Real-Time Solution of Time-Varying Yau Filtering Problems via Direct Method and Gaussian Approximation

Xiuqiong Chen, Ji Shi, and Stephen S.-T. Yau, *Fellow, IEEE*

Dedicated to Professor Peter Caines on the occasion of his 73th birthday.

***Abstract*—Direct method for Yau filtering system has been studied since 1990s and all these work are limited in time-invariant systems. In this work, we extend the direct method so that it is applicable to time-varying cases. We need less assumptions compared with our previous work. The novelty of this work is that we propose several transformations on the forward Kolmogorov equation so that it can be solved by means of solving some ordinary differential equations if the initial distribution is Gaussian. The corresponding results for any non-Gaussian initial distributions can be obtained via Gaussian approximation. It can be seen that our new scheme direct method can treat nearly most general Yau filtering problems under natural assumptions. Our algorithm has been compared with the extended Kalman filter, multilevel particle filter, and ensemble Kalman filter by numerical examples and the simulation results show the efficiency of our method.**

***Index Terms*—Direct method, Duncan-Mortensen-Zakai (DMZ) equation, Gaussian approximation, nonlinear filtering (NLF), time-varying Yau system.**

I. Introduction

How to estimate the state of a stochastic dynamical system from noisy observations taken on the state is of central significance in engineering and filter is a powerful tool to estimate unobservable stochastic processes that arise in many applied fields including communication, target tracking, and mathematical finance. The continuous time-varying filtering problem can be stated as follows:

$$\begin{cases} dx_t = f(x_t,t)dt + g(t)dv_t \\ dy_t = h(x_t,t)dt + dw_t \end{cases} \tag{1}$$

where $x_t, f \in \mathbb{R}^{n\times 1}$, g is an $n \times r$ matrix, v_t is an $r-$vector Brownian motion process with $E[dv_t dv_t^T] = \tilde{Q}(t)dt$ and $\tilde{Q}(t) > 0$, $y_t, h \in \mathbb{R}^{m\times 1}$, and w_t is an $m-$vector Brownian motion process with $E[dw_t dw_t^T] = S(t)dt$ and $S(t) > 0$. Here we refer x_t as the state of the system at time t, $f(x_t,t)$ as the drift term, $\tilde{Q}(t), S(t)$ as the covariance of the noises, and y_t as the observation at time t with $y_0 = 0$.

Interest in filtering problem can be dated back almost two centuries to the work of Gauss and later, the names of Wiener and Kalman are associated with advances in filtering theory. The most influential work in filtering theory are the classical Kalman filter (KF) [16], which was published in 1960, and its continuous counterpart Kalman–Bucy filter [17]. Since most systems considered in real applications are nonlinear, there have been a lot of work which extend the filtering results to the nonlinear filtering (NLF) problems, such as the extended KF (EKF), ensemble KF (EnKF) [11], and particle filter (PF) [9], [13]. In fact, EKF, which is the simplest filter for NLF systems, performs poorly when the dynamic system is significantly nonlinear and is very sensitive to initial value due to Taylor approximation. EnKF is the NLF theory unifying the data assimilation and ensemble generation problem, and has been key foci of prediction and predictability research for numerical weather and ocean prediction applications [2], [3], [19]. PF is also one of the most popular method nowadays, which can be referred to in [4] and [5], and references therein. And more recently, it uses diffusion processes to model continuous-time phenomena [12]. Besides, multilevel Monte Carlo framework is extended to PF, which is called the multilevel PF (MLPF) in [15].

Since our interest lies especially in the conditional mean, which is the minimum variance estimate, another way to NLF problem is to derive the conditional probability of the state. It is known that the unnormalized probability density function of the state satisfies the Duncan–Mortensen–Zakai (DMZ) equation [10], [23], [36]. And the third author proposed an algorithm [21] to solve general NLF problems using DMZ equations in real-time manner. "Real-time" means that the decision of the states is made on the spot instantaneously, while the observation data keep coming in. Obviously, the real-time property is of much significance in practical applications. For example, it is hard to implement PF in real time due to time-consuming Monte Carlo simulation.

However, we usually cannot get the explicit solution of the DMZ equation in most situations and there are two methods to solve it explicitly to the best of our knowledge for the past quarter of a century. One of them is to use Lie algebraic method proposed by Brockett [6] and Mitter [22], and the details of this method were worked out in [26]. The basic idea is to solve the DMZ equation by solving a finite system of ordinary differential equations (ODE), Kolmogorov equation, and several first-order linear partial differential equations (PDE). However, one must know the basis of the estimation algebra. Yau and his coworkers [8], [28], [33] have completely classified all finite dimensional estimation algebras of maximal rank. In particular, they have proved that for all finite dimensional filters, the observation terms $h_i(x), 1 \le i \le m$ in (1), must be polynomials of degree one.

The other approach to solve DMZ equation is the direct method which works particularly well for the Yau filtering system, i.e., $f(x,t)$ in (1) is of the form $f(x,t) = Lx + l + \nabla\phi(x)$ where L and l are

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X. Q. Chen is with the Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China (e-mail: cxq14@mails.tsinghua.edu.cn).

