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An implicit leap-frog discontinuous Galerkin method for the time-domain Maxwell's equations in metamaterials ☆

Jichun Li [a,*,1], Jiajia Wang Waters [a], Eric A. Machorro [b]

[a] Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, NV 89154-4020, USA
[b] National Security Technologies, LLC, P.O. Box 98521, MS NLV 070, Las Vegas, NV 89193-8521, USA

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ABSTRACT

Numerical simulation of metamaterials play a very important role in the design of invisibility cloak, and sub-wavelength imaging. In this paper, we propose a leap-frog discontinuous Galerkin method to solve the time-dependent Maxwell's equations in metamaterials. Conditional stability and error estimates are proved for the scheme. The proposed algorithm is implemented and numerical results supporting the analysis are provided.

1. Introduction

The metamaterials often refer to some artificially constructed electromagnetic materials whose refraction index is negative. Such negative refraction index metamaterials were first successfully demonstrated in 2001 [31]. Due to many potential interesting applications in various fields such as design of invisibility cloak, sub-wavelength imaging, antenna and radar technolgy, the study of metamaterials has attracted a great attention of scientists and engineers since 2000. Numerical simulation plays a very important role in the study of metamaterial and its applications due to its cost effectiveness compared to the physical experiments. However, simulations are mostly restricted to either the classic finite-difference time-domain (FDTD) method [12] or commercial packages such as COMSOL Multiphysics Finite Element Analysis Software. It is known that the FDTD method has a big disadvantage for solving problems with complex geometries. Hence it would be quite interesting and useful to develop efficient and robust finite element methods for modeling metamaterials.

The discontinuous Galerkin (DG) method, originally introduced by Reed and Hill back in 1973, has become one of the most popular methods used for solving various differential equations (e.g., [1,2,5,7,17,18,20,21,27,29,32,35]). The DG method has a great flexibility in mesh construction by allowing conforming or non-conforming meshes and using different orders of basis functions in different elements. In the past decade, there has been a growing interest in developing DG methods for solving Maxwell's equations in free space [3,6,8–11,14,16,28,30]. Very recently, there were some DG investigations [23,26,19,33] carried out for Maxwell's equations in dispersive media, whose permittivity depends on the wave frequency. However, to our best knowledge, the study of DG methods for solving Maxwell's equations in metamaterials is quite limited.

This paper continues our recent initial effort [22] on developing DG methods for solving Maxwell's equations in metamaterials. In [22], we extended the DG method developed by Hesthaven and Warburton [14,15] for Maxwell's equations in free space to metamaterials. Preliminary numerical results were performed and good convergence rates were observed, but without any theoretical analysis. Here we extend the framework of [14,15] to develop a leap-frog type DG method for solving the time-domain Maxwell's equations in metamaterials. Detailed stability results and error estimates for the scheme are carried out, and numerical results consistent with the theoretical analysis are provided.

The rest of the paper is organized as follows. In Section 2, we present our leap-frog DG method, and prove the stability of the

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* Corresponding author. Tel.: +1 702 895 0355.
E-mail address: jichun@unlv.nevada.edu (J. Li).
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scheme. Then in Section 3, we prove the optimal error estimate for the scheme. Numerical results supporting the analysis are presented in Section 4. Finally, we conclude the paper in Section 5.

2. The leap-frog DG method

2.1. The governing equations

The governing equations for modeling wave propagation in metamaterials described by the Drude model has been derived in our early work [25]:

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{J}, \quad \text{in } \Omega \times (0,T), \tag{1}$$

$$\mu_0 \frac{\partial \boldsymbol{H}}{\partial t} = -\nabla \times \boldsymbol{E} - \boldsymbol{K}, \quad \text{in } \Omega \times (0,T), \tag{2}$$

$$\frac{\partial \boldsymbol{J}}{\partial t} + \Gamma_e \boldsymbol{J} = \epsilon_0 \omega_{pe}^2 \boldsymbol{E}, \quad \text{in } \Omega \times (0,T), \tag{3}$$

$$\frac{\partial \boldsymbol{K}}{\partial t} + \Gamma_m \boldsymbol{K} = \mu_0 \omega_{pm}^2 \boldsymbol{H}, \quad \text{in } \Omega \times (0,T), \tag{4}$$

where $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{H}(\boldsymbol{x},t)$ are the electric and magnetic fields, $\boldsymbol{J}(\boldsymbol{x},t)$ and $\boldsymbol{K}(\boldsymbol{x},t)$ are the induced electric and magnetic currents, ϵ_0 and μ_0 are the permittivity and permeability in free space, respectively, ω_{pe} and ω_{pm} are the electric and magnetic plasma frequencies, respectively, Γ_e and Γ_m are the electric and magnetic damping frequencies, respectively. For simplicity, we assume that Ω is a bounded polyhedal domain of R^3 (note that our analysis below holds true for R^2 also), and the system (1)–(4) is supplemented with the perfect conducting boundary condition (PEC):

$$\mathbf{n} \times \boldsymbol{E} = 0 \quad \text{on } \partial\Omega, \tag{5}$$

and initial conditions

$$\boldsymbol{E}(\boldsymbol{x},0) = \boldsymbol{E}_0(\boldsymbol{x}), \ \ \boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}),$$
$$\boldsymbol{J}(\boldsymbol{x},0) = \boldsymbol{J}_0(\boldsymbol{x}), \ \ \boldsymbol{K}(\boldsymbol{x},0) = \boldsymbol{K}_0(\boldsymbol{x}), \tag{6}$$

where $\boldsymbol{n}$ denotes the unit outward normal to $\partial\Omega$, and $\boldsymbol{E}_0, \boldsymbol{H}_0, \boldsymbol{J}_0, \boldsymbol{K}_0$ are some given functions.

