rates [2, 3] in two ways. First, we present flocking estimates for general communication rates which can be singular when two particles are very close enough. Secondly, we remove the conditional assumption in [2, 3] on the initial configuration in critical case, where the communication rate behaves like $|\boldsymbol{x}|^{-1}$, as $|\boldsymbol{x}|\to\infty$. We show that the standard deviations of particle phase-space positions are dominated by the system of dissipative differential inequalities (SDDI):

$$\left|\frac{dX}{dt}\right| \le V, \qquad \frac{dV}{dt} \le -\phi(X)V, \tag{1.1}$$

where (X,V) are nonnegative functions and ϕ is a nonnegative measurable function.

A simple phase plane analysis (see section 3) provides a unified condition on initial states in which the convergence to a consensus will occur. For the mean-field limit for large particle systems with flocking dissipation, we derive a bounding function for the growth on the size of a velocity support. This enables us to establish a stability estimate of measure valued solutions to the C-S system in Kantorovich-Rubinstein distance. For the existence of a measure valued solution, we refine the argument given

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in [13, 17]. The novelty of this paper is to present a direct and simple but surprisingly a complete flocking analysis employing a Lyapunov functional to the SDDI (1.1). This is a contrasted difference from Cucker-Smale's original proof [2, 3] and its modified approach in [8] using a bootstrapping argument. We refer to Degond and Motsch's works [4, 5, 6] for recent kinetic description of Viscek type model for flocking.

The rest of the paper is divided into two main parts (flocking estimates and mean-field limit) after this introduction. The first four sections are devoted to the dissipation estimates for the SDDI system, which yield simplified and strong estimates for the dynamics of the C-S system. More precisely, in section 2, we briefly review the C-S system and provide the reduction of *Cucker-Smale dynamics* to the dynamics of the SSDI. In section 3, we present a general frame work on the dissipation estimate of the SDDI using explicit Lyapunov functionals, and we also present several conditions regarding the initial configurations and communication rate functions for the flocking formation. In section 4, we study the flocking estimates for dissipative systems with the algebraically decaying communication rates with singularity or non-singularity at the origin as a direct application of the results in section 3, and we also briefly discuss the comparison between Cucker-Smale's result with main results in section 4. Finally the last two sections deal with kinetic mean-field limit of the C-S system. In section 5, we study the particle approximation [15] to the kinetic C-S model introduced in [8]. Section 6 is devoted to the existence of measure valued solutions to the kinetic C-S model.

Notation: Throughout the rest part of the paper, we denote $C(T,N,\dots)$ to be a generic positive constant depending only on T,N, etc., and for any vector $\mathbf{b}$, $\mathbf{b}^i$ represents i-th component of it.

2. Preliminaries

In this section, we discuss the reduction of the C-S dynamics to the SDDI (1.1) and present a definition for the time-asymptotic flocking.

Consider an interacting particle system consisting of N identical autonomous agents with unit mass [2, 3, 8, 9, 16]. Let $(\boldsymbol{x}_i(t),\boldsymbol{v}_i(t))\in\mathbb{R}^d_x\times\mathbb{R}^d_v$ be the phase space position of the ith particle, $1\le i\le N$, governed by the general Cucker-Smale dynamical system:

$$\frac{d\boldsymbol{x}_i}{dt}=\boldsymbol{v}_i, \qquad \frac{d\boldsymbol{v}_i}{dt}=\frac{\lambda}{N}\sum_{j=1}^N\psi(|\boldsymbol{x}_j-\boldsymbol{x}_i|)(\boldsymbol{v}_j-\boldsymbol{v}_i), \quad t>0, \tag{2.1}$$

with initial data

$$\boldsymbol{x}_i(0)=\boldsymbol{x}_{i0}, \qquad \boldsymbol{v}_i(0)=\boldsymbol{v}_{i0}. \tag{2.2}$$

Here λ and ψ are a nonnegative coupling strength and the mutual communication rate between autonomous agents, respectively.

Note that the vector field $(\boldsymbol{v},\lambda\mathbf{F})$, $\quad\mathbf{F}^i=\frac{1}{N}\sum_{j=1}^N\psi(|\boldsymbol{x}_j-\boldsymbol{x}_i|)(\boldsymbol{v}_j-\boldsymbol{v}_i))$ associated with (2.1) satisfies dissipative condition:

$$\nabla_{(\boldsymbol{x},\boldsymbol{v})}\cdot(\boldsymbol{v},\lambda\mathbf{F})=\lambda\sum_{i=1}^N\nabla_{\boldsymbol{v}_i}\cdot\mathbf{F}^i=-\frac{\lambda}{N}\sum_{i,j=1}^N\psi(|\boldsymbol{x}_j-\boldsymbol{x}_i|)\le 0.$$

This implies that the C-S model (2.1) is a dissipative dynamical system. We first set the center of mass system $(\boldsymbol{x}_c, \boldsymbol{v}_c)$:

$$\boldsymbol{x}_c := \frac{1}{N}\sum_{k=1}^{N}\boldsymbol{x}_k, \qquad \boldsymbol{v}_c := \frac{1}{N}\sum_{k=1}^{N}\boldsymbol{v}_k. \tag{2.3}$$

Then the system (2.1) implies that

$$\frac{d\boldsymbol{x}_c}{dt} = \boldsymbol{v}_c, \qquad \frac{d\boldsymbol{v}_c}{dt} = 0,$$

which gives the explicit solution $\boldsymbol{x}_c(t) = \boldsymbol{x}_c(0) + t\boldsymbol{v}_c(0), \boldsymbol{v}_c(t) = \boldsymbol{v}_c(0)$. Without loss of generality, we may assume that the center of mass coordinate of the system is fixed at zero in phase space at time t:

$$\boldsymbol{x}_c(t) = 0, \quad \boldsymbol{v}_c(t) = 0, \tag{2.4}$$

which is equivalent to the relations:

$$\sum_{i=1}^{N}\boldsymbol{x}_i(t) = 0, \qquad \sum_{i=1}^{N}\boldsymbol{v}_i(t) = 0, \quad t \geq 0. \tag{2.5}$$

If necessary, instead of $(\boldsymbol{x}_i, \boldsymbol{v}_i)$ we may consider new variables $(\hat{\boldsymbol{x}}_i, \hat{\boldsymbol{v}}_i) := (\boldsymbol{x}_i - \boldsymbol{x}_c, \boldsymbol{v}_i - \boldsymbol{v}_c)$ which correspond to the fluctuations around the center of mass system.

2.1. Reduction of the C-S dynamics to the SDDI's dynamics. In this part, we explain how the dynamics of the C-S system (2.1) with (2.4) can be determined by the corresponding dynamics of the SDDI, which will be discussed in next section. We now consider the system (2.1) with (2.4). It follows from (2.1) that we have

$$\frac{d}{dt}\sum_{i=1}^{N}\|\boldsymbol{v}_i\|^2 = -\frac{\lambda}{N}\sum_{1\leq i,j\leq N}\psi(|\boldsymbol{x}_j - \boldsymbol{x}_i|)||\boldsymbol{v}_j - \boldsymbol{v}_i||^2. \tag{2.6}$$

Here $||\cdot||$ is the standard l_2-norm in $\mathbb{R}^d$.

We set

$$\boldsymbol{x} := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) \in \mathbb{R}^{Nd}, \qquad \boldsymbol{v} := (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N) \in \mathbb{R}^{Nd},$$

and

$$||\boldsymbol{x}|| = \Big(\sum_{i=1}^{N}||\boldsymbol{x}_i||^2\Big)^{\frac{1}{2}}, \qquad ||\boldsymbol{v}|| = \Big(\sum_{i=1}^{N}||\boldsymbol{v}_i||^2\Big)^{\frac{1}{2}}.$$

Note that $||\boldsymbol{x}||$ and $||\boldsymbol{v}||$ denote quantities proportional to the *standard deviations* of $\boldsymbol{x}_i$ and $\boldsymbol{v}_i$, respectively.

Lemma 2.1. *Let* $(\boldsymbol{x}_i, \boldsymbol{v}_i)$ *be the solution to* (2.1) *with a nonnegative and non-increasing function* ψ. *Assume that* (2.4) *holds. Then* $||\boldsymbol{x}\|$ *and* $\|\boldsymbol{v}\|$ *satisfy the SDDI* (1.1) *with* $\phi(s) = \psi(2s)$*:*

$$\left|\frac{d\|\boldsymbol{x}\|}{dt}\right| \leq \|\boldsymbol{v}\|, \qquad \frac{d\|\boldsymbol{v}\|}{dt} \leq -\frac{\lambda}{N}\psi(2\|\boldsymbol{x}\|)\|\boldsymbol{v}\|. \tag{2.7}$$

Proof. We take an inner product the C-S system (2.1) with $\pm 2\boldsymbol{x}_i$ and use the Cauchy-Schwartz inequality to see

$$\pm\frac{d\|\boldsymbol{x}\|^2}{dt} = \pm 2\left\langle \frac{d\boldsymbol{x}}{dt}, \boldsymbol{x} \right\rangle = \pm 2\langle \boldsymbol{v}, \boldsymbol{x}\rangle \leq 2\|\boldsymbol{x}\|\|\boldsymbol{v}\|,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product in $\mathbb{R}^{Nd}$. This gives the first inequality in (2.7). Next, we use (2.6) and the nonnegativity and non-increasing properties of ψ, and the fact

$$\max_{1\leq i,j\leq N} |\boldsymbol{x}_i - \boldsymbol{x}_j| \leq 2\|\boldsymbol{x}\|,$$

to find

$$\frac{d\|\boldsymbol{v}\|^2}{dt} \leq -\frac{\lambda}{N}\psi(2\|\boldsymbol{x}\|)\sum_{1\leq i,j\leq N} ||\boldsymbol{v}_j - \boldsymbol{v}_i||^2 = -\frac{2\lambda}{N}\psi(2\|\boldsymbol{x}\|)\|\boldsymbol{v}\|^2.$$

This gives the second inequality in (2.7). Here we used (2.5) to see

$$\sum_{1\leq i,j\leq N} \|\boldsymbol{v}_i - \boldsymbol{v}_j\|^2 = 2\sum_{i=1}^{N} ||v_i||^2 - 2\left\langle \sum_{i=1}^{N}\boldsymbol{v}_i, \sum_{j=1}^{N}\boldsymbol{v}_j \right\rangle = 2\|\boldsymbol{v}\|^2. \qquad \square$$

The above reduction to the SDDI can be recast in a more abstract form. Let E be a vector space over $\mathbb{R}$ with an inner product $\langle\cdot,\cdot\rangle$ and its corresponding norm $\|\cdot\|$. Let $(\boldsymbol{x}_i)_{i=1}^n, (\boldsymbol{v}_i)_{i=1}^n \in \mathbb{R}^{2d}$ be the phase space coordinate of i-th autonomous agents among N agents. In this case, E is simply the N-particle phase space $\mathbb{R}^{Nd}$. Consider the following dynamical system in $E\times E$:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}, \qquad \frac{d\boldsymbol{v}}{dt} = -L(\boldsymbol{x})\boldsymbol{v}, \qquad t>0, \tag{2.8}$$

with initial data

$$\boldsymbol{x}(0) = \boldsymbol{x}_0, \qquad \boldsymbol{v}(0) = \boldsymbol{v}_0. \tag{2.9}$$

where $L(\boldsymbol{x}): E \to E$ is a linear operator. We assume that the linear operator $L(x)$ satisfies a coercivity condition: there is a positive and non-increasing function $\psi(s)$ such that

$$\langle L(\boldsymbol{x})\boldsymbol{v}, \boldsymbol{v}\rangle \geq \psi(\|\boldsymbol{x}\|)\|\boldsymbol{v}\|^2. \tag{2.10}$$

Here ψ is a nonnegative measurable function. We set Ψ to be the primitive function of ψ:

$$\Psi'(s) = \psi(s), \quad s \geq 0.$$

Then it is easy to see that Ψ is an increasing function. We next show that the norms $(||\boldsymbol{x}||, ||\boldsymbol{v}||)$ of the solution $(\boldsymbol{x}, \boldsymbol{v})$ to the system (2.8) with (2.10) satisfies the SDDI. It follows from (2.8) that

$$\frac{d\|\boldsymbol{x}\|^2}{dt} = 2\langle \boldsymbol{x}, \boldsymbol{v}\rangle \leq 2\|\boldsymbol{x}\|\|\boldsymbol{v}\|,$$

$$\frac{d\|\boldsymbol{v}\|^2}{dt} = -2\langle L(\boldsymbol{x})\boldsymbol{v},\boldsymbol{v}\rangle \leq -2\psi(\|\boldsymbol{x}\|)\|\boldsymbol{v}\|^2.$$

These yield the SDDI:

$$\left|\frac{d\|\boldsymbol{x}\|}{dt}\right| \leq \|\boldsymbol{v}\|, \quad \frac{d\|\boldsymbol{v}\|}{dt} \leq -\psi(\|\boldsymbol{x}\|)\|\boldsymbol{v}\|. \tag{2.11}$$

Before we close this section, we present the definition of the time-asymptotic flocking as follows.

DEFINITION 2.2. *The Cucker-Smale system* (2.1) *has a time-asymptotic flocking if and only if the solutions* $\{\boldsymbol{x}_i,\boldsymbol{v}_i\}, i=1,\ldots,N$ *to* (2.1) *satisfy the following two conditions:*

1. *The velocity fluctuations go to zero time-asymptotically (velocity alignment):*

$$\lim_{t\to+\infty}\sum_{i=1}^N ||\boldsymbol{v}_i(t)-\boldsymbol{v}_c(t)||^2 = 0.$$

2. *The position fluctuations are uniformly bounded in time* t *(forming a group):*

$$\sup_{0\leq t<\infty}\sum_{i=1}^N ||\boldsymbol{x}_i(t)-\boldsymbol{x}_c(t)||^2 < \infty.$$

3. A flocking theorem

Note that the SDDI (2.11) admits natural Lyapunov functionals $\mathcal{E}_\pm(\boldsymbol{x},\boldsymbol{v})$ which can be viewed as energy functionals for (2.11):

$$\mathcal{E}_\pm(||\boldsymbol{x}||,||\boldsymbol{v}||) := ||\boldsymbol{v}|| \pm \Psi(||\boldsymbol{x}||). \tag{3.1}$$

The first and second terms in the $\mathcal{E}_\pm(||\boldsymbol{x}||,||\boldsymbol{v}||)$ can be regarded as the kinetic and internal (potential) energies respectively. The next lemma shows that the functionals $\mathcal{E}_\pm(||\boldsymbol{x}||,||\boldsymbol{v}||)$ are non-increasing along the solutions $(||\boldsymbol{x}||,||\boldsymbol{v}||)$ of (2.11).