J. Shi is with the Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China (e-mail: shij13@mails.tsinghua.edu.cn).

S. S.-T. Yau is with the Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China (e-mail: yau@uic.edu).

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matrices with proper dimensions and $\phi(x)$ is a C^∞ function. This method was introduced in [27] and generalized in [14], [29], and [30]. Unlike the Lie algebra method, direct method does not need to integrate n first-order linear PDEs. However, for the direct methods in [14], [27], [29], and [30], they need to assume that the observation terms $h_i(x), 1 \leq i \leq m$ in (1) are degree one polynomials. In [31], Yau and Lai solved DMZ equation by solving a series of ODEs when the initial distribution is Gaussian. In [32], Yau and Yau transformed the DMZ equation to time varying Schrödinger equation in very general cases where observation terms are of linear growths.

However, all these existing results about direct method are for time-invariant systems and they need to assume that $g(t)\tilde{Q}(t)g^T(t)$ is an identity matrix, and these conditions are also needed for the recent work [25]. More recently, we extended the related results to time-varying situations with some restrictions to the filtering system [7]. In this work, we remove these restrictions and extend the direct method to nearly most general Yau filtering systems and the only two assumptions for the filtering system seem very natural. Here we study the time-varying Yau filtering system with arbitrary initial distributions by two steps. First, we transform the DMZ equation to the Kolmogorov forward equation (KFE), and then obtain the explicit solution to the KFE by solving some ODEs when the initial distribution is Gaussian. Second, in the non-Gaussian cases, we approximate the non-Gaussian distribution by several Gaussian distributions by the use of Gaussian approximation proposed in [25], and then continue this procedure using results obtained in the first step.

This paper is organized as follows. In Section II, we recall some basic concepts and existing results with respect to (w.r.t.) the filtering problem. In Section III, we give the explicit solution of the KFE when the initial distribution is Gaussian. Section IV is devoted to obtain the numerical results of the KFE with arbitrary initial distribution with the assistance of Gaussian approximation. We present numerical simulation results in Section V and draw our conclusion in the last section.

II. Basic Filtering Problems

In the considered continuous time-varying filtering system (1), we assume that $G(t) \triangleq g(t)\tilde{Q}(t)g^T(t)$ is C^∞ smooth, $f(x,t)$ and $h(x,t)$ are C^∞ smooth in both state and time. For the sake of clarity, we shall explain some notations first: A_{ij} denotes the ijentry of an arbitrary matrix A, a_i denotes the ith element of an arbitrary vector a, and A^T denotes the transposition of A.

In terms of the density function $\rho(t,x)$ of x_t conditioned on the observation history $\mathscr{F}_t \triangleq \{y_s : \ 0 \leq s \leq t\}$, we know that it must satisfy the normalization condition, i.e.,

$$\int \rho(t,x)dx = 1. \tag{2}$$

Actually, if there is any function $\sigma(t,x)$ which satisfies

$$\rho(t,x) \propto \sigma(t,x) \text{ w.r.t. } x \tag{3}$$

then we can compute $\rho(t,x)$ by normalization

$$\rho(t,x) = \frac{\sigma(t,x)}{\int \sigma(t,x)dx}. \tag{4}$$

In [10], we know that the unnormalized density function $\sigma(t,x)$ of x_t conditioned on the observation history $\mathscr{F}_t$ satisfies the DMZ equation

$$\begin{cases} d\sigma(t,x) = \left[\frac{1}{2}\sum_{i,j=1}^n G_{ij}(t)\frac{\partial^2\sigma}{\partial x_i \partial x_j}(t,x) - \sum_{i=1}^n f_i \frac{\partial \sigma}{\partial x_i}(t,x) \right. \\ \qquad \left. -\sigma(t,x)\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(t,x)\right] dt \\ \qquad +\sigma(t,x)h^T(x,t)S^{-1}(t)dy_t \\ \sigma(0,x) = \ \sigma_0(x) \end{cases} \tag{5}$$

where $\sigma_0(x)$ is the probability density of the initial state x_0. For each arrived observation, making an invertible exponential transformation [24]

$$u(t,x) = \exp\left[-h^T(x,t)S^{-1}(t)y_t\right]\sigma(t,x) \tag{6}$$

the DMZ equation is transformed into a deterministic PDE with stochastic coefficients

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\sum_{i,j=1}^n G_{ij}(t)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) \\ \qquad +\sum_{i=1}^n \left(\sum_{j=1}^n G_{ij}(t)\frac{\partial \tilde{K}}{\partial x_j} - f_i\right)\frac{\partial u}{\partial x_i}(t,x) \\ \qquad +\left(-\frac{\partial}{\partial t}\left(h^T S^{-1}\right)^T y_t \right. \\ \qquad +\frac{1}{2}\sum_{i,j=1}^n G_{ij}(t)\left[\frac{\partial^2 \tilde{K}}{\partial x_i \partial x_j} + \frac{\partial \tilde{K}}{\partial x_i}\frac{\partial \tilde{K}}{\partial x_j}\right] \\ \qquad -\sum_{i=1}^n f_i \frac{\partial \tilde{K}}{\partial x_i}(t,x) - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(t,x) \\ \qquad \left. -\frac{1}{2}\left(h^T S^{-1} h\right)\right) u(t,x) \\ u(0,x) = \sigma_0(x) \end{cases} \tag{7}$$