2.2. Notation and the DG scheme

We assume that the bounded Lipschitz polyhedral domain Ω is partitioned into disjoint tetrahedral elements T_i such that $\overline{\Omega} = \cup_i T_i$. For each internal face $a_{ik} = T_i \bigcap T_k$, we denote $\mathbf{n}_{ik}$ the unit normal, oriented from T_i towards T_k. We denote h the maximum mesh size, ν_i the set of indices of the neighboring elements of the T_i, F_h^{int} the union of internal faces. Furthermore, we denote the jump terms

$$[\boldsymbol{E}_i] = \boldsymbol{E}_i^+ - \boldsymbol{E}_i^-, \quad [\boldsymbol{H}_i] = \boldsymbol{H}_i^+ - \boldsymbol{H}_i^-,$$

where superscripts '+' and '−' refer to field values from the neighbor element and the local element itself, respectively.

We introduce the discontinuous finite element space:

$$V_h = \{\boldsymbol{v}_h \in L^2(\Omega)^3 : \boldsymbol{v}_h|_{T_i} \in (P_k(T_i))^3 \text{ for any } T_i \in \overline{\Omega}\}, \tag{7}$$

i.e., the basis function is a discontinous polynomial of degree k over each element.

To define a fully discrete scheme, we divide the time interval $(0,T)$ into M uniform subintervals by points $0 = t_0 < t_1 < \cdots < t_M = T$, where $t_k = k\tau$ and τ is the time step size. Moreover, we define $\boldsymbol{E}_i^n = \boldsymbol{E}(\cdot, t_n)$ as the approximate field on element T_i, and $\boldsymbol{E}_h$ as the global approximate field, i.e., $\boldsymbol{E}_h|_{T_i} = \boldsymbol{E}_i$. Similar notation holds for other fields $\boldsymbol{H}_h, \boldsymbol{J}_h$ and $\boldsymbol{K}_h$. Below we also use the average notation

$$\boldsymbol{E}_i^{[n+\frac{1}{2}]} = \left(\boldsymbol{E}_i^n + \boldsymbol{E}_i^{n+1}\right)\Big/2, \quad \boldsymbol{H}_i^{[n+1]} = \left(\boldsymbol{H}_i^{n+\frac{1}{2}} + \boldsymbol{H}_i^{n+\frac{3}{2}}\right)\Big/2.$$

With the above preparation, now we can construct our leap-frog DG method. Multiplying (1)–(4) by test functions $\boldsymbol{u}_i, \boldsymbol{v}_i, \boldsymbol{\phi}_i, \boldsymbol{\psi}_i$ respectively, integrating the resultants over each element T_i, and choosing the upwind flux for the first two equations, we obtain the following leap-frog DG scheme: given initial approximations $\boldsymbol{E}_i^0, \boldsymbol{K}_i^0, \boldsymbol{H}_i^{\frac{1}{2}}, \boldsymbol{J}_i^{\frac{1}{2}}$, for $n = 0,1,\ldots$, find $\boldsymbol{E}_i^{n+1}, \boldsymbol{K}_i^{n+1}, \boldsymbol{H}_i^{n+\frac{3}{2}}, \boldsymbol{J}_i^{n+\frac{3}{2}} \in V_h$ such that

$$\int_{T_i} \epsilon_0 \frac{\boldsymbol{E}_i^{n+1} - \boldsymbol{E}_i^n}{\tau} \cdot \boldsymbol{u}_i = \int_{T_i} \boldsymbol{u}_i \cdot \nabla \times \boldsymbol{H}_i^{n+\frac{1}{2}} - \int_{T_i} \boldsymbol{J}_i^{n+\frac{1}{2}} \cdot \boldsymbol{u}_i + \sum_{k\in\nu_i} \int_{a_{ik}} \boldsymbol{u}_i \cdot \frac{1}{2}\mathbf{n}_{ik} \times \left(\left[\boldsymbol{H}_i^{n+\frac{1}{2}}\right] - \mathbf{n}_{ik} \times \left[\boldsymbol{E}_i^{[n+\frac{1}{2}]}\right]\right), \tag{8}$$

$$\int_{T_i} \mu_0 \frac{\boldsymbol{H}_i^{n+\frac{3}{2}} - \boldsymbol{H}_i^{n+\frac{1}{2}}}{\tau} \cdot \boldsymbol{v}_i = -\int_{T_i} \boldsymbol{v}_i \cdot \nabla \times \boldsymbol{E}_i^{n+1} - \int_{T_i} \boldsymbol{K}_i^{n+1} \cdot \boldsymbol{v}_i - \sum_{k\in\nu_i} \int_{a_{ik}} \boldsymbol{v}_i \cdot \frac{1}{2}\mathbf{n}_{ik} \times \left(\mathbf{n}_{ik} \times \left[\boldsymbol{H}_i^{[n+1]}\right] + \left[\boldsymbol{E}_i^{n+1}\right]\right), \tag{9}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2} \int_{T_i} \frac{\boldsymbol{J}_i^{n+\frac{3}{2}} - \boldsymbol{J}_i^{n+\frac{1}{2}}}{\tau} \cdot \boldsymbol{\phi}_i + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2} \int_{T_i} \frac{\boldsymbol{J}_i^{n+\frac{3}{2}} + \boldsymbol{J}_i^{n+\frac{1}{2}}}{2} \cdot \boldsymbol{\phi}_i = \int_{T_i} \boldsymbol{E}_i^{n+1} \cdot \boldsymbol{\phi}_i, \tag{10}$$

$$\frac{1}{\mu_0\omega_{pm}^2} \int_{T_i} \frac{\boldsymbol{K}_i^{n+1} - \boldsymbol{K}_i^n}{\tau} \cdot \boldsymbol{\psi}_i + \frac{\Gamma_m}{\mu_0\omega_{pm}^2} \int_{T_i} \frac{\boldsymbol{K}_i^{n+1} + \boldsymbol{K}_i^n}{2} \cdot \boldsymbol{\psi}_i = \int_{T_i} \boldsymbol{H}_i^{n+\frac{1}{2}} \cdot \boldsymbol{\psi}_i. \tag{11}$$

