SIAM J. NUMER ANAL
Vol. 33, No. 4, pp. 1597–1621, August 1996

PROJECTION METHOD II: GODUNOV–RYABENKI ANALYSIS*

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Abstract. This is the second of a series of papers on the subject of projection methods for viscous incompressible flow calculations. The purpose of the present paper is to explain why the accuracy of the velocity approximation is not affected by (1) the numerical boundary layers in the approximation of pressure and the intermediate velocity field and (2) the noncommutativity of the projection operator and the laplacian. This is done by using a Godunov–Ryabenki type of analysis in a rigorous fashion. By doing so, we hope to be able to convey the message that normal mode analysis is basically sufficient for understanding the stability and accuracy of a finite-difference method for the Navier–Stokes equation even in the presence of boundaries. As an example, we analyze the second-order projection method based on pressure increment formulations used by van Kan and Bell, Colella, and Glaz. The leading order error term in this case is of $O(\Delta t)$ and behaves as high frequency oscillations over the whole domain, compared with the $O(\Delta t^{1/2})$ numerical boundary layers found in the second-order Kim–Moin method.

Key words. viscous incompressible flows, projection method, convergence, numerical boundary layers, Godunov–Ryabenki analysis

AMS subject classifications. 65M06, 76M20

1. Introduction.

This is the second of a series of papers on the subject of projection methods for viscous incompressible flow calculations. The purpose of the present paper is to explain why the accuracy of the velocity approximation is not affected by (1) the numerical boundary layers in the approximation of pressure and the intermediate velocity field and (2) the noncommutativity of the projection operator and the laplacian. This is done by using a Godunov–Ryabenki type of analysis in a rigorous fashion. By doing so, we hope to be able to convey the message that normal mode analysis is basically sufficient for understanding the stability and accuracy of a finite-difference method for Navier–Stokes equations even with the presence of boundaries.

Projection method in the presence of solid boundaries has been the focus of much discussion. The main issue is whether the accuracy in the interior of the domain is polluted by large errors made at the boundary from imposing inconsistent boundary conditions. Indeed a crude analysis [3, 1] suggests that both the formally first- and second-order accurate (this is the accuracy for periodic problems) projection methods could deteriorate to 1/4-order accurate when solid boundaries are present. This is due to the fact that in this case, the projection operator no longer commutes with various other operators involved, as was the case with periodic boundary conditions. On the other hand, numerical evidence seems to indicate that the projected velocity field has full accuracy all the way up to the boundary [3, 1]. The mechanism according to which the full accuracy is retained has been a mystery for more than 25 years.

Several issues were resolved in our previous paper [4] where we characterized explicitly the structure of the numerical boundary layers in the first-order projection method and the second-order Kim and Moin method. As a consequence we proved that the projected velocity has full accuracy (namely first-order for the formally first-order method and second order for Kim and Moin's method) all the way up to the boundary. We also studied the effect on the overall accuracy of choosing different numerical boundary conditions at the projection step.

*Received by the editors March 14, 1994; accepted for publication (in revised form) October 19, 1994.

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In the present paper, we explain the numerical phenomena mentioned above by resorting to a classical tool in numerical analysis: the normal mode analysis. With boundary conditions, this is usually referred to as the Godunov–Ryabenki analysis. As we will see, there are two principle factors which contribute to the full accuracy of the projected velocity field. The first is that the boundary layer mode created by the inconsistent boundary condition is orthogonal to the space of divergence-free vector fields. Consequently, the projected velocity field does not contain any numerical boundary layers. The second is that the commutator terms (resulting from the noncommutativity of the projection operator $\mathcal{P}$ with the laplacian, etc.) have a very special structure (see §3.3). As a result, although the magnitude of the commutator terms can be quite large, it does not accumulate since the approximate evolution operator acts efficiently to suppress it.

Compared with other approaches, normal mode analysis has the advantage of being much more explicit. As an example, we analyze the second-order projection method based on pressure increment formulations used by van Kan and Bell, Colella, and Glaz. The leading order error term in this case is of $O(\Delta t)$ and behaves as high frequency oscillations over the whole domain, in contrast to the $O(\Delta t^{1/2})$ numerical boundary layers found in the second-order Kim–Moin method.

The standard calculation of the normal mode analysis was carried out in [11]. These results give explicit and concrete information about the behavior of the classical projection method. Nevertheless, [11] did not go further to the full nonlinear equations and identify the two main factors mentioned previously. Consequently, the result of [11] was often viewed as being restricted to a linear model and not rigorous. Indeed for hyperbolic equations there is a gap between the predictions of normal mode analysis and the true behavior of a numerical method. Identifying that gap was a major task of classical numerical analysis [15, 7, 12, 13, 5]. In this paper, we answer that criticism by translating the predictions of normal mode analysis into rigorous theorems. So although the main result in this paper can be obtained using other approaches [4, 14, 16] (indeed sharper and more general results were proved in [4, 14, 16]), we feel the proof we present in this paper is more explicit and addresses directly the issues that have been puzzling for quite some time.

The rigorous side of the normal mode analysis can be understood as follows. We know from Strang's theorem that for general nonlinear problems, as long as the exact solutions are smooth, L^2-stability of the linearized scheme implies convergence in the L^∞-norm for the full nonlinear problem with maximum accuracy. In his paper [17], Strang only dealt with 2-level explicit schemes without boundary. The generalization to multilevel scheme in the presence of boundary was done by Michelson [10]. For parabolic equations, this can be reduced even further to the study of the (frozen coefficient) leading-order equations which are then amenable to normal mode analysis. This is the content of an earlier result of F. John [8] which also deals only with the boundary-free case. The generalization to problems involving boundaries was recently done by Kreiss and Wu [20]. In the present context, the problem is reduced to the study of the Stokes equation. We emphasize that we will not actually use Strang's trick of constructing asymptotic solutions with high accuracy, as a device to overcome the difficulties in obtaining an L^∞ estimate from L^2-stability. We believe that for equations of parabolic type, this is not necessary and should be avoided since it usually requires far more regularity than is actually needed. To implement the program described above, we use a discrete semigroup formulation which allows us to fully explore the regularizing effect of the parabolic equations.

This paper is organized as follows. In §2, we review the Godunov–Ryabenki analysis for the first-order projection method. In §3, we prove our main theorem which basically asserts that everything predicted by the Godunov–Ryabenki analysis for the Stokes equation holds for the full Navier–Stokes equation. Since the proof is a bit technical, an outline is given in the beginning of §3. In §4, we study the second-order projection methods based on pressure increment formulation using normal mode analysis. Finally, the Appendix contains the proof of some technical results needed in §3.

2. Godunov–Ryabenki analysis. In this section, we review the standard Godunov–Ryabenki analysis. We begin by writing down the viscous incompressible Navier–Stokes equation (NSE):

$$\begin{cases} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \Delta \boldsymbol{u}\,, \\ \nabla\cdot\boldsymbol{u} = 0\,, \end{cases} \tag{2.1}$$

on a domain Ω. We will focus on the most commonly used boundary condition, the no-slip boundary condition:

$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega\,. \tag{2.2}$$

One way of solving (2.1) is the backward Euler scheme:

$$\begin{cases} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} + (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n + \nabla p^{n+1} = \Delta \boldsymbol{u}^{n+1}\,, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0, \end{cases} \tag{2.3}$$

plus some spatial discretization and the boundary condition (2.2). However, this is a rather inefficient method since at each time step, one has to solve a coupled system of Stokes-type equations for $(\boldsymbol{u}^{n+1}, p^{n+1})$.

Projection method is a way of discretizing (2.1) in time so that the computation of velocity and pressure becomes decoupled. As a result, at each time step, we only need to solve a few Poisson-type equations separately for velocity and pressure, instead of the coupled system (2.3). This is done by first ignoring the incompressibility constraint, computing an intermediate velocity field $\boldsymbol{u}^*$ using the momentum equation, and then projecting $\boldsymbol{u}^*$ back to the space of incompressible vector fields to obtain $\boldsymbol{u}^{n+1}$ and p^{n+1}. We write the analogous projection method of (2.3) as follows [2, 18]:

Step 1. the evolution step

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} + (\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n = \Delta \boldsymbol{u}^*\,, \\ \boldsymbol{u}^* = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Step 2. the projection step

$$\begin{cases} \boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \Delta t\, \nabla p^{n+1}\,, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0\,, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega\,. \end{cases} \tag{2.5}$$

Equations (2.5) are equivalent to solving a Poisson equation for pressure together with a Neumann boundary condition

$$\begin{cases} \Delta p^{n+1} = \dfrac{\nabla\cdot\boldsymbol{u}^*}{\Delta t}\,, \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = 0 \qquad \text{on } \partial\Omega\,. \end{cases} \tag{2.6}$$

In this case (2.5) is simply the standard Helmholtz decomposition for the vector field $\boldsymbol{u}^*$. One could argue that other types of boundary conditions for (2.6) are equally plausible. The effect of choosing different numerical boundary conditions is discussed in [4]. Notice that the boundary condition for pressure in (2.6) is inconsistent with the Navier–Stokes equation (NSE) (2.1)–(2.2). If we take the inner product of (2.1) with the unit normal vector at $\partial\Omega$, we arrive at

$$\frac{\partial p}{\partial \boldsymbol{n}} = \boldsymbol{n}\cdot\Delta\boldsymbol{u} \qquad \text{on } \partial\Omega,$$

which is in general not zero. Therefore we expect that numerical boundary layers must be present if the method converges sufficiently strongly.

In the presence of physical boundaries, the projection method exhibits a number of interesting numerical phenomena including numerical boundary layers and boundary-excited high frequency oscillations. This made the task of analyzing such methods much more difficult than the periodic case. However, most of these phenomena can be understood by studying a simple model problem, the linear Stokes equation in a channel, for which an exact solution can be explicitly obtained, even for the numerical scheme.

Consider the Stokes equation on $\Omega = [-1, 1] \times [0, 2\pi]$:

$$\begin{cases} \partial_t \boldsymbol{u} + \nabla p = \Delta \boldsymbol{u}, \\ \nabla\cdot\boldsymbol{u} = 0. \end{cases} \tag{2.7}$$

We impose a periodic boundary condition at the boundaries $\{y = 0\}$ and $\{y = 2\pi\}$, and a no-slip boundary condition at the boundaries $\{x = -1\}$ and $\{x = 1\}$. After a Fourier transform in the y variable, the problem is reduced to a family of one-dimensional problems indexed by $k \in Z$:

$$\begin{cases} \partial_t u + \partial_x p = (\partial_x^2 - k^2)u, \\ \partial_t v + ikp = (\partial_x^2 - k^2)v, \\ \partial_x u + ikv = 0, \\ u(\pm 1, t) = v(\pm 1, t) = 0. \end{cases} \tag{2.8}$$

Here and in what follows, we use the notation $\boldsymbol{u} = (u, v)$.

We will use the following notation in the rest of the paper:

$$\Delta_k \boldsymbol{u} = (\partial_x^2 - k^2)\boldsymbol{u}, \qquad \nabla_k p = \begin{pmatrix} \partial_x p \\ ikp \end{pmatrix},$$

$$\nabla_k\cdot\boldsymbol{u} = \partial_x u + ikv, \qquad \operatorname{curl}_k \boldsymbol{u} = -iku + \partial_x v,$$

$$u_k(x) = \int_0^{2\pi} u(x, y)e^{-iky}\, dy,$$

and for two complex vector-valued functions $\boldsymbol{f}$ and $\boldsymbol{g}$, $\boldsymbol{f} = (f_1, f_2)$, $\boldsymbol{g} = (g_1, g_2)$ (these are functions of x), we define the inner product:

$$\langle \boldsymbol{f}(x), \boldsymbol{g}(x)\rangle = \int_{-1}^{1} (\bar{f}_1(x)g_1(x) + \bar{f}_2(x)g_2(x))\, dx,$$

where overbar means complex conjugate.

