



## A note on one-dimensional time fractional ODEs

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## ABSTRACT

In this note, we prove or re-prove several important results regarding one dimensional time fractional ODEs following our previous work Feng et al. [15]. Here we use the definition of Caputo derivative proposed in Li and Liu (2017) [5,7] based on a convolution group. In particular, we establish generalized comparison principles consistent with the new definition of Caputo derivatives. In addition, we establish the full asymptotic behaviors of the solutions for  $D_c^\gamma u = Au^p$ . Lastly, we provide a simplified proof for the strict monotonicity and stability in initial values for the time fractional differential equations with weak assumptions.

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## 1. Introduction

The fractional calculus in time has been used widely in physics and engineering for memory effect, viscoelasticity, porous media etc. [1–5]. There is a huge amount of literature discussing time fractional differential equations. For instance, one can find some results in [3,6] using the classic Caputo derivatives. In this paper, we study the following time fractional ODE:

$$D_c^\gamma u = f(t, u), \quad u(0) = u_0, \quad (1.1)$$

for  $\gamma \in (0, 1)$  and  $f$  measurable. Here  $D_c^\gamma u$  is the generalized Caputo derivative introduced in [7,8]. As we will see later, this generalized definition is theoretically more convenient, since it allows us to take advantage of the underlying group structure.

As in [7], we use the following distributions  $\{g_\beta\}$  as convolution kernels for  $\beta \in (-1, 0)$ :

$$g_\beta(t) = \frac{1}{\Gamma(1+\beta)} D(\theta(t)t^\beta).$$

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Here  $\theta(t)$  is the standard Heaviside step function,  $\Gamma(\cdot)$  is the gamma function, and  $D$  means the distributional derivative on  $\mathbb{R}$ . Indeed,  $g_\beta$  can be defined for  $\beta \in \mathbb{R}$  (see [7]) so that  $\{g_\beta : \beta \in \mathbb{R}\}$  forms a convolution group. In particular, we have

$$g_{\beta_1} * g_{\beta_2} = g_{\beta_1 + \beta_2}. \quad (1.2)$$

Here since the support of  $g_{\beta_i}$  ( $i = 1, 2$ ) is bounded from left, the convolution is well-defined. Now we are able to give the generalized definition of fractional derivatives:

**Definition 1.1** ([7,8]). Let  $0 < \gamma < 1$ . Consider  $u \in L^1_{\text{loc}}[0, T)$ . Given  $u_0 \in \mathbb{R}$ , we define the  $\gamma$ -th order generalized Caputo derivative of  $u$ , associated with initial value  $u_0$ , to be a distribution in  $\mathcal{D}'(-\infty, T)$  with support in  $[0, T)$ , given by

$$D_c^\gamma u = g_{-\gamma} * \left( (u - u_0)\theta(t) \right).$$

If  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$ , we call  $D_c^\gamma u$  the Caputo derivative of  $u$ .

As in [7], if the function  $u$  is absolutely continuous, the generalized definition reduces to the classical definition. However, the generalized definition is theoretically useful because it reveals the underlying group structure (see Proposition 1.1).

**Definition 1.2.** Let  $T > 0$ . A function  $u \in L^1_{\text{loc}}[0, T)$  is a weak solution to (1.1) on  $[0, T)$  with initial value  $u_0$ , if  $f(t, u(t)) \in \mathcal{D}'(-\infty, T)$  and the equality holds in the distributional sense. We call a weak solution  $u$  a strong solution if (i).  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$ ; (ii). both  $D_c^\gamma u$  and  $f(t, u(t))$  are locally integrable on  $[0, T)$ .

By the group property (1.2), we have

**Proposition 1.1** ([7]). Suppose  $f \in L^\infty_{\text{loc}}([0, \infty) \times \mathbb{R}; \mathbb{R})$ . Fix  $T > 0$ . Then,  $u(t) \in L^1_{\text{loc}}[0, T)$  with initial value  $u_0$  is a strong solution of (1.1) on  $(0, T)$  if and only if  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$  and it solves the following integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s, u(s)) ds, \quad \forall t \in (0, T). \quad (1.3)$$

Using this integral formulation, the following has been shown in [7].