For a metallic boundary face a_{ik}, the boundary condition $\mathbf{n}_{ik} \times \boldsymbol{E}|_{a_{ik}} = 0$ is implemented as

$$\boldsymbol{E}_k^n\big|_{a_{ik}} = -\boldsymbol{E}_i^n\big|_{a_{ik}}, \tag{12}$$

$$\boldsymbol{H}_k^{n+\frac{1}{2}}\Big|_{a_{ik}} = \boldsymbol{H}_i^{n+\frac{1}{2}}\Big|_{a_{ik}}. \tag{13}$$

Remark 2.1. We want to remark that both (8) and (9) are implicit, since the upwind fluxes involve unknowns $\boldsymbol{E}_i^{n+1}$ and $\boldsymbol{H}_i^{n+\frac{3}{2}}$ on the right hand sides of (8) and (9), respectively. Hence our leap-frog DG scheme is less efficient than a standard Runge-Kutta DG (RKDG) [14,15]. However, a complete stability and error analysis for a fully explicit RKDG method (even for Maxwell's equations in vacuum) is still open. Development of more efficient fully-discrete DG schemes and theoretical analysis are needed.

For our scheme (8)–(11), we can prove the following conditional stability.

Theorem 2.1. *Denote $C_v = 1/\sqrt{\epsilon_0\mu_0}$ for the wave propagation speed in free space. Under the CFL condition*

$$\tau \leqslant \min\left\{\frac{1}{8}, \frac{h}{5C_{inv}^2 C_v}, \frac{h}{5C_{inv}C_v}, \frac{1}{2\omega_{pm}}, \frac{1}{2\omega_{pe}}\right\}, \tag{14}$$

where $C_{inv} > 0$ is the constant appearing in the standard inverse estimates [4]:

$$|u|_{0,\partial T_i} \leqslant C_{inv} h_{T_i}^{-\frac{1}{2}} \|u\|_{0,T_i}, \quad |u|_{1,T_i} \leqslant C_{inv} h_{T_i}^{-1} \|u\|_{0,T_i}, \quad \forall u \in V_h, \tag{15}$$

here and below $|u|_{k,T_i}$ and $\|u\|_{k,T_i}$ denote the semi-norm and norm for a function u in the Sobolev space $H^k(T_i)$, respectively, then the scheme (8)–(11) is stable and has the following stability:

$$\epsilon_0\|\boldsymbol{E}_h^n\|_{0,\Omega}^2 + \mu_0\left\|\boldsymbol{H}_h^{n+\frac{1}{2}}\right\|_{0,\Omega}^2 + \frac{1}{\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}_h^{n+\frac{1}{2}}\right\|_{0,\Omega}^2 + \frac{1}{\mu_0\omega_{pm}^2}\|\boldsymbol{K}_h^n\|_{0,\Omega}^2$$
$$\leqslant C\left(\left\|\boldsymbol{E}_h^0\right\|_{0,\Omega}^2 + \left\|\boldsymbol{H}_h^{\frac{1}{2}}\right\|_{0,\Omega}^2 + \left\|\boldsymbol{J}_h^{\frac{1}{2}}\right\|_{0,\Omega}^2 + \left\|\boldsymbol{K}_h^0\right\|_{0,\Omega}^2\right),$$

where the constant $C>0$ depends on the physical parameters $\epsilon_0,\mu_0,\omega_{pe},\omega_{pm},\Gamma_e$ and Γ_m, but is independent of the time step size τ and mesh size h.

Proof. Let us denote $EGY_i^n = EGY_{i1}^n + EGY_{i2}^n$, where

$$EGY_{i1}^n = \frac{1}{2}\int_{T_i}\epsilon_0 \boldsymbol{E}_i^n\cdot\boldsymbol{E}_i^n + \frac{1}{2}\int_{T_i}\mu_0\boldsymbol{H}_i^{n+\frac12}\cdot\boldsymbol{H}_i^{n+\frac12}, \tag{16}$$

$$EGY_{i2}^n = \frac{1}{2}\int_{T_i}\frac{1}{\epsilon_0\omega_{pe}^2}\boldsymbol{J}_i^{n+\frac12}\cdot\boldsymbol{J}_i^{n+\frac12} + \frac{1}{2}\int_{T_i}\frac{1}{\mu_0\omega_{pm}^2}\boldsymbol{K}_i^n\cdot\boldsymbol{K}_i^n. \tag{17}$$

Choosing $\boldsymbol{u}_i = \tau\boldsymbol{E}_i^{[n+\frac12]}$ in (8) and $\boldsymbol{v}_i = \tau\boldsymbol{H}_i^{[n+1]}$ in (9), then adding the resultants together, we have

$$\begin{aligned} EGY_{i1}^{n+1} &= EGY_{i1}^n - \tau\int_{T_i}\frac{\boldsymbol{H}_i^{n+\frac32}+\boldsymbol{H}_i^{n+\frac12}}{2}\cdot\nabla\times\boldsymbol{E}_i^{n+1} - \tau\int_{T_i}\boldsymbol{H}_i^{[n+1]} \\ &\quad\cdot\boldsymbol{K}_i^{n+1} - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik} \\ &\quad\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right] + \left[\boldsymbol{E}_i^{n+1}\right]\right) + \tau\int_{T_i}\frac{\boldsymbol{E}_i^n+\boldsymbol{E}_i^{n+1}}{2}\cdot\nabla \\ &\quad\times\boldsymbol{H}_i^{n+\frac12} - \tau\int_{T_i}\boldsymbol{J}_i^{n+\frac12}\cdot\boldsymbol{E}_i^{[n+\frac12]} + \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac12]}\cdot\mathbf{n}_{ik} \\ &\quad\times\left(\left[\boldsymbol{H}_i^{n+\frac12}\right] - \mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac12]}\right]\right). \end{aligned} \tag{18}$$