The normal mode analysis of this problem and the associated projection method (2.4)–(2.5) was carried out [11]. Following is a review of their results. We will use the same notations.

The normal mode solutions of (2.8) are of the form

$$(\boldsymbol{u}, p)(x,t) = e^{\sigma t}\,(\widehat{\boldsymbol{u}}, \widehat{p})(x)\,. \tag{2.9}$$

For these solutions, (2.8) becomes

$$\begin{cases} \sigma\widehat{\boldsymbol{u}} + \nabla_k \widehat{p} = \Delta_k \widehat{\boldsymbol{u}}\,, \\ \nabla_k \cdot \widehat{\boldsymbol{u}} = 0\,, \\ \widehat{\boldsymbol{u}}(\pm 1) = 0\,. \end{cases} \tag{2.10}$$

This is a system of linear ordinary differential equations (ODEs) with constant coefficients whose solutions can be found exactly. The symmetric solutions are

$$\begin{cases} \widehat{u}(x) = \cos\mu x - \cos\mu \dfrac{\cosh kx}{\cosh k}\,, \\[2ex] \widehat{v}(x) = \dfrac{\mu}{ik}\sin\mu x + \dfrac{1}{i}\cos\mu \dfrac{\sinh kx}{\cosh k}\,, \\[2ex] \widehat{p}(x) = \dfrac{\sigma}{k}\cos\mu \dfrac{\sinh kx}{\cosh k}, \end{cases} \tag{2.11}$$

where μ satisfies

$$\mu \tan\mu + k\tanh k = 0 \tag{2.12}$$

and

$$\sigma = -k^2 - \mu^2\,. \tag{2.13}$$

In the interval $((j-\frac{1}{2})\pi, (j+\frac{1}{2})\pi)$, there is a unique solution μ_{jk} to (2.12). We will denote the solution in (2.11) as $\widehat{u}_{jk}$, $\widehat{v}_{jk}$, and $\widehat{p}_{jk}$, etc.

The antisymmetric solutions are

$$\begin{cases} \widehat{u}(x) = \sin\mu x - \sin\mu \dfrac{\sinh kx}{\sinh k}\,, \\[2ex] \widehat{v}(x) = -\dfrac{\mu}{ik}\cos\mu x + \dfrac{1}{i}\sin\mu \dfrac{\cosh kx}{\sinh k}\,, \\[2ex] \widehat{p}(x) = \dfrac{\sigma}{k}\sin\mu \dfrac{\cosh kx}{\sinh k}, \end{cases} \tag{2.14}$$

where μ satisfies

$$\mu\cot\mu - k\coth k = 0 \tag{2.15}$$

and $\sigma = -k^2 - \mu^2$. For (2.8) the projection method (2.4)–(2.5) takes the following form:

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} = \Delta_k \boldsymbol{u}^*\,, \\[2ex] \boldsymbol{u}^*(\pm 1, t) = 0\,, \end{cases} \tag{2.16}$$

$$\begin{cases} \boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \Delta t\,\nabla_k p^{n+1}\,, \\ \nabla_k \cdot \boldsymbol{u}^{n+1} = 0\,, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}(\pm 1, t) = 0\,. \end{cases} \tag{2.17}$$

The normal mode solutions of these equations are of the form

$$(\boldsymbol{u}^n, p^n) = \kappa^n (\widetilde{\boldsymbol{u}}, \widetilde{p}), \qquad \boldsymbol{u}^* = \kappa^{n+1} \widetilde{\boldsymbol{u}}^*. \tag{2.18}$$

Again the set of equations obtained by substituting (2.18) into (2.16) and (2.17) can be solved and we get the following.

Symmetric modes:

(2.19)
$$\begin{cases} \widetilde{u}(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k}, \\ \widetilde{v}(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}, \\ \widetilde{u}^*(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k} - \beta\Delta t\cos\widetilde{\mu}\left(\dfrac{\cosh\lambda x}{\cosh\lambda} - \dfrac{\cosh kx}{\cosh k}\right), \\ \widetilde{v}^*(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k} - ik\beta\Delta t\cos\widetilde{\mu}\left(\dfrac{1}{\lambda}\dfrac{\sinh\lambda x}{\cosh\lambda} - \dfrac{1}{k}\dfrac{\sinh kx}{\cosh k}\right), \\ \widetilde{p}(x) = -\beta\cos\widetilde{\mu}\left(\dfrac{1}{\lambda}\dfrac{\sinh\lambda x}{\cosh\lambda} - \dfrac{1}{k}\dfrac{\sinh kx}{\cosh k}\right), \end{cases}$$

where

$$\lambda = (k^2 + \Delta t^{-1})^{1/2}, \qquad \beta = -k^2 - \widetilde{\mu}^2, \tag{2.20}$$

and $\widetilde{\mu}$, β, and λ satisfy

$$\widetilde{\mu}\tan\widetilde{\mu} + k\tanh k = \beta k\,\Delta t\left(\tanh k - \frac{k}{\lambda}\tanh\lambda\right). \tag{2.21}$$

In each interval $((j - \frac{1}{2})\pi, (j + \frac{1}{2})\pi)$, there is a unique solution $\widetilde{\mu}_{jk}$, which gives rise to β_{jk} and λ_k. We will denote the solution in (2.19) as $\widetilde{u}_{jk}$, $\widetilde{v}_{jk}$, $\widetilde{u}^*_{jk}$, $\widetilde{v}^*_{jk}$, and $\widetilde{p}_{jk}$. It is easy to check that

$$\kappa_{j,k} = \frac{1}{1 - \beta_{j,k}\Delta t}. \tag{2.22}$$

Antisymmetric modes:

(2.23)
$$\begin{cases} \widetilde{u}(x) = \sin\widetilde{\mu}x - \sin\widetilde{\mu}\dfrac{\sinh kx}{\sinh k}, \\ \widetilde{v}(x) = -\dfrac{\mu}{ik}\cos\widetilde{\mu}x + \dfrac{1}{i}\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k}, \\ \widetilde{u}^*(x) = \sin\widetilde{\mu}x - \sin\widetilde{\mu}\dfrac{\sinh kx}{\sinh k} - \beta\Delta t\sin\widetilde{\mu}\left(\dfrac{\sinh\lambda x}{\sinh\lambda} - \dfrac{\sinh kx}{\sinh k}\right), \\ \widetilde{v}^*(x) = -\dfrac{\widetilde{\mu}}{ik}\cos\widetilde{\mu}x + \dfrac{1}{i}\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k} - \beta\Delta t\sin\widetilde{\mu}\left(\dfrac{1}{\lambda}\dfrac{\cosh\lambda x}{\sinh\lambda} - \dfrac{1}{k}\dfrac{\cosh kx}{\sinh k}\right), \\ \widetilde{p}(x) = -\beta\sin\widetilde{\mu}\left(\dfrac{1}{\lambda}\dfrac{\cosh\lambda x}{\sinh\lambda} - \dfrac{1}{k}\dfrac{\cosh kx}{\sinh k}\right), \end{cases}$$

where $\lambda = (k^2 + \Delta t^{-1})^{1/2}$ and $\beta = -k^2 - \widetilde{\mu}^2$ and satisfies

$$\widetilde{\mu}\cot\widetilde{\mu} - k\coth k = \beta k\,\Delta t\left(\coth k - \frac{k}{\lambda}\coth\lambda\right). \tag{2.24}$$

What was said about the symmetric modes also holds for the antisymmetric ones.

It is clear from these formulas that the numerical scheme (2.16)–(2.17) has a couple of new modes not shared by the original partial differential equation (PDE). These are the boundary layer modes represented by λ. It is also important to notice that the boundary layer mode in $(\widetilde{u}^*, \widetilde{v}^*)$ is an exact gradient. Therefore it does not contribute to $(\widetilde{u}, \widetilde{v})$ which is the divergence-free part of $(\widetilde{u}^*, \widetilde{v}^*)$. This is the primary reason why the projected velocity still has $O(\Delta t)$ accuracy up to the boundary, whereas the approximate pressure is only $O(\Delta t^{1/2})$ accurate at the boundary.

From another point of view, it is well known that the linearized Navier–Stokes equation—the Orr–Sommerfeld equation—is of fourth order. In comparison, the normal mode equation of (2.16)–(2.17) is of sixth order with small coefficient Δt at its leading order. This can be most clearly seen from the equations for pressure. Equation (2.10) implies that

$$\Delta_k p = 0, \tag{2.25}$$

whereas (2.16)–(2.17) gives

$$(I - \Delta t \Delta_k)\, \Delta_k p = 0\,. \tag{2.26}$$

This clearly indicates that we are dealing with a singular perturbation problem and we expect to have a boundary layer of width $O(\Delta t^{1/2})$ in the approximation of pressure. We remark that (2.26) was derived previously by Gresho [6].

In the following we will often omit the subscripts j, k for notational simplicity. We will also use the following notations:

$$|||\boldsymbol{u}|||_{H^\ell} = \sup_{0 \le t \le T} \|\boldsymbol{u}(\cdot, t)\|_{H^\ell} \qquad \text{and} \qquad |||\boldsymbol{u}^n|||_{H^\ell} = \sup_{0 \le t^n \le T} \|\boldsymbol{u}^n\|_{H^\ell},$$

where $t^n = n\Delta t$. We will use $\|\cdot\|$ to denote L^2-norm in space.

3. Convergence results.

3.1. Statement of the theorem and outline of the proof. Let $T > 0$ be fixed and $\mathcal{S}(\cdot)$ be the solution operator of the linear Stokes equation (2.7). The main result we want to prove is the following theorem.

THEOREM 1. *Assume that $\boldsymbol{u} \in L^\infty([0, T]; H^5)$ and $\partial_t \mathcal{S}(t)\boldsymbol{u} \in L^\infty([0, T]; H^3)$ is the solution of Navier–Stokes equation* (2.1)–(2.2) *on domain $\Omega = [-1, 1] \times [0, 2\pi]$ with no-slip boundary condition at $x = \pm 1$ and periodic boundary condition at $y = 0, 2\pi$. Then we have*

$$\max_{0 \le t^n \le T} \|\boldsymbol{u}(t^n, \cdot) - \boldsymbol{u}^n(\cdot)\| \le C \Delta t\, (|||\boldsymbol{u}|||_{H^5}^2 + |||\partial_t \mathcal{S}(t)\boldsymbol{u}|||_{H^3}), \tag{3.1}$$

where C is independent of Δt and $\boldsymbol{u}$.

Outline of the proof. We assume a priori that

$$\max_{0 \le t^n \le T} (\|\boldsymbol{u}^n\|_{H^1 \cap L^\infty} + \|\partial_y \boldsymbol{u}^n\|_{H^1 \cap L^\infty}) \le M, \tag{3.2}$$

where M is a constant independent of Δt. There are two ways of proving this a priori estimate. One is given in [4]. The other is a direct proof based on the semigroup type of argument that we will use later. Here we will not give the details of that since it is rather standard and tedious.