**Proposition 1.2.** Suppose  $f : [0, \infty) \times (\alpha, \beta) \rightarrow \mathbb{R}$  is continuous and locally Lipschitz continuous in  $u$ . For any given initial value  $u_0 \in (\alpha, \beta)$ , there is a unique strong solution, which either exists globally on  $[0, \infty)$  or approaches the boundary of  $(\alpha, \beta)$  in finite time. Moreover, this solution is continuous on the interval of existence.

Below in Section 2, we will establish some generalized comparison principles consistent with the new definition of Caputo derivatives. In Section 3, we establish the full asymptotic behaviors of the solutions for  $D_c^\gamma u = Au^p$ . In Section 4, we provide a new proof for the strict monotonicity and stability in initial values with weak assumptions.

## 2. Generalized comparison principles

The comparison principles are important in the analysis of time fractional PDEs (See [9]). There are many versions of comparison principles proved in literature using various definitions of Caputo derivatives. In [7], the authors assumed  $f(t, \cdot)$  to be non-decreasing. In [10, Lemma 2.6],  $f(t, \cdot)$  was assumed to be non-increasing. In [11, Theorem 2.3], there is no assumption on the monotonicity of  $f(t, \cdot)$ , but the function  $v$  is assumed to be  $C^1$  so that the pointwise value of  $D_c^\gamma v$  can be defined. Combining these ideas and establishing a crucial lemma (Lemma 2.1), we prove some generalized comparison principles in this section. Similar to [7], we define the inequality in the distributional sense:

**Definition 2.1.** Let  $U$  be an open interval. We say  $f \in \mathcal{D}'(U)$  is a nonpositive (nonnegative) distribution if for any  $\varphi \in C_c^\infty(U)$  with  $\varphi \geq 0$ , we have  $\langle f, \varphi \rangle \leq 0$  ( $\langle f, \varphi \rangle \geq 0$ ). We say  $f_1 \leq f_2$  in the distributional sense for  $f_1, f_2 \in \mathcal{D}'(U)$ , if  $f_1 - f_2$  is nonpositive. We say  $f_1 \geq f_2$  in the distributional sense if  $f_1 - f_2$  is nonnegative.

In order to prove the comparison principle, we first prove the following auxiliary lemma:

**Lemma 2.1.** Suppose  $u \in L^1_{\text{loc}}[0, T)$  and  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$ . If there exists a function  $f \in L^1_{\text{loc}}(0, T)$  such that on interval  $(0, T)$  we have in the distributional sense that  $D_c^\gamma u \leq f$ , then for any given  $A \in \mathbb{R}$ , we have in the distributional sense

$$D_c^\gamma (u - A)^+ \leq \chi(u \geq A)f, \quad \text{on } (0, T).$$

**Proof.** First, recall the following result in [7, Proposition 3.11]: if  $u \in C[0, T) \cap C^1(0, T)$  and  $u \mapsto E(u)$  is  $C^1$  and convex, we have

$$D_c^\gamma E(u) \leq E'(u)D_c^\gamma u.$$

Now let us consider  $\eta \in C_c^\infty(-1, 0)$  with  $\eta \geq 0$  and  $\int \eta dt = 1$ . Define  $\eta^\epsilon(t) = \frac{1}{\epsilon} \eta(\frac{t}{\epsilon})$  and  $u^\epsilon = \eta^\epsilon * u$ . As showed in [7, Proposition 3.11],  $u^\epsilon(0) \rightarrow u_0$  and  $u^\epsilon(t) \rightarrow u(t)$  in  $L^1_{\text{loc}}[0, T)$ .