Using the identity

$$\int_{T_i}\boldsymbol{H}_i^{n+\frac12}\cdot\nabla\times\boldsymbol{E}_i^{n+1} = \int_{T_i}\nabla\times\boldsymbol{H}_i^{n+\frac12}\cdot\boldsymbol{E}_i^{n+1} + \int_{\partial T_i}\boldsymbol{E}_i^{n+1}\cdot\left(\boldsymbol{H}_i^{n+\frac12}\times\mathbf{n}_i\right)$$

and the jump definition $[\boldsymbol{E}_i] = \boldsymbol{E}_i^+ - \boldsymbol{E}_i^- = \boldsymbol{E}_k - \boldsymbol{E}_i$, $[\boldsymbol{H}_i] = \boldsymbol{H}_i^+ - \boldsymbol{H}_i^- = \boldsymbol{H}_k - \boldsymbol{H}_i$ in (18), we have

$$\begin{aligned} EGY_{i1}^{n+1} &= EGY_{i1}^n - \frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac32}\cdot\nabla\times\boldsymbol{E}_i^{n+1} - \frac{\tau}{2}\int_{\partial T_i}\boldsymbol{E}_i^{n+1}\cdot\left(\boldsymbol{H}_i^{n+\frac12}\times\mathbf{n}_i\right) \\ &\quad - \tau\int_{T_i}\boldsymbol{H}_i^{[n+1]}\cdot\boldsymbol{K}_i^{n+1} - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1} \\ &\quad + \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_i^{n+1} - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik} \\ &\quad \times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right]\right) + \frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^n\cdot\nabla\times\boldsymbol{H}_i^{n+\frac12} - \tau\int_{T_i}\boldsymbol{J}_i^{n+\frac12}\cdot\boldsymbol{E}_i^{[n+\frac12]} \\ &\quad + \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac12]}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac12} - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac12]}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac12} \\ &\quad - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac12]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac12]}\right]\right) \\ &= EGY_{i1}^n - \frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac32}\cdot\nabla\times\boldsymbol{E}_i^{n+1} - \frac{\tau}{2}\int_{\partial T_i}\boldsymbol{E}_i^{n+1}\cdot\left(\boldsymbol{H}_i^{n+\frac12}\times\mathbf{n}_i\right) \\ &\quad - \tau\int_{T_i}\boldsymbol{H}_i^{[n+1]}\cdot\boldsymbol{K}_i^{n+1} - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac32}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1} \\ &\quad - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac12}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1} + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac32}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_i^{n+1} \\ &\quad + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac12}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_i^{n+1} - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik} \\ &\quad \times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right]\right) + \frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^n\cdot\nabla\times\boldsymbol{H}_i^{n+\frac12} - \tau\int_{T_i}\boldsymbol{J}_i^{n+\frac12}\cdot\boldsymbol{E}_i^{[n+\frac12]} \\ &\quad + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac12} + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac12} \\ &\quad - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac12} - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac12} \\ &\quad - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac12]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac12]}\right]\right). \end{aligned} \tag{19}$$

Using the identity

$$\begin{aligned} \frac{\tau}{2}\int_{\partial T_i}\boldsymbol{E}_i^{n+1}\cdot\left(\boldsymbol{H}_i^{n+\frac12}\times\mathbf{n}_i\right) &= \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac12}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_i^{n+1} - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^{n+1} \\ &\quad\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac12} \end{aligned}$$

in (19), and moving the 6th term to the last and the 12th term to the second last, we obtain

$$\begin{aligned} EGY_{i1}^{n+1} &= EGY_{i1}^n - \frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac32}\cdot\nabla\times\boldsymbol{E}_i^{n+1} - \tau\int_{T_i}\boldsymbol{H}_i^{[n+1]}\cdot\boldsymbol{K}_i^{n+1} \\ &\quad - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac32}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1} + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac32}\cdot\mathbf{n}_{ik} \\ &\quad \times\boldsymbol{E}_i^{n+1} - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right]\right) + \frac{\tau}{2} \\ &\quad \times\int_{T_i}\boldsymbol{E}_i^n\cdot\nabla\times\boldsymbol{H}_i^{n+\frac12} - \tau\int_{T_i}\boldsymbol{J}_i^{n+\frac12}\cdot\boldsymbol{E}_i^{[n+\frac12]} + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n \\ &\quad \cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac12} - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac12} - \sum_{k\in v_i}\frac{\tau}{2} \\ &\quad \times\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac12]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac12]}\right]\right) + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^{n+1} \\ &\quad \cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac12} - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac12}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1}. \end{aligned} \tag{20}$$