Write (2.4) as

$$\boldsymbol{u}^* = (I - \Delta t \Delta)^{-1} (\boldsymbol{u}^n - \Delta t (\boldsymbol{u}^n \cdot \nabla) \boldsymbol{u}^n) \tag{3.3}$$

and denote by $\mathcal{P}$ the projection operator in the Helmholtz decomposition:

$$\boldsymbol{u} = \boldsymbol{v} + \nabla g\,, \qquad \boldsymbol{v} = \mathcal{P}\boldsymbol{u}, \tag{3.4}$$

where

$$\nabla\cdot \boldsymbol{v} = 0\,, \qquad \boldsymbol{v}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega\,. \tag{3.5}$$

Then the projection method can be written as

$$\boldsymbol{u}^{n+1} = \mathcal{P}(I - \Delta t\Delta)^{-1}(\boldsymbol{u}^n - \Delta t(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n)\,. \tag{3.6}$$

On the other hand, we can write (2.1) in an integral form:

$$\boldsymbol{u}(t) = \mathcal{S}(t)\boldsymbol{u}_0 - \int_0^t \mathcal{S}(t-\tau)(\boldsymbol{u}(\tau)\cdot\nabla)\boldsymbol{u}(\tau)\,d\tau\,. \tag{3.7}$$

We then have

$$\boldsymbol{u}(t^{n+1}) = \mathcal{S}(\Delta t)\boldsymbol{u}(t^n) - \int_{t^n}^{t^{n+1}} \mathcal{S}(t^{n+1}-\tau)(\boldsymbol{u}(\tau)\cdot\nabla)\boldsymbol{u}(\tau)\,d\tau\,. \tag{3.8}$$

To compare (3.8) with (3.6), let us denote

$$\mathcal{S}_{\Delta t} = \mathcal{P}(I - \Delta t\Delta)^{-1}\mathcal{P}\,. \tag{3.9}$$

Equation (3.6) now becomes

$$\boldsymbol{u}^{n+1} = \mathcal{S}_{\Delta t}\,\boldsymbol{u}^n - \Delta t\,\mathcal{S}_{\Delta t}\,(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n - \Delta t\,\mathcal{P}(I-\Delta t\Delta)^{-1}(I-\mathcal{P})(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n\,. \tag{3.10}$$

Let $\boldsymbol{e}^n = \boldsymbol{u}(t^n) - \boldsymbol{u}^n$. From (3.8) and (3.10), we obtain

$$\begin{aligned} \boldsymbol{e}^{n+1} &= \mathcal{S}_{\Delta t}\,\boldsymbol{e}^n - \Delta t\mathcal{S}_{\Delta t}\,[(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{e}^n + (\boldsymbol{e}^n\cdot\nabla)\boldsymbol{u}(t^n)] \\ &\quad + E^n + \Delta t\,\mathcal{P}(I-\Delta t\Delta)^{-1}(I-\mathcal{P})(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} E^n &\equiv (\mathcal{S}(\Delta t) - \mathcal{S}_{\Delta t})\big[\boldsymbol{u}(t^n) - \Delta t\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}(t^n)\big] \\ &\quad + \int_{t^n}^{t^{n+1}} \mathcal{S}(t^{n+1}-\tau)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}(\tau)\,d\tau - \Delta t\mathcal{S}(\Delta t)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}(t^n)\,. \end{aligned} \tag{3.12}$$

The last term in (3.11) is the commutator term and is the main source of difficulty. Its magnitude can be of order Δt so standard estimates would give a $O(1)$ bound for the error in maximum norm. To obtain a better estimate, we need to take into account the cumulative effect of the approximate solution operator.

The recurrence formula (3.11) gives

$$\begin{aligned} \boldsymbol{e}^n &= \mathcal{S}_{\Delta t}^n\,\boldsymbol{e}^0 - \Delta t\sum_{\ell=0}^{n-1}\mathcal{S}_{\Delta t}^{n-\ell}\,[(\boldsymbol{u}^\ell\cdot\nabla)\boldsymbol{e}^\ell + (\boldsymbol{e}^\ell\cdot\nabla)\boldsymbol{u}(t^\ell)] \\ &\quad + \sum_{\ell=0}^{n-1}\mathcal{S}_{\Delta t}^{n-\ell}\,E^\ell + \Delta t\sum_{\ell=0}^{n-1}\mathcal{S}_{\Delta t}^{n-\ell}\,\mathcal{P}(I-\Delta t\Delta)^{-1}(I-\mathcal{P})(\boldsymbol{u}^\ell\cdot\nabla)\boldsymbol{u}^\ell\,. \end{aligned} \tag{3.13}$$

This is the basic equation we will work with. In the following we will need some technical lemmas whose proofs will be given in the Appendix.

LEMMA 1. *Assume that* $\boldsymbol{a}\in W^{1,\infty}$, $\nabla\cdot\boldsymbol{a} = 0$, *and* $\boldsymbol{a}\cdot\boldsymbol{n} = 0$, *on* $\partial\Omega$. *Then*

$$\|\mathcal{S}_{\Delta t}^m\,(\boldsymbol{a}\cdot\nabla)\boldsymbol{u}\| \le C(m\Delta t)^{-3/4}\,\|\boldsymbol{a}\|_{L^\infty}\,\|\boldsymbol{u}\|, \tag{3.14}$$

$$\|\mathcal{S}_{\Delta t}^m\,(\boldsymbol{u}\cdot\nabla)\boldsymbol{a}\| \le C(m\Delta t)^{-3/4}\,\|\boldsymbol{a}\|_{W^{1,\infty}}\,\|\boldsymbol{u}\|, \tag{3.15}$$

where C *is independent of* m, Δt, $\boldsymbol{a}$, *and* $\boldsymbol{u}$.

Applying Lemma 1, we get

$$
\begin{aligned}
\|e^n\| &\le \|e^0\| + C\Delta t \sum_{\ell=0}^{n-1} ((n-\ell)\Delta t)^{-3/4} \|e^\ell\| + \sum_{\ell=0}^{n-1} \|\mathcal{S}_{\Delta t}^{n-\ell} E^\ell\| \\
&\quad + \Delta t \sum_{\ell=0}^{n-1} \|\mathcal{S}_{\Delta t}^{n-\ell} \mathcal{P}(I-\Delta t\Delta)^{-1}(I-\mathcal{P})(u^\ell\cdot\nabla)u^\ell\| .
\end{aligned} \tag{3.16}
$$

The truncation error terms can be estimated using the following lemma.

LEMMA 2. *Let $\mathcal{S} = \mathcal{S}(\Delta t)$. Then we have*

$$\sum_{m=1}^{n} \|\mathcal{S}_{\Delta t}^{m} (\mathcal{S} - \mathcal{S}_{\Delta t}) u\| \le C\Delta t\, (\|u\|_{H^5} + \|\partial_t \mathcal{S}(t) u\|_{H^3}) , \tag{3.17}$$

$$\sum_{m=1}^{n} \|\mathcal{S}_{\Delta t}^{m} (\mathcal{S} - \mathcal{S}_{\Delta t})(u\cdot\nabla)u\| \le C\, \|u\|_{H^5}^2, \tag{3.18}$$

and

$$\left\| \int_{t^n}^{t^{n+1}} \mathcal{S}(t^{n+1}-\tau)(u\cdot\nabla u(\tau))\, d\tau - \Delta t \mathcal{S}(\Delta t)(u\cdot\nabla u(t^n)) \right\| \le C\Delta t^2\, \|\!|u|\!\|_{H^5}^2 . \tag{3.19}$$

Before going into the details of the proof, we remark that since both $\mathcal{S}(\Delta t)$ and $\mathcal{S}_{\Delta t}$ are self-adjoint operators in the space $L^2(\mathrm{div}) = \{u \in L^2(\Omega),\quad \nabla\cdot u = 0\}$, their complete sets of eigenfunctions $\{\widehat{u}_{jk}\, e^{iky}\}_{jk}$, $\{\widetilde{u}_{jk}\, e^{iky}\}_{jk}$ form complete orthogonal bases in $L^2(\mathrm{div})$.

3.2. Proof of Theorem 1. From here on, we will only deal with the symmetric modes. The reader can easily check that the argument works equally well when the antisymmetric modes are also taken into account.

The main problem is to estimate of the effect of the commutator terms. We write

$$J^n \equiv \mathcal{P}(I-\Delta t\Delta)^{-1}(1-\mathcal{P})(u^n\cdot\nabla)u^n = \sum_{j,k} \gamma_{jk}\, \widetilde{u}_{jk}(x)\, e^{iky} . \tag{3.20}$$

To compute the coefficient γ_{jk}, we have from the Helmholtz decomposition

$$(u^n\cdot\nabla)u^n = \mathcal{P}(u^n\cdot\nabla)u^n + \nabla q ,$$

where q satisfies

$$\begin{cases} \Delta q = \nabla\cdot((u^n\cdot\nabla)u^n) , \\[2mm] \dfrac{\partial q}{\partial n} = 0 \qquad \text{on } \partial\Omega . \end{cases} \tag{3.21}$$

Denote

$$w \equiv \mathcal{P}(I-\Delta t\Delta)^{-1}\nabla q . \tag{3.22}$$

Expand q and w as

$$q(x,y) = \sum_k q_k(x)\, e^{iky}, \qquad w(x,y) = \sum_k w_k(x)\, e^{iky} .$$

Then for each k we have

$$w_k(x) = \mathcal{P}(I-\Delta t\Delta_k)^{-1}\nabla_k q_k(x) , \tag{3.23}$$

$$\gamma_{jk} = \frac{\langle w_k(x)\, ,\, \widetilde{u}_{jk}(x)\rangle}{\langle \widetilde{u}_{jk}(x)\, ,\, \widetilde{u}_{jk}(x)\rangle} . \tag{3.24}$$

Let us compute

$$\begin{aligned}\langle w_k\,,\,\widetilde{u}_{jk}\rangle &= \langle\widetilde{u}_{jk}\,,\,(I-\Delta t\Delta_k)^{-1}\nabla_k q_k\rangle = \langle\nabla_k q_k\,,\,(I-\Delta t\Delta_k)^{-1}\widetilde{u}_{jk}\rangle\\ &= \int_{-1}^{1}\overline{\nabla_k q_k(x)}\cdot(I-\Delta t\Delta_k)^{-1}\widetilde{u}_{jk}\,dx\\ &= -\int_{-1}^{1} q_k(x)\,(\partial_x(I-\Delta t\Delta_k)^{-1}\\ &\qquad -(I-\Delta t\Delta_k)^{-1}\partial_x)\left(\cos\widetilde{\mu}x-\cos\widetilde{\mu}\frac{\cosh kx}{\cosh k}\right)dx\,.\end{aligned}$$

It is straightforward to compute

$$(I-\Delta t\Delta_k)^{-1}\left(\frac{\cos\widetilde{\mu}x}{\cos\widetilde{\mu}}-\frac{\cosh kx}{\cosh k}\right)=\kappa_{jk}\frac{\cos\widetilde{\mu}x}{\cos\widetilde{\mu}}-\frac{\cosh kx}{\cosh k}+(1-\kappa_{jk})\frac{\cosh\lambda x}{\cosh\lambda}$$

and

$$\begin{aligned}&(I-\Delta t\Delta_k)^{-1}\left(-\widetilde{\mu}\frac{\sin\widetilde{\mu}x}{\cos\widetilde{\mu}}-k\frac{\sinh kx}{\cosh k}\right)\\ &\quad=-\kappa_{jk}\,\widetilde{\mu}\,\frac{\sin\widetilde{\mu}x}{\cos\widetilde{\mu}}-k\frac{\sinh kx}{\cosh k}+(k\tanh k+\kappa_{jk}\,\widetilde{\mu}\tan\widetilde{\mu})\frac{\sinh\lambda x}{\sinh\lambda}\,.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\partial_x(I-\Delta t\Delta_k)^{-1}-(I-\Delta t\Delta_k)^{-1}\partial_x\right)\left(\frac{\cos\widetilde{\mu}x}{\cos\widetilde{\mu}}-\frac{\cosh kx}{\cosh k}\right)\\ &\quad=\left((1-\kappa_{jk})\lambda\tanh\lambda-k\tanh k-\kappa_{jk}\,\widetilde{\mu}\tan\widetilde{\mu}\right)\frac{\sinh\lambda x}{\sinh\lambda}\,.\end{aligned}$$

Using (2.21) we get

$$(1-\kappa_{jk})\lambda\tanh\lambda-k\tanh k-\kappa_{jk}\widetilde{\mu}\tan\widetilde{\mu}=\frac{-\beta_{jk}\kappa_{jk}}{\lambda_k}\tanh\lambda\,.$$