Denote  $E(u) = (u - A)^+$  and define  $E^\delta(u) = (E * \eta^\delta)(u)$ . Clearly,  $(E^\delta)'(u) = \eta^\delta * \chi(u \geq A)$  is nonnegative and increasing, which implies that  $E^\delta$  is a convex increasing function. Then, we have

$$D_c^\gamma E^\delta(u^\epsilon) \leq (E^\delta)'|_{u^\epsilon} D_c^\gamma u^\epsilon. \tag{2.1}$$

It is not hard to see  $\limsup_{\epsilon \rightarrow 0} (E^\delta)'|_{u^\epsilon} D_c^\gamma u^\epsilon \leq (E^\delta)'|_u f(t)$ . Since  $E^\delta(u^\epsilon)$  converges to  $E^\delta(u)$  in  $L^1_{\text{loc}}$  and  $E^\delta(u^\epsilon(0))$  converges to  $E^\delta(u_0)$ , according to Definition 1.1,  $D_c^\gamma E^\delta(u^\epsilon) \rightarrow D_c^\gamma E^\delta(u)$  as distributions. Moreover, notice that the inequality is preserved in the distributional sense (Definition 2.1). We have  $D_c^\gamma E^\delta(u) \leq (E^\delta)'|_u f(t)$ . Taking  $\delta \rightarrow 0$ , similarly we have  $D_c^\gamma E^\delta(u)$  converges as distributions to  $D_c^\gamma (u - A)^+$ . Then the right hand side of (2.1) converges to  $\chi(u \geq A)f(t)$ , and the inequality is preserved in the distributional sense.  $\square$

As is well-known, if  $u \in H^1(0, T)$ ,  $D(u - A)^+ = \chi(u - A)Du$ . Since Caputo derivative is nonlocal, the equality is no longer true in general. However, we have similar inequalities and Lemma 2.1 provides an answer.

**Corollary 2.1.** Suppose  $u(t)$  is a locally integrable function with  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u(s) - u_0| ds = 0$ . Let  $A \in \mathbb{R}$  and  $t_1 \in (0, T)$  is a Lebesgue point. If  $u \leq A$  for a.e.  $t \leq t_1$ , and on the interval  $(t_1, T)$  we have  $D_c^\gamma u \leq 0$  in the distributional sense, then we have  $u \leq A$ , a.e.  $(0, T)$ .

Let  $u^\epsilon$  be the mollification in the proof of Lemma 2.1. Consider  $v^\epsilon = u^\epsilon - \frac{C(\epsilon)\theta(t)}{\Gamma(1+\gamma)}t^\gamma$  such that  $v^\epsilon \leq A$  for  $t \in [0, t_1 + \epsilon]$ .  $C(\epsilon) \rightarrow 0$  since  $t_1$  is a Lebesgue point. Applying Lemma 2.1,  $D_c^\gamma(v^\epsilon - A)^+ \leq \chi(t \geq t_1 + \epsilon)(D_c^\gamma u^\epsilon - C(\epsilon)) \leq \chi(t \geq t_1 + \epsilon)(D_c^\gamma u^\epsilon - \eta_\epsilon * D_c^\gamma u)$ . Taking  $\epsilon \rightarrow 0$  yields  $D_c^\gamma(u - A)^+ \leq 0$ . The details are left to readers. Now several versions of comparison principles can be stated as follows:

**Theorem 2.1.**

- (i) Suppose  $u_i \in L^1_{loc}[0, T]$  with  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u_i(s) - u_{i,0}| ds = 0$  ( $i = 1, 2$ ). Suppose  $u_1(t) \leq u_2(t)$  on  $[0, t_1]$  for a Lebesgue point  $t_1$ , and the  $\gamma$ -th Caputo derivatives of  $u_1, u_2$  on  $[0, t_1]$  are locally integrable. Define

$$h_i(t) = u_{i,0} + \frac{1}{\Gamma(\gamma)} \int_0^{t \wedge t_1} (t - s)^{\gamma-1} D_c^\gamma u_i(s) ds, \quad i = 1, 2.$$

Then,  $h_1(t) \leq h_2(t)$  for all  $t \in [0, T]$ . Moreover, assume there exists a measurable function  $f(t, u)$  such that (i)  $f(\cdot, u_i(\cdot))$  ( $i = 1, 2$ ) is locally integrable on  $[t_1, T]$ ; (ii)  $f(t, \cdot)$  is non-decreasing on  $[t_1, T]$ ; (iii)  $D_c^\gamma u_1 \leq f(t, u_1)$  and  $D_c^\gamma u_2 \geq f(t, u_2)$  in the distributional sense on  $(t_1, T)$ , then  $u_1 \leq u_2$  a.e. on  $[0, T]$ .