in which

$$\tilde{K}(x,t) = h^T(x,t)S^{-1}(t)y_t. \tag{8}$$

We shall call (7) "pathwise-robust" DMZ equation in this paper. However, the exact solution to (7), generally speaking, does not have a closed form. Therefore, many mathematicians try to seek an efficient algorithm to construct a good approximation. Let us assume that the observations arrive at discrete instants, therefore we construct the approximation as in [21] and [35], and get the robust DMZ equation (9) in each time interval.

Let us denote the observation time sequence as $\mathcal{P}_k = \{0 = \tau_0 < \tau_1 < \cdots < \tau_N = T\}$. Let u_k be the solution of the robust DMZ equation with $y_t = y_{\tau_{k-1}}$ on the time interval $\tau_{k-1} \leq t \leq \tau_k, \ k = 1, 2, \ldots, N$

$$\begin{cases} \frac{\partial u_k}{\partial t}(t,x) = \frac{1}{2}\sum_{i,j=1}^n G_{ij}(t)\frac{\partial^2 u_k}{\partial x_i \partial x_j}(t,x) \\ \qquad +\sum_{i=1}^n \left(\sum_{j=1}^n G_{ij}(t)\frac{\partial \tilde{K}}{\partial x_j} - f_i\right)\frac{\partial u_k}{\partial x_i}(t,x) \\ \qquad +\left(-\frac{\partial}{\partial t}\left(h^T S^{-1}\right)^T y_{\tau_{k-1}} \right. \\ \qquad +\frac{1}{2}\sum_{i,j=1}^n G_{ij}(t)\left[\frac{\partial^2 K}{\partial x_i \partial x_j} + \frac{\partial \tilde{K}}{\partial x_i}\frac{\partial \tilde{K}}{\partial x_j}\right] \\ \qquad -\sum_{i=1}^n f_i \frac{\partial \tilde{K}}{\partial x_i}(t,x) - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(t,x) \\ \qquad \left. -\frac{1}{2}\left(h^T S^{-1} h\right)\right) u_k(t,x) \\ u_1(0,x) = \sigma_0(x) \\ u_k(\tau_{k-1},x) = u_{k-1}(\tau_{k-1},x), \ k = 2, 3, \ldots, N \end{cases} \tag{9}$$

with

$$\tilde{K}(x,t) = h^T(x,t)S^{-1}(t)y_{\tau_{k-1}}. \tag{10}$$

Define the norm of $\mathcal{P}_k$ by $|\mathcal{P}_k| = \sup_{1\le k\le N}(\tau_k - \tau_{k-1})$. By [34], we know that in both point-wise sense and L^2 sense

$$u(\tau,x) = \lim_{|\mathcal{P}_k|\to 0} u_k(\tau,x). \tag{11}$$

Therefore, $u_k(t,x)$ is a good approximation of $u(t,x)$ in the interval $[\tau_{k-1},\tau_k]$. We only need to seek the solution of DMZ equation (9).

In [21], Luo and Yau proposed an on- and off-line algorithm to solve the NLF problems in real time, which has been verified numerically as an effective tool in very low dimension. The key observation is that the heavy computation of solving PDE can be moved to off-line by the following proposition.

Proposition 1: (Proposition 2.1, [21]) For each $\tau_{k-1}\le t\le \tau_k$, $k=1,2,\ldots,N$, $u_k(t,x)$ satisfies (9) if and only if

$$\tilde{u}_k(t,x) = \exp\left[h^T(x,t)S^{-1}(t)y_{\tau_{k-1}}\right]u_k(t,x) \tag{12}$$

satisfies the KFE

$$\begin{cases} \dfrac{\partial \tilde{u}_k}{\partial t}(t,x) = \dfrac{1}{2}\sum_{i,j=1}^n G_{ij}(t)\dfrac{\partial^2 \tilde{u}_k}{\partial x_i \partial x_j}(t,x) - \sum_{i=1}^n f_i \dfrac{\partial \tilde{u}_k}{\partial x_i}(t,x) \\ \qquad - \left(\sum_{i=1}^n \dfrac{\partial f_i}{\partial x_i}(t,x) + \dfrac{1}{2}h^T S^{-1} h\right)\tilde{u}_k(t,x) \\ \tilde{u}_1(0,x) = \sigma_0(x) \\ \tilde{u}_k(\tau_{k-1},x) = \exp\left[h^T(x,\tau_{k-1})S^{-1}(\tau_{k-1})\left(y_{\tau_{k-1}} - y_{\tau_{k-2}}\right)\right] \\ \qquad \cdot \tilde{u}_{k-1}(\tau_{k-1},x),\ k=2,3,\ldots,N. \end{cases} \tag{13}$$

III. Explicit Solution of the DMZ Equation in Terms of Solutions of ODEs

The results for time-invariant Yau filtering system [14], [27], [34] have been extended in [7] and [25]. In this section, we aim to extend the results to the more general time-varying Yau filtering systems

$$f(x,t) = L(t)x + l(t) + \nabla_x \widetilde{\phi}(t,x) \tag{14}$$

where $L(t) = (l_{ij}(t)), 1\le i,j\le n, l^T(t) = (l_1(t),\ldots,l_n(t))$, and $\widetilde{\phi}(t,x)$ is C^∞ w.r.t. x on $\mathbb{R}^n$. For the conciseness of notation, we shall omit the t in l and L in the sequel if no confusion will arise.