Hence we obtain

$$\langle w_k\,,\,\widetilde{u}_{jk}\rangle=\frac{-\beta_{jk}\kappa_{jk}\cos\widetilde{\mu}}{\lambda_k}\int_{-1}^{1}q_k(x)\frac{\sinh\lambda x}{\cosh\lambda}\,dx\,. \tag{3.25}$$

Using the fact that $\widetilde{u}_{jk}(x)$ is divergence-free, we have from integrating by parts that

$$\begin{aligned}\langle\widetilde{u}_{jk}\,,\,\widetilde{u}_{jk}\rangle &= -\frac{\widetilde{\beta}}{k^2}\int_{-1}^{1}\widetilde{u}_{jk}\cos\mu x\,dx\\ &= -\frac{\widetilde{\beta}}{k^2}\left(1+\frac{k^2-\mu^2}{2\mu(k^2+\mu^2)}\sin 2\mu-\frac{2k}{k^2+\mu^2}\cos^2\mu\tanh k\right).\end{aligned} \tag{3.26}$$

Substituting (3.25) and (3.26) back into (3.24), we obtain

$$|\gamma_{jk}|\le C\frac{k\,\kappa_{jk}}{\lambda}\left|\int_{-1}^{1}k\,q_k(x)\frac{\sinh\lambda x}{\cosh\lambda}\,dx\right|\,. \tag{3.27}$$

Next we write $\boldsymbol{u}^n = (u, v)$ and

(3.28) $$\nabla\cdot((\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n) = \partial_x^2(u^2) + 2\partial_x\partial_y(vu) + \partial_y^2(v^2) = \sum_k f_k(x)\, e^{iky}\,.$$

We have from (3.28) and (3.21) that

(3.29) $$\begin{cases} -\Delta_k q_k = f_k(x)\,, \\ \partial_x q_k(\pm 1) = 0\,. \end{cases}$$

Integrating by parts and using (3.29) gives

(3.30)
$$\begin{aligned}\int_{-1}^{1} kq_k(x)\frac{\sinh\lambda x}{\cosh\lambda}\,dx &= \lambda\Delta t(kq_k(1) + kq_k(-1)) + \Delta t\int_{-1}^{1}\Delta_k kq_k(x)\frac{\sinh\lambda x}{\cosh\lambda}\,dx \\ &= \lambda\Delta t(kq_k(1) + kq_k(-1)) - \Delta t\int_{-1}^{1} kf_k(x)\frac{\sinh\lambda x}{\cosh\lambda}\,dx\,.\end{aligned}$$

Since

$$\begin{aligned} ikf_k(x) &= ik\int_{-\pi}^{\pi}\left(\partial_x^2(u^2) + 2\partial_x\partial_y(vu) + \partial_y^2(v^2)\right)\, e^{-iky}\,dy \\ &= \partial_x^2(\partial_y u^2)_k + 2ik\partial_x(\partial_y(vu))_k - k^2(\partial_y v^2)_k\,,\end{aligned}$$

we have

$$\begin{aligned}\int_{-1}^{1} kf_k(x)\frac{\sinh\lambda x}{\cosh\lambda}\,dx &= \int\left(\lambda^2(\partial_y u^2)_k + 2i\lambda k(\partial_y(vu))_k - k^2(\partial_y v^2)_k\right)\frac{\sinh\lambda x}{\cosh\lambda}\,dx \\ &\le C\,\lambda_k\,\|\partial_y(\boldsymbol{u}^n\otimes\boldsymbol{u}^n)_k\|\,.\end{aligned}$$

Here we have used tensor product notation $\boldsymbol{u}\otimes\boldsymbol{u} \equiv \boldsymbol{u}\boldsymbol{u}^T$. Using Sobolev inequality, we obtain

$$k^2 q_k(1)^2 \le Ck^2\int_0^1 (q_k(x)^2 + (\partial_x q_k(x))^2)\,dx \le C\,\|\partial_y(\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n)_k\|^2\,.$$

Consequently we get

(3.31) $$\left|\int_{-1}^{1} kq_k(x)\frac{\sinh\lambda x}{\cosh\lambda}\,dx\right| \le C\,\lambda_k\,\Delta t\,(\|\partial_y(\boldsymbol{u}^n\otimes\boldsymbol{u}^n)_k\| + \|\partial_y(\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n)_k\|)\,.$$

Coming back to (3.27), we obtain

(3.32) $$|\gamma_{jk}| \le C\,k\Delta t\,\kappa_{jk}(\|\partial_y(\boldsymbol{u}^n\otimes\boldsymbol{u}^n)_k\| + \|\partial_y(\boldsymbol{u}^n\cdot\nabla\boldsymbol{u}^n)_k\|)\,.$$

Since

$$\mathcal{S}_{\Delta t}^m\, J^n = \sum_{j,k}\gamma_{jk}\,\kappa_{jk}^m\,\tilde{\boldsymbol{u}}_{jk}(x)\, e^{iky}$$

and

$$|\kappa_{jk}^{m+1}(j^2 + k^2)\Delta t| \le \frac{C}{m}\,, \qquad \sum_j \kappa_{jk}^{m+1}\sqrt{m\Delta t} \le C\,,$$

we have

$$
\begin{aligned}
\|\mathcal{S}_{\Delta t}^m J^n\|^2 &= \sum_{j,k} \gamma_{jk}^2 \kappa_{jk}^{2m} \|\boldsymbol{u}_{jk}\|^2 \\
&\le C \sum_{j,k} \kappa_{jk}^{2m+2} (j^2+k^2)\Delta t^2 (\|\partial_y(\boldsymbol{u}^n \otimes \boldsymbol{u}^n)_k\|^2 + \|\partial_y(\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n)_k\|^2) \\
&\le C \frac{\sqrt{\Delta t}}{m^{3/2}} \sum_{j,k} \kappa_{jk}^{m+1} (m\Delta t)^{1/2} (\|\partial_y(\boldsymbol{u}^n \otimes \boldsymbol{u}^n)_k\|^2 + \|\partial_y(\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n)_k\|^2) \\
&\le C \frac{\sqrt{\Delta t}}{m^{3/2}} \sum_{k} (\|\partial_y(\boldsymbol{u}^n \otimes \boldsymbol{u}^n)_k\|^2 + \|\partial_y(\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n)_k\|^2) \\
&\le C \frac{\sqrt{\Delta t}}{m^{3/2}} \left(\|\partial_y(\boldsymbol{u}^n \otimes \boldsymbol{u}^n)\|^2 + \|\partial_y(\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n)\|^2\right) .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sum_m \|\mathcal{S}_{\Delta t}^m J^n\| &\le C \Delta t^{1/4} \sum_{m \le T/\Delta t} m^{-3/4} (\|\partial_y(\boldsymbol{u}^n \otimes \boldsymbol{u}^n)\|^2 + \|\partial_y(\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n)\|^2) \\
&\le C \|\boldsymbol{u}^n\|_{H^1 \cap L^\infty}^2 \|\partial_y \boldsymbol{u}^n\|_{H^1 \cap L^\infty}^2 \le C M^2 .
\end{aligned} \tag{3.33}
$$

Together with (3.16)–(3.19), we obtain

$$
\|\boldsymbol{e}^n\| \le \|\boldsymbol{e}^0\| + C\Delta t \sum_{\ell=0}^{n-1} ((n-\ell)\Delta t)^{-3/4} \|\boldsymbol{e}^\ell\| + C\Delta t\, (|||\boldsymbol{u}|||_{H^5}^2 + |||\partial_t \mathcal{S}(t)\boldsymbol{u}|||_{H^3}) . \tag{3.34}
$$

By Gronwall's inequality, we get

$$
\|\boldsymbol{e}^n\| \le C\, \Delta t\, (|||\boldsymbol{u}|||_{H^5}^2 + |||\partial_t \mathcal{S}(t)\boldsymbol{u}|||_{H^3}) . \tag{3.35}
$$

This completes the proof of Theorem 1. □

3.3. Structure of the commutator term. It is of interest to find out the detailed structure of the commutator term. We start from the Helmholtz decomposition

$$
(I - \Delta t \Delta)^{-1} \nabla q = \mathcal{P}(I - \Delta t \Delta)^{-1} \nabla q + \nabla r = \boldsymbol{w} + \nabla r
$$

for some function r. Hence

$$
\nabla q = (I - \Delta t \Delta)\boldsymbol{w} + (I - \Delta t \Delta)\nabla r
$$

or

$$
\nabla(q - (I - \Delta t \Delta) r) = (I - \Delta t \Delta)\boldsymbol{w} .
$$

Taking *curl* on both sides gives

$$
(I - \Delta t \Delta) \operatorname{curl} \boldsymbol{w} = 0 .
$$

Therefore, for each k, we have

$$
\begin{aligned}
\boldsymbol{w}_k(x) &= \mathcal{P}(I - \Delta t \Delta_k)^{-1} \nabla_k q_k(x) , \\
(I - \Delta t \Delta_k) &\operatorname{curl}_k \boldsymbol{w}_k = 0 .
\end{aligned} \tag{3.36}
$$

The solution to (3.36) is given by

$$\operatorname{curl}_k \boldsymbol{w}_k = A\cosh\lambda x + B\sinh\lambda x$$

for some constant A and B. Hence $\boldsymbol{w}_k$ must be of the form

$$\boldsymbol{w}_k(x) = (\eta_k\,\boldsymbol{h}_k(x) + \bar{\eta}_k\,\bar{\boldsymbol{h}}_k(x)),$$

where

$$\boldsymbol{h}_k(x) = \begin{pmatrix} \frac{\cosh kx}{\cosh k} - \frac{\cosh\lambda x}{\cosh\lambda} \\ i\frac{\sinh kx}{\cosh k} - \frac{i\lambda}{k}\frac{\sinh\lambda x}{\cosh\lambda} \end{pmatrix}, \quad \bar{\boldsymbol{h}}_k(x) = \begin{pmatrix} \frac{\sinh kx}{\sinh k} - \frac{\sinh\lambda x}{\sinh\lambda} \\ i\frac{\cosh kx}{\sinh k} - \frac{i\lambda}{k}\frac{\cosh\lambda x}{\sinh\lambda} \end{pmatrix}. \tag{3.37}$$

To compute η_k, we use the orthogonality of e^{iky} for different k, and the orthogonality of $\boldsymbol{h}_k$ and $\bar{\boldsymbol{h}}_k$, to get

$$\begin{aligned}\eta_k \int |\boldsymbol{h}_k|^2 &= \int_{-1}^{1} \boldsymbol{h}_k\cdot(I - \Delta t\Delta_k)^{-1}\nabla_k q_k(x)\\ &= \int_{-1}^{1} \nabla_k q_k(x)\cdot(I - \Delta t\Delta_k)^{-1}\boldsymbol{h}_k\\ &= \int_{-1}^{1} q_k(x)\left(\partial_x(I - \Delta t\Delta_k)^{-1} - (I - \Delta t\Delta_k)^{-1}\partial_x\right)\left(\frac{\cosh kx}{\cosh k} - \frac{\cosh\lambda x}{\cosh\lambda}\right).\end{aligned}$$

It is straightforward to compute

$$\begin{aligned}&(I - \Delta t\Delta_k)^{-1}\left(\frac{\cosh kx}{\cosh k} - \frac{\cosh\lambda x}{\cosh\lambda}\right)\\ &\quad= \frac{\cosh kx}{\cosh k} - \left(1 + \frac{\tanh\lambda}{2\lambda\Delta t}\right)\frac{\cosh\lambda x}{\cosh\lambda} + \frac{1}{2\lambda\Delta t}\frac{x\sinh\lambda x}{\cosh\lambda}\end{aligned}$$

and

$$\begin{aligned}&(I - \Delta t\Delta_k)^{-1}\left(k\frac{\sinh kx}{\cosh k} - \lambda\frac{\sinh\lambda x}{\cosh\lambda}\right)\\ &\quad= k\frac{\sinh kx}{\cosh k} - \left(\frac{1}{2\Delta t} + k\tanh k\right)\frac{\sinh\lambda x}{\sinh\lambda} + \frac{1}{2\Delta t}\frac{x\cosh\lambda x}{\cosh\lambda}.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\partial_x(I - \Delta t\Delta_k)^{-1} - (I - \Delta t\Delta_k)^{-1}\partial_x\right)\left(\frac{\cosh kx}{\cosh k} - \frac{\cosh\lambda x}{\cosh\lambda}\right)\\ &\quad= \frac{1 - \lambda(2\lambda\Delta t + \tanh\lambda) + \lambda(1 + 2k\Delta t\tanh k)\coth\lambda}{2\lambda\Delta t}\,\frac{\sinh\lambda x}{\cosh\lambda}.\end{aligned}$$

Hence

$$\eta_k = c_k\int_{-1}^{1} q_k(x)\frac{\sinh\lambda x}{\cosh\lambda}\,dx \tag{3.38}$$

for some constant c_k. Similarly,

$$\bar{\eta}_k = \bar{c}_k\int_{-1}^{1} q_k(x)\frac{\cosh\lambda x}{\sinh\lambda}\,dx. \tag{3.39}$$

In summary, we have the following theorem.

THEOREM 2. *Let*

$$q = \sum_k q_k(x)\, e^{iky}\,.$$

Then we have

$$\mathcal{P}(I - \Delta t\,\Delta)^{-1}\nabla q = \sum_k [\eta_k\, \boldsymbol{h}_k(x) + \bar{\eta}_k\, \bar{\boldsymbol{h}}_k(x)]\, e^{iky}\,.$$

This result says that for each k, the commutator term lives in a two-dimensional space spanned by $\boldsymbol{h}_k$ and $\bar{\boldsymbol{h}}_k$.

4. Second-order projection method based on the pressure increment formulation. There are at least three different ways of getting formally second-order projection methods. These are, respectively, projection methods based on: (1) accurate boundary conditions for the intermediate velocity field [9]; (2) accurate pressure boundary conditions [11]; (3) pressure increment formulation [1, 19]. In [4], we proved second-order convergence of the first type of projection methods in L^∞-norms and characterized the numerical boundary layers. Similar results are expected to hold for the second type of projection methods. Here we will analyze the third type of projection methods which turns out to exhibit completely different numerical behavior.

We concentrate on the version used by Bell, Colella, and Glaz [1].

Step 1. the evolution step

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} + (\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2} + \nabla p^{n-1/2} = \Delta \dfrac{\boldsymbol{u}^* + \boldsymbol{u}^n}{2}\,, \\ \boldsymbol{u}^* = 0 \qquad \text{on } \partial\Omega\,. \end{cases} \tag{4.1}$$

Step 2. the projection step

$$\begin{cases} \boldsymbol{u}^* = \boldsymbol{u}^{n+1} + \Delta t\, \nabla(p^{n+1/2} - p^{n-1/2})\,, \\ \nabla\cdot\boldsymbol{u}^{n+1} = 0\,, \\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n} = 0 \qquad \text{on } \partial\Omega\,. \end{cases} \tag{4.2}$$

The cost of solving (4.1) and (4.2) is basically the same as for the first-order projection method. The nonlinear convection term $(\boldsymbol{u}^{n+1/2}\cdot\nabla)\boldsymbol{u}^{n+1/2}$ can be treated in many ways, such as the explicit Adams–Bashforth formula, $\frac{3}{2}(\boldsymbol{u}^n\cdot\nabla)\boldsymbol{u}^n - \frac{1}{2}(\boldsymbol{u}^{n-1}\cdot\nabla)\boldsymbol{u}^{n-1}$, which is the one used in use flux (slope)-limited finite difference methods.

To carry out the normal mode analysis, we will adopt the same geometry and boundary condition as we had earlier, and consider the linear Stokes equation. Equation (4.1) becomes

$$\begin{cases} \dfrac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} = \Delta_k \dfrac{\boldsymbol{u}^* + \boldsymbol{u}^n}{2} - \nabla_k p^n\,, \\ \boldsymbol{u}^*(\pm 1) = 0, \end{cases} \tag{4.3}$$

and (4.2) becomes

$$\begin{cases} \boldsymbol{u}^{n+1} - \boldsymbol{u}^* = -\Delta t\, \nabla_k q^n\,, \\ \nabla_k\cdot\boldsymbol{u}^{n+1} = 0\,, \\ p^{n+1} = p^n + q^{n+1}\,, \\ (\boldsymbol{u}^{n+1}\cdot\boldsymbol{n})(\pm 1) = 0\,. \end{cases} \tag{4.4}$$

For simplicity in notation, we shifted the index of p and q by $\frac{1}{2}$. The normal mode solutions of these equations are of the form

$$(\boldsymbol{u}^{n+1}, p^{n+1}, q^{n+1}, \boldsymbol{u}^*) = (\widetilde{\boldsymbol{u}}, \widetilde{p}, \widetilde{q}, \widetilde{\boldsymbol{u}}^*)\, \kappa^{n+1}\,. \tag{4.5}$$

Again we can solve explicitly the set of equations for $(\widetilde{\boldsymbol{u}}, \widetilde{p}, \widetilde{q}, \widetilde{\boldsymbol{u}}^*)$, and we get the following.

Symmetric modes:

(4.6)

$$\begin{cases} \widetilde{u}(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k}, \\ \widetilde{v}(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k}, \\ \widetilde{u}^*(x) = \cos\widetilde{\mu}x - \cos\widetilde{\mu}\dfrac{\cosh kx}{\cosh k} - \left(\dfrac{2\beta\Delta t}{2+\beta\Delta t}\right)^2\cos\widetilde{\mu}\left(\dfrac{\cos\lambda x}{\cos\lambda} - \dfrac{\cosh kx}{\cosh k}\right), \\ \widetilde{v}^*(x) = \dfrac{\widetilde{\mu}}{ik}\sin\widetilde{\mu}x + \dfrac{1}{i}\cos\widetilde{\mu}\dfrac{\sinh kx}{\cosh k} - ik\left(\dfrac{2\beta\Delta t}{2+\beta\Delta t}\right)^2\cos\widetilde{\mu}\left(\dfrac{1}{\lambda}\dfrac{\sin\lambda x}{\cos\lambda} - \dfrac{1}{k}\dfrac{\sinh kx}{\cosh k}\right), \\ \widetilde{p}(x) = -\dfrac{4\beta+2\beta^2\Delta t}{(2+\beta\Delta t)^2}\cos\widetilde{\mu}\left(\dfrac{1}{\lambda}\dfrac{\sin\lambda x}{\cos\lambda} - \dfrac{1}{k}\dfrac{\sinh kx}{\cosh k}\right), \end{cases}$$

where

$$\lambda = \left(\frac{2\kappa}{(1-\kappa)\Delta t} - k^2\right)^{1/2}, \quad \beta = -k^2 - \widetilde{\mu}^2. \tag{4.7}$$

$\widetilde{\mu}$, β, and λ satisfy

$$\widetilde{\mu}\tan\widetilde{\mu} + k\tanh k = k\left(\frac{2\beta\Delta t}{2+\beta\Delta t}\right)^2\left(\tanh k - \frac{k}{\lambda}\tan\lambda\right). \tag{4.8}$$

There is a unique solution of (4.8), $\widetilde{\mu}_{jk}$, in each interval $((j-\frac{1}{2})\pi, (j+\frac{1}{2})\pi)$. We will denote the solution in (4.6) as $\widetilde{u}_{jk}$, $\widetilde{v}_{jk}$, $\widetilde{u}^*_{jk}$, $\widetilde{v}^*_{jk}$, and $\widetilde{p}_{jk}$. It is easy to check that

$$\kappa_{j,k} = \frac{2+\beta_{j,k}\Delta t}{2-\beta_{j,k}\Delta t}. \tag{4.9}$$

We also have the antisymmetric modes:

(4.10)

$$\begin{cases} \widetilde{u}(x) = \sin\widetilde{\mu}x - \sin\widetilde{\mu}\dfrac{\sinh kx}{\sinh k}, \\ \widetilde{v}(x) = -\dfrac{\widetilde{\mu}}{ik}\cos\widetilde{\mu}x + \dfrac{1}{i}\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k}, \\ \widetilde{u}^*(x) = \sin\widetilde{\mu}x - \sin\widetilde{\mu}\dfrac{\sinh kx}{\sinh k} - \left(\dfrac{2\beta\Delta t}{2+\beta\Delta t}\right)^2\sin\widetilde{\mu}\left(\dfrac{\sin\lambda x}{\sin\lambda} - \dfrac{\sinh kx}{\sinh k}\right), \\ \widetilde{v}^*(x) = -\dfrac{\widetilde{\mu}}{ik}\cos\widetilde{\mu}x + \dfrac{1}{i}\sin\widetilde{\mu}\dfrac{\cosh kx}{\sinh k} - ik\left(\dfrac{2\beta\Delta t}{2+\beta\Delta t}\right)^2\sin\widetilde{\mu}\left(\dfrac{1}{\lambda}\dfrac{\cos\lambda x}{\sin\lambda} - \dfrac{1}{k}\dfrac{\cosh kx}{\sinh k}\right), \\ \widetilde{p}(x) = -\dfrac{4\beta+2\beta^2\Delta t}{(2+\beta\Delta t)^2}\sin\widetilde{\mu}\left(\dfrac{1}{\lambda}\dfrac{\cos\lambda x}{\sin\lambda} - \dfrac{1}{k}\dfrac{\cosh kx}{\sinh k}\right), \end{cases}$$

where

$$\lambda = \left(\frac{2\kappa}{(1-\kappa)\Delta t} - k^2\right)^{1/2}, \quad \beta = -k^2 - \widetilde{\mu}^2. \tag{4.11}$$

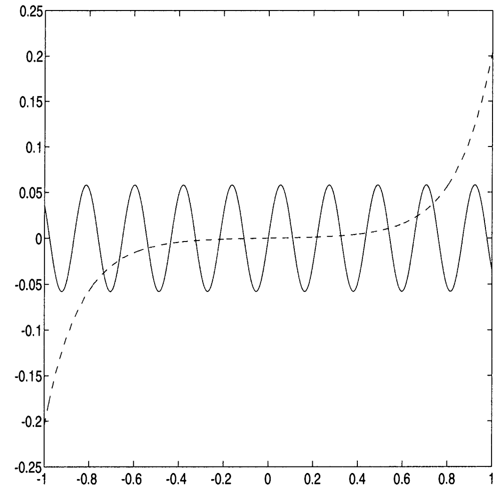

FIG. 1. *Relative error in pressure for the second-order projection method based on pressure increment formulation (solid line) and the Kim–Moin method (dashed line). The fundamental symmetric mode $j, k = 1$ was taken as the exact solution. Spatial variables are not discretized, so the results plotted are obtained from normal mode analysis, not direct computations. Parameters: viscosity* $= 0.1$, $\Delta t = 0.05$.

$\widetilde{\mu}$, β, and λ satisfy

$$\widetilde{\mu}\cot\widetilde{\mu} + k\coth k = k\left(\frac{2\beta\Delta t}{2+\beta\Delta t}\right)^2\left(\coth k - \frac{k}{\lambda}\cot\lambda\right). \tag{4.12}$$

Again there is a unique solution of (4.12), $\widetilde{\mu}$, in each interval $((j-\frac{1}{2})\pi, (j+\frac{1}{2})\pi)$.

Several things can be observed from (4.6)–(4.12). First, the normal modes of (4.3)–(4.4) for the projected velocity approximate the exact ones (2.11)–(2.15) to second-order accuracy in Δt, the growth rate $\kappa_{j,k}$ approximates the growth rate $e^{\sigma_{j,k}\Delta t}$ of the Stokes equation to third-order accuracy in Δt. This implies that for the linear Stokes equation, the accuracy of this projection method is indeed second order for the projected velocity.

Secondly, from (4.6) and (4.10) we see that there is a fundamental change of character in the numerical profile of p. The spurious numerical mode represented by λ, introduced by the projection procedure, is of the type of high frequency oscillations with wavelength and magnitude of order Δt. This should be compared to the spurious modes in Kim and Moin's method which is of a boundary layer type with width $O(\sqrt{\Delta t})$ and magnitude $O(\sqrt{\Delta t})$. This comparison is made in Figure 1 where we plot the error in pressure for the fundamental symmetric mode $\mu = 2.883356$, $k = 1$.

However in actual computations, the spatial discretization, which was neglected in the analysis presented above, has an important effect in these spurious numerical modes. If one uses a finite-difference method in space together with (4.1)–(4.2), then the intrinsic numerical diffusion in the finite-difference method will damp out the oscillations. Their effect will be limited to a region near the boundary. This is shown in Figures 2 and 3. On the other hand, we also expect if we use a spectral method in space, then the structure of the spurious numerical modes will be close to the one shown in Figure 1 since spectral methods are infinite order accurate and do not introduce numerical dissipations.

Appendix. Proof of the technical lemmas.

Proof of Lemma 1. Expand $(\boldsymbol{a}\cdot\nabla)\boldsymbol{u}$ as

$$(\boldsymbol{a}\cdot\nabla)\boldsymbol{u} = \sum_{jk}\alpha_{jk}\,\widetilde{\boldsymbol{u}}_{jk}\,e^{iky}, \quad \alpha_{jk}\,\|\widetilde{\boldsymbol{u}}_{jk}\|^2 = \langle(\boldsymbol{a}\cdot\nabla\boldsymbol{u})_k\,,\,\widetilde{\boldsymbol{u}}_{jk}\rangle\,. \tag{5.1}$$

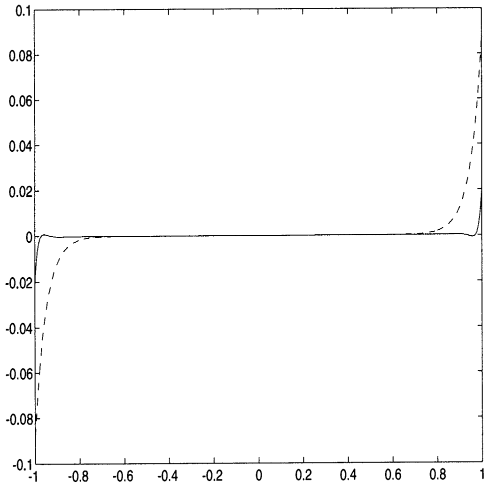

FIG. 2. *Relative error in pressure for the second-order projection method based on pressure increment formulation (solid line) and the Kim–Moin method (dashed line). The fundamental symmetric mode $j, k = 1$ was taken as the exact solution. Spatial variables are discretized using straightforward second-order centered differencing. Parameters: viscosity* $= 0.1$, $\Delta x = 0.001$, $\Delta t = 0.05$, $t = 1$.

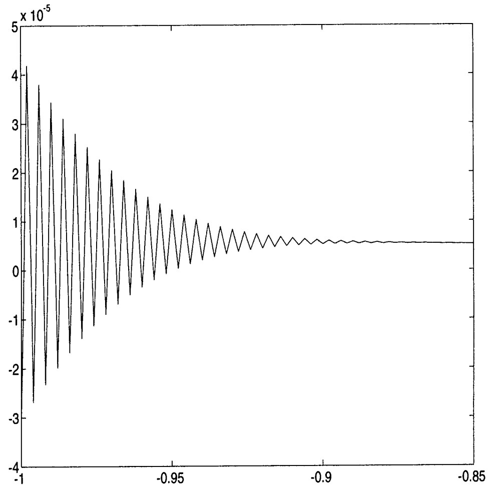

FIG. 3. *Detailed view of the pressure error near the boundary in the second-order projection method based on pressure increment formulation. This figure shows in more detail the oscillatory nature of the error. Parameters: viscosity* $= 0.001$, $\Delta x = 0.001$, $\Delta t = 0.001$, $t = 1$. *Notice that the numerical parameters are different from the previous figure. Here we choose a set of parameters in order to show more drastically the oscillatory nature of the error. Notice also that the magnitude of the error is quite small. In practice, the oscillations can hardly be noticed if a finite-difference method is used in the spatial discretization.*

Integrating by parts and using the fact that $\nabla\cdot\boldsymbol{a} = 0$ and $\boldsymbol{a}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$, we have

$$\begin{aligned}\alpha_{jk}\,\|\widetilde{\boldsymbol{u}}_{jk}\|^2 &= -\langle(\boldsymbol{a}\boldsymbol{u})_k\,,\ \partial_x\widetilde{\boldsymbol{u}}_{jk}\rangle - ik\langle(\boldsymbol{b}\boldsymbol{u})_k\,,\ \widetilde{\boldsymbol{u}}_{jk}\rangle\\ &\le \|(\boldsymbol{a}\boldsymbol{u})_k\|\ \|\partial_x\widetilde{\boldsymbol{u}}_{jk}\| + |k|\ \|(\boldsymbol{b}\boldsymbol{u})_k\|\ \|\widetilde{\boldsymbol{u}}_{jk}\|\,.\end{aligned}$$

Since

$$\|\partial_x\widetilde{\boldsymbol{u}}_{jk}\|^2 \le C\,\frac{j^2(j^2+k^2)}{k^2}\,,$$

therefore

$$\alpha_{jk}^2 \, \|\widetilde{\boldsymbol{u}}_{jk}\|^2 \le C \, (j^2 \, \|(a\boldsymbol{u})_k\|^2 + k^2 \, \|(b\boldsymbol{u})_k\|^2) \, . \tag{5.2}$$

Hence

$$\begin{aligned} \|\mathcal{S}_{\Delta t}^m (\boldsymbol{a}\cdot\nabla)\boldsymbol{u}\|^2 &= \sum_{jk} \kappa_{jk}^{2m} \, \alpha_{jk}^2 \, \|\widetilde{\boldsymbol{u}}_{jk}\|^2 \\ &\le \sum_{jk} \kappa_{jk}^{2m} \, (j^2 \, \|(a\boldsymbol{u})_k\|^2 + k^2 \, \|(b\boldsymbol{u})_k\|^2) \, . \end{aligned} \tag{5.3}$$

Using the fact that

$$\kappa_{jk}^m \, j^2 \le \frac{C}{m\Delta t} \quad \text{and} \quad \sum_j \kappa_{jk}^m \sqrt{m\Delta t} \le C \, ,$$

we get

$$\begin{aligned} \sum_{jk} \kappa_{jk}^{2m} \, j^2 \, \|(a\boldsymbol{u})_k\|^2 &\le C \, (m\Delta t)^{-3/2} \sum_{jk} \kappa_{jk}^m (m\Delta t)^{1/2} \, \|(a\boldsymbol{u})_k\|^2 \\ &\le C \, (m\Delta t)^{-3/2} \sum_k \|(a\boldsymbol{u})_k\|^2 = C \, (m\Delta t)^{-3/2} \, \|a\boldsymbol{u}\|^2 \, . \end{aligned}$$

Similarly we have

$$\sum_{jk} \kappa_{jk}^{2m} \, k^2 \, \|(b\boldsymbol{u})_k\|^2 \le C \, (m\Delta t)^{-3/2} \, \|b\boldsymbol{u}\|^2.$$

Therefore

$$\|\mathcal{S}_{\Delta t}^m (\boldsymbol{a}\cdot\nabla)\boldsymbol{u}\|^2 \le C \, (m\Delta t)^{-3/2} \, (\|a\boldsymbol{u}\|^2 + \|b\boldsymbol{u}\|^2) \le C \, (m\Delta t)^{-3/2} \, \|\boldsymbol{a}\|_{L^\infty}^2 \, \|\boldsymbol{u}\|^2 \, .$$

This leads to (3.14). Equation (3.15) is straightforward to prove. This completes the proof of Lemma 1. □

The following three lemmas will be used in the proof of Lemma 2.

LEMMA 3.

$$|\mu_{jk} - \widetilde{\mu}_{jk}| \le C \, \min(1, (|j|+1) \, |k| \, \Delta t) \, , \tag{5.4}$$

$$\|\widetilde{\boldsymbol{u}}_{jk}(x) - \widehat{\boldsymbol{u}}_{jk}(x)\| \le C \frac{\sqrt{j^2+k^2}}{|k|} \, |\mu_{jk} - \widetilde{\mu}_{jk}|, \tag{5.5}$$

where C is independent of j, k, and Δt.

Proof. We only need to show (5.4) for the case when $(j^2+k^2)\Delta t \le 1$. The other case is trivial. To prove (5.4), we subtract (2.12) from (2.21) to get

$$|\widetilde{\mu} \tan \widetilde{\mu} - \mu \tan \mu| = \left| \beta k \Delta t \left(\tanh k - \frac{k}{\lambda} \tanh \lambda \right) \right| \le 2 \, |\beta k \Delta t| \, . \tag{5.6}$$

Here we used $|k/\lambda| \le 1$ and $|\tanh k|$, $|\tanh \lambda| \le 1$. Taylor expansion gives

$$\widetilde{\mu} \tan \widetilde{\mu} - \mu \tan \mu = (x \tan x)'|_{x=\xi} \, (\widetilde{\mu} - \mu), \tag{5.7}$$

where ξ is a number between $\widetilde{\mu}$ and μ.

We estimate from below $(x\tan x)'|_{x=\xi}$. It is enough to consider the case $k,\ j\geq 0$. Since

$$\widetilde{\mu}\tan\widetilde{\mu}-\mu\tan\mu=\beta k\Delta t\left(\tanh k-\frac{k}{\lambda}\tanh\lambda\right)<0$$

and

(5.8)
$$(x\tan x)'=\tan x+\frac{x}{\cos^2 x}>0\,,\quad \text{for } x>\frac{1}{2},$$

we know that $\widetilde{\mu}_{jk}\leq\mu_{jk}$ for $j\geq 1$. On the other hand, since

(5.9)
$$(x\tan x)''=\frac{2}{\cos^2 x}(1+x\tan x)<0\,,\quad \text{for } \widetilde{\mu}\leq x\leq\mu\,,$$

we get

(5.10)
$$(x\tan x)'|_{x=\xi}\geq(x\tan x)'|_{x=\mu}=\tan\mu+\frac{\mu}{\cos^2\mu}=-\frac{k}{\mu}\tanh k+\mu\left(1+\frac{k^2}{\mu^2}\tanh^2 k\right).$$

Clearly, if $k=0$ then $\mu\tan\mu=\widetilde{\mu}\tan\widetilde{\mu}=0$. Hence (5.4) holds. For the case $k=O(1)$ and $j=0$, we can easily check that (5.4) holds.