- (ii) Suppose  $u_i \in L^1_{loc}[0, T]$  with  $\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t |u_i(s) - u_{i,0}| ds = 0$  ( $i = 1, 2$ ). If  $u_1 \leq u_2$  on  $[0, t_1]$  for a Lebesgue point  $t_1$  and  $D_c^\gamma(u_1 - u_2) \leq f(t, u_1) - f(t, u_2)$  as distributions on  $(t_1, T)$ , with  $f(t, \cdot)$  being non-increasing on  $(t_1, T)$  and  $f(\cdot, u_i(\cdot))$  ( $i = 1, 2$ ) being locally integrable on  $[t_1, T]$ , then  $u_1 \leq u_2$  a.e. on  $[0, T]$ .
- (iii) Suppose  $u(t)$  is a continuous function on  $[0, T]$ . If  $u(t_1) = \sup_{0 \leq s \leq t_1} u(s)$  for some  $t_1 \in (0, T)$  and  $f(t) = D_c^\gamma u(t)$  is a continuous function, then  $f(t_1) \geq 0$ .

**Proof.** (i) Clearly,  $D_c^\gamma h_i = D_c^\gamma u_i$  for  $t \leq t_1$  and  $D_c^\gamma h_i = 0$  for  $t > t_1$ . Let  $u = h_1 - h_2$ ,  $A = 0$  in Corollary 2.1, we find  $h_1 \leq h_2$ . On  $[t_1, T]$ , we have

$$u_1(t) \leq h_1(t) + \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t - s)^{\gamma-1} f(s, u_1) ds, \quad u_2(t) \geq h_2(t) + \frac{1}{\Gamma(\gamma)} \int_{t_1}^t (t - s)^{\gamma-1} f(s, u_2) ds.$$

As  $h_1(t) \leq h_2(t)$  and  $f(t, \cdot)$  is non-decreasing, one has  $u_1(t) \leq u_2(t)$  (see [7, Theorem 4.10]).

(ii) Apply Lemma 2.1 for  $u_1 - u_2$  and  $A = 0$ . (The proof is similar as in Corollary 2.1.)

(iii) Consider  $u^\epsilon(t) = u(t) + \frac{\epsilon\theta(t)}{\Gamma(1+\gamma)}t^\gamma$ , where  $\epsilon > 0$ . Then,  $t_1$  is the unique maximizer of  $u^\epsilon$  on  $[0, t_1]$ . Let  $f^\epsilon = D_c^\gamma u^\epsilon = f + \epsilon$ . It suffices to show

$$f^\epsilon(t_1) \geq 0, \quad \forall \epsilon > 0. \tag{2.2}$$

Otherwise, there is an  $\epsilon_0 > 0$  such that  $f^{\epsilon_0}(t_1) < 0$ . Since  $f^{\epsilon_0}$  is continuous, we can find  $\delta > 0$  such that on  $[t_1 - \delta, t_1]$   $f^{\epsilon_0}$  is negative and  $u^{\epsilon_0}(t) \leq u^{\epsilon_0}(t_1 - \delta)$  for  $t \leq t_1 - \delta$ . Applying Corollary 2.1, we have  $u^{\epsilon_0}(t) \leq u^{\epsilon_0}(t_1 - \delta)$  for  $t \in [t_1 - \delta, t_1]$ , which is a contradiction. Taking  $\epsilon \rightarrow 0$  then gives the result.  $\square$

**Remark 2.1.** Though the conditions here are weaker under the new definition of Caputo derivative, (ii) is essentially [10, Lemma 2.6] and (iii) is well-known for  $C^1$  functions (see, for example [11,12]).

Now, we establish a generalized Grönwall inequality (or another version of comparison principle), consistent with the new definition of Caputo derivative. The main construction is inspired by [11].