Now we give the first assumption.

Assumption 1: $G(t)$ is a positive definite matrix.

Under this assumption, we can obtain the following proposition.

Proposition 2: [7] Suppose $\tilde{u}_k(t,x)$ is the solution to (13) in the interval $[\tau_{k-1},\tau_k]$, $k=1,2,\ldots,N$, and $f(x,t)$ is of the form (14). Let

$$\tilde{u}_k(t,x) = e^{\phi(t,x)}\tilde{v}_k(t,x) \tag{15}$$

where $\phi(t,x)$ satisfies $\nabla_x\phi(t,x) = G^{-1}(t)\nabla_x\widetilde{\phi}(t,x)$, then we have the following equation for $\tilde{v}_k(t,x)$:

$$\begin{cases} \dfrac{\partial \tilde{v}_k}{\partial t}(t,x) = & \dfrac{1}{2}\sum_{i,j=1}^n G_{ij}(t)\dfrac{\partial^2 \tilde{v}_k}{\partial x_i\partial x_j}(t,x) \\ & -(Lx+l)^T\nabla\tilde{v}_k(t,x) - \dfrac{1}{2}q(t,x)\tilde{v}_k(t,x) \\ \tilde{v}_1(0,x) = & \sigma_0(x)e^{-\phi(0,x)} \\ \tilde{v}_k(\tau_{k-1},x) = & \exp\left[h^T(x,\tau_{k-1})S^{-1}(\tau_{k-1})(y_{\tau_{k-1}} - y_{\tau_{k-2}})\right] \\ & \cdot\tilde{v}_{k-1}(\tau_{k-1},x),\ k=2,3,\ldots,N \end{cases} \tag{16}$$

where

$$\begin{aligned} q(t,x) = & -\sum_{i,j=1}^n G_{ij}(t)\frac{\partial^2\phi}{\partial x_i\partial x_j}(t,x) \\ & + \nabla_x\phi^T(t,x)G(t)\nabla_x\phi(t,x) \\ & + 2(Lx+l)^T\nabla_x\phi(t,x) \\ & + 2\sum_{i=1}^n \frac{\partial^2\widetilde{\phi}(t,x)}{\partial^2 x_i^2} + 2\frac{\partial\phi(t,x)}{\partial t} \\ & + \sum_{p,l=1}^n S_{pl}^{-1}(t)h_p(x,t)h_l(x,t) + 2tr(L). \end{aligned} \tag{17}$$

Since $G(t)$ is positive definite, then we can find a positive definite matrix $F(t)>0$ such that

$$G(t) = F(t)F^T(t) \tag{18}$$

according to Cholesky decomposition.

Theorem 1: Under Assumption 1, suppose $\tilde{v}_k(t,x)$ is a solution of (16) and let

$$\tilde{v}_k(t,x) = v_k(t,z) \tag{19}$$

where

$$\begin{aligned} z &= B(t)x \\ \text{and}\quad B(t) &= F^{-1}(t). \end{aligned} \tag{20}$$

Then $v_k(t,z)$ is the solution of the following equation:

$$\begin{cases} \dfrac{\partial v_k}{\partial t}(t,z) = \dfrac{1}{2}\Delta v_k(t,z) - \dfrac{1}{2}q\left(t,F(t)z\right)v_k(t,z) \\ \qquad -\left[\left(\dfrac{dB}{dt}B^{-1} + BLB^{-1}\right)z + Bl\right]^T\nabla v_k(t,z) \\ v_1(0,z) = \sigma_0(F(0)z)\exp\left(-\phi(0,F(0)z)\right) \\ v_k(\tau_{k-1},z) = \exp\left[h^T\left(F(\tau_{k-1})z,\tau_{k-1}\right)S^{-1}(\tau_{k-1})\right. \\ \qquad \left.\cdot(y_{\tau_{k-1}} - y_{\tau_{k-2}})\right]v_{k-1}(\tau_{k-1},z) \\ \qquad k=2,3,\ldots,N. \end{cases} \tag{21}$$