Next we consider the case $|k|\gg 1$ and $j=0$. In this case, we know that $|\mu|,\ |\widetilde{\mu}|>C$ for some positive constant C. We have from (5.10) that

$$|(x\tan x)'|_{x=\xi}|\geq Ck^2\,.$$

This gives

$$|\widetilde{\mu}_{jk}-\mu_{jk}|\leq C\Delta t\,|k|^3/k^2=C\Delta t\,|k|\,.$$

Now consider case $k\neq 0$ and $j\neq 0$. Using (5.10) and fact that $(e-1)/(e+1)<|\tanh k|<1$,

$$|(x\tan x)'|_{x=\xi}|\geq C(|j|+k^2/|j|)\,.$$

Hence

(5.11)
$$|\widetilde{\mu}_{jk}-\mu_{jk}|\leq C\Delta t\,|jk|\,.$$

Noticing that $|\mu_{jk}-\widetilde{\mu}_{jk}|\leq\pi$, we obtain (5.4) directly from (5.11). Finally using the divergence-free property of $\boldsymbol{u}_{jk}$ and $\widetilde{\boldsymbol{u}}_{jk}$, we get

$$\begin{aligned}\|\widetilde{\boldsymbol{u}}_{jk}(x)-\boldsymbol{u}_{jk}(x)\|^2&=-\frac{1}{k^2}\int(\widetilde{u}_{jk}(x)-u_{jk}(x))\,\Delta_k(\widetilde{u}_{jk}(x)-u_{jk}(x))\,dx\\&=-\int\left(\cos\widetilde{\mu}x-\cos\mu x+(\cos\widetilde{\mu}-\cos\mu)\frac{\cosh kx}{\cosh k}\right)\left(\frac{\widetilde{\beta}}{k^2}\cos\widetilde{\mu}x-\frac{\beta}{k^2}\cos\mu x\right)\\&\leq\frac{j^2+k^2}{k^2}\,|\widetilde{\mu}-\mu|^2,\end{aligned}$$

which gives (5.5). This completes the proof of Lemma 3. □

LEMMA 4. *Let* $\boldsymbol{u}$ *be a divergence-free vector field satisfying* $\boldsymbol{u}\,|_{x=\pm1}=0$, *and*

(5.12)
$$\alpha_{jk}=\frac{1}{M_{jk}}\langle\boldsymbol{u}_k(x)\,,\,\widehat{\boldsymbol{u}}_{jk}(x)\rangle\,,\qquad \widetilde{\alpha}_{jk}=\frac{1}{\widetilde{M}_{jk}}\langle\boldsymbol{u}_k(x)\,,\,\widetilde{\boldsymbol{u}}_{jk}(x)\rangle,$$

where M_{jk} and $\widetilde{M}_{jk}$ are the normalizing constants

$$M_{jk} = \langle \widehat{\boldsymbol{u}}_{jk}(x) , \widehat{\boldsymbol{u}}_{jk}(x) \rangle , \quad \widetilde{M}_{jk} = \langle \widetilde{\boldsymbol{u}}_{jk}(x) , \widetilde{\boldsymbol{u}}_{jk}(x) \rangle . \tag{5.13}$$

Then we have

$$|\alpha_{jk}| \, \|\widehat{\boldsymbol{u}}_{jk}\| \le \frac{C}{j^3 k^3} \Big(\|(\partial_x^4 \partial_y^2 u)_k\| + \|(\partial_x^3 \partial_y^3 u)_k\| + \|(\partial_x^2 \partial_y^3 u)_k\| \Big) \tag{5.14}$$

and

$$|\alpha_{jk}| \, \|\widehat{\boldsymbol{u}}_{jk}\| \le C \min\Big(\frac{1}{j^2 k^2} , \frac{1}{|j|^3 |k|} \Big) \Big(\|(\partial_x^4 u)_k\| + \|(\partial_x^3 \partial_y u)_k\| + \|(\partial_x^2 \partial_y^2 u)_k\| \Big) , \tag{5.15}$$

and similarly if we replace α_{jk} and $\widehat{\boldsymbol{u}}_{jk}$ by $\widetilde{\alpha}_{jk}$ and $\widetilde{\boldsymbol{u}}_{jk}$, respectively. We also have

$$|\alpha_{jk} - \widetilde{\alpha}_{jk}| \, \|\widetilde{\boldsymbol{u}}_{jk}\| \le \frac{C}{j^3 k^3} \, |\mu_{jk} - \widetilde{\mu}_{jk}| \Big(\|(\partial_x^4 \partial_y^2 u)_k\| + \|(\partial_x^3 \partial_y^3 u)_k\| \Big), \tag{5.16}$$

where C is independent of j, k, and Δt.

Proof. Using divergence-free property of $\boldsymbol{u}$ and $\widehat{\boldsymbol{u}}_{jk}$, and integrating by parts, we have

$$\langle \boldsymbol{u}_k(x) , \widehat{\boldsymbol{u}}_{jk}(x) \rangle = -\frac{\beta}{k^2} \int_{-1}^{1} u_k(x) \, \cos \mu x \, dx . \tag{5.17}$$

Since $\partial_x u_k + ikv_k = 0$ and $(u, v)_k \,|_{x=\pm 1} = 0$ we get

$$(\partial_y^2 u)_k(\pm 1) = \partial_x (\partial_y^2 u)_k(\pm 1) = 0 .$$

Integrating by parts three times, we obtain

$$\begin{aligned} \int_{-1}^{1} u_k(x) \cos \mu x \, dx &= -\frac{1}{k^2} \int_{-1}^{1} (\partial_y^2 u)_k(x) \cos \mu x \, dx \\ &= -\frac{1}{k^2 \mu^2} \int_{-1}^{1} \partial_x^2 (\partial_y^2 u)_k(x) \cos \mu x \, dx \\ &= -\frac{2 \sin \mu}{\mu^3 k^2} (\partial_x^2 \partial_y^2 u)_k(1) + \frac{1}{\mu^3 k^2} \int_{-1}^{1} (\partial_x^3 \partial_y^2 u)_k(x) \sin \mu x \, dx . \end{aligned} \tag{5.18}$$

Using Hölder and Sobolev inequalities, we have

$$|(\partial_x^2 \partial_y u)_k(1)| \le C \Big(\|(\partial_x^2 \partial_y u)_k\| + \|(\partial_x^3 \partial_y u)_k\| \Big) .$$

From (2.12) we see that $|\sin \mu| \le |k/\mu|$. This gives

$$\left| \frac{2 \sin \mu}{\mu^3 k^2} (\partial_x^2 \partial_y^2 u)_k(1) \right| \le \frac{C}{\mu^4 k^2} \Big(\|(\partial_x^2 \partial_y u)_k\| + \|(\partial_x^3 \partial_y u)_k\| \Big) . \tag{5.19}$$

To estimate the second term in (5.18), we integrate by parts once again to obtain

$$\int_{-1}^{1} (\partial_x^3 \partial_y^2 u)_k(x) \sin \mu x \, dx = -\frac{2 \cos \mu}{\mu} (\partial_x^3 \partial_y^2 u)_k(1) + \frac{1}{\mu} \int_{-1}^{1} (\partial_x^4 \partial_y^2 u)_k(x) \cos \mu x \, dx .$$

Therefore, we have

$$(5.20)\qquad \left|\int_{-1}^{1}(\partial_x^3\partial_y^2 u)_k(x)\sin\mu x\,dx\right| \le \frac{C}{|j|}\left(\|(\partial_x^4\partial_y^2 u)_k\| + \|(\partial_x^3\partial_y^2 u)_k\|\right).$$

Combining (5.18), (5.19), and (5.20) we obtain

$$(5.21)\qquad \left|\int_{-1}^{1} u_k(x)\cos\mu x\,dx\right| \le \frac{C}{j^4k^2}\left(\|(\partial_x^4\partial_y^2 u)_k\| + \|(\partial_x^3\partial_y^2 u)_k\|\right).$$

Similarly, we have

$$(5.22)\qquad \left|\int_{-1}^{1} u_k(x)\cos\mu x\,dx\right| \le \frac{C}{j^3k^3}\left(\|(\partial_x^3\partial_y^3 u)_k\| + \|(\partial_x^2\partial_y^3 u)_k\|\right).$$

Therefore, we have from (3.26), (5.12), (5.17), and (5.21)

$$\begin{aligned}
&|\alpha_{jk}|\,\|\widehat{\boldsymbol{u}}_{jk}\| \\
&\le C\frac{\sqrt{j^2+k^2}}{|k|}\min\left(\frac{1}{j^4k^2}, \frac{1}{|j|^3|k|^3}\right)\left(\|(\partial_x^4\partial_y^2 u)_k\| + \|(\partial_x^3\partial_y^3 u)_k\| + \|(\partial_x^2\partial_y^3 u)_k\|\right) \\
&\le \frac{C}{j^3k^3}\left(\|(\partial_x^4\partial_y^2 u)_k\| + \|(\partial_x^3\partial_y^3 u)_k\| + \|(\partial_x^2\partial_y^3 u)_k\|\right).
\end{aligned}$$

This proves (5.14). Equation (5.15) can be proved similarly. To prove (5.16), we write

$$(5.23)\qquad \alpha_{jk} - \widetilde{\alpha}_{jk} = \frac{\beta}{k^2 M_{jk}}\int_{-1}^{1}(\cos\widetilde{\mu}x - \cos\mu x)u_k(x)\,dx + \frac{\beta\widetilde{M}_{jk} - \widetilde{\beta}M_{jk}}{\widetilde{\beta}M_{jk}}\widetilde{\alpha}_{jk}.$$

Similar to (5.21) and (5.22) we have

$$\left|\int_{-1}^{1}(\cos\mu x - \cos\widetilde{\mu}x)u_k(x)\,dx\right| \le \frac{C}{j^4k^2}\,|\widetilde{\mu}-\mu|\,(\|(\partial_x^4\partial_y^2 u)_k\| + \|(\partial_x^3\partial_y^2 u)_k\|)$$

and

$$\left|\int_{-1}^{1}(\cos\mu x - \cos\widetilde{\mu}x)u_k(x)\,dx\right| \le \frac{C}{j^3k^3}\,|\widetilde{\mu}-\mu|\,(\|(\partial_x^3\partial_y^3 u)_k\| + \|(\partial_x^2\partial_y^3 u)_k\|).$$

From (5.12) and (3.26) we have

$$\left|\frac{\beta\widetilde{M}_{jk} - \widetilde{\beta}M_{jk}}{\widetilde{\beta}M_{jk}}\right| \le C\,|\mu_{jk} - \widetilde{\mu}_{jk}|.$$

Now (5.16) follows easily. □

LEMMA 5.

$$(5.24)\qquad |||\partial_t \mathcal{S}(t)\boldsymbol{u}||| \le C\,\|\boldsymbol{u}\|_{H^4}.$$

Proof. We expand $\boldsymbol{u}$ as

$$\boldsymbol{u} = \sum_{jk}\alpha_{jk}\widehat{\boldsymbol{u}}_{jk}(x)\,e^{iky}.$$

Then

$$\|\partial_t \mathcal{S}(t)\boldsymbol{u}\|^2 = \sum_{jk} \sigma_{jk}^2\, \alpha_{jk}^2\, \|\widehat{\boldsymbol{u}}_{jk}(x)\|^2\, e^{2\sigma_{jk}t}\,.$$

Using Lemma 4, we have

$$\begin{aligned}\|\partial_t \mathcal{S}(t)\boldsymbol{u}\|^2 &\le C \sum_{jk} \sigma_{jk}^2 \min\left(\frac{1}{j^6k^2}, \frac{1}{j^2k^4}\right) (\|(\partial_x^4 u)_k\|^2 + \|(\partial_x^3\partial_y u)_k\|^2 + \|(\partial_x^2\partial_y u)_k\|^2)\\ &\le C \sum_{jk} \frac{1}{j^2} (\|(\partial_x^4 u)_k\|^2 + \|(\partial_x^3\partial_y u)_k\|^2 + \|(\partial_x^2\partial_y u)_k\|^2) \le C\, \|u\|_{H^4}^2\,. \qquad \square\end{aligned}$$