**Theorem 2.2.** Suppose  $f(t, u)$  is continuous and locally Lipschitz in  $u$ . Let  $v(t)$  be a continuous function. If  $D_c^\gamma v \leq f(t, v)$  in the distributional sense, and  $D_c^\gamma u = f(t, u)$ , with  $v_0 \leq u_0$ . Then,  $v \leq u$  on the common interval. Similarly, if we have  $D_c^\gamma v \geq f(t, v)$  as distributions and  $v_0 \geq u_0$ , then  $v \geq u$  on the common interval.

**Proof.** We only prove the first claim (the proof for the other is similar). By Proposition 1.2,  $D_c^\gamma u = f(t, u)$  with initial value  $u(0) = u_0$  has a unique solution on the interval  $[0, T_b)$ , where  $T_b$  is the largest time of existence. Moreover,  $u$  is continuous on  $[0, T_b)$ .

Fix  $T \in (0, T_b)$ . Pick  $M$  large enough so that  $u(t)$  and  $v(t)$  fall into  $[0, T] \times [-M, M]$ . Let  $L$  be the Lipschitz constant of  $f(t, \cdot)$  for the region  $[0, T] \times [-2M, 2M]$ . Consider

$$v^\epsilon = v - \epsilon w.$$

Here  $w = E_\gamma(2Lt^\gamma)$  is the solution to  $D_c^\gamma w = 2Lw$  with initial value 1, where  $E_\gamma(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\gamma+1)}$  is the Mittag-Leffler function [13,14]. Clearly, if  $\epsilon$  is sufficiently small,  $v^\epsilon$  falls into  $[0, T] \times [-2M, 2M]$ . Then, we find that in the distributional sense

$$D_c^\gamma v^\epsilon = D_c^\gamma v - \epsilon 2Lw \leq f(t, v) - \epsilon 2Lw \leq f(t, v^\epsilon) - \epsilon Lw.$$

We claim that for all such small  $\epsilon$ ,

$$v^\epsilon(t) \leq u(t), \forall t \in [0, T]. \tag{2.3}$$

If not, define

$$t_1 = \sup\{t \in (0, T] : v^\epsilon(s) \leq u(s), \forall s \in [0, t]\}.$$

Since  $v^\epsilon(0) = v_0 - \epsilon < u_0$ , by continuity we have  $t_1 > 0$ . By assumption, (2.3) is not true, and we have  $t_1 < T$ . Consequently, there exists  $\delta_1 > 0$ , such that  $v^\epsilon(t_1) = u(t_1)$  and  $v^\epsilon(t) > u(t)$  for  $t \in (t_1, t_1 + \delta_1)$ . Moreover,

$$D_c^\gamma (v^\epsilon - u) \leq f(t, v^\epsilon) - \epsilon Lw - f(t, u).$$

By continuity, for some  $\delta_2 \in (0, \delta_1)$ ,  $D_c^\gamma (v^\epsilon - u)$  is a nonpositive distribution on the interval  $(t_1, t_1 + \delta_2)$ . By Corollary 2.1, we have  $v^\epsilon(t) \leq u(t)$  for  $t \in (t_1, t_1 + \delta_2)$ , which is a contradiction. Hence, (2.3) is true. Taking  $\epsilon \rightarrow 0$  in (2.3) yields the result on  $[0, T]$ . Since  $T$  is arbitrary, the result is true.  $\square$

### 3. Asymptotic behaviors for a class of fractional ODEs

In this section, we study the solution curves to the following autonomous fractional ODEs:

$$D_c^\gamma u = Au^p, \quad u(0) = u_0 > 0. \tag{3.1}$$

The monotonicity of the solutions to (3.1) and some partial results for the asymptotic behaviors have been established in our previous work [15]. The asymptotic behaviors of the solutions for the  $A < 0, p > 0$  case have also been discussed in [10, Theorem 7.1]. However, the discussion on all the range of  $A$  and  $p$  is not complete. Here, we will give a complete description on asymptotic behaviors of the solution curves.