LEMMA 3.1. *Suppose* $(||\boldsymbol{x}||,||\boldsymbol{v}||)$ *satisfy the SDDI* (2.11) *with* $\psi \geq 0$. *Then we have*

(i) $\mathcal{E}_\pm(||\boldsymbol{x}(t)||,||\boldsymbol{v}(t)||) \leq \mathcal{E}_\pm(||\boldsymbol{x}_0||,||\boldsymbol{v}_0||), \quad t\geq 0.$

(ii) $\|\boldsymbol{v}(t)\| + \left|\int_{||\boldsymbol{x}_0||}^{||\boldsymbol{x}(t)||}\psi(s)ds\right| \leq \|\boldsymbol{v}_0\|.$

Proof.

(i) We now use (2.11) to obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_\pm(||\boldsymbol{x}(t)||,||\boldsymbol{v}(t)||) &= \frac{d}{dt}(\|\boldsymbol{v}(t)\| \pm \Psi(\|\boldsymbol{x}(t)\|))\\ &= \frac{d\|\boldsymbol{v}\|}{dt} \pm \psi(\|\boldsymbol{x}\|)\frac{d\|\boldsymbol{x}\|}{dt}\\ &\leq \psi(\|\boldsymbol{x}\|)\left(-\|\boldsymbol{v}\| \pm \frac{d\|\boldsymbol{x}\|}{dt}\right)\\ &\leq 0.\end{aligned}$$

(ii) It follows from (i) that

$$\|\boldsymbol{v}(t)\| - \|\boldsymbol{v}_0\| \leq -(\Psi(\|\boldsymbol{x}(t)\|) - \Psi(\|\boldsymbol{x}_0\|)), \quad \|\boldsymbol{v}(t)\| - \|\boldsymbol{v}_0\| \leq (\Psi(\|\boldsymbol{x}(t)\|) - \Psi(\|\boldsymbol{x}_0\|)).$$

Hence we have

$$\|\boldsymbol{v}(t)\| - \|\boldsymbol{v}_0\| \le -|\Psi(\|\boldsymbol{x}(t)\|) - \Psi(\|\boldsymbol{x}_0\|)| = -\Big|\int_{||\boldsymbol{x}_0||}^{||\boldsymbol{x}(t)||} \psi(s).ds\Big|. \qquad \square$$

As a direct application of Lemma 3.1, we obtain the following bounds on $||\boldsymbol{x}||$ and $||\boldsymbol{v}||$ satisfying the SDDI.

THEOREM 3.2. *Suppose* $(||\boldsymbol{x}||, ||\boldsymbol{v}||)$ *satisfy the SDDI* (2.11) *with* $\psi \ge 0$. *Then the following estimates hold.*

(i) If

$$\|\boldsymbol{v}_0\| < \int_0^{\|\boldsymbol{x}_0\|} \psi(s) ds,$$

then there is a $x_m \ge 0$ *such that*

$$\|\boldsymbol{v}_0\| = \int_{x_m}^{\|\boldsymbol{x}_0\|} \psi(s) ds, \qquad \|\boldsymbol{x}(t)\| \ge x_m, \qquad t \ge 0.$$

(ii) If

$$\|\boldsymbol{v}_0\| < \int_{\|\boldsymbol{x}_0\|}^{\infty} \psi(s) ds,$$

then there is a $x_M \ge 0$ *such that*

$$\|\boldsymbol{v}_0\| = \int_{\|\boldsymbol{x}_0\|}^{x_M} \psi(s) ds, \quad \|\boldsymbol{x}(t)\| \le x_M, \quad \|\boldsymbol{v}(t)\| \le \|\boldsymbol{v}_0\| e^{-\psi(x_M)t}.$$

Proof.

(i) Since ψ is a nonnegative measurable function,

$$\int_{\delta}^{||\boldsymbol{x}_0||} \psi(s) ds \qquad \text{is a non-increasing } \textit{continuous} \text{ function in } \delta \ge 0.$$

Hence if $\|\boldsymbol{v}_0\| < \int_0^{\|\boldsymbol{x}_0\|} \psi(s) ds$, let $x_m > 0$ to be the smallest value such that

$$\|\boldsymbol{v}_0\| = \int_{x_m}^{\|\boldsymbol{x}_0\|} \psi(s) ds. \tag{3.2}$$

For the second assertion, we use result (ii) in Lemma 3.1, i.e., for any solution $(\boldsymbol{x}, \boldsymbol{v})$ to (2.8) we have

$$\Big|\int_{||\boldsymbol{x}_0||}^{||\boldsymbol{x}(t)||} \psi(s) ds\Big| \le \|\boldsymbol{v}_0\|, \qquad t \ge 0. \tag{3.3}$$

Suppose there exists a $t \in (0, \infty)$ such that

$$||\boldsymbol{x}(t)|| < x_m.$$

Then we can choose a time $t_* \in (0, \infty)$ such that

$$\|\boldsymbol{v}_0\| < \int_{||\boldsymbol{x}(t_*)||}^{\|\boldsymbol{x}_0\|} \psi(s) ds, \qquad ||\boldsymbol{x}(t_*)|| < x_m, \qquad \min_{0 \le s \le t_*} ||\boldsymbol{x}(s)|| \ge \frac{x_m}{2}.$$

We now consider Ψ defined as follows.

$$\Psi(s) := \int_{\|\boldsymbol{x}_0\|}^{s} \psi(\tau) d\tau. \tag{3.4}$$

Then for such t_* and Ψ, we have

$$\|\boldsymbol{v}_0\| = \int_{x_m}^{\|\boldsymbol{x}_0\|} \psi(s) ds < \int_{||\boldsymbol{x}(t_*)||}^{\|\boldsymbol{x}_0\|} \psi(s) ds,$$

which is contradictory to (3.3).

(ii) We use the same argument as (i), i.e.,

$$\text{If} \ \ \|\boldsymbol{v}_0\| < \int_{\|\boldsymbol{x}_0\|}^{\infty} \psi(s) ds,$$

then we choose the largest value $x_M \geq 0$ to satisfy

$$\|\boldsymbol{v}_0\| := \int_{\|\boldsymbol{x}_0\|}^{x_M} \psi(s) ds.$$

Similar to (i), it is easy to see that

$$\|\boldsymbol{x}(t)\| \leq x_M.$$

On the other hand, we use the above upper bound for $||\boldsymbol{x}(t)||$ to find

$$\frac{d\|\boldsymbol{v}\|}{dt} \leq -\psi(x_M)\|\boldsymbol{v}\|.$$

This yields the decay estimate for $||\boldsymbol{v}(t)||$:

$$\|\boldsymbol{v}(t)\| \leq \|\boldsymbol{v}_0\| e^{-\psi(x_M)t}.$$

□

In next section, we consider the SDDI (2.11) equipped with explicit singular and regular ψ's decaying algebraically at infinity, and apply Theorem 3.1 to get the detailed information on the size of spatial support and time-decay estimates of $||\boldsymbol{v}||$.

4. Application of the flocking theorem to the Cucker-Smale system

In this section, we consider two explicit communication rate functions ψ:

$$\psi_1(s) := \frac{\alpha}{s^\beta}, \qquad \psi_2(s) := \frac{\alpha}{(1+s^2)^{\frac{\beta}{2}}}, \qquad \alpha > 0, \ \ \beta \geq 0.$$

The second communication rate function ψ_2 has been employed in previous literatures [2, 3, 8, 16] on flocking. In the following two subsections, we present several estimates for x_m and x_M together with explicit dissipation estimates.

4.1. Singular communication rate. In this part, we consider the singular ψ decaying algebraically at infinity:

$$\psi_1(s) = \frac{\alpha}{s^\beta}, \qquad \alpha > 0, \ \ \beta \geq 0.$$

Since the flocking dynamics of ψ_1 completely depends on $\beta \in [0,1]$ or $\beta \in (1,\infty)$, we separate its presentation. For the long-range communication rate ψ_1 with $\beta \in [0,1]$, we have global flocking regardless of initial configurations. In contrast, for the case of short range case with $\beta\in(1,\infty)$, we have a conditional flocking saying that only certain classes of initial configurations reduce to the flocking state.

Proposition 4.1. (Unconditional flocking) *Suppose* $(||\boldsymbol{x}||,||\boldsymbol{v}||)$ *satisfy the SDDI* (2.11) *with* $\psi=\psi_1$*:*

$$\beta\in[0,1], \quad ||\boldsymbol{x}_0||\neq 0.$$

Then there exist x_m *and* x_M *independent of* t *satisfying*

$$x_m\le\|\boldsymbol{x}(t)\|\le x_M, \qquad \|\boldsymbol{v}(t)\|\le\|\boldsymbol{v}_0\|e^{-\psi(x_M)t},$$

where x_m *and* x_M *are explicitly given by*

$$x_m:=\begin{cases}\left(\max\left\{0,\|\boldsymbol{x}_0\|^{1-\beta}-\frac{1-\beta}{\alpha}\|\boldsymbol{v}_0\|\right\}\right)^{\frac{1}{1-\beta}}, & \beta\in[0,1),\\ \|\boldsymbol{x}_0\|e^{-\frac{\|\boldsymbol{v}_0\|}{\alpha}}, & \beta=1,\end{cases} \qquad \text{and}$$

$$x_M:=\begin{cases}\left(\|\boldsymbol{x}_0\|^{1-\beta}+\frac{1-\beta}{\alpha}\|\boldsymbol{v}_0\|\right)^{\frac{1}{1-\beta}}, & \beta\in[0,1),\\ \|\boldsymbol{x}_0\|e^{\frac{\|\boldsymbol{v}_0\|}{\alpha}}, & \beta=1.\end{cases}$$

Proof. If $||\boldsymbol{v}_0||=0$, then we have $||\boldsymbol{v}(t)||\equiv 0$ and $||\boldsymbol{x}(t)||\equiv||\boldsymbol{x}_0||$. So the lemma holds. Now we show the lemma also holds for the case of $||\boldsymbol{v}_0||\neq 0$.

(i) (Estimates of x_M): For $\beta\in[0,1]$, since

$$\int_{\|\boldsymbol{x}_0\|}^{\infty}\psi_1(s)ds=\int_{\|\boldsymbol{x}_0\|}^{\infty}\frac{\alpha}{s^\beta}ds=\infty,$$

it follows from Theorem 3.2 that there exists a $x_M>0$ such that

$$\|\boldsymbol{v}_0\|=\int_{\|\boldsymbol{x}_0\|}^{x_M}\psi_1(s)ds, \quad \|\boldsymbol{x}(t)\|\le x_M, \quad \|\boldsymbol{v}(t)\|\le\|\boldsymbol{v}_0\|e^{-\psi(x_M)t}.$$

Below, we give a upper bound for x_M using the above defining relation.

Case 1 $(0\le\beta<1)$: By direct calculation, we have

$$\begin{aligned}\|\boldsymbol{v}_0\|&=\int_{\|\boldsymbol{x}_0\|}^{x_M}\psi(s)ds=\frac{\alpha}{1-\beta}(x_M^{1-\beta}-||\boldsymbol{x}_0||^{1-\beta}), \quad \text{i.e.,}\\ x_M&=\left(\|\boldsymbol{x}_0\|^{1-\beta}+\frac{1-\beta}{\alpha}\|\boldsymbol{v}_0\|\right)^{\frac{1}{1-\beta}}.\end{aligned}$$

Case 2 $(\beta=1)$: In this case, we have

$$\|\boldsymbol{v}_0\|=\int_{\|\boldsymbol{x}_0\|}^{x_M}\psi_1(s)ds=\alpha\ln\frac{x_M}{||\boldsymbol{x}_0||}, \quad \text{i.e.,} \quad x_M=||\boldsymbol{x}_0||e^{\frac{||\boldsymbol{v}_0||}{\alpha}}.$$

(ii) (Estimates of x_m): As in (i), we separate the estimate into two cases.

Case 1 $(0\le\beta<1)$: We again use Theorem 3.2 to obtain the estimate of x_m. Note that

$$\int_0^{\|\boldsymbol{x}_0\|}\psi_1(s)ds=\frac{\alpha}{1-\beta}\|\boldsymbol{x}_0\|^{1-\beta}, \qquad \int_{x_m}^{\|\boldsymbol{x}_0\|}\psi_1(s)ds=\frac{\alpha}{1-\beta}(\|\boldsymbol{x}_0\|^{1-\beta}-x_m^{1-\beta}).$$

If $\|\boldsymbol{v}_0\| < \frac{\alpha}{1-\beta}\|\boldsymbol{x}_0\|^{1-\beta}$, then the defining condition for x_m yields the explicit representation for x_m:

$$\|\boldsymbol{v}_0\| = \int_{x_m}^{\|\boldsymbol{x}_0\|} \psi_1(s)ds, \quad \text{i.e.} \quad \|\boldsymbol{v}_0\| = \frac{\alpha}{1-\beta}(\|\boldsymbol{x}_0\|^{1-\beta} - x_m^{1-\beta}).$$

Hence we have

$$x_m = \left(\|\boldsymbol{x}_0\|^{1-\beta} - \frac{1-\beta}{\alpha}\|\boldsymbol{v}_0\|\right)^{\frac{1}{1-\beta}}.$$

Otherwise, we simply take $x_m = 0$.