Proof: Through direct computations, we have

$$\begin{aligned} \frac{\partial\tilde{v}_k}{\partial t}(t,x) &= \frac{\partial v_k}{\partial t}(t,z) + \sum_{i,j=1}^n \frac{\partial v_k}{\partial z_i}(t,z)\frac{dB_{ij}(t)}{dt}x_j \\ &= \frac{\partial v_k}{\partial t}(t,z) + \left(\frac{dB(t)}{dt}x\right)^T\nabla v_k(t,z) \end{aligned} \tag{22}$$

$$\frac{\partial\tilde{v}_k}{\partial x_i}(t,x) = \sum_{l=1}^n \frac{\partial v_k}{\partial z_l}(t,z)B_{li} \tag{23}$$

and

$$\frac{\partial^2\tilde{v}_k}{\partial x_i\partial x_j}(t,x) = \sum_{p,l=1}^n \frac{\partial^2 v_k}{\partial z_p\partial z_l}(t,z)B_{pi}B_{lj}. \tag{24}$$

Then we can get the compact form

$$\nabla\tilde{v}_k(t,x) = B^T\nabla v_k(t,x) \tag{25}$$

and

$$\begin{aligned}&\sum_{i,j=1}^{n} G_{ij}(t)\frac{\partial^2 \tilde{v}_k}{\partial x_i \partial x_j}(t,x)\\&= \sum_{i,j=1}^{n}\sum_{p,l=1}^{n} B_{il}G_{lp}B_{jp}\frac{\partial^2 v_k}{\partial z_i \partial z_j}(t,z)\\&= \sum_{i,j=1}^{n}\left(BGB^T\right)_{ij}\frac{\partial^2 v_k}{\partial z_i \partial z_j}(t,z)\\&= \triangle v_k(t,x) \end{aligned} \tag{26}$$

since $BGB^T = I$ by (18) and (20), and I is the $n \times n$ identity matrix.

Substitute (19), (25), and (26) into (16), we can easily arrive at (21), which is the desired result. □

Let

$$\tilde{q}(t,z) = q(t, F(t)z) \tag{27}$$

and we rewrite (21) as in the following form before we proceed:

$$\begin{cases} \dfrac{\partial v_k}{\partial t}(t,x) = \dfrac{1}{2}\Delta v_k(t,x) - \dfrac{1}{2}\tilde{q}(t,x)v_k(t,x) \\ \qquad - \left[\left(\dfrac{dB}{dt}B^{-1} + BLB^{-1}\right)x + Bl\right]^T \nabla v_k(t,x) \\ v_1(0,x) = \sigma_0(F(0)x)\exp\left(-\phi(0,F(0)x)\right) \\ v_k(\tau_{k-1},x) = \exp\left[h^T(F(\tau_{k-1})x,\tau_{k-1})S^{-1}(\tau_{k-1})\right. \\ \qquad \left.(y_{\tau_{k-1}} - y_{\tau_{k-2}})\right] v_{k-1}(\tau_{k-1},x) \\ \qquad k = 2,3,\ldots,N. \end{cases} \tag{28}$$

Now we continue to seek the explicit solution of the KFE (28).

When the KFE has Gaussian initial value, Yau and Lai [31] wrote down its solution by means of the solution of certain ODEs, Inspired by this, we consider how to get the solution of (28) under the assumption that the initial value $v_k(\tau_{k-1},x)$ is Gaussian. In what follows, we shall need the following natural assumption.

Assumption 2: $\tilde{q}(t,x)$ defined in (27) is quadratic w.r.t. x.

And it is natural that $\tilde{q}(t,x)$ can be written in the following form:

$$-\frac{1}{2}\tilde{q}(t,x) = x^T Q(t)x + p^T(t)x + r(t) \tag{29}$$

where $Q(t)$ is an $n \times n$ symmetric matrix, $p(t)$ is an $n \times 1$ vector, and $r(t)$ is a scalar.

Now we can draw the similar conclusion which is summarized in the following theorem.

Theorem 2: Under Assumptions 1 and 2, consider the following KFE with Gaussian initial distribution:

$$\begin{cases} \dfrac{\partial v_k}{\partial t}(t,x) = \dfrac{1}{2}\Delta v_k(t,x) - \dfrac{1}{2}\tilde{q}(t,x)v_k(t,x) \\ \qquad - \left[\left(\dfrac{dB}{dt}B^{-1} + BLB^{-1}\right)x + Bl\right]^T \nabla v_k(t,x) \\ v_k(\tau_{k-1},x) = \exp\left\{x^T A(\tau_{k-1})x + b^T(\tau_{k-1})x + c(\tau_{k-1})\right\} \end{cases} \tag{30}$$

where $A(\tau_{k-1})$ is an $n \times n$ symmetric matrix, $b(\tau_{k-1})$ is an $n \times 1$ vector, $x^T = (x_1, x_2, \ldots, x_n)$ is a row vector, and $c(\tau_{k-1})$ is a scalar. Then the solution of (30) is of the following form:

$$v_k(t,x) = \exp\left\{x^T A(t)x + b^T(t)x + c(t)\right\} \tag{31}$$

where A(t) is an $n \times n$ symmetric matrix valued function of t, $b(t)$ is an $n \times 1$ vector valued function of t, $c(t)$ is a scalar valued function of t and satisfy the following system of nonlinear ODEs:

$$\begin{aligned} \frac{dA(t)}{dt} &= 2A^2(t) - 2A(t)D(t) + Q(t) \\ \frac{db^T(t)}{dt} &= 2b^T(t)A(t) - b^T(t)D(t) - 2d^T(t)A(t) + p^T(t) \\ \frac{dc(t)}{dt} &= trA(t) + \frac{1}{2}b^T b(t) - d^T(t)b(t) + r(t) \end{aligned} \tag{32}$$