Choosing $\boldsymbol{\phi}_i = \frac{\tau}{2}\left(\boldsymbol{J}_i^{n+\frac32}+\boldsymbol{J}_i^{n+\frac12}\right)$ in (10) and $\boldsymbol{\psi}_i = \frac{\tau}{2}\left(\boldsymbol{K}_i^{n+1}+\boldsymbol{K}_i^n\right)$ in (11), and adding the resultants, we obtain

$$\begin{aligned} EGY_{i2}^{n+1} &= EGY_{i2}^n - \frac{\Gamma_e\tau}{\epsilon_0\omega_{pe}^2}\int_{T_i}\boldsymbol{J}_i^{[n+1]}\cdot\boldsymbol{J}_i^{[n+1]} + \frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^{n+1}\cdot\boldsymbol{J}_i^{n+\frac32} + \frac{\tau}{2} \\ &\quad \times\int_{T_i}\boldsymbol{E}_i^{n+1}\cdot\boldsymbol{J}_i^{n+\frac12} - \frac{\Gamma_m\tau}{\mu_0\omega_{pm}^2}\int_{T_i}\boldsymbol{K}_i^{[n+\frac12]}\cdot\boldsymbol{K}_i^{[n+\frac12]} + \frac{\tau}{2} \\ &\quad \times\int_{T_i}\boldsymbol{H}_i^{n+\frac12}\cdot\boldsymbol{K}_i^{n+1} + \frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac12}\cdot\boldsymbol{K}_i^n. \end{aligned} \tag{21}$$

Adding (20) to (21), then expanding $\boldsymbol{H}_i^{[n+1]},\boldsymbol{E}_i^{[n+\frac12]}$ and simplifying the results, we have

$$\begin{aligned} EGY_i^{n+1} &= EGY_{i1}^{n+1} + EGY_{i2}^{n+1} \\ &= EGY_i^n - \frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac32}\cdot\nabla\times\boldsymbol{E}_i^{n+1} - \frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac32}\cdot\boldsymbol{K}_i^{n+1} \\ &\quad - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac32}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1} + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac32}\cdot\mathbf{n}_{ik} \\ &\quad \times\boldsymbol{E}_i^{n+1} - \sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right]\right) + \frac{\tau}{2} \\ &\quad \times\int_{T_i}\boldsymbol{E}_i^n\cdot\nabla\times\boldsymbol{H}_i^{n+\frac12} - \frac{\tau}{2}\int_{T_i}\boldsymbol{J}_i^{n+\frac12}\cdot\boldsymbol{E}_i^n + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n \\ &\quad \cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac12} - \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac12} - \sum_{k\in v_i}\frac{\tau}{2} \\ &\quad \times\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac12]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac12]}\right]\right) + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^{n+1} \\ &\quad \cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac12} + \sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_k^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac12} - \frac{\Gamma_e\tau}{\epsilon_0\omega_{pe}^2} \\ &\quad \times\int_{T_i}\boldsymbol{J}_i^{[n+1]}\cdot\boldsymbol{J}_i^{[n+1]} + \frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^{n+1}\cdot\boldsymbol{J}_i^{n+\frac32} - \frac{\Gamma_m\tau}{\mu_0\omega_{pm}^2} \\ &\quad \times\int_{T_i}\boldsymbol{K}_i^{[n+\frac12]}\cdot\boldsymbol{K}_i^{[n+\frac12]} + \frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac12}\cdot\boldsymbol{K}_i^n. \end{aligned} \tag{22}$$

Substituting the identity

$$-\frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac{3}{2}}\cdot\nabla\times\boldsymbol{E}_i^{n+1}=-\frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^{n+1}\cdot\nabla\times\boldsymbol{H}_i^{n+\frac{3}{2}}-\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac{3}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_i^{n+1}$$

into (22), we can rewrite (22) as

$$\begin{aligned}EGY_i^{n+1}&+\frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^{n+1}\cdot\nabla\times\boldsymbol{H}_i^{n+\frac{3}{2}}-\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac{3}{2}}\\&+\frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac{3}{2}}\cdot\boldsymbol{K}_i^{n+1}-\frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^{n+1}\cdot\boldsymbol{J}_i^{n+\frac{3}{2}}\\&=EGY_i^n+\frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^n\cdot\nabla\times\boldsymbol{H}_i^{n+\frac{1}{2}}-\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac{1}{2}}\\&+\frac{\tau}{2}\int_{T_i}\boldsymbol{H}_i^{n+\frac{1}{2}}\cdot\boldsymbol{K}_i^n-\frac{\tau}{2}\int_{T_i}\boldsymbol{J}_i^{n+\frac{1}{2}}\cdot\boldsymbol{E}_i^n-\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac{3}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1}\\&-\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right]\right)\\&+\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac{1}{2}}-\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac{1}{2}]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac{1}{2}]}\right]\right)\\&+\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac{1}{2}}+\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_k^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac{1}{2}}\\&-\frac{\Gamma_e\tau}{\epsilon_0\omega_{pe}^2}\int_{T_i}\boldsymbol{J}_i^{[n+1]}\cdot\boldsymbol{J}_i^{[n+1]}-\frac{\Gamma_m\tau}{\mu_0\omega_{pm}^2}\int_{T_i}\boldsymbol{K}_i^{[n+\frac{1}{2}]}\cdot\boldsymbol{K}_i^{[n+\frac{1}{2}]}.\end{aligned}\tag{23}$$

Let us denote

$$F1_i^n=EGY_i^n+\frac{\tau}{2}\int_{T_i}\boldsymbol{E}_i^n\cdot\nabla\times\boldsymbol{H}_i^{n+\frac{1}{2}}-\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac{1}{2}}+\frac{\tau}{2}\times\int_{T_i}\boldsymbol{H}_i^{n+\frac{1}{2}}\cdot\boldsymbol{K}_i^n-\frac{\tau}{2}\int_{T_i}\boldsymbol{J}_i^{n+\frac{1}{2}}\cdot\boldsymbol{E}_i^n.$$