Case 2 ($\beta = 1$): Since

$$\int_0^{\|\boldsymbol{x}_0\|} \psi_1(s)ds = \infty,$$

we use Theorem 3.2 directly to get

$$\|\boldsymbol{v}_0\| = \int_{x_m}^{\|\boldsymbol{x}_0\|} \frac{\alpha}{s}ds, \qquad x_m = \|\boldsymbol{x}_0\| e^{-\frac{\|\boldsymbol{v}_0\|}{\alpha}}.$$

∎

Below, we consider the case $\beta > 1$, which is the integrable case at $s = \infty$.

Proposition 4.2. (Conditional flocking) *Suppose* $(||\boldsymbol{x}||, ||\boldsymbol{v}||)$ *satisfy the SDDI* (2.11) *with* $\psi = \psi_1$:

$$\beta \in (1,\infty), \quad ||\boldsymbol{x}_0|| \neq 0, \quad \|\boldsymbol{x}_0\|^{1-\beta} > \frac{\beta-1}{\alpha}\|\boldsymbol{v}_0\|.$$

Then there exist x_m *and* x_M *independent of* t *satisfying*

$$x_m \le \|\boldsymbol{x}(t)\| \le x_M, \qquad \|\boldsymbol{v}(t)\| \le \|\boldsymbol{v}_0\| e^{-\psi(x_M)t},$$

where x_m *and* x_M *are given by*

$$x_m := \left(\|\boldsymbol{x}_0\|^{1-\beta} + \frac{\beta-1}{\alpha}\|\boldsymbol{v}_0\|)\right)^{\frac{1}{1-\beta}}, \quad x_M := \left(\|\boldsymbol{x}_0\|^{1-\beta} - \frac{\beta-1}{\alpha}\|\boldsymbol{v}_0\|\right)^{\frac{1}{1-\beta}}.$$

Proof. As in the proof of Proposition 4.1, the lemma holds for $||\boldsymbol{v}_0|| = 0$. So we only need to prove the lemma for the case of $||\boldsymbol{v}_0|| \neq 0$.

(i) Note that

$$\int_{\|\boldsymbol{x}_0\|}^{\infty} \frac{\alpha}{s^\beta}ds = \frac{\alpha}{\beta-1}\|\boldsymbol{x}_0\|^{1-\beta} < \infty.$$

We again use Theorem 3.2 to find x_M: If

$$\|\boldsymbol{v}_0\| < \frac{\alpha}{\beta-1}\|\boldsymbol{x}_0\|^{1-\beta},$$

then there exists $x_M > 0$ such that

$$\|\boldsymbol{v}_0\| = \int_{\|\boldsymbol{x}_0\|}^{x_M} \frac{\alpha}{s^\beta}ds = \frac{\alpha}{\beta-1}(\|\boldsymbol{x}_0\|^{1-\beta} - x_M^{1-\beta}),$$

which gives the desired result.
(ii) Since

$$\int_0^{\|\boldsymbol{x}_0\|} \frac{\alpha}{s^\beta} ds = \infty,$$

we can find x_m from the defining relation:

$$||\boldsymbol{v}_0|| = \int_{x_m}^{\|\boldsymbol{x}_0\|} \frac{\alpha}{s^\beta} ds = \frac{\alpha}{\beta-1}(x_m^{1-\beta} - \|\boldsymbol{x}_0\|^{1-\beta}).$$ □

4.2. Regular communication rate. In this part, we will consider regular force case at $s=0$. For definiteness, we set

$$\psi_2(s) := \frac{\alpha}{(1+s^2)^{\frac{\beta}{2}}}, \qquad \alpha > 0, \quad \beta \geq 0.$$

In the following, we will use the fact that

$$\int_a^b (1+s^2)^{-\frac{\beta}{2}} ds \geq \int_a^b s(1+s^2)^{-\frac{\beta+1}{2}} ds = \frac{1}{1-\beta}(1+s^2)^{\frac{1-\beta}{2}} |_a^b, \quad \beta \neq 1,$$
$$\int_a^b (1+s^2)^{-\frac{\beta}{2}} ds = \ln\left(s + \sqrt{1+s^2}\right) |_a^b, \qquad \beta = 1.$$

Below, we present two parallel propositions with Propositions 4.1 and 4.2 without proofs.

Proposition 4.3. (Unconditional flocking) *Suppose* $(||\boldsymbol{x}||, ||\boldsymbol{v}||)$ *satisfy the SDDI* (2.11) *with* $\psi = \psi_2$*:*

$$\beta \in [0,1].$$

Then there exist x_m *and* x_M *independent of* t *satisfying*

$$x_m \leq \|\boldsymbol{x}(t)\| \leq x_M, \qquad \|\boldsymbol{v}(t)\| \leq \|\boldsymbol{v}_0\| e^{-\psi(x_M)t},$$

where x_m *and* x_M *are given by the solution of the following relations:*

$$\begin{cases} (1+x_m^2)^{\frac{1-\beta}{2}} \geq \max\left\{1, (1+||\boldsymbol{x}_0||^2)^{\frac{1-\beta}{2}} - \frac{1-\beta}{\alpha}||\boldsymbol{v}_0||\right\}, & \beta \in [0,1), \\ x_m + \sqrt{1+x_m^2} = \max\{1, (\|\boldsymbol{x}_0\| + \sqrt{1+\|\boldsymbol{x}_0\|^2}) e^{-\|\boldsymbol{v}_0\|/\alpha}\}, \ \beta = 1. \end{cases}$$

and

$$\begin{cases} (1+x_M^2)^{\frac{1-\beta}{2}} \leq (1+\|\boldsymbol{x}_0\|^2)^{\frac{1-\beta}{2}} + \frac{1-\beta}{\alpha}\|\boldsymbol{v}_0\|, & \beta \in [0,1), \\ x_M + \sqrt{1+x_M^2} = \left(\|\boldsymbol{x}_0\| + \sqrt{1+\|\boldsymbol{x}_0\|^2}\right) e^{\frac{\|\boldsymbol{v}_0\|}{\alpha}}, \ \beta = 1, \end{cases}$$

Proof. The proof is similar to that of Proposition 4.1. Hence we omit its detailed proof. □

Proposition 4.4. (Conditional flocking) *Suppose* $(||\boldsymbol{x}||, ||\boldsymbol{v}||)$ *satisfy the SDDI* (2.11) *with* $\psi = \psi_2$*:*

$$\beta \in (1,\infty), \quad (1+\|\boldsymbol{x}_0\|^2)^{\frac{1-\beta}{2}} > \frac{(\beta-1)}{\alpha}\|\boldsymbol{v}_0\|.$$

Then there exist x_m *and* x_M *independent of* t *satisfying*

$$x_m \le \|\boldsymbol{x}(t)\| \le x_M, \qquad \|\boldsymbol{v}(t)\| \le \|\boldsymbol{v}_0\| e^{-\psi(x_M)t},$$

where x_m *and* x_M *are given by the solution of the following relations:*

$$(1+x_m^2)^{\frac{\beta-1}{2}} \ge \Big[\min\Big\{1, (1+\|\boldsymbol{x}_0\|^2)^{\frac{1-\beta}{2}} + \frac{(\beta-1)}{\alpha}\|\boldsymbol{v}_0\|\Big\}\Big]^{-1},$$
$$(1+x_M^2)^{\frac{\beta-1}{2}} \le \Big[(1+\|\boldsymbol{x}_0\|^2)^{\frac{1-\beta}{2}} - \frac{(\beta-1)}{\alpha}\|\boldsymbol{v}_0\|\Big]^{-1}.$$

4.3. Comparison with Cucker-Smale's results. In this part, we briefly summarize improved flocking estimates on the C-S model (2.1) with $\psi_i, i=1,2$, and compare our results with these given in [2, 3] for the communication rate ψ_2. Since the C-S model can be reduced to the SDDI (see section 2.1), we can use the estimates in Propositions 4.3 and 4.4. The main results on flocking phenomena can be summarized as follows:

THEOREM 4.5. (Unconditional flocking) *Assume that the communication rate is* $\psi = \psi_i$, $i=1,2$ *with* $\beta \in [0,1]$. *Let* $(\boldsymbol{x},\boldsymbol{v})$ *be a solution to* (2.1) *with* $\boldsymbol{x}_0 \neq 0$. *Then there exist positive constants* x_m *and* x_M *independent of* t *satisfying*

$$x_m \le \|\hat{\boldsymbol{x}}(t)\| \le x_M, \qquad \|\hat{\boldsymbol{v}}(t)\| \le \|\hat{\boldsymbol{v}}_0\| e^{-\psi(x_M)t}, \qquad t \ge 0.$$

REMARK 4.1. The results given in the above theorem improves the flocking results in Cucker-Smale [2, 3, 8, 16] which only deal with regular communication rate ψ_2 in two aspects: Improved flocking estimates for critical case ($\beta = 1$) and singular communication rate ψ_1. More precisely,

(i) For $0 \le \beta < 1$, unconditional flocking with exponential decay in the variance of velocity was obtained both in [2] and [3].

(ii) For the system (2.1) with regular communication rate ψ_2 with $\beta = 1$, conditional flocking for initial configuration satisfying

$$\|\boldsymbol{v}_0\|^2 \le C(N,\lambda),$$

was obtained in [2, 3], whereas Ha-Tadmor [8] improved Cucker-Smale's conditional flocking for the unconditional flocking with algebraic decay rates. Hence the result given in this theorem also improves the results [2, 3, 8].

THEOREM 4.6. (Conditional flocking) *Let* $(\boldsymbol{x},\boldsymbol{v})$ *be a solution to* (2.1) *with* $\boldsymbol{x}_0 \neq 0$, *and assume that* ψ *takes one of the form* ψ_i *with* $\beta \in (1,\infty)$. *Suppose the initial configuration* $(\boldsymbol{x}_0, \boldsymbol{v}_0)$ *satisfies*

$$\begin{cases} \|\boldsymbol{x}_0\|^{1-\beta} > \frac{\beta-1}{\alpha}\|\boldsymbol{v}_0\|, & \psi = \psi_1, \\ (1+\|\boldsymbol{x}_0\|^2)^{\frac{1-\beta}{2}} > \frac{\beta-1}{\alpha}\|\boldsymbol{v}_0\|, & \psi = \psi_2. \end{cases}$$

Then there exist positive constants x_m *and* x_M *independent of* t *satisfying*

$$x_m \le \|\hat{\boldsymbol{x}}(t)\| \le x_M, \qquad \|\hat{\boldsymbol{v}}(t)\| \le \|\hat{\boldsymbol{v}}_0\| e^{-\psi(x_M)t}, \qquad t \ge 0.$$

REMARK 4.2. For the short-range communication rate ψ_2 with $\beta > 1$, i.e., in the Cucker-Smale's context

$$K = \frac{\lambda}{N}, \quad \sigma = 1,$$

the sufficient condition for flocking formation given in [2, 3] can be rephrased as follows.

$$A^{\frac{1}{\beta-1}}\left[\left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}}-\left(\frac{1}{\beta}\right)^{\frac{\beta}{\beta-1}}\right]>B,$$

where

$$A=\frac{\nu^2\lambda^2}{8N^3||\boldsymbol{v}_0||^2},\qquad B=1+2N\|\boldsymbol{x}_0\|^2,$$

which can be recast as

$$(1+2N\|\boldsymbol{x}_0\|^2)^{\frac{1-\beta}{2}}>\frac{(2N)^{\frac{3}{2}}||\boldsymbol{v}_0||}{\nu\lambda}\left[\left(\frac{1}{\beta}\right)^{\frac{1}{\beta-1}}-\left(\frac{1}{\beta}\right)^{\frac{\beta}{\beta-1}}\right]^{\frac{1-\beta}{2}}. \tag{4.1}$$

Here $\nu=\nu(N)$ is a positive constant bounded by

$$\frac{1}{3N}\le\nu(N)\le 2N(N-1),\qquad \text{for } N\ge 2.$$

The discrepancy between the condition in Theorem 4.6 and (4.1) might be due to different a priori estimates.

5. Measure valued solution and stability estimate

In this section, we present a global existence of measure valued solutions for the kinetic C-S model [8] and their stability estimate in Kantorovich-Rubinstein distance. Recall C-S particle model: For $i=1,\dots,N$,

$$\begin{cases}\dfrac{d\boldsymbol{x}_i}{dt}=\boldsymbol{v}_i,\qquad \dfrac{d\boldsymbol{v}_i}{dt}=\lambda\displaystyle\sum_{j=1}^{N}m_j\psi(|\boldsymbol{x}_j-\boldsymbol{x}_i|)(\boldsymbol{v}_j-\boldsymbol{v}_i),\qquad m_j\ :\ \text{constant},\\ \boldsymbol{x}_i(0)=\boldsymbol{x}_{i0},\quad \boldsymbol{v}_i(0)=\boldsymbol{v}_{i0},\end{cases} \tag{5.1}$$

and the corresponding kinetic C-S model:

$$\begin{cases}\partial_t f+\text{div}_{\boldsymbol{x}}(\boldsymbol{v}f)+\lambda\text{div}_{\boldsymbol{v}}(\mathbf{F}[f]f)=0,\\ f(\boldsymbol{x},\boldsymbol{v},0)=f_0(\boldsymbol{x},\boldsymbol{v}),\end{cases} \tag{5.2}$$

where f is the one-particle distribution function, and $\mathbf{F}[f]$ is given by the following representation:

$$\mathbf{F}[f](\boldsymbol{x},\boldsymbol{v},t):=-\int_{\mathbb{R}^{2d}}\psi(|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{v}-\boldsymbol{v}_*)f(\boldsymbol{y},\boldsymbol{v}_*,t)d\boldsymbol{v}_*d\boldsymbol{x}.$$

The kinetic model (5.2) was introduced by Ha and Tadmor [8] using the method of BBGKY hierarchy from the C-S particle model as a mesoscopic description for flocking.

5.1. Measure valued solution and moment estimates.

In this part, we review the notion of a measure valued solution to (5.2) and present several a priori estimates for the particle trajectory. In the time-asymptotic limit ($t\to\infty$), the distribution function $f=f(\boldsymbol{x},\boldsymbol{v},t)$ will concentrate on the velocity mean value (see [2, 3, 8]), hence the natural solution space for the kinetic Equ. (5.2) will be the space of nonnegative measures, including Dirac measures. We will prove that the kinetic C-S model (5.2) is well-posed in the space of Radon measures.