with

$$D(t) = \frac{dB}{dt}B^{-1} + BLB^{-1}, d(t) = B(t)l(t). \tag{33}$$

Proof: Let A_i be the ith row of A. By direct computations, we have

$$\frac{\partial v_k}{\partial t}(t,x) = v_k(t,x)\left(x^T\frac{dA(t)}{dt}x + \frac{db^T(t)}{dt}x + \frac{dc(t)}{dt}\right) \tag{34}$$

$$\frac{\partial v_k}{\partial x_i}(t,x) = v_k(t,x)\left(A_i x + A_i^T x + b_i\right) \tag{35}$$

then it can be easily seen

$$\begin{aligned}\nabla_x v_k(t,x) &= v_k(t,x)\left(Ax + A^T x + b\right)\\ &= v_k(t,x)(2Ax + b)\end{aligned} \tag{36}$$

since $A(t)$ is symmetric. Following (35), we get

$$\triangle v_k(t,x) = v_k(t,x)\left(4x^T A^2 x + 4x^T Ab + b^T b + 2tr(A)\right). \tag{37}$$

Put (34), (36), and (37) into (30), we have

$$\begin{aligned} &x^T\frac{dA(t)}{dt}x + \frac{db^T(t)}{dt}x + \frac{dc(t)}{dt}\\ &= 2x^T A^2 x + 2x^T Ab + \frac{1}{2}b^T b + tr(A) - 2x^T ADx\\ &\quad - \left(b^T D + 2d^T A\right)x - d^T b + x^T Qx + p^T x + r \end{aligned} \tag{38}$$

and compare the coefficients of both sides of (38), then we can easily reach the desired result. □

However, the initial value $v_k(\tau_{k-1},x)$ in (28) in every step usually cannot be Gaussian. Therefore we need to derive its Gaussian approximation [25] which will be discussed in the next section.

IV. Numerical Algorithm

In this section, we derive Gaussian approximation of the initial value $v_k(\tau_{k-1},x)$ in (28) in every step and then get the numerical results of general (28). First, we introduce the Gaussian approximation algorithm derived in [25].

A. Gauss Approximation

Given a probability density $\phi(x)$ and the threshold E, [25] proposed a numerical algorithm to get a Gaussian approximation $\tilde{\phi}(x) = \sum_{i=1}^{\widetilde{N}} \alpha_i \mathcal{N}(\mu_i, \sigma_i)$ which satisfies $\max_x |\phi(x) - \tilde{\phi}(x)| \le E$, and $\widetilde{N}, \alpha_i, \mu_i, \sigma_i$ are determined by probability density $\phi(x)$ and the threshold E. This Gaussian approximation method is summarized in Algorithm 1.

B. Numerical Solution

Now we use Theorem 2 and Algorithm 1 to derive the numerical solution of the KFE (28) and then get the conditional density function $\sigma(t,x)$ as well as the conditional mean of the state. The detailed steps are shown in Algorithm 2.

Algorithm 1: Gaussian approximation.

1: Let $f(x) = \phi(x)$ and the threshold $E = \alpha * \max \phi(x)$, where α is a given small number.
2: Fitting the peaks of $f(x)$ which are larger than E with gaussian distributions. Suppose the sum of gaussian distributions in this step is $g(x)$.
3: Let $f_1(x) = f(x) - g(x)$. If $f_1(x)$ has no peaks whose values are larger than E, then go to step 4. Otherwise, let $f(x) = f_1(x)$ and go to step 2.
4: Let $f_2(x) = -f_1(x)$. If $f_2(x)$ has no peaks which are larger than E, then done. Otherwise, let $f(x) = f_2(x)$ and go to step 2.

Algorithm 2: Direct method.

1: **Initialization**: give $T, \Delta t, \sigma_0(x)$ and the parameter α in Algorithm 1. Let $N = \frac{T}{\Delta t}$ and $\{0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_N = T\}$.
2: **for** $k = 1 : N$ **do**
3: Using Algorithm 1 to get the Gaussian approximation $v_k(\tau_{k-1}, x) \approx \sum_{i=1}^{N(k)} \alpha_{k,i} \mathcal{N}(\mu_{k,i}, \sigma_{k,i})$.
4: For each Gaussian distribution $\mathcal{N}(\mu_{k,i}, \sigma_{k,i})$, suppose the solution of (30) with initial condition $\mathcal{N}(\mu_{k,i}, \sigma_{k,i})$ is $v_{k,i}(\tau_k, x)$. Solving (32), we obtain $v_{k,i}(\tau_k, x)$. Then $v_k(\tau_k, x) = \sum_{i=1}^{N(k)} \alpha_{k,i} \hat{u}_{k,i}(\tau_k, x)$.
5: Calculate $v_{k+1}(\tau_k, x)$ by $v_k(\tau_k, x)$ and (28).
6: Calculate $\tilde{v}_k(t_k, x), \tilde{u}_k(t_k, x)$ by (15) and (19).
7: Calculate $u_k(t_k, x), \sigma(t_k, x)$ by (6) and (12).
8: Calculate $\rho(t_k, x)$ by (4).
9: Calculate the conditional expectation of the state x_{t_k}.
10: **end for**