By Proposition 1.2, the strong solution  $u$  to (3.1) exists on  $[0, T_b)$  for  $T_b \in (0, \infty]$ . If  $T_b < \infty$ , either  $\lim_{t \rightarrow T_b^-} u(t) = 0$  or  $\lim_{t \rightarrow T_b^-} u(t) = \infty$ . We give a complete description regarding the solutions curves to (3.1):

**Theorem 3.1.** Consider (3.1). If  $A = 0$ , then  $u(t) = u_0$ . If  $A > 0$ , then all the solutions are strictly increasing on  $(0, T_b)$ . If  $A < 0$ , then all solutions are strictly decreasing before they touch 0.

- (i) Suppose  $A > 0$ . If  $p > 1$ , then  $T_b < \infty$  and  $u(t) \sim \left[ \frac{\Gamma(\frac{p\gamma}{p-1})}{A\Gamma(\frac{\gamma}{p-1})} \right]^{\frac{1}{p-1}} (T_b - t)^{-\frac{\gamma}{p-1}}$ , as  $t \rightarrow T_b^-$ . If  $p = 1$ , then  $u(t) = u_0 E_\gamma(At^\gamma)$ . If  $p < 1$ , then there exist  $c_1 > 0$  and  $c_2 > 0$  such that  $c_1 t^{\frac{\gamma}{1-p}} \leq u(t) \leq c_2 t^{\frac{\gamma}{1-p}}$ ,  $t \geq 1$ .
- (ii) Suppose  $A < 0$ . If  $p < 0$ , the solution curve touches  $u = 0$  in finite time where the right hand side blows up. If  $p = 0$ , then  $u = u_0 + Ag_{1+\gamma}$ . If  $p > 0$ , then  $T_b = \infty$ , and there exist  $c_1 > 0, c_2 > 0$  such that  $c_1 t^{-\frac{\gamma}{p}} \leq u(t) \leq c_2 t^{-\frac{\gamma}{p}}$ ,  $t \geq 1$ .

**Proof.** The  $A = 0$  or  $p = 0$  cases are trivial. The monotonicity has been proved in [15]. The  $A > 0, p > 1$  case has also been discussed there. Indeed, there is also an accurate estimate of  $T_b$  in [15]. The  $p = 1$  case is trivial. The  $A < 0, p > 0$  case has been discussed in [10, Theorem 7.1]. In fact, they established a version of comparison principle and used a subsolution and a supersolution to get  $c_1 t^{-\frac{\gamma}{p}} \leq u(t) \leq c_2 t^{-\frac{\gamma}{p}}, t \geq 1$ . For the case  $A < 0, p < 0$ , since the solution is decreasing, we have  $D_c^\gamma u \leq Au_0^p < 0$  before  $u$  touches zero. Hence, the claim follows.

Now, we establish the results for  $A > 0, p < 1$  case. First, let us construct the sub-solution as follows:

$$\omega(t) = \begin{cases} u_0, & t \in [0, t_0], \\ at^{\frac{\gamma}{1-p}}, & t \geq t_0. \end{cases}$$

Here  $a > 0$  is to be determined and  $t_0$  is determined by  $at_0^{\frac{\gamma}{1-p}} = u_0$ . Clearly,  $\omega$  is absolutely continuous on any finite interval. For  $t < t_0, D_c^\gamma \omega = 0 \leq A\omega^p$ . For  $t \geq t_0$ , we have

$$D_c^\gamma \omega = \frac{a\gamma}{(1-p)\Gamma(1-\gamma)} \int_{t_0}^t \frac{\tau^{\frac{\gamma}{1-p}-1}}{(t-\tau)^\gamma} d\tau < \frac{a\gamma B(\frac{\gamma}{1-p}, 1-\gamma)}{(1-p)\Gamma(1-\gamma)} t^{\frac{\gamma p}{1-p}} = \frac{a\Gamma(\gamma/(1-p)+1)}{\Gamma(\gamma p/(1-p)+1)} t^{\frac{p\gamma}{1-p}},$$

where  $B(\cdot, \cdot)$  is the Beta function. Clearly, if we choose  $a > 0$  such that  $\frac{a\Gamma(\gamma/(1-p)+1)}{\Gamma(\gamma p/(1-p)+1)} \leq Aa^p$ , then  $D_c^\gamma \omega \leq A\omega^p$ . Such  $a$  exists because  $p < 1$ .