Let $\mathcal{M}(\mathbb{R}^{2d})$ be the set of nonnegative Radon measures on the phase space $\mathbb{R}^{2d}$, which can be understood as nonnegative bounded linear functionals on $C_0(\mathbb{R}^{2d})$. For a Radon measure $\nu \in \mathcal{M}(\mathbb{R}^{2d})$, we use a standard duality relation:

$$\langle \nu, g \rangle := \int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v})\nu(d\boldsymbol{x},d\boldsymbol{v}), \qquad g \in C_0(\mathbb{R}^{2d}).$$

The definition of a measure-valued solution to (5.2) is given as follows.

Definition 5.1. *For $T \in [0,\infty)$, let $\mu \in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d})$ be a measure valued solution to* (5.2) *with initial Radon measure $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ if and only if μ satisfies the following conditions:*

1. μ is weakly continuous:

$$\langle \mu_t, g \rangle \quad \text{is continuous as a function of } t, \qquad \forall\ g \in C_0(\mathbb{R}^{2d}).$$

2. μ satisfies the integral equation: $\forall\ g \in C_0^1(\mathbb{R}^{2d} \times [0,T))$,

$$\langle \mu_t, g(\cdot,\cdot,t)\rangle - \langle \mu_0, g(\cdot,\cdot,0)\rangle = \int_0^t \langle \mu_s, \partial_s g + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} g + \lambda \mathbf{F}\cdot\nabla_{\boldsymbol{v}} g\rangle\, ds, \tag{5.3}$$

where $\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_s)$ is a forcing term defined as follows:

$$\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_s) := -\int_{\mathbb{R}^{2d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{v}-\boldsymbol{v}_*)\mu_s(d\boldsymbol{y},d\boldsymbol{v}_*). \tag{5.4}$$

Remark 5.1.

1. If $f \in L^1(\mathbb{R}^{2d}\times[0,T))$ is a weak solution (in the sense of distributions) to (5.2), then $\mu_t(d\boldsymbol{x},d\boldsymbol{v}) = f(\boldsymbol{x},\boldsymbol{v},t)d\boldsymbol{x}d\boldsymbol{v}$ is a measure valued solution to (5.2).

2. If μ is a measure valued solution to (5.2) and μ_t be the absolutely continuous measure with respect to Lebesgue measure whose distribution function is given by $f \in L^1(\mathbb{R}^{2d}\times[0,T))$, i.e., $\mu_t(d\boldsymbol{x},d\boldsymbol{v}) = f(\boldsymbol{x},\boldsymbol{v},t)d\boldsymbol{x}d\boldsymbol{v}$, then f is a distributional weak solution to the (5.2).

3. For any solutions $\{(\boldsymbol{x}_i,\boldsymbol{v}_i)\}_{i=1}^N$ to particle system (5.1), the discrete measure

$$\mu_t := \sum_{i=1}^N m_i \delta(\boldsymbol{x}-\boldsymbol{x}_i(t)) \otimes \delta(\boldsymbol{v}-\boldsymbol{v}_i(t)), \qquad m_i\ :\ \text{constant},$$

is a measure valued solution.

4. Recall that $\mathrm{spt}(\mu)$ (the support of a measure μ) is the closure of the set consisting of all points $(\boldsymbol{x},\boldsymbol{v})$ in $\mathbb{R}^{2d}$ such that $\mu(B_r((\boldsymbol{x},\boldsymbol{v})))>0,\ \forall\ r>0$. For a finite measure with a compact support, we can use $g \in C^1(\mathbb{R}^{2d})$ as a test function in (5.3).

5. Let μ be a measure valued solution to (5.2) with initial Radon measure μ_0 (see section 6 for existence issue), then for any $g \in C_0^1(\mathbb{R}^{2d})$, we have

$$\frac{d}{dt}\langle \mu_t, g\rangle = \langle \mu_t, \boldsymbol{v}\cdot\nabla_x g\rangle + \lambda\langle \mu_t, \mathbf{F}\cdot\nabla_{\boldsymbol{v}} g\rangle,$$

as long as the left hand side of the above relation is well-defined. In fact, for the measure valued solution μ in Theorem 6.2, $\frac{d}{dt}\langle \mu_t, g\rangle$ exists a.e.

6. We will revisit the formulation (5.3) from the viewpoint of particle trajectories in Lemma 5.5.

LEMMA 5.2. *Let $\mu \in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d}))$ be a nonnegative measure valued solution to (5.2) such that μ_t has a compact support for t a.e. Then we have*

$$(i)\ \ \frac{d}{dt}\int_{\mathbb{R}^{2d}} \mu_t(d\boldsymbol{x},d\boldsymbol{v}) = 0, \quad \frac{d}{dt}\int_{\mathbb{R}^{2d}} \boldsymbol{v}\mu_t(d\boldsymbol{x},d\boldsymbol{v}) = 0.$$

$$(ii)\ \ \frac{d}{dt}\int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2\mu_t(d\boldsymbol{x},d\boldsymbol{v}) = -\int_{\mathbb{R}^{2d}}\Big[\int_{\mathbb{R}^{2d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)|\boldsymbol{v}-\boldsymbol{v}_*|^2\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)\Big]\mu_t(d\boldsymbol{x},d\boldsymbol{v}).$$

Proof. The time derivative of velocity moments can be checked directly by taking $g(\boldsymbol{x},\boldsymbol{v},t) = 1, \boldsymbol{v}^i, |\boldsymbol{v}|^2$ in (5.3) respectively (see Remark 5.1 (4)), i.e.,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^{2d}} \boldsymbol{v}^i\mu_t(d\boldsymbol{x},d\boldsymbol{v}) &= \lambda\langle\mu_t, \mathbf{F}\cdot\nabla_{\boldsymbol{v}}\boldsymbol{v}^i\rangle = \lambda\langle\mu_t,\mathbf{F}^i\rangle \\ &= -\lambda\int_{\mathbb{R}^{4d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{v}^i-\boldsymbol{v}^i_*)\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)\mu_t(d\boldsymbol{x},d\boldsymbol{v}) = 0,\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2\mu_t(d\boldsymbol{x},d\boldsymbol{v}) &= 2\lambda\langle\mu_t,\mathbf{F}\cdot\boldsymbol{v}\rangle \\ &= -2\lambda\int_{\mathbb{R}^{4d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)\boldsymbol{v}\cdot(\boldsymbol{v}-\boldsymbol{v}_*)\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)\mu_t(d\boldsymbol{x},d\boldsymbol{v}) \\ &= -\lambda\int_{\mathbb{R}^{4d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)|\boldsymbol{v}-\boldsymbol{v}_*|^2\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)\mu_t(d\boldsymbol{x},d\boldsymbol{v}).\end{aligned}$$

Here we used the change of variables $(\boldsymbol{x},\boldsymbol{v}) \leftrightarrow (\boldsymbol{y},\boldsymbol{v}_*)$ and Fubini's theorem. ∎

5.2. A priori estimates on the particle trajectory. In this part, we present several a priori estimates for the particle trajectory which are crucial in the next section on the existence of a measure valued solution and the mean-field limit.

We first note that the forcing term $F(\boldsymbol{x},\boldsymbol{v},\mu_t)$ in (5.4) can be rewritten as

$$\begin{aligned}F(\boldsymbol{x},\boldsymbol{v},\mu_t) &= -\int_{\mathbb{R}^{2d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)(\boldsymbol{v}-\boldsymbol{v}_*)\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*) \\ &= \Big[\int_{\mathbb{R}^{2d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)\boldsymbol{v}_*\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)\Big] - \Big[\int_{\mathbb{R}^{2d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)\Big]\boldsymbol{v} \\ &:= \mathbf{a}(\boldsymbol{x},\mu_t) - b(\boldsymbol{x},\mu_t)\boldsymbol{v}. \end{aligned} \tag{5.5}$$

LEMMA 5.3. *Assume the communication rate ψ takes the form of*

$$\psi(s) = \frac{1}{(1+s^2)^{\frac{\beta}{2}}}, \qquad \beta \ge 0,$$

and let $\mu \in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d}))$ be a measure valued function with the following properties.

1. *A compact support for each time slice: for some nonnegative locally bounded functions $R(t), P(t)$,*

$$spt(\mu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0).$$

2. Uniform boundedness of the first two moments:

$$\int_{\mathbb{R}^{2d}} \mu_t(d\boldsymbol{x},d\boldsymbol{v}) \le m_0 < \infty, \qquad \int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2 \mu_t(d\boldsymbol{x},d\boldsymbol{v}) \le m_2 < \infty.$$

Then $\mathbf{a}(\boldsymbol{x},\mu_t)$ *and* $b(\boldsymbol{x},\mu_t)$ *satisfy*

(i) $|\mathbf{a}(\boldsymbol{x},\mu_t)| \le (m_0 m_2)^{\frac{1}{2}}, \quad |\mathbf{a}(\boldsymbol{x},\mu_t) - \mathbf{a}(\boldsymbol{y},\mu_t)| \le \beta |\boldsymbol{x}-\boldsymbol{y}|(m_0 m_2)^{\frac{1}{2}}.$

(ii) $|b(\boldsymbol{x},\mu_t)| \le m_0, \quad |b(\boldsymbol{x},\mu_t) - b(\boldsymbol{y},\mu_t)| \le \beta|\boldsymbol{x}-\boldsymbol{y}|.$

(iii) $|\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_t) - \mathbf{F}(\boldsymbol{y},\boldsymbol{v}_*,\mu_t)| \le \Big[\beta\Big((m_0m_2)^{\frac{1}{2}} + P(t)\Big) + m_0\Big] |(\boldsymbol{x},\boldsymbol{v}) - (\boldsymbol{y},\boldsymbol{v}_*)|.$

Here $B_r(0)$ *denotes the ball with a radius* r *and a center* 0*.*

Proof. Let $(\boldsymbol{x},\boldsymbol{v}) \in \mathrm{spt}(\mu_t)$.

(i) We use $|\psi| \le 1$ and Lemma 5.2 to obtain

$$\begin{aligned}
|\mathbf{a}(\boldsymbol{x},\mu_t)| &\le \int_{\mathbb{R}^{2d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)|\boldsymbol{v}_*|\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*) \\
&\le \Big(\int_{\mathbb{R}^{2d}} \mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^{2d}} |\boldsymbol{v}_*|^2 \mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)\Big)^{\frac{1}{2}} \\
&\le (m_0m_2)^{\frac{1}{2}}.
\end{aligned}$$

For the Lipschitz continuity of $\mathbf{a}$, we use $|\psi'| \le \beta$ and the mean value theorem

$$|\psi(|\boldsymbol{x}-\boldsymbol{z}|) - \psi(|\boldsymbol{y}-\boldsymbol{z}|)| \le \beta|\boldsymbol{x}-\boldsymbol{y}|$$

to find

$$\begin{aligned}
|\mathbf{a}(\boldsymbol{x},\mu_t) - \mathbf{a}(\boldsymbol{y},\mu_t)| &\le \int_{\mathbb{R}^{2d}} |\psi(|\boldsymbol{x}-\boldsymbol{z}|) - \psi(|\boldsymbol{y}-\boldsymbol{z}|)||\boldsymbol{v}_*|\mu_t(d\boldsymbol{z},d\boldsymbol{v}_*) \\
&\le \beta|\boldsymbol{x}-\boldsymbol{y}|(m_0m_2)^{\frac{1}{2}}.
\end{aligned}$$

(ii) By direct estimates, we have

$$\begin{aligned}
|b(\boldsymbol{x},\mu_t)| &\le \int_{\mathbb{R}^{2d}} \psi(|\boldsymbol{x}-\boldsymbol{y}|)\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*) \le m_0, \\
|b(\boldsymbol{x},\mu_t) - b(\boldsymbol{y},\mu_t)| &\le \int_{\mathbb{R}^{2d}} |\psi(|\boldsymbol{x}-\bar{\boldsymbol{y}}|) - \psi(|\boldsymbol{y}-\bar{\boldsymbol{y}}|)|\mu_t(d\bar{\boldsymbol{y}},d\boldsymbol{v}_*) \le \beta|\boldsymbol{x}-\boldsymbol{y}|.
\end{aligned}$$

(iii) We use the estimates (i) and (ii) to obtain

$$\begin{aligned}
|\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_t) - \mathbf{F}(\boldsymbol{y},\boldsymbol{v}_*,\mu_t)| &\le |\mathbf{a}(\boldsymbol{x},\mu_t) - \mathbf{a}(\boldsymbol{y},\mu_t)| + |b(\boldsymbol{x},\mu_t)\boldsymbol{v} - b(\boldsymbol{y},\mu_t)\boldsymbol{v}_*| \\
&\le |\mathbf{a}(\boldsymbol{x},\mu_t) - \mathbf{a}(\boldsymbol{y},\mu_t)| + |b(\boldsymbol{x},\mu_t) - b(\boldsymbol{y},\mu_t)||\boldsymbol{v}| \\
&\quad + |b(\boldsymbol{y},\mu_t)||\boldsymbol{v}-\boldsymbol{v}_*| \\
&\le \beta|\boldsymbol{x}-\boldsymbol{y}|(m_0m_2)^{\frac{1}{2}} + \beta|\boldsymbol{x}-\boldsymbol{y}|P(t) + m_0|\boldsymbol{v}-\boldsymbol{v}_*| \\
&= \beta\Big[(m_0m_2)^{\frac{1}{2}} + P(t)\Big]|\boldsymbol{x}-\boldsymbol{y}| + m_0|\boldsymbol{v}-\boldsymbol{v}_*| \\
&\le \Big[\beta\Big((m_0m_2)^{\frac{1}{2}} + P(t)\Big) + m_0\Big]|(\boldsymbol{x},\boldsymbol{v}) - (\boldsymbol{y},\boldsymbol{v}_*)|.
\end{aligned}$$