V. Simulation

In this section, we use two numerical examples to verify the efficiency of the proposed Algorithm 2, and the filtering system here is as follows:

$$\begin{cases} dx_t = f(x_t, t)dt + dv_t \\ dy_1(t) = x_t \sin(x_t)dt + dw_1(t) \\ dy_2(t) = x_t \cos(x_t)dt + dw_2(t) \\ \sigma_0(x) = \exp(-x \sin x - \frac{1}{2} x \cos x - x^2 + 3x + 2). \end{cases} \tag{39}$$

Here, $v(t)$, $w_1(t)$, and $w_2(t)$ are scalar independent Brownian motions, $E(v(t)v(t)) = \sigma_1^2$, $E(w_1(t)w_1(t)) = \sigma_2^2$, and $E(w_2(t)w_2(t)) = \sigma_3^2$.

To compare the average performance of different methods, we introduce the mean of the squared estimation error (MSE) defined in [20]. The MSE for M repeated realizations at instant τ_k is defined as follows:

$$\frac{1}{M} \sum_{i=1}^{M} (x_{\tau_k}^i - \hat{x}_{\tau_k}^i)^2 \tag{40}$$

where $x_{\tau_k}^i$ is the real state at instant τ_k in the ith realization and $\hat{x}_{\tau_k}^i$ is the estimation of $x_{\tau_k}^i$ by different filtering methods.

In the following two examples, the real dynamic system (39) is approximated by Euler's method in [18] with time step $\Delta t = 0.1$ where Δt is the sampling interval for observations and dy at the discrete instants is computed by (39) directly.

TABLE I
Number of the Particles Used in MLPF for Example 1

Level	1	2	3	4
particle number	2250	2000	1750	1500
Level	5	6	7	8
particle number	1250	1000	750	500

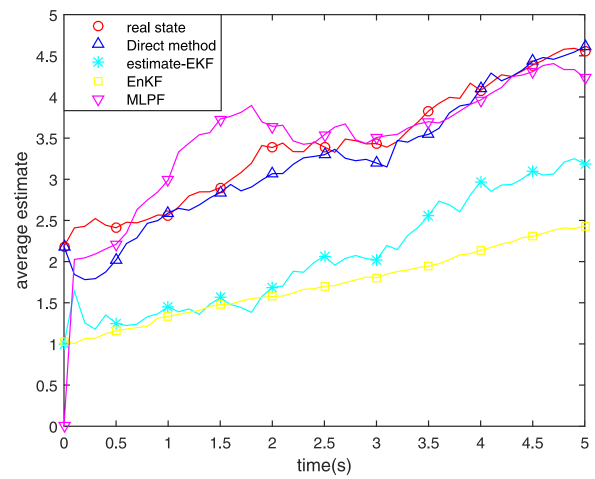

Fig. 1. Average estimation results based on 20 simulations of Example 1.

Example 1: In this example, $f(x_t, t) = c(x_t + 1)$ where $c = 0.1$ is a constant and $\sigma_1 = 1 + 0.1 \sin 2t, \sigma_2 = \sigma_3 = 1$. It can be easily checked that the $G(t)$ defined for this system is not identity matrix and therefore we cannot use the result in the previous work [25] to solve this problem while the direct method proposed in this paper.

The initial values in Algorithm 2 are $T = 5, \Delta t = 0.1, \alpha = 0.1$, and the initial value of real state is 2.16. The EKF was numerically implemented by ode45 function in MATLAB with mean 0 and covariance 5. Besides, when implementing the MLPF, we approximate the dynamic system (39) by the Euler's method for MLPF at levels $L = 1, \ldots, 8$ [15] and the number of particles in each level is shown in Table I. Besides, we use 2500 particles at level $L = 0$. EnKF considered here is with 80 ensembles. The average estimations over $M = 20$ simulations and the MSE for different methods are displayed in Figs. 1 and 2, respectively. It can be seen easily that the direct method performs better than the EnKF and MLPF, and it can be also concluded from Fig. 2 that the estimation error is bounded. On the other hand, the MLPF is rather time consuming since it needs a lot of particles in the Monte Carlo simulations and the average time cost of different methods is shown in Table II.

In summary, our direct method gains advantage over the EKF, EnKF, and MLPF in real applications.