For the super-solution, let us consider

$$v(t) = \begin{cases} u_0 + B_1 \frac{t^\gamma}{\Gamma(1+\gamma)}, & t \in [0, 1], \\ B_2 t^{\frac{\gamma}{1-p}}, & t \geq 1. \end{cases}$$

$B_2$  is determined by  $B_2 = u_0 + \frac{B_1}{\Gamma(1+\gamma)}$ . This choice of  $B_2$  makes  $v$  absolutely continuous on any finite interval. We now determine  $B_1$ . On  $[0, 1]$ , one has  $D_c^\gamma v = B_1$ . For  $t > 1$ , we have

$$D_c^\gamma v = \frac{B_1\gamma}{B(1+\gamma, 1-\gamma)} \int_0^1 \frac{\tau^{\gamma-1}}{(t-\tau)^\gamma} d\tau + \frac{B_2}{\Gamma(1-\gamma)} \frac{\gamma}{1-p} \int_1^t \frac{\tau^{\frac{\gamma}{1-p}-1}}{(t-\tau)^\gamma} d\tau.$$

On  $[1, 2]$ , one has  $D_c^\gamma v > \frac{B_1\gamma}{B(1+\gamma, 1-\gamma)} \int_0^1 \frac{\tau^{\gamma-1}}{(2-\tau)^\gamma} d\tau = B_1 C_1(\gamma)$ . For  $t > 2$ , we have

$$D_c^\gamma v > B_2 \frac{1}{\Gamma(1-\gamma)} \frac{\gamma}{1-p} t^{\frac{\gamma p}{1-p}} \int_{\frac{1}{2}}^1 \frac{\tau^{\frac{\gamma}{1-p}-1}}{(1-\tau)^\gamma} d\tau \geq B_2 t^{\frac{\gamma p}{1-p}} C_2(p, \gamma).$$

It is clear that there exists  $M_1(A, p, \gamma)$  such that as long as  $B_2 \geq M_1, D_c^\gamma v \geq Av^p$  for  $t \geq 2$  since  $p < 1$ . For  $v$  to be a super-solution, one needs

$$u_0 + B_1 \frac{1}{\Gamma(1+\gamma)} \geq M_1, \quad B_1 \min(1, C_1(\gamma)) \geq A \max\left(u_0^p, \left(u_0 + \frac{B_1}{\Gamma(1+\gamma)}\right)^p 2^{\frac{p\gamma}{1-p}}\right).$$

Such  $B_1$  exists since  $p < 1$ . Hence, applying comparison principle Theorem 2.2 yields the result.  $\square$

#### 4. Strict monotonicity and stability in initial values

It is well-known that solution curves for well-behaved ODEs do not touch each other. However, for fractional ODEs, similar results are not trivial since the dynamics is non-Markovian. By the comparison principles (or generalized Grönwall inequality), if  $f(t, u)$  in (1.1) is continuous and locally Lipschitz in  $u, u(0) < v(0)$  implies  $u(t) \leq v(t)$  for  $t \geq 0$ . However we do not have strict inequality. In [3, Theorem 6.12], the

strict inequality has been established following a series of contraction techniques. Using our new definition of Caputo derivative, we provide a new proof of that solutions are strict monotone in initial values, by assuming  $f \in L^\infty_{\text{loc}}$ .