Here we used $\max\{|\boldsymbol{x}-\boldsymbol{y}|, |\boldsymbol{v}-\boldsymbol{v}_*|\} \le |(\boldsymbol{x},\boldsymbol{v}) - (\boldsymbol{y},\boldsymbol{v}_*)|$. ∎

For $(\boldsymbol{x},\boldsymbol{v},t)\in\mathbb{R}^d\times\mathbb{R}^d\times[0,T)$ and $\mu\in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d}))$, we denote $(\mathbf{X}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}),\mathbf{V}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}))$ to be the particle trajectory passing through $(\boldsymbol{x},\boldsymbol{v})$ at time t, i.e.,

$$\frac{d}{ds}\mathbf{X}_\mu(s;t,\boldsymbol{x},\boldsymbol{v})=\mathbf{V}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}),\quad s\in[0,T), \tag{5.6}$$

$$\frac{d}{ds}\mathbf{V}_\mu(s;t,\boldsymbol{x},\boldsymbol{v})=-\lambda\int_{\mathbb{R}^{2d}}\psi(|\mathbf{X}_\mu(s;t,\boldsymbol{x},\boldsymbol{v})-\boldsymbol{y}|)(\mathbf{V}_\mu(s;t,\boldsymbol{x},\boldsymbol{v})-\boldsymbol{v}_*)\mu_s(d\boldsymbol{y},d\boldsymbol{v}_*), \tag{5.7}$$

subject to initial data

$$\mathbf{X}_\mu(t;t,\boldsymbol{x},\boldsymbol{v})=\boldsymbol{x},\qquad \mathbf{V}_\mu(t;t,\boldsymbol{x},\boldsymbol{v})=\boldsymbol{v}.$$

For notational simplicity, we use the simplified notation

$$[\boldsymbol{x}_\mu(s),\boldsymbol{v}_\mu(s)]:=[\boldsymbol{x}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}),\boldsymbol{v}_\mu(s;t,\boldsymbol{x},\boldsymbol{v})].$$

LEMMA 5.4. *Let $\mu\in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d}))$ be a measure valued function with the following properties.*

1. *A compact support for each time slice: For some nonnegative locally bounded functions $R(t),P(t)$,*

$$spt(\mu_t)\subset B_{R(t)}(0)\times B_{P(t)}(0).$$

2. *Uniform boundedness of the first two moments:*

$$\int_{\mathbb{R}^{2d}}\mu_t(d\boldsymbol{x},d\boldsymbol{v})\le m_0<\infty,\qquad \int_{\mathbb{R}^{2d}}|\boldsymbol{v}|^2\mu_t(d\boldsymbol{x},d\boldsymbol{v})\le m_2<\infty.$$

Then we have

(i) For any fixed $(\boldsymbol{x},\boldsymbol{v},t)\in\mathbb{R}^d\times\mathbb{R}^d\times[0,T)$, there is a unique global particle trajectory $[\boldsymbol{x}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}),\boldsymbol{v}_\mu(s;t,\boldsymbol{x},\boldsymbol{v})]$ which is a C^1-function of $s\in[0,T]$ and admits an unique inverse map in the form of

$$\boldsymbol{x}:=\mathbf{X}_\mu(t;s,\bar{\boldsymbol{x}},\bar{\boldsymbol{v}}),\qquad \boldsymbol{v}:=\mathbf{V}_\mu(t;s,\bar{\boldsymbol{x}},\bar{\boldsymbol{v}}),$$

where

$$\bar{\boldsymbol{x}}:=\mathbf{X}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}),\qquad \bar{\boldsymbol{v}}:=\mathbf{V}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}).$$

(ii) Let $\hat{P}(t)$ and $\hat{R}(t)$ be the bounds for supports of the velocity and position variables respectively. Then we have

$$\hat{P}(t)\le\hat{P}(0)+\lambda(m_0m_2)^{\frac{1}{2}}t,\qquad \hat{R}(t)\le\hat{R}(0)+\hat{P}(0)t+\frac{\lambda}{2}(m_0m_2)^{\frac{1}{2}}t^2.$$

Proof.

(i) Consider the vector field on the phase-space generated by $(\boldsymbol{v},\lambda\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_t))$. Then this vector field satisfies the following two properties

1. Continuity in time t due to the weak continuity of μ_t.
2. Global Lipschitz continuity in the $\boldsymbol{x}$ and $\boldsymbol{v}$ variables due to (iii) of Lemma 5.3.

Hence the standard ordinary differential equation theory implies that there exists the unique global C^1 particle trajectory passing through $(\boldsymbol{x},\boldsymbol{v})$ at time t generated by the above vector field, and its inverse map is just a backward trajectory. Hence (i) holds.

(ii) Since the estimate for $\hat{R}(t)$ follows from the estimate for $\hat{P}(t)$, we only consider the growth estimate of $\hat{P}(t)$. For simplicity in presentation, we denote

$$[\boldsymbol{x}(s),\boldsymbol{v}(s)] := [\boldsymbol{x}_\mu(s;0,\boldsymbol{x},\boldsymbol{v}),\boldsymbol{v}_\mu(s;0,\boldsymbol{x},\boldsymbol{v})],$$

with $(\boldsymbol{x},\boldsymbol{v})\in B_{\hat{R}(0)}(0)\times B_{\hat{P}(0)}(0)$.

We now consider the following equation for $\boldsymbol{v}$:

$$\frac{d\boldsymbol{v}(s)}{ds} = \lambda \mathbf{a}(\boldsymbol{x}(s),\mu_s) - \lambda b(\boldsymbol{x}(s),\mu_s)\boldsymbol{v}(s).$$

We integrate the equation to get the integral equation:

$$\boldsymbol{v}(t) = \boldsymbol{v}_0 e^{-\lambda\int_0^t b(\boldsymbol{x}(s),\mu_s)ds} + \lambda\int_0^t e^{-\lambda\int_s^t b(\boldsymbol{x}(\tau),\mu_\tau)d\tau}\mathbf{a}(\boldsymbol{x}(s),\mu_s)ds.$$

Then we use $b\ge 0$ and the estimates in Lemma 5.3 to obtain

$$|\boldsymbol{v}(t)| \le |\boldsymbol{v}_0| + \lambda\int_0^t |\mathbf{a}(\boldsymbol{x}(s),\mu_s)|ds \le \hat{P}(0) + \lambda(m_0m_2)^{\frac{1}{2}}t.$$

Hence we obtain the upper bound on the size of the velocity support of μ:

$$\hat{P}(t) \le \hat{P}(0) + \lambda(m_0m_2)^{\frac{1}{2}}t.$$

Now we can directly obtain a bound on $\boldsymbol{x}(t)$:

$$|\boldsymbol{x}(t)| \le |\boldsymbol{x}(0)| + \int_0^t |\boldsymbol{v}(s)|ds \le \hat{R}(0) + \hat{P}(0)t + \frac{\lambda}{2}(m_0m_2)^{\frac{1}{2}}t^2.$$

This completes the proof of Lemma. □

Lemma 5.5. *For any $T\in(0,\infty]$, let $\mu\in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d}))$ be a measure-valued solution of* (5.2) *satisfying* (5.3)–(5.4). *Then for any test function $h\in C_0^1(\mathbb{R}^{2d})$, we have*

$$\int_{\mathbb{R}^{2d}} h(\boldsymbol{x},\boldsymbol{v})\mu_t(d\boldsymbol{x},d\boldsymbol{v}) = \int_{\mathbb{R}^{2d}} h(\mathbf{X}_\mu(t;s,\boldsymbol{x},\boldsymbol{v}),\mathbf{V}_\mu(t;s,\boldsymbol{x},\boldsymbol{v}))\mu_s(d\boldsymbol{x},d\boldsymbol{v}).$$

Here $[\mathbf{X}_\mu(t;s,\boldsymbol{x},\boldsymbol{v}),\mathbf{V}_\mu(t;s,\boldsymbol{x},\boldsymbol{v})]$ is the particle trajectory passing through $(\boldsymbol{x},\boldsymbol{v})$ at time s.

Proof. Recall that the defining relation (5.3) can be rewritten as

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v},t)\mu_t(d\boldsymbol{x},d\boldsymbol{v}) - \int_{\mathbb{R}^{2d}} g(\boldsymbol{y},\boldsymbol{v}_*,s)\mu_s(d\boldsymbol{y},d\boldsymbol{v}_*)\\ &= \int_s^t\int_{\mathbb{R}^{2d}}\Big(\partial_\tau g + \bar{\boldsymbol{v}}\cdot\nabla_{\bar{\boldsymbol{y}}}g + \lambda\mathbf{F}(\bar{\boldsymbol{y}},\bar{\boldsymbol{v}})\cdot\nabla_{\bar{\boldsymbol{v}}}g\Big)\mu_s(d\bar{\boldsymbol{y}},d\bar{\boldsymbol{v}})d\tau. \qquad (5.8)\end{aligned}$$

We now choose a test function g so that the right hand side of (5.8) vanishes. For any $h\in C_0^1(\mathbb{R}^{2d})$ and fixed t, we set

$$g(\bar{\boldsymbol{x}},\bar{\boldsymbol{v}},\tau) := h(\mathbf{X}_\mu(t;\tau,\bar{\boldsymbol{x}},\bar{\boldsymbol{v}}),\mathbf{V}_\mu(t;\tau,\bar{\boldsymbol{x}},\bar{\boldsymbol{v}})).$$

Then, by (i) in Lemma 5.4, we have

$$g(\mathbf{X}_\mu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\mathbf{V}_\mu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\tau)=h(\boldsymbol{x},\boldsymbol{v}). \tag{5.9}$$

Direct differentiation of the above relation (5.9) with respect to τ and Lemma 5.4 imply

$$g\in C_0^1(\mathbb{R}^{2d}\times[0,T)), \qquad \partial_\tau g+\bar{\boldsymbol{v}}\cdot\nabla_{\bar{\boldsymbol{y}}}g+\lambda\mathbf{F}(\bar{\boldsymbol{y}},\bar{\boldsymbol{v}})\cdot\nabla_{\bar{\boldsymbol{v}}}g=0.$$

Hence relation (5.8) implies that

$$\int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v},t)\mu_t(d\boldsymbol{x},d\boldsymbol{v})=\int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v},s)\mu_s(d\boldsymbol{x},d\boldsymbol{v})$$

or

$$\int_{\mathbb{R}^{2d}} h(\boldsymbol{x},\boldsymbol{v})\mu_t(d\boldsymbol{x},d\boldsymbol{v})=\int_{\mathbb{R}^{2d}} h(\mathbf{X}_\mu(t;s,\boldsymbol{x},\boldsymbol{v}),\mathbf{V}_\mu(t;s,\boldsymbol{x},\boldsymbol{v}))\mu_s(d\boldsymbol{x},d\boldsymbol{v}). \qquad \square$$

5.3. Stability estimate in bounded Lipschitz distance. In this part, we derive a stability estimate for measure valued solutions to (5.2) in a bounded Lipschitz distance. This stability estimate is crucial in the mean-field limit of the C-S particle model in next section.

We review the definition of the bounded Lipschitz distance in [13, 17]. We first define the admissible set Ω of test functions:

$$\Omega:=\Big\{g:\mathbb{R}^{2d}\to\mathbb{R}\ :\ ||g||_{L^\infty}\le 1,\ \ \mathrm{Lip}(g):=\sup_{\boldsymbol{z}_1\neq\boldsymbol{z}_2\in\mathbb{R}^{2d}}\frac{|g(\boldsymbol{z}_1)-g(\boldsymbol{z}_2)|}{|\boldsymbol{z}_1-\boldsymbol{z}_2|}\le 1\Big\}.$$

Definition 5.6. [13, 17] *Let $\mu,\nu\in\mathcal{M}(\mathbb{R}^{2d})$ be two Radon measures. Then the bounded Lipschitz distance $d(\mu,\nu)$ between μ and ν is given by*

$$d(\mu,\nu):=\sup_{g\in\Omega}\Big|\int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v})\mu(d\boldsymbol{x},d\boldsymbol{v})-\int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v})\nu(d\boldsymbol{x},d\boldsymbol{v})\Big|.$$

Remark 5.2.

1. The space of Radon measures $\mathcal{M}(\mathbb{R}^{2d})$ equipped with the metric $d(\cdot,\cdot)$ is a complete metric space.

2. The bounded Lipschitz distance d is equivalent to the Wasserstein-1 distance (Kantorovich-Rubinstein distance) W_1 (see [7, 17]):

$$W_1(\mu,\nu):=\inf_{\gamma\in\Pi(\mu,\nu)}\int_{\mathbb{R}^{2d}}|\boldsymbol{z}_1-\boldsymbol{z}_2|\gamma(d\boldsymbol{z}_1,d\boldsymbol{z}_2),$$

where $\Pi(\mu,\nu)$ is the set of all product measures on $\mathbb{R}^{2d}\times\mathbb{R}^{2d}$ such that their marginals are μ and ν.

3. For any $g\in C_0(\mathbb{R}^{2d})$ with $|g(\boldsymbol{z})|\le a$ and $\mathrm{Lip}(g)\le b$, we have

$$\Big|\int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v})\mu(d\boldsymbol{x},d\boldsymbol{v})-\int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v})\nu(d\boldsymbol{x},d\boldsymbol{v})\Big|\le\max\{a,b\}d(\mu,\nu). \tag{5.10}$$

We present a series of estimates regarding the dynamics of particle trajectories.