Example 2: In the second example, $f(x_t, t) = x_t + 1 + \frac{dF}{dx}$, where

$$F(x) = \int_{-\infty}^{x} \left\{ \frac{e^{-(z-\frac{1}{2})^2}}{\int_{-\infty}^{z} e^{-(y-\frac{1}{2})^2} dy} - 3/2 \right\} dz \tag{41}$$

and $\sigma_1 = 1, \sigma_2 = \sigma_3 = 1 + \sin 2t/10$. Then the $\tilde{q}$ is time varying. The initial values in Algorithm 2 are $T = 1.5, \Delta t = 0.1, \alpha = 0.01$, and the integral in (41) is numerically approximated by Gaussian quadrature method with 60 points [1]. Similarly, the EKF was numerically realized by ode45 function in MATLAB with initial state 1 and initial covariance 3. And we approximate the dynamic system (39) by the Euler's method

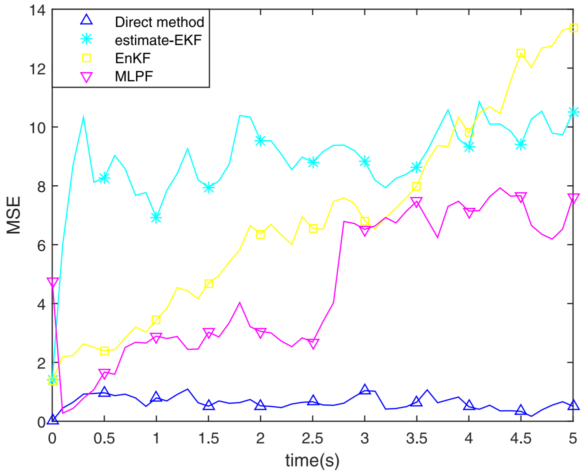

Fig. 2. MSE of different methods based on 20 simulations of Example 1.

TABLE II
AVERAGE TIME COST OF DIFFERENT METHODS FOR EXAMPLE 1

Methods	Direct method	EKF	MLPF	EnKF
time(s)	1.7400	0.1586	49.3457	0.1812

TABLE III
NUMBER OF THE PARTICLES USED IN MLPF FOR EXAMPLE 2

Level	1	2	3	4
particle number	4500	4000	3500	3000
Level	5	6	7	8
particle number	2500	2000	1500	1000

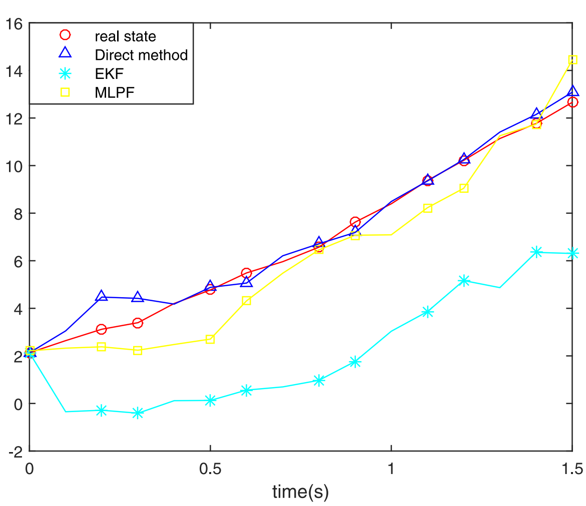

Fig. 3. Estimation result of different methods based on one simulation of Example 2.

for MLPF at levels $L = 1, \ldots, 8$ and the number of particles in each level is shown inTable III with 5000 particles at level $L = 0$. The performance in one realization can be seen from Fig. 3 and the MSE over $M = 10$ simulations is displayed in Fig 4. It is obvious that the direct method can track the real state better compared with EKF and MLPF. Besides, the average time cost over 10 simulations are shown in TABLE IV. It can be seen that the MLPF is even more time consuming and this is because we need to compute the integral in (41) for every particle when sampling the dynamic system. Considering the tradeoff between MSE and running time, our direct method is a better choice in real applications.

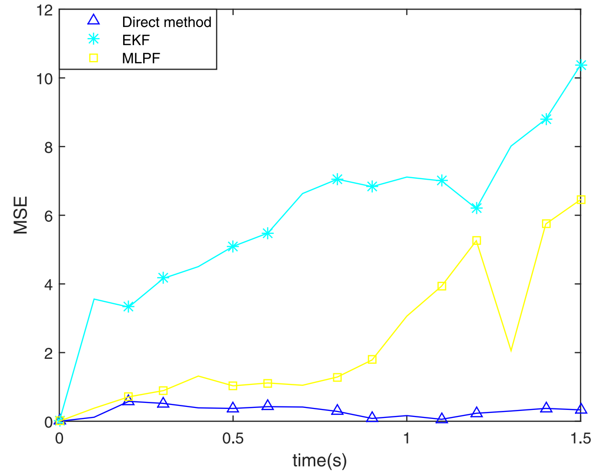

Fig. 4. MSE of different methods based on 10 simulations of Example 2.

TABLE IV
AVERAGE TIME COST OF DIFFERENT METHODS FOR EXAMPLE 2

Methods	Direct method	EKF	MLPF
time(s)	0.5045	0.0712	212.5157

VI. CONCLUSION

In this paper, we extended the direct method to a more general case so that it can be applicable to time-varying Yau systems with arbitrary initial distributions. Through several transformations, the DMZ equation can be transformed into a series of ODEs when the initial distribution is Gaussian and then the non-Gaussian cases can be solved by the Gaussian approximation. The numerical results show that the proposed method performs better than the EKF, MLPF, and EnKF considering about both MSE and running time.

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