The following lemma (a variant of [15, Lemma 3.4] or [16, Theorem 1]), is important:

**Lemma 4.1.** *Let  $r_\lambda(t) = -\frac{d}{dt}E_\gamma(-\lambda\Gamma(\gamma)t^\gamma)$  be the resolvent for kernel  $\lambda t^{\gamma-1}$  (in other words,  $r_\lambda(t) + \lambda \int_0^t (t-s)^{\gamma-1}r_\lambda(s)ds = \lambda t^{\gamma-1}$ ). Let  $T > 0$ . Assume  $h \in L^1[0, T]$ ,  $h > 0$  a.e., satisfying*

$$h(t) - \int_0^t r_\lambda(t-s)h(s)ds > 0, \text{ a.e., } \forall \lambda > 0.$$

*Suppose  $v \in L^\infty[0, T]$ , then the integral equation*

$$y(t) + \int_0^t (t-s)^{\gamma-1}v(s)y(s)ds = h(t) \tag{4.1}$$

*has a unique solution  $y(t) \in L^1[0, T]$ . Moreover,  $y(t) > 0$ , a.e.*

The proof is exactly the same as [15, Lemma 3.4], though we only assume  $v \in L^\infty[0, T]$  here. Next, we provide a new proof for the strict monotonicity in initial value. We also prove the stability of solutions with respect to initial values.

**Theorem 4.1.** *Assume that  $f(\cdot, \cdot) \in L^\infty_{\text{loc}}([0, \infty) \times \mathbb{R})$ . Moreover, assume for every compact set  $K$ , there is  $L_K > 0$  such that  $|f(t, u) - f(t, v)| \leq L_K|u - v|$  for a.e.  $(t, u), (t, v) \in K$ . Then, for a given initial value  $u_0$ , the solution in  $L^\infty_{\text{loc}}[0, T_b)$  is unique. Further, we have*

- *Any two solutions  $u_i \in L^\infty_{\text{loc}}[0, T_b^i)$  ( $i = 1, 2$ ) with initial values  $u_{1,0} < u_{2,0}$  satisfy  $u_1(t) < u_2(t)$  on  $[0, \min(T_b^1, T_b^2))$ .*
- *For any  $T > 0$ ,  $M > 0$ , there exists  $C(M, T) > 0$  such that any two solutions with  $\|u_i\|_{L^\infty[0, T]} \leq M$  ( $i = 1, 2$ ) and initial values  $u_{1,0}, u_{2,0}$  satisfy*

$$\|u_1 - u_2\|_{L^\infty[0, T]} \leq C(M, T)|u_{1,0} - u_{2,0}|.$$

**Proof.** Fix  $T \in (0, \min(T_b^1, T_b^2))$ . There exists  $K$  compact such that for a.e  $t \in [0, T]$ ,  $(t, u_i(t)) \in K$ . By Proposition 1.1, one has

$$u_i(t) = u_{i,0} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1}f(s, u_i(s)) ds.$$

The boundedness of  $f(s, u_i(s))$  implies that  $u_i(t) \in C[0, T]$ . If  $u_{1,0} = u_{2,0}$ , by taking the difference,  $|u_1(t) - u_2(t)| \leq C \int_0^t (t-s)^{\gamma-1}|u_1(s) - u_2(s)| ds$  and the uniqueness therefore follows.

Now, assume  $u_{1,0} \neq u_{2,0}$ . Define  $y(t) = (u_2(t) - u_1(t))/(u_{2,0} - u_{1,0})$ , we have

$$y(t) + \int_0^t (t-s)^{\gamma-1}v(s)y(s) ds = 1, \text{ where } v(s) = -\frac{1}{\Gamma(\gamma)} \frac{f(s, u_2(s)) - f(s, u_1(s))}{u_2(s) - u_1(s)}.$$

If  $u_1(s) = u_2(s)$ , we define  $v(s) = 0$ . Note that  $|v| \leq L_K/\Gamma(\gamma)$  a.e. for  $t \in (0, T)$ . By setting  $h = 1$  in Lemma 4.1, one has

$$1 - \int_0^t r_\lambda(t-s) ds = E_\gamma(-\lambda\Gamma(\gamma)t^\gamma) > 0.$$

By Lemma 4.1,  $y(t) > 0$ . Since  $y$  is continuous, satisfying

$$y(t) \leq 1 + \int_0^t (t-s)^{\gamma-1}\|v\|_{L^\infty[0, T]}y(s) ds,$$

we have  $y(t) \leq C(\|v\|_{L^\infty}, T)$  by [15, Proposition 5]. This verifies the last claim.  $\square$

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