Lemma 5.7. *Let $\mu,\nu \in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d}))$ be two measure valued functions with compact supports for each time slice:*

$$spt(\mu_t),\ spt(\nu_t) \subset B_{R(t)}(0)\times B_{P(t)}(0),$$

where $R(t)$ and $P(t)$ are nonnegative locally bounded functions. Then $\mathbf{a}(\boldsymbol{x},\mu_t)$ and $b(\boldsymbol{x},\mu_t)$ in (5.5) satisfy

$$\text{(i) } |\mathbf{a}(\boldsymbol{x},\mu_t)-\mathbf{a}(\boldsymbol{x},\nu_t)| \le d\max\{P(t),\beta P(t)+1\}d(\mu_t,\nu_t).$$

$$\text{(ii) } |b(\boldsymbol{x},\mu_t)-b(\boldsymbol{x},\nu_t)| \le \max\{1,\beta\}d(\mu_t,\nu_t).$$

Proof.
(i) Let $(\boldsymbol{y},\boldsymbol{v}_*)\in \mathrm{spt}(\mu_t)\cup\mathrm{spt}(\nu_t)$. Then we have

$$\begin{aligned}&|\psi(|\boldsymbol{x}-\boldsymbol{y}|)\boldsymbol{v}_*^i| \le P(t) \quad \text{and}\\ &|\psi(|\boldsymbol{x}-\boldsymbol{y}_1|)\boldsymbol{v}_1^i-\psi(|\boldsymbol{x}-\boldsymbol{y}_2|)\boldsymbol{v}_2^i| \le (\beta P(t)+1)|(\boldsymbol{y}_1,\boldsymbol{v}_1)-(\boldsymbol{y}_2,\boldsymbol{v}_2)|.\end{aligned}$$

It follows from (5.10) that we have

$$\begin{aligned}|\mathbf{a}^i(\boldsymbol{x},\mu_t)-\mathbf{a}^i(\boldsymbol{x},\nu_t)| &= \Big|\int_{\mathbb{R}^{2d}}\psi(|\boldsymbol{x}-\boldsymbol{y}|)\boldsymbol{v}_*^i(\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)-\nu_t(d\boldsymbol{y},d\boldsymbol{v}_*))\Big|\\ &\le \max\{P(t),\beta P(t)+1\}d(\mu_t,\nu_t).\end{aligned}$$

This yields the desired result.

(ii) Note that

$$\psi(|\boldsymbol{x}-\boldsymbol{y}|)\le 1 \quad \text{and} \quad |\psi(|\boldsymbol{x}-\boldsymbol{y}_1|)-\psi(|\boldsymbol{x}-\boldsymbol{y}_2|)| \le \beta|\boldsymbol{y}_1-\boldsymbol{y}_2|,$$

hence we again use (5.10) to obtain

$$\begin{aligned}|b(\boldsymbol{x},\mu_t)-b(\boldsymbol{x},\nu_t)| &\le \Big|\int_{\mathbb{R}^{2d}}\psi(|\boldsymbol{x}-\boldsymbol{y}|)(\mu_t(d\boldsymbol{y},d\boldsymbol{v}_*)-\nu_t(d\boldsymbol{y},d\boldsymbol{v}_*))\Big|\\ &\le \max\{1,\beta\}d(\mu_t,\nu_t).\end{aligned}$$

∎

Lemma 5.8. *Let $\mu,\nu \in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d}))$ be two measure valued functions with compact supports*

$$spt(\mu_t),\ spt(\nu_t) \subset B_{R(t)}(0)\times B_{P(t)}(0),$$

and finite moments

$$\begin{aligned}&\int_{\mathbb{R}^{2d}}\mu_t(d\boldsymbol{x},d\boldsymbol{v})\le m_0, \qquad \int_{\mathbb{R}^{2d}}|\boldsymbol{v}|^2\mu_t(d\boldsymbol{x},d\boldsymbol{v})\le m_2,\\ &\int_{\mathbb{R}^{2d}}\nu_t(d\boldsymbol{x},d\boldsymbol{v})\le m_0, \qquad \int_{\mathbb{R}^{2d}}|\boldsymbol{v}|^2\nu_t(d\boldsymbol{x},d\boldsymbol{v})\le m_2.\end{aligned}$$

Then for any $0\le s\le T$, we have

$$\begin{aligned}(i)\quad &|\mathbf{a}(\mathbf{X}_\mu(s),\mu_s)-\mathbf{a}(\mathbf{X}_\nu(s),\nu_s)|\\ &\le \beta(m_0m_2)^{\frac{1}{2}}|(\mathbf{X}_\mu(s)-\mathbf{X}_\nu(s)|+d\max\{P(\max\{s,t\}),\beta P(\max\{s,t\})+1\}d(\mu_s,\nu_s).\end{aligned}$$

(ii) $|b(\mathbf{X}_\mu(s),\mu_s) - b(\mathbf{X}_\nu(s),\nu_s)|$
$\leq \beta m_0 |(\mathbf{X}_\mu(s) - \mathbf{X}_\nu(s)| + \max\{1,\beta\} d(\mu_s,\nu_s).$

Here we used the simplified notations:

$$\mathbf{X}_\mu(s) := \mathbf{X}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}), \qquad \mathbf{X}_\nu(s) := \mathbf{X}_\nu(s;t,\boldsymbol{x},\boldsymbol{v}).$$

Proof.

(i) By definition of $\mathbf{a}$, we have

$$\begin{aligned}
&|\mathbf{a}(\mathbf{X}_\mu,\mu_s) - \mathbf{a}(\mathbf{X}_\nu,\nu_s)| \\
&\quad = \Big| \int_{\mathbb{R}^{2d}} \psi(|\mathbf{X}_\mu - \boldsymbol{y}|) \boldsymbol{v}_* \mu_s(d\boldsymbol{y},d\boldsymbol{v}_*) - \int_{\mathbb{R}^{2d}} \psi(|\mathbf{X}_\nu - \boldsymbol{y}|) \boldsymbol{v}_* \nu_s(d\boldsymbol{y},d\boldsymbol{v}_*) \Big| \\
&\quad \leq \Big| \int_{\mathbb{R}^{2d}} \Big(\psi(|\mathbf{X}_\mu - \boldsymbol{y}|) - \psi(|\mathbf{X}_\nu - \boldsymbol{y}|) \Big) \boldsymbol{v}_* \mu_s(d\boldsymbol{y},d\boldsymbol{v}_*) \Big| \\
&\qquad + \Big| \int_{\mathbb{R}^{2d}} \psi(|\mathbf{X}_\nu - \boldsymbol{y}|) \boldsymbol{v}_* \mu_s(d\boldsymbol{y},d\boldsymbol{v}_*) - \int_{\mathbb{R}^{2d}} \psi(|\mathbf{X}_\nu - \boldsymbol{y}|) \boldsymbol{v}_* \nu_s(d\boldsymbol{y},d\boldsymbol{v}_*) \Big| \\
&\quad := \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

The term $\mathcal{I}_1$ can be treated as follows.

$$\mathcal{I}_1 \leq \beta |\mathbf{X}_\mu - \mathbf{X}_\nu| \int_{\mathbb{R}^{2d}} |\boldsymbol{v}_*| \mu_s(d\boldsymbol{y},d\boldsymbol{v}_*) \leq \beta |\mathbf{X}_\mu - \mathbf{X}_\nu| (m_0 m_2)^{\frac{1}{2}}.$$

On the other hand, for the estimate of $\mathcal{I}_2$, we use the same argument as in Lemma 5.7 (i) to obtain

$$\mathcal{I}_2 \leq d \max\{P(\max\{s,t\}),\ \beta P(\max\{s,t\}) + 1\} d(\mu_s,\nu_s).$$

(ii) Similar to (i), we have

$$\begin{aligned}
&|b(\mathbf{X}_\mu,\mu_s) - b(\mathbf{X}_\nu,\nu_s)| \\
&\quad \leq \int_{\mathbb{R}^{2d}} |\psi(|\mathbf{X}_\mu - \boldsymbol{y}|) - \psi(|\mathbf{X}_\nu - \boldsymbol{y}|)| \mu_s(d\boldsymbol{y},d\boldsymbol{v}_*) \\
&\qquad + \Big| \int_{\mathbb{R}^{2d}} \psi(|\mathbf{X}_\nu - \boldsymbol{y}|) \mu_s(d\boldsymbol{y},d\boldsymbol{v}_*) - \int_{\mathbb{R}^{2d}} \psi(|\mathbf{X}_\nu - \boldsymbol{y}|) \nu_s(d\boldsymbol{y},d\boldsymbol{v}_*) \Big| \\
&\quad \leq \beta |\mathbf{X}_\mu - \mathbf{X}_\nu| m_0 + \max\{1,\beta\} d(\mu_s,\nu_s).
\end{aligned}$$

□

LEMMA 5.9. *Let $\mu,\nu \in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d}))$ be two measure valued solutions with compact supports*

$$spt(\mu_t),\ spt(\nu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0), \quad \textit{for some positive functions}\ \ R(t),\ P(t),$$

and uniform bounded moments

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \mu_t(d\boldsymbol{x},d\boldsymbol{v}) \leq m_0, \qquad \int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2 \mu_t(d\boldsymbol{x},d\boldsymbol{v}) \leq m_2, \\
&\int_{\mathbb{R}^{2d}} \nu_t(d\boldsymbol{x},d\boldsymbol{v}) \leq m_0, \qquad \int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2 \nu_t(d\boldsymbol{x},d\boldsymbol{v}) \leq m_2.
\end{aligned}$$

Then for any $0 \leq s \leq T$, we have

$$|\mathbf{X}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}) - \mathbf{X}_\nu(s;t,\boldsymbol{x},\boldsymbol{v})| + |\mathbf{V}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}) - \mathbf{V}_\nu(s;t,\boldsymbol{x},\boldsymbol{v}))|$$

$$\leq \int_{\min\{s,t\}}^{\max\{s,t\}} \alpha(\tau;s) d(\mu_\tau,\nu_\tau) d\tau,$$

where $\alpha = \alpha(\tau;s)$ *is a smooth function depending only on* $T,\lambda,\beta,d,m_0,m_2,P(\cdot)$, *and* $R(\cdot)$.

Proof. We set

$$\boldsymbol{y}(s) := \mathbf{X}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}) - \mathbf{X}_\nu(s;t,\boldsymbol{x},\boldsymbol{v}), \qquad \boldsymbol{v}(s) := \mathbf{V}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}) - \mathbf{V}_\nu(s;t,\boldsymbol{x},\boldsymbol{v}).$$

We consider the case $0 < t < s$. The other case is exactly the same except the change of sign. It follows from (5.6) that we have

$$\frac{d\boldsymbol{y}(\tau)}{d\tau} = \boldsymbol{v}(\tau), \qquad \boldsymbol{y}(t) = 0.$$

We integrate the above equation from $\tau = t$ to $\tau = s$ to find

$$|\boldsymbol{y}(s)| \leq \int_t^s |\boldsymbol{v}(\tau)| d\tau.$$

On the other hand, $\boldsymbol{v}(\tau)$ satisfies

$$\begin{aligned} &\frac{d\boldsymbol{v}(\tau)}{d\tau} + \lambda b(\mathbf{X}_\mu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\mu_\tau)\boldsymbol{v}(\tau) \\ &= \lambda[\mathbf{a}(\mathbf{X}_\mu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\mu_\tau) - \mathbf{a}(\mathbf{X}_\nu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\nu_\tau)] \\ &\quad -\lambda[b(\mathbf{X}_\mu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\mu_\tau) - b(\mathbf{X}_\nu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\nu_\tau)]\mathbf{V}_\nu(\tau;t,\boldsymbol{x},\boldsymbol{v}), \\ &\boldsymbol{v}(t) = 0. \end{aligned}$$

We use the method of integrating factor and $e^{-\lambda b(\mathbf{X}_\mu)} \leq 1$ to obtain

$$\begin{aligned} |\boldsymbol{v}(s)| &\leq \int_t^s \lambda |\mathbf{a}(\mathbf{X}_\mu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\mu_\tau) - \mathbf{a}(\mathbf{X}_\nu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\nu_\tau)| d\tau \\ &\quad + \int_t^s \lambda |b(\mathbf{X}_\mu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\mu_\tau) - b(\mathbf{X}_\nu(\tau;t,\boldsymbol{x},\boldsymbol{v}),\nu_\tau)||\mathbf{V}_\nu(\tau;t,\boldsymbol{x},\boldsymbol{v})| d\tau \\ &\leq \int_t^s \lambda \Big[d\max\Big\{P(\tau),\beta P(\tau)+1\Big\} + \max\{1,\beta\}P(\tau)\Big] d(\mu_\tau,\nu_\tau) d\tau \\ &\quad + \int_t^s \lambda\beta \Big[m_0 P(\tau) + (m_0 m_2)^{1/2}\Big] |\boldsymbol{y}(\tau)| d\tau \\ &\leq \int_t^s \lambda \Big[d + (1+d)P(\tau)\max\{1,\beta\}\Big] d(\mu_\tau,\nu_\tau) d\tau \\ &\quad + \int_t^s \lambda\beta \Big[m_0 P(\tau) + (m_0 m_2)^{1/2}\Big] |\boldsymbol{y}(\tau)| d\tau. \end{aligned} \tag{5.11}$$

Here we used the fact that

$$d\max\{P(\tau),\beta P(\tau)+1\} + \max\{1,\beta\}P(\tau) \leq d + (1+d)P(\tau)\max\{1,\beta\}.$$

We set

$$\begin{aligned} &z(s) := |\boldsymbol{v}(s)| + |\boldsymbol{y}(s)|, \quad A(\tau) := \lambda[d + (1+d)P(\tau)\max\{1,\beta\}], \\ &B(\tau) := \max\Big\{1, \lambda\Big[m_0 P(\tau) + (m_0 m_2)^{1/2}\Big]\Big\}. \end{aligned}$$

Then z satisfies the Gronwall type inequality:

$$z(s) \le \int_t^s A(\tau) d(\mu_\tau \nu_\tau)\, d\tau + \int_t^s B(\tau) z(\tau)\, d\tau,$$

hence the Gronwall Lemma implies that

$$z(s) \le \int_t^s A(\tau) e^{\int_\tau^s B(r)dr} d(\mu_\tau, \nu_\tau)\, d\tau \equiv \int_t^s \alpha(\tau; s) d(\mu_\tau, \nu_\tau)\, d\tau.$$