Proof of Lemma 2. Proof of (3.17). Write

$$\boldsymbol{u}(x,y) = \sum_{j,k} \alpha_{jk}\, \widehat{\boldsymbol{u}}_{jk}(x)\, e^{iky} = \sum_{j,k} \widetilde{\alpha}_{jk}\, \widetilde{\boldsymbol{u}}_{jk}(x)\, e^{iky} \tag{5.25}$$

and

$$\begin{aligned}\mathcal{S}_{\Delta t}^m (\mathcal{S} - \mathcal{S}_{\Delta t})\boldsymbol{u} &= \sum_{j,k} \alpha_{jk} \left(\exp(\sigma_{jk}\,\Delta t) - 1\right) (\mathcal{S}_{\Delta t}^m - \kappa_{jk}^m)\, \widehat{\boldsymbol{u}}_{jk}(x)\, e^{iky}\\ &\quad + \sum_{j,k} \kappa_{jk}^m \left(\alpha_{jk}\, \widehat{\boldsymbol{u}}_{jk}(x) - \widetilde{\alpha}_{jk}\, \widetilde{\boldsymbol{u}}_{jk}(x)\right) \left(\exp(\sigma_{jk}\,\Delta t) - 1\right) e^{iky}\\ &\quad + \sum_{j,k} \kappa_{jk}^m\, \widetilde{\alpha}_{jk} \left(\exp(\sigma_{jk}\,\Delta t) - \kappa_{jk}\right) \widetilde{\boldsymbol{u}}_{jk}(x)\, e^{iky}\\ &\equiv I_1 + I_2 + I_3\,.\end{aligned} \tag{5.26}$$

Let

$$\sum_{jk} \alpha_{jk} \left(\exp(\sigma_{jk}\,\Delta t) - 1\right) \widehat{\boldsymbol{u}}_{jk}(x)\, e^{iky} = \left(\mathcal{S}(\Delta t) - I\right) \boldsymbol{u} \equiv \overline{\boldsymbol{u}}(x,y)\,. \tag{5.27}$$

Denote by η_{jk} and $\widetilde{\eta}_{jk}$ the Fourier coefficients of $\overline{\boldsymbol{u}}$ in $\{\widehat{\boldsymbol{u}}_{jk}\}$ and $\{\widetilde{\boldsymbol{u}}_{jk}\}$, respectively. We have

$$\sum_{j,k} \kappa_{jk}^m\, \alpha_{jk} \left(\exp(\sigma_{jk}\,\Delta t) - 1\right) \widehat{\boldsymbol{u}}_{jk}(x)\, e^{iky} = \sum_{j,k} \kappa_{jk}^m\, \eta_{jk}\, \widehat{\boldsymbol{u}}_{jk}(x)\, e^{iky}\,. \tag{5.28}$$

Expanding $\widehat{\boldsymbol{u}}_{jk}$ in the basis $\{\widetilde{\boldsymbol{u}}_{\ell k}\}$,

$$\widehat{\boldsymbol{u}}_{jk}(x) = \sum_{\ell} \gamma_{j\ell}\, \widetilde{\boldsymbol{u}}_{\ell k}(x)\,,$$

we get

$$\begin{aligned}&\sum_{j,k} \alpha_{jk} \left(\exp(\sigma_{jk}\,\Delta t) - 1\right) \mathcal{S}_{\Delta t}^m\, \widehat{\boldsymbol{u}}_{jk}(x)\\ &\quad = \sum_{j,k} \kappa_{\ell k}^m\, \alpha_{jk} \left(\exp(\sigma_{jk}\,\Delta t) - 1\right) \sum_{\ell} \gamma_{j\ell}\, \widetilde{\boldsymbol{u}}_{\ell k}(x)\\ &\quad = \sum_{\ell,k} \kappa_{\ell k}^m\, \widetilde{\boldsymbol{u}}_{\ell k}(x) \sum_{j} \gamma_{j\ell}\, \alpha_{jk} \left(\exp(\sigma_{jk}\,\Delta t) - 1\right) = \sum_{\ell,k} \kappa_{\ell k}^m\, \widetilde{\eta}_{\ell k}\, \widetilde{\boldsymbol{u}}_{\ell k}(x)\,.\end{aligned} \tag{5.29}$$

Using Lemmas 3 and 4, we have

$$\begin{aligned}\|\eta_{jk}\, \widehat{\boldsymbol{u}}_{jk}(x) - \widetilde{\eta}_{jk}\, \widetilde{\boldsymbol{u}}_{jk}(x)\| &\le |\eta_{jk} - \widetilde{\eta}_{jk}|\, \|\widehat{\boldsymbol{u}}_{jk}\| + |\widetilde{\eta}_{jk}|\, \|\widehat{\boldsymbol{u}}_{jk} - \widetilde{\boldsymbol{u}}_{jk}\|\\ &\le C\, \frac{1}{j^2 k} \min(1, |jk|\Delta t)\, (\|(\partial_x^3 \bar{u})_k\| + \|(\partial_x^2\partial_y \bar{u})_k\|)\,.\end{aligned} \tag{5.30}$$

Therefore

$$\begin{aligned}\sum_m \|I_1\| &\le \sum_m \left\| \sum_{j,k} \kappa_{jk}^m \left(\eta_{jk}\,\widehat{\boldsymbol{u}}_{jk}(x) - \widetilde{\eta}_{jk}\,\widetilde{\boldsymbol{u}}_{jk}(x)\right) e^{iky} \right\| \\ &\le C \sum_{j,k} \frac{1}{1-\kappa_{jk}} \frac{1}{j^2 k} \min(1, |jk|\Delta t)\,(\|(\partial_x^3 \bar{u})_k\| + \|(\partial_x^2 \partial_y \bar{u})_k\|) \\ &\le C \sum_{jk} \frac{1}{j^2 k} \|(\partial_x^3 \bar{u})_k, (\partial_x^2 \partial_y \bar{u})_k\| \le C\,\|\bar{u}\|_{H^3},\end{aligned}$$

where we used

$$\frac{1}{1-\kappa_{jk}} \min(1, |jk|\Delta t) \le C\,.$$

Since

$$\|\bar{\boldsymbol{u}}\|_{H^3} = \|(\mathcal{S}(\Delta t) - I)\boldsymbol{u}\|_{H^3} \le \Delta t\,|\!|\!|\partial_t \mathcal{S}(t)\boldsymbol{u}|\!|\!|_{H^3}\,,$$

we get

$$\sum_m \|I_1\| \le C\Delta t\,|\!|\!|\partial_t \mathcal{S}(t)\boldsymbol{u}|\!|\!|_{H^3}\,. \tag{5.31}$$

To estimate I_2, we have, similar to (5.30), that

$$\|\alpha_{jk}\,\widehat{\boldsymbol{u}}_{jk} - \widetilde{\alpha}_{jk}\,\widetilde{\boldsymbol{u}}_{jk}\| \le \frac{C\Delta t}{j^2|k|}\left(\|(\partial_x^4 \partial_y u)_k\| + \|(\partial_x^3 \partial_y^2 u)_k\| \right).$$

Hence

$$\begin{aligned}\sum_m \|I_2\| &\le \sum_{j,k} \frac{1}{1-\kappa_{jk}} |\exp(\sigma_{jk}\,\Delta t) - 1|\, \|\widetilde{\alpha}_{jk}\widetilde{\boldsymbol{u}}_{jk}(x) - \alpha_{jk}\boldsymbol{u}_{jk}(x)\| \\ &\le C\Delta t \sum_{jk} \frac{1}{j^2|k|} \|(\partial_x^4 \partial_y u)_k, (\partial_x^3 \partial_y^2 u)_k\| \le C\Delta t\,\|u\|_{H^5},\end{aligned} \tag{5.32}$$

where we have used

$$\frac{1}{1-\kappa_{jk}} |\exp(\sigma_{jk}\,\Delta t) - 1| \le C\,.$$

Finally

$$\begin{aligned}\|I_3\|^2 &= \sum_{j,k} \kappa_{jk}^{2m} |\exp(\sigma_{jk}\,\Delta t) - \kappa_{jk}|^2\, \widetilde{\alpha}_{jk}^2\, \|\widetilde{\boldsymbol{u}}_{jk}(x)\|^2 \\ &\le C \sum_{j,k} \kappa_{jk}^{2m} (j^2+k^2)^4 \Delta t^4 \min\left(\frac{1}{j^6 k^2}, \frac{1}{j^2 k^6}\right) \\ &\quad \times \left(\|(\partial_x^4 u)_k\|^2 + \|(\partial_x^3 \partial_y^1 u)_k\|^2 + \|(\partial_x^2 \partial_y^2 u)_k\|^2 + \|(\partial_x \partial_y^3 u)_k\|^2 \right).\end{aligned}$$

Using

$$\kappa_{jk}^{2m} (j^2+k^2)^4 \Delta t^4 \le C\Delta t^{9/4} m^{-5/4} (j^{9/2} + k^{9/2})\,,$$

we get

$$\begin{aligned}\|I_3\|^2 &\le C\Delta t^{9/4} m^{-5/4} \left(\sum_{j\le k} \frac{1}{j^2 k^{3/2}} + \sum_{j\ge k} \frac{1}{j^{3/2}k^2}\right) \\ &\quad \times\left(\|(\partial_x^4 u)_k\|^2 + \|(\partial_x^3\partial_y^1 u)_k\|^2 + \|(\partial_x^2\partial_y^2 u)_k\|^2 + \|(\partial_x\partial_y^3 u)_k\|^2\right) \\ &\le C\Delta t^{9/4} m^{-5/4} \|\boldsymbol{u}\|_{H^5}^2 .\end{aligned}$$

Hence

$$\sum_m \|I_3\| \le C\Delta t^{9/8} \|\boldsymbol{u}\|_{H^5} \sum_{0\le m\le T/\Delta t} m^{-5/8} \le C\Delta t \|\boldsymbol{u}\|_{H^5} .$$

Proof of (3.18). Clearly

$$\|(\mathcal{S}-\mathcal{S}_{\Delta t})\boldsymbol{u}\| \le \|(\mathcal{S}-I)\boldsymbol{u}\| + \|(\mathcal{S}_{\Delta t}-I)\boldsymbol{u}\| .$$

By Lemma 5, one has

$$\|(\mathcal{S}-I)\boldsymbol{u}\| \le C\Delta t\|\boldsymbol{u}\|_{H^4} .$$

Similarly, we have

$$\|(\mathcal{S}_{\Delta t}-I)\boldsymbol{u}\| \le C\Delta t \|\boldsymbol{u}\|_{H^4} .$$

Hence

$$\sum_m \|\mathcal{S}_{\Delta t}^m (\mathcal{S}-\mathcal{S}_{\Delta t})\boldsymbol{u}\| \le C \|\boldsymbol{u}\|_{H^4}.$$

Since

$$\|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{H^4} \le C \|\boldsymbol{u}\|_{H^5}^2$$

we obtain (3.18).

Proof of (3.19). We have

$$\begin{aligned}&\int_{t^n}^{t^{n+1}} \mathcal{S}(t^{n+1}-\tau)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}(\tau)\,d\tau - \Delta t\mathcal{S}(\Delta t)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}(t^n) \\ &\quad = \int_{t^n}^{t^{n+1}} \tau[\mathcal{S}(t^{n+1}-\tau) - \mathcal{S}(\Delta t)](\boldsymbol{u}\cdot\nabla)\boldsymbol{u}(t^n)\,d\tau\end{aligned}$$

and hence from Lemma 5

$$\begin{aligned}&\left\|\int_{t^n}^{t^{n+1}} \mathcal{S}(t^{n+1}-\tau)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}(\tau)\,d\tau - \Delta t\mathcal{S}(\Delta t)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}(t^n)\right\| \\ &\quad \le C\Delta t^2 \|\!|\partial_t\mathcal{S}(t)(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|\!| \le C\Delta t^2 \|(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\|_{H^4} \le C\Delta t^2 \|\boldsymbol{u}\|_{H^5}^2 .\end{aligned}$$

This completes the proof of Lemma 2. □

Acknowledgment. We thank Alexander Chorin for bringing this problem to our attention and for stimulating discussions.

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