Then we can rewrite (23) as

$$\begin{aligned}F1_i^{n+1}&=F1_i^n-\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{H}_i^{n+\frac{3}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1}+\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\\&\times\boldsymbol{H}_k^{n+\frac{1}{2}}-\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{H}_i^{[n+1]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right]\right)\\&-\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\boldsymbol{E}_i^{[n+\frac{1}{2}]}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac{1}{2}]}\right]\right)+\sum_{k\in v_i}\frac{\tau}{4}\\&\times\int_{a_{ik}}\boldsymbol{E}_i^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac{1}{2}}+\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_k^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac{1}{2}}\\&-\frac{\Gamma_e\tau}{\epsilon_0\omega_{pe}^2}\int_{T_i}\boldsymbol{J}_i^{[n+1]}\cdot\boldsymbol{J}_i^{[n+1]}-\frac{\Gamma_m\tau}{\mu_0\omega_{pm}^2}\int_{T_i}\boldsymbol{K}_i^{[n+\frac{1}{2}]}\cdot\boldsymbol{K}_i^{[n+\frac{1}{2}]}.\end{aligned}\tag{24}$$

Now summing up (24) over all elements T_i of Ω, and noting that all terms of

$$\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_i^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac{1}{2}}+\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}}\boldsymbol{E}_k^{n+1}\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_i^{n+\frac{1}{2}}$$

vanish on both the internal faces F_h^{int} and physical boundary $\partial\Omega$, we obtain

$$\begin{aligned}F1_{\Omega_h}^{n+1}&=F1_{\Omega_h}^n-\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{H}_i^{n+\frac{3}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^{n+1}\\&+\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{E}_i^n\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{n+\frac{1}{2}}-\sum_{f_i\in F_h^{int}}\frac{\tau}{2}\int_{f_i}\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right]\right)\\&\cdot\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[n+1]}\right]\right)-\sum_{f_i\in F_h^{int}}\frac{\tau}{2}\int_{f_i}\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac{1}{2}]}\right]\right)\cdot\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{[n+\frac{1}{2}]}\right]\right)\\&-\frac{\Gamma_e\tau}{\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}_i^{[n+1]}\right\|_{0,\Omega}^2-\frac{\Gamma_m\tau}{\mu_0\omega_{pm}^2}\left\|\boldsymbol{K}_i^{[n+\frac{1}{2}]}\right\|_{0,\Omega}^2.\end{aligned}\tag{25}$$

Summing (25) from $n = 0$ to $m-1$ (for any $m \geqslant 1$), we obtain

$$\begin{aligned}F1_{\Omega_h}^m&=F1_{\Omega_h}^0-\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{H}_i^{m+\frac{1}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^m+\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\\&\times\int_{a_{ik}\in F_h^{int}}\boldsymbol{E}_i^0\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{\frac{1}{2}}-\sum_{j=0}^m\sum_{f_i\in F_h^{int}}\frac{\tau}{2}\int_{f_i}\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[j]}\right]\right)\\&\cdot\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{[j]}\right]\right)-\sum_{j=0}^m\sum_{f_i\in F_h^{int}}\frac{\tau}{2}\int_{f_i}\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{j-\frac{1}{2}}\right]\right)\\&\cdot\left(\mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{j-\frac{1}{2}}\right]\right)-\sum_{j=0}^m\frac{\Gamma_e\tau}{\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}_i^{[j]}\right\|_{0,\Omega}^2\\&-\sum_{j=0}^m\frac{\Gamma_m\tau}{\mu_0\omega_{pm}^2}\left\|\boldsymbol{K}_i^{[j-\frac{1}{2}]}\right\|_{0,\Omega}^2,\end{aligned}\tag{26}$$

which leads to

$$F1_{\Omega_h}^m\leqslant F1_{\Omega_h}^0-\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{H}_i^{m+\frac{1}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^m+\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{E}_i^0\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{\frac{1}{2}}.$$

Recall the definition of $F1_{\Omega_h}^m$, we obtain

$$\begin{aligned}F1_{\Omega_h}^m&\equiv\frac{1}{2}\epsilon_0\|\boldsymbol{E}^m\|_{0,\Omega}^2+\frac{1}{2}\mu_0\left\|\boldsymbol{H}^{m+\frac{1}{2}}\right\|_{0,\Omega}^2+\frac{1}{2\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}^{m+\frac{1}{2}}\right\|_{0,\Omega}^2\\&+\frac{1}{2\mu_0\omega_{pm}^2}\|\boldsymbol{K}^m\|_{0,\Omega}^2+\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{E}^m\cdot\nabla\times\boldsymbol{H}^{m+\frac{1}{2}}\\&-\frac{\tau}{4}\sum_{T_i}\int_{\partial T_i}\boldsymbol{E}_i^m\cdot\mathbf{n}_i\times\boldsymbol{H}_i^{m+\frac{1}{2}}+\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{H}^{m+\frac{1}{2}}\cdot\boldsymbol{K}^m-\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{J}^{m+\frac{1}{2}}\cdot\boldsymbol{E}^m\\&\leqslant F1_{\Omega_h}^0-\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{H}_i^{m+\frac{1}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^m\\&+\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{E}_i^0\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{\frac{1}{2}}\\&\equiv\frac{1}{2}\epsilon_0\|\boldsymbol{E}^0\|_{0,\Omega}^2+\frac{1}{2}\mu_0\left\|\boldsymbol{H}^{\frac{1}{2}}\right\|_{0,\Omega}^2+\frac{1}{2\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}^{\frac{1}{2}}\right\|_{0,\Omega}^2+\frac{1}{2\mu_0\omega_{pm}^2}\|\boldsymbol{K}^0\|_{0,\Omega}^2\\&+\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{E}^0\cdot\nabla\times\boldsymbol{H}^{\frac{1}{2}}-\frac{\tau}{4}\sum_{T_i}\int_{\partial T_i}\boldsymbol{E}_i^0\cdot\mathbf{n}_i\times\boldsymbol{H}_i^{\frac{1}{2}}+\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{H}^{\frac{1}{2}}\cdot\boldsymbol{K}^0\\&-\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{J}^{\frac{1}{2}}\cdot\boldsymbol{E}^0-\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{H}_i^{m+\frac{1}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^m\\&+\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{E}_i^0\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{\frac{1}{2}}.\end{aligned}\tag{27}$$

Simplifying the above equation, we obtain

$$\begin{aligned}&\frac{1}{2}\epsilon_0\|\boldsymbol{E}^m\|_{0,\Omega}^2+\frac{1}{2}\mu_0\left\|\boldsymbol{H}^{m+\frac{1}{2}}\right\|_{0,\Omega}^2+\frac{1}{2\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}^{m+\frac{1}{2}}\right\|_{0,\Omega}^2+\frac{1}{2\mu_0\omega_{pm}^2}\|\boldsymbol{K}^m\|_{0,\Omega}^2\\&\leqslant\frac{1}{2}\epsilon_0\|\boldsymbol{E}^0\|_{0,\Omega}^2+\frac{1}{2}\mu_0\left\|\boldsymbol{H}^{\frac{1}{2}}\right\|_{0,\Omega}^2+\frac{1}{2\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}^{\frac{1}{2}}\right\|_{0,\Omega}^2+\frac{1}{2\mu_0\omega_{pm}^2}\|\boldsymbol{K}^0\|_{0,\Omega}^2\\&-\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{E}^m\cdot\nabla\times\boldsymbol{H}^{m+\frac{1}{2}}+\frac{\tau}{4}\sum_{T_i}\int_{\partial T_i}\boldsymbol{E}_i^m\cdot\mathbf{n}_i\times\boldsymbol{H}_i^{m+\frac{1}{2}}-\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{H}^{m+\frac{1}{2}}\cdot\boldsymbol{K}^m\\&+\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{J}^{m+\frac{1}{2}}\cdot\boldsymbol{E}^m+\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{E}^0\cdot\nabla\times\boldsymbol{H}^{\frac{1}{2}}-\frac{\tau}{4}\sum_{T_i}\int_{\partial T_i}\boldsymbol{E}_i^0\cdot\mathbf{n}_i\times\boldsymbol{H}_i^{\frac{1}{2}}\\&+\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{H}^{\frac{1}{2}}\cdot\boldsymbol{K}^0-\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{J}^{\frac{1}{2}}\cdot\boldsymbol{E}^0-\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{H}_i^{m+\frac{1}{2}}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^m\\&+\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{E}_i^0\cdot\mathbf{n}_{ik}\times\boldsymbol{H}_k^{\frac{1}{2}}.\end{aligned}\tag{28}$$

By the standard inverse estimates (15), we have

$$\left|\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{E}^m\cdot\nabla\times\boldsymbol{H}^{m+\frac12}\right| \leqslant \frac{C_{inv}\tau h^{-1}}{4\sqrt{\epsilon_0\mu_0}}\left(\epsilon_0\|\boldsymbol{E}^m\|_{0,\Omega}^2+\mu_0\left\|\boldsymbol{H}^{m+\frac12}\right\|_{0,\Omega}^2\right),$$

$$\left|\frac{\tau}{4}\sum_{T_i}\int_{\partial T_i}\boldsymbol{E}_i^m\cdot\mathbf{n}_i\times\boldsymbol{H}_i^{m+\frac12}\right| \leqslant \frac{C_{inv}^2\tau h^{-1}}{8\sqrt{\epsilon_0\mu_0}}\left(\epsilon_0\|\boldsymbol{E}^m|_{0,\Omega}^2+\mu_0\left\|\boldsymbol{H}^{m+\frac12}\right\|_{0,\Omega}^2\right),$$

$$\left|\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{H}^{m+\frac12}\cdot\boldsymbol{K}^m\right| \leqslant \frac{\omega_{pm}\tau}{4}\left(\mu_0\left\|\boldsymbol{H}^{m+\frac12}\right\|_{0,\Omega}^2+\frac{1}{\mu_0\omega_{pm}^2}\|\boldsymbol{K}^m|_{0,\Omega}^2\right),$$

$$\left|\frac{\tau}{2}\int_{\Omega_h}\boldsymbol{J}^{m+\frac12}\cdot\boldsymbol{E}^m\right| \leqslant \frac{\omega_{pe}\tau}{4}\left(\epsilon_0\|\boldsymbol{E}^m\|_{0,\Omega}^2+\frac{1}{\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}^{m+\frac12}\right|_{0,\Omega}^2\right),$$

$$\left|\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{4}\int_{a_{ik}\in F_h^{int}}\boldsymbol{H}_i^{m+\frac12}\cdot\mathbf{n}_{ik}\times\boldsymbol{E}_k^m\right| \leqslant \frac{C_{inv}^2\tau h^{-1}}{8\sqrt{\epsilon_0\mu_0}}\left(\epsilon_0\|\boldsymbol{E}^m\|_{0,\Omega}^2+\mu_0\left\|\boldsymbol{H}^{m+\frac12}\right\|_{0,\Omega}^2\right).$$

By choosing the time step τ small enough so that the right hand side terms can be controlled by the corresponding terms on the left hand side of (28), we can obtain a stability result. An exemplary choice is

$$\tau\leqslant\min\left\{\frac18,\frac{h}{5C_{inv}^2C_v},\frac{h}{5C_{inv}C_v},\frac{1}{2\omega_{pm}},\frac{1}{2\omega_{pe}}\right\},$$

and substituting all above estimates into (28), we have

$$\epsilon_0\|\boldsymbol{E}^n\|_{0,\Omega}^2+\mu_0\left\|\boldsymbol{H}^{n+\frac12}\right\|_{0,\Omega}^2+\frac{1}{\epsilon_0\omega_{pe}^2}\left\|\boldsymbol{J}^{n+\frac12}\right\|_{0,\Omega}^2+\frac{1}{\mu_0\omega_{pm}^2}\|\boldsymbol{K}^n\|_{0,\Omega}^2 \leqslant C\left[\|\boldsymbol{E}^0\|_{0,\Omega}^2+\left\|\boldsymbol{H}^{\frac12}\right\|_{0,\Omega}^2+\left\|\boldsymbol{J}^{\frac12}\right\|_{0,\Omega}^2+\|\boldsymbol{K}^0\|_{0,\Omega}^2\right],$$

where the physical parameters ϵ_0, μ_0, ω_{pe}, ω_{pm}, Γ_e and Γ_m have been absorbed into the generic constant C. □

Remark 2.2. The tight estimate of the constant C_{inv} in the inverse inequalities (15) depends on the element shape and the order of the basis functions. Harari and Hughes [13] explicitly derived the constant C_{inv} for some inverse inequalities. A sharp bound for the constant in the first inverse inequality of (15) is proved in [34].

3. The error estimate

Before we prove the error estimate, we need some lemmas.

Lemma 3.1 24, Lemma 5.1. *Denote $u^j = u(.,j\tau)$. For any $u\in H^2(0,T;L^2(\Omega))$, we have*

(i) $$\left\|u^j-\frac1\tau\int_{t_{j-\frac12}}^{t_{j+\frac12}}u(s)\,ds\right\|_0^2\leqslant\frac{\tau^3}{4}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\|u_{tt}(s)\|_0^2\,ds,$$

(ii) $$\left\|u^{j-\frac12}-\frac1\tau\int_{t_{j-1}}^{t_j}u(s)\,ds\right\|_0^2\leqslant\frac{\tau^3}{4}\int_{t_{j-1}}^{t_j}\|u_{tt}(s)\|_0^2\,ds,$$

(iii) $$\left\|\frac12(u^j+u^{j+1})-\frac1\tau\int_{t_j}^{t_{j+1}}u(s)\,ds\right\|_0^2\leqslant\frac{\tau^3}{4}\int_{t_j}^{t_{j+1}}\|u_{tt}(s)\|_0^2\,ds,$$

(iv) $$\left\|\frac12\left(u^{j-\frac12}+u^{j+\frac12}\right)-\frac1\tau\int_{t_{j-\frac12}}^{t_{j+\frac12}}u(s)\,ds\right\|_0^2\leqslant\frac{\tau^3}{4}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\|u_{tt}(s)\|_0^2\,ds.$$

Let P_h denote the standard L^2-projection onto V_h or V_h^0, which is the subspace of V_h with the boundary condition $\boldsymbol{n}\times\boldsymbol{E}=0$ imposed. It is known that the projection error estimate

$$\|u-P_hu\|_{0,T}\leqslant Ch_T^{\min\{s,k\}+1}\|u\|_{s+1,T},\tag{29}$$

holds true for any element T, and $u\in H^{s+1}(T)$.

Lemma 3.2. *For any functions $\eta_i^{j-\frac12},\eta_i^{j+\frac12},\xi_i^j,\xi_i^{j-1}\in V_h$, we have*

$$\begin{aligned}
&-\sum_i\int_{\partial T_i}\mathbf{n}_i\times\eta_i^{j-\frac12}\cdot\left(\xi_i^j+\xi_i^{j-1}\right)-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\left(\xi_i^j+\xi_i^{j-1}\right)\cdot\frac12\mathbf{n}_{ik}\\
&\times\left(\left[\eta_i^{j-\frac12}\right]-\mathbf{n}_{ik}\times\left[\frac{\xi_i^j+\xi_i^{j-1}}{2}\right]\right)+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\left(\eta_i^{j+\frac12}+\eta_i^{j-\frac12}\right)\cdot\frac12\mathbf{n}_{ik}\\
&\times\left(\mathbf{n}_{ik}\times\left[\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\right]+\left[\xi_i^j\right]\right)=\sum_i\int_{\partial T_i}\frac12\mathbf{n}_i\times\xi_i^{j-1}\cdot\eta_i^{j-\frac12}\\
&-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\xi_i^{j-1}\cdot\frac12\mathbf{n}_{ik}\times\eta_k^{j-\frac12}+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\eta_i^{j+\frac12}\cdot\frac12\mathbf{n}_{ik}\times\xi_k^j\\
&-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\eta_i^{j+\frac12}\cdot\frac12\mathbf{n}_{ik}\times\xi_i^j+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\frac{\xi_i^j+\xi_i^{j-1}}{2}\cdot\mathbf{n}_{ik}\\
&\times\left(\mathbf{n}_{ik}\times\left[\frac{\xi_i^j+\xi_i^{j-1}}{2}\right]\right)+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\cdot\mathbf{n}_{ik}\\
&\times\left(\mathbf{n}_{ik}\times\left[\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\right]\right).
\end{aligned}\tag{30}$$

Proof. Using the jump definition

$$\left[\eta_i^{j-\frac12}\right]=\eta_k^{j-\frac12}-\eta_i^{j-\frac12},\quad\left[\xi_i^j\right]=\xi_k^j-\xi_i^j$$

in the left hand-side (LHS) of (30), we have

$$\begin{aligned}
\text{LHS}=&-\sum_i\int_{\partial T_i}\mathbf{n}_i\times\eta_i^{j-\frac12}\cdot\left(\xi_i^j+\xi_i^{j-1}\right)-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\left(\xi_i^j+\xi_i^{j-1}\right)\cdot\frac12\mathbf{n}_{ik}\times\eta_k^{j-\frac12}\