Here we used a simplified notation:

$$\alpha(\tau; s) := A(\tau) e^{\int_\tau^s B(r)dr}.$$

Therefore, we have

$$\begin{aligned} &|\mathbf{X}_\mu(s;t,\boldsymbol{x},\boldsymbol{v}) - \mathbf{X}_\nu(s;t,\boldsymbol{x},\boldsymbol{v})| + |\mathbf{V}_\mu(s;t,\boldsymbol{x},\boldsymbol{v})) - \mathbf{V}_\nu(s;t,\boldsymbol{x},\boldsymbol{v}))| \\ &= |\boldsymbol{v}(s)| + |\boldsymbol{y}(s)| \le z(s) \le \int_t^s \alpha(\tau;s) d(\mu_\tau, \nu_\tau)\, d\tau. \end{aligned}$$

□

Proposition 5.10. *Let $\mu, \nu \in L^\infty([0,T); \mathcal{M}(\mathbb{R}^{2d}))$ be two measure valued solutions corresponding initial data μ_0, ν_0 with compact supports:*

$$spt(\mu_t),\ spt(\nu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0), \quad \textit{for some positive functions } R(t),\ P(t),$$

and uniform bounded moments:

$$\int_{\mathbb{R}^{2d}} \mu_0(d\boldsymbol{x}, d\boldsymbol{v}) \le m_0, \qquad \int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2 \mu_0(d\boldsymbol{x}, d\boldsymbol{v}) \le m_2,$$

$$\int_{\mathbb{R}^{2d}} \nu_0(d\boldsymbol{x}, d\boldsymbol{v}) \le m_0, \qquad \int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2 \nu_0(d\boldsymbol{x}, d\boldsymbol{v}) \le m_2.$$

Then there exists a nonnegative function $C_2 = C(T, d, \lambda, \beta, P(\cdot), R(\cdot), m_0, m_2)$ satisfying

$$d(\mu_t, \nu_t) \le C_2 d(\mu_0, \nu_0), \qquad t \in [0, T).$$

Proof. Let $g \in C_0(\mathbb{R}^{2d})$ a test function in Ω, i.e.,

$$||g||_{L^\infty} \le 1 \quad \text{and} \quad \mathrm{Lip}(g) \le 1.$$

Then we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v}) \mu_t(d\boldsymbol{x}, d\boldsymbol{v}) - \int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v}) \nu_t(d\boldsymbol{x}, d\boldsymbol{v}) \right| \\ \le &\int_{\mathbb{R}^{2d}} |g(\mathbf{X}_\mu(t;0,\boldsymbol{x},\boldsymbol{v}), \mathbf{V}_\mu(t;0,\boldsymbol{x},\boldsymbol{v})) - g(\mathbf{X}_\nu(t;0,\boldsymbol{x},\boldsymbol{v}), \mathbf{V}_\nu(t;0,\boldsymbol{x},\boldsymbol{v}))| \mu_0(d\boldsymbol{x}, d\boldsymbol{v}) \\ + &\left| \int_{\mathbb{R}^{2d}} g(\mathbf{X}_\nu(t;0,\boldsymbol{x},\boldsymbol{v}), \mathbf{V}_\nu(t;0,\boldsymbol{x},\boldsymbol{v})) \mu_0(d\boldsymbol{x}, d\boldsymbol{v}) \right. \\ &\left. - \int_{\mathbb{R}^{2d}} g(\mathbf{X}_\nu(t;0,\boldsymbol{x},\boldsymbol{v}), \mathbf{V}_\nu(t;0,\boldsymbol{x},\boldsymbol{v})) \nu_0(d\boldsymbol{x}, d\boldsymbol{v}) \right| \end{aligned}$$

$$\leq \int_{\mathbb{R}^{2d}} (|\mathbf{X}_\mu(t;0,\boldsymbol{x},\boldsymbol{v}) - \mathbf{X}_\nu(t;0,\boldsymbol{x},\boldsymbol{v})| + |\mathbf{V}_\mu(t;0,\boldsymbol{x},\boldsymbol{v}) - \mathbf{V}_\nu(t;0,\boldsymbol{x},\boldsymbol{v})|)\,\mu_0(d\boldsymbol{x},d\boldsymbol{v}) + d(\mu_0,\nu_0)$$
$$\leq m_0 \int_0^t \alpha(\tau;t) d(\mu_\tau,\nu_\tau)\,d\tau + d(\mu_0,\nu_0).$$

Therefore we have

$$d(\mu_t,\nu_t) \leq m_0 \int_0^t \alpha(\tau;T) d(\mu_\tau,\nu_\tau)\,d\tau + d(\mu_0,\nu_0).$$

The Gronwall lemma gives

$$d(\mu_t,\nu_t) \leq d(\mu_0,\nu_0) e^{m_0 \int_0^t \alpha(\tau;T)d\tau}. \qquad \square$$

6. Mean-field limit and the existence of measure valued solutions

In this section, we show the existence of measure valued solutions to (5.2) with initial Radon measure μ_0. For this, we use a standard method for the Vlasov equation with a bounded and Lipshitz kernel in [13, 17]. However unlike to the case in Neuzert and Sphohn's case in [13, 17], the kinetic model (5.2) does not have a bounded kernel due to the relative velocity factor $(\boldsymbol{v}-\boldsymbol{v}_*)$. Hence we need to control the growth of velocity support (Lemma 5.4) to construct measure valued solutions.

6.1. A particle method. For the particle method or particle in cell (PIC), the initial Radon measure μ_0 is approximated by a finite sum of Delta measures. For example, for given $h>0$ and lattice points $i,j \in \mathbb{Z}^d$, we set a phase space box $R^h(i,j)$ and its center $(\boldsymbol{x}_i,\boldsymbol{v}_j)$:

$$\boldsymbol{x}_i := ih, \qquad \boldsymbol{v}_j := jh, \qquad \text{and}$$
$$R^h(i,j) := \Pi_{k=1}^d \Big[(i_k - \frac{1}{2})h, (i_k + \frac{1}{2})h\Big] \times \Pi_{k=1}^d \Big[(j_k - \frac{1}{2})h, (j_k + \frac{1}{2})h\Big].$$

Hence the whole phase space $\mathbb{R}^{2d}$ is the countable union of $R^h(i,j)$:

$$\mathbb{R}^{2d} := \cup_{i,j\in\mathbb{Z}^d} R^h(i,j).$$

Suppose initial Radon measure μ_0 has a compact support, i.e.,

$$\exists \;\; R_0, P_0 < \infty \qquad \text{such that} \quad \operatorname{spt}(\mu_0) \subset B_{R_0}(0) \times B_{P_0}(0).$$

We construct the initial approximation μ_0^h as

$$\mu_0^h := \sum_{i,j} m_{ij} \delta(\boldsymbol{x}-\boldsymbol{x}_i) \otimes \delta(\boldsymbol{v}-\boldsymbol{v}_j), \qquad m_{ij} := \int_{R^h(i,j)} \mu_0(d\boldsymbol{x},d\boldsymbol{v}). \tag{6.1}$$

Note that since μ_0 has a compact support, the sum in the above definition (6.1) is in fact a finite sum.

LEMMA 6.1. *Let μ_0 be a given initial Radon measure on $\mathbb{R}^{2d}$ with a compact support:*

$$spt(\mu_0) \subset B_{R_0}(0) \times B_{P_0}(0),$$

and let μ_0^h be the initial approximation given by (6.1) *with a uniform grid size h. Then we have*

$$d(\mu_0^h,\mu_0) \leq \frac{\sqrt{d}}{2} ||\mu_0|| h.$$

Here $d(\cdot,\cdot)$ is the bounded Lipschitz distance introduced in section 5.3, and $||\mu_0|| := \int_{\mathbb{R}^{2d}} \mu_0(d\boldsymbol{x}, d\boldsymbol{v})$.

Proof. Let $g \in \Omega$ be a test function with the properties $(|g| \le 1,\ \mathrm{Lip}(g) \le 1)$. Then we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v})\mu_0(d\boldsymbol{x},d\boldsymbol{v}) - \int_{\mathbb{R}^{2d}} g(\boldsymbol{x},\boldsymbol{v})\mu_0^h(d\boldsymbol{x},d\boldsymbol{v}) \Big| \\
&\quad \le \sum_{i,j} \Big| \int_{R^h(i,j)} g(\boldsymbol{x},\boldsymbol{v})\mu_0(d\boldsymbol{x},d\boldsymbol{v}) - \int_{R^h(i,j)} g(\boldsymbol{x},\boldsymbol{v})\mu_0^h(d\boldsymbol{x},d\boldsymbol{v}) \Big| \\
&\quad = \sum_{i,j} \Big| \int_{R^h(i,j)} g(\boldsymbol{x},\boldsymbol{v})\mu_0(d\boldsymbol{x},d\boldsymbol{v}) - \int_{R^h(i,j)} g(\boldsymbol{x}_i,\boldsymbol{v}_j)\mu_0(d\boldsymbol{x},d\boldsymbol{v}) \Big| \\
&\quad \le \sum_{i,j} \int_{R^h(i,j)} |g(\boldsymbol{x},\boldsymbol{v}) - g(\boldsymbol{x}_i,\boldsymbol{v}_j)|\mu_0(d\boldsymbol{x},d\boldsymbol{v}) \\
&\quad \le \sum_{i,j} \int_{R^h(i,j)} |(\boldsymbol{x},\boldsymbol{v}) - (\boldsymbol{x}_i,\boldsymbol{v}_j)|\mu_0(d\boldsymbol{x},d\boldsymbol{v}) \\
&\quad \le \frac{\sqrt{d}}{2}||\mu_0||h.
\end{aligned}$$

∎

REMARK 6.1.

1. We will use single index i, $i = 1,\dots,N$ instead of double index, and rewrite μ_0^h in (6.1) as

$$\mu_0^h := \sum_{i=1}^{N} m_i \delta(\boldsymbol{x} - \boldsymbol{x}_i) \otimes \delta(\boldsymbol{v} - \boldsymbol{v}_i).$$

2. Note that the sequence $\{\mu^h\}$ satisfies

$$d(\mu_0^{h_1}, \mu_0^{h_2}) \le d(\mu_0^{h_1}, \mu_0) + d(\mu_0, \mu_0^{h_2}) \le \sqrt{d}||\mu_0|| \max\{h_1, h_2\},$$

which implies that $\{\mu_0^h\}$ is a Cauchy sequence in the complete metric space $(\mathcal{M}(\mathbb{R}^{2d}), d(\cdot,\cdot))$.

The particle method is exactly identical with the Cucker-Smale dynamics:

$$\frac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \qquad \frac{d\boldsymbol{v}_i}{dt} = \lambda \sum_{j=1}^{N} m_j \psi(|\boldsymbol{x}_i - \boldsymbol{x}_j|)(\boldsymbol{v}_j - \boldsymbol{v}_i), \qquad i = 1,\dots,N. \tag{6.2}$$

With solutions $(\boldsymbol{x}_i(t), \boldsymbol{v}_i(t))$ of (6.2), we can define an approximate solution:

$$\mu_t^h := \sum_{i=1}^{N} m_i \delta(\boldsymbol{x} - \boldsymbol{x}_i(t)) \otimes \delta(\boldsymbol{v} - \boldsymbol{v}_i(t)). \tag{6.3}$$

It is clear that bounds for supports of the velocity and position variables as $\hat{P}^h(t)$ and $\hat{R}^h(t)$ is also bounds of support of measure (6.3) μ_t^h, i.e.,

$$\mathrm{spt}(\mu_t^h) \subset B_{\hat{R}^h(t)}(0) \times B_{\hat{P}^h(t)}(0). \tag{6.4}$$

6.2. Convergence of a particle method. In this part, we present a convergence of approximate solutions to the measure valued solution of (5.2) and error estimate for the particle method which is first order accurate.

THEOREM 6.2. *Let $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ be a Radon measure with a compact support. Let μ_t^h be the approximate solution constructed by the particle method with a grid size h. Let (R_0, P_0) be the radius of the initial compact support, i.e.,*

$$spt(\mu_0) \subset B_{R_0}(0) \times B_{P_0}(0),$$

and let m_0, m_2 be the moments of the initial measure

$$m_0 = \int_{\mathbb{R}^{2d}} \mu_0(d\boldsymbol{x}, d\boldsymbol{v}), \qquad m_2 = \int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2 \mu_0(d\boldsymbol{x}, d\boldsymbol{v}).$$

Then there is a measure valued solution $\mu \in L^\infty([0,T); \mathcal{M}(\mathbb{R}^{2d})$ to (5.2) with initial data μ_0 such that

1. *μ_t is the weak-$*$ limit of the approximate solutions μ_t^h as $h \to 0+$, and we have the error estimate*

$$d(\mu_t, \mu_t^h) \le C_3 h,$$

where $C_3 = C_3(T, d, \lambda, \beta, P_0, m_0, m_2)$ is a positive constant depending on the arguments specified.

2. *μ is weakly Lip continuous and has bounded first two moments for each time slice*

$$\int_{\mathbb{R}^{2d}} \mu_t(d\boldsymbol{x}, d\boldsymbol{v}) \le m_0, \qquad \int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2 \mu_t(d\boldsymbol{x}, d\boldsymbol{v}) \le m_2.$$

Moreover μ has compact support for each time slice:

$$spt(\mu_t) \subset B_{R(t)}(0) \times B_{P(t)}(0),$$

where $R(t)$ and $P(t)$ are positive bounded functions satisfying the growth estimates

$$P(t) \le P_0 + \lambda (m_0 m_2)^{\frac{1}{2}} t, \qquad R(t) \le R_0 + P_0 t + \frac{\lambda}{2} (m_0 m_2)^{\frac{1}{2}} t^2$$

3. *μ is unique in the class of measure valued solution to (5.2) with initial data μ_0 and compact supports for each time slice.*

Proof. We present the proof in several steps.
Step A (bounds for moments and compact support): Denote

$$m_0^h(t) = \int_{\mathbb{R}^{2d}} \mu_t^h(d\boldsymbol{x}, d\boldsymbol{v}), \qquad m_2^h(t) = \int_{\mathbb{R}^{2d}} |\boldsymbol{v}|^2 \mu_t^h(d\boldsymbol{x}, d\boldsymbol{v}),$$

and let $(R^h(t), P^h(t))$ be the radius of the initial compact support, i.e.,

$$\mathrm{spt}(\mu_t^h) \subset B_{R^h(t)}(0) \times B_{P^h(h)}(0).$$

As explained in (6.4), $P^h(t)$ and $R^h(t)$ can also be bounds for supports of the velocity and position variables $\hat{P}^h(t)$ and $\hat{R}^h(t)$ respectively. From Remark 5.1 (3), we know that μ_t^h is a measure valued solution. We now apply Lemmas 5.2 and 5.4 to μ_t^h,

$$m_0^h(t) = m_0^h(0), \quad m_2^h(t) \le m_2^h(0),$$

$$P^h(t) \le P^h(0) + \lambda(m_0^h(0) m_2^h(0))^{\frac{1}{2}} t, \quad R^h(t) \le R^h(0) + P^h(0)t + \frac{\lambda}{2}(m_0^h m_2^h)^{\frac{1}{2}} t^2$$

Due to the construction of initial approximation μ_0^h, it is easy to prove that

$$m_0^h(0) \le m_0 + Ch, \quad m_2^h(0) \le m_2 + Ch, \quad R^h(0) \le R_0 + Ch, \quad P^h(0) \le P_0 + Ch. \tag{6.5}$$

Hence

$$P^h(t) \le P_0 + \lambda(m_0 m_2)^{\frac{1}{2}} t + Ch(1+t), \tag{6.6}$$

$$R^h(t) \le R_0 + P_0 t + \frac{\lambda}{2}(m_0 m_2)^{\frac{1}{2}} t^2 + Ch(1+t+t^2). \tag{6.7}$$

Step B (Existence of weak-$*$ limit and error estimate): For fixed $t \in [0,T)$, it follows from Proposition 5.10 and Remark 6.1 that

$$d(\mu_t^{h_1}, \mu_t^{h_2}) \le C_2 d(\mu_0^{h_1}, \mu_0^{h_2}) \le C_2 \sqrt{d} ||\mu_0|| \max\{h_1, h_2\}.$$

Due to (6.6) and (6.7) and the fact that $m_0 = ||\mu_0||$, we know that C_2 depends only on $T, d, \lambda, \beta, P(0), m_0$ and m_2. Hence, we set

$$C_3(T,d,\lambda,\beta,P_0,m_0,m_2) := C_2 \sqrt{d} ||\mu_0||$$

to obtain

$$d(\mu_t^{h_1}, \mu_t^{h_2}) \le C_3 \max\{h_1, h_2\}.$$

Therefore the sequence of approximate solutions $\{\mu_t^h\}$ is a Cauchy sequence in the complete metric space $(\mathcal{M}(\mathbb{R}^{2d}), d(\cdot,\cdot))$. This guarantees the existence of a limit measure $\mu_t \in \mathcal{M}(\mathbb{R}^{2d})$. By taking $h_1 = h$ fixed and and $h_2 \to 0$, we have a first order accurate error estimate

$$d(\mu_t^h, \mu_t) \le C_3 h.$$

Since $d(\cdot,\cdot)$-convergence is also equivalent to weak-$*$ convergence[17], we know that μ_t is the weak$*$-limit of μ_t^h. From (6.5)–(6.7), we have

$$m_0(t) = m_0, \quad m_2(t) \le m_2,$$

$$P(t) \le P_0 + \lambda(m_0 m_2)^{\frac{1}{2}} t, \quad R(t) \le R_0 + P_0 t + \frac{\lambda}{2}(m_0 m_2)^{\frac{1}{2}} t^2.$$

Step C (Weak Lipschitz continuity): We need to check the weak continuity of the map $t \to \mu_t$. We first note that the vector field $(\boldsymbol{v}, \lambda \mathbf{F}(\mu_t^h))$ is bounded in the time-strip $\mathbb{R}^{2d} \times [0,T)$:

$$\begin{aligned} |(\boldsymbol{v}, \lambda \mathbf{F}(\mu_t^h))| &\le |\boldsymbol{v}| + \lambda |\mathbf{F}(\mu_t^h)| \le |\boldsymbol{v}| + \lambda |\mathbf{a}(\boldsymbol{x}, \mu_t^h)| + \lambda |b(\boldsymbol{x}, \mu_t^h)| |\boldsymbol{v}| \\ &\le P^h(t) + \lambda (m_0^h m_2^h)^{\frac{1}{2}} + \lambda m_0^h P^h(t) \le C(T,d,\lambda,\beta,P_0,m_0,m_2) < \infty. \end{aligned}$$

Here we used Lemmas 5.3 and 5.4. This leads to

$$|\mathbf{X}^h(t+\Delta t) - \mathbf{X}^h(t)| + |\mathbf{V}^h(t+\Delta t) - \mathbf{V}^h(t)| \le C(T,d,\lambda,\beta,P_0,m_0,m_2)\Delta t.$$

On the other hand, note that for any $g\in C_0^1(\mathbb{R}^{2d})$, we have from Lemma 5.5 that

$$\begin{aligned}|\langle\mu^h_{t+\Delta t},g\rangle-\langle\mu^h_t,g\rangle| &= \Big|\int_{\mathbb{R}^{2d}}(g(\mathbf{X}^h(t+\Delta t;t,\boldsymbol{x},\boldsymbol{v}),\mathbf{V}^h(t+\Delta t;t,\boldsymbol{x},\boldsymbol{v}))\\ &\quad -g(\boldsymbol{x},\boldsymbol{v}))\mu^h_t(d\boldsymbol{x},d\boldsymbol{v})\Big|\\ &\le ||g||_{C^1}m^h_0\Big(|\mathbf{X}^h(t+\Delta t)-\mathbf{X}^h(t)|+|\mathbf{V}^h(t+\Delta t)-\mathbf{V}^h(t)|\Big)\\ &\qquad\le C(T,d,\lambda,\beta,P_0,m_0,m_2,g)\Delta t.\end{aligned}$$

As $\Delta t\to 0$, we see that μ is weakly Lipschitz continuous in t. Since μ^h satisfy required bounds, the limit μ also satisfy required bounds.

Step D (μ satisfies (5.3)): In the following, we show that this limit measure μ is a measure valued solution to (5.2).

Note that μ^h is a measure valued solution, hence it satisfies the equation (5.3):

$$\langle\mu^h_t,g(\cdot,\cdot,t)\rangle-\langle\mu^h_0,g(\cdot,\cdot,0)\rangle=\int_0^t\langle\mu^h_s,\partial_s g+\boldsymbol{v}\cdot\nabla_x g+\lambda\mathbf{F}(\mu^h_s)\cdot\nabla_{\boldsymbol{v}}g\rangle\,ds. \tag{6.8}$$

Since $d(\cdot,\cdot)$-convergence is equivalent to weak-$*$ convergence, it is easy to see that the first two terms in (6.8) converge to the corresponding terms for weak$*$-limit μ_t, i.e.,

$$\langle\mu^h_t,g(\cdot,\cdot,t)\rangle-\langle\mu^h_0,g(\cdot,\cdot,0)\rangle\to\langle\mu_t,g(\cdot,\cdot,t)\rangle-\langle\mu_0,g(\cdot,\cdot,0)\rangle\qquad\text{as}\quad h\to 0+.$$

It remains to show that the terms in the right hand side of the equation (6.8) converge to the corresponding terms for μ.

Claim: For any test function $g\in C_0^1(\mathbb{R}^{2d}\times[0,T))$,

$$\int_0^t\langle\mu^h_s,\partial_s g+\boldsymbol{v}\cdot\nabla_x g+\lambda\mathbf{F}(\mu^h_s)\cdot\nabla_{\boldsymbol{v}}g\rangle\,ds\to\int_0^t\langle\mu_s,\partial_s g+\boldsymbol{v}\cdot\nabla_x g+\lambda\mathbf{F}(\mu_s)\cdot\nabla_{\boldsymbol{v}}g\rangle\,ds,$$

as $h\to 0+$.

The proof of claim. The above relation directly follows from the following strong result. For $t\in[0,T)$, we prove that

$$\begin{aligned}&\Big|\langle\mu^h_t,\partial_s g+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}g+\lambda\mathbf{F}(\mu^h_t)\nabla_{\boldsymbol{v}}g\rangle-\langle\mu_t,\partial_s g+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}g+\lambda\mathbf{F}(\mu_t)\nabla_{\boldsymbol{v}}g\rangle\Big|\\ &\le C_4h.\end{aligned}$$

Here $C_4=C_4(T,d,\lambda,\beta,P_0,m_0,m_2,g)$ is a positive constant.

Note that

$$|\langle\mu^h_t,\partial_s g+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}g\rangle-\langle\mu_t,\partial_s g+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}g\rangle|\le||\partial_s g+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}g||_{C^1}d(\mu^h_t,\mu_t)\le Ch,$$

and hence it is enough to show that

$$|\langle\mu^h_s,\lambda\mathbf{F}(\mu^h_s)\cdot\nabla_{\boldsymbol{v}}g\rangle-\langle\mu_s,\lambda\mathbf{F}(\mu^h_s)\cdot\nabla_{\boldsymbol{v}}g\rangle|\le Ch. \tag{6.9}$$

By definition of $\mathbf{F}$, we have

$$\begin{aligned}
&|\langle \mu_s^h, \lambda \mathbf{F}(\mu_s^h)\cdot\nabla_{\boldsymbol{v}} g\rangle - \langle \mu_s, \lambda\mathbf{F}(\mu_s)\cdot\nabla_{\boldsymbol{v}} g\rangle| \\
\le\ & \lambda\Big|\int_{\mathbb{R}^{2d}} (\mathbf{F}(\mu_s^h)-\mathbf{F}(\mu_t))\cdot\nabla_{\boldsymbol{v}} g\ \mu_s^h(d\boldsymbol{x},d\boldsymbol{v})\Big| \\
&+\lambda\ \Big|\int_{\mathbb{R}^{2d}} [\mathbf{F}(\mu_t)\cdot\nabla_{\boldsymbol{v}} g]\mu_s^h(d\boldsymbol{x},d\boldsymbol{v}) - \int_{\mathbb{R}^{2d}} [\mathbf{F}(\mu_t)\cdot\nabla_{\boldsymbol{v}} g]\ \mu_s(d\boldsymbol{x},d\boldsymbol{v})\Big| \\
:=\ & \mathcal{J}_1+\mathcal{J}_2.
\end{aligned}$$

We next estimate $\mathcal{J}_i,\ \ i=1,2$ as follows.

Case 1 ($\mathcal{J}_1$): We use Lemma 5.4 and Lemma 5.7 to see that

$$\begin{aligned}
|\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_t)-\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_t^h)| &\le |\mathbf{a}(\boldsymbol{x},\mu_t)-\mathbf{a}(\boldsymbol{x},\mu_t^h)|+|b(\boldsymbol{x},\mu_t)-b(\boldsymbol{x},\mu_t^h)||\boldsymbol{v}| \\
&\le d\max\{P(t),\beta P(t)+1\}d(\mu_t,\mu_t^h)+\max\{1,\beta\}P(t)d(\mu_t,\mu_t^h) \\
&\le Cd(\mu_t,\mu_t^h) \\
&\le Ch.
\end{aligned}$$

This yields

$$\mathcal{J}_1 \le m_0\lambda Ch.$$

Case 2 ($\mathcal{J}_2$): In this case, we use Lemma 5.3 to obtain

$$\begin{aligned}
&||\mathbf{F}(\mu_t)\cdot\nabla_{\boldsymbol{v}} g||_{L^\infty} \le |\mathbf{F}(\mu_t)||\nabla_{\boldsymbol{v}} g| \le C||\nabla_{\boldsymbol{v}} g||_{L^\infty}\Big[(m_0m_2)^{\frac{1}{2}}+m_0P(t)\Big] \le C, \\
&|\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_t)\cdot\nabla_{\boldsymbol{v}} g(\boldsymbol{x},\boldsymbol{v}) - \mathbf{F}(\boldsymbol{y},\boldsymbol{v}_*,\mu_t)\cdot\nabla_{\boldsymbol{v}} g(\boldsymbol{y},\boldsymbol{v}_*)| \\
&\le |\mathbf{F}(\boldsymbol{x},\boldsymbol{v},\mu_t)-\mathbf{F}(\boldsymbol{y},\boldsymbol{v}_*,\mu_t)|||\nabla_{\boldsymbol{v}} g||_{L^\infty} + ||\mathbf{F}||_{L^\infty}|\nabla_{\boldsymbol{v}} g(\boldsymbol{x},\boldsymbol{v})-\nabla_{\boldsymbol{v}} g(\boldsymbol{y},\boldsymbol{v}_*)| \\
&\le \Big\{2\Big[\beta\Big((m_0m_2)^{\frac{1}{2}}+P(t)\Big)+m_0\Big]||\nabla_{\boldsymbol{v}} g||_{L^\infty}+||\mathbf{F}||_{L^\infty}||g||_{C^2}\Big\}|(\boldsymbol{x},\boldsymbol{v})-(\boldsymbol{y},\boldsymbol{v}_*)| \\
&\le C|(\boldsymbol{x},\boldsymbol{v})-(\boldsymbol{y},\boldsymbol{v}_*)|.
\end{aligned}$$

Therefore, thanks to Remark 5.2 (3), the term $\mathcal{J}_2$ can be estimated as follows.

$$\mathcal{J}_2 \le Cd(\mu_t,\mu_t^h) \le Ch.$$

Hence Case 1 and 2 verify that μ satisfies the equation (5.3).

(Uniqueness part): Let $\nu\in L^\infty([0,T);\mathcal{M}(\mathbb{R}^{2d})$ be a measure valued solution to (5.2) corresponding to same initial data μ_0 with a compact support for each time slice:

$$\mathrm{spt}(\nu_t)\subset B_{R(t)}(0)\times B_{P(t)}(0).$$

Then we apply the stability estimate in Proposition 5.10 to find

$$d(\mu_t,\nu_t)=0, \qquad \text{i.e.} \qquad \mu_t=\nu_t, \qquad \forall\ t>0.$$

This yields $\mu=\nu$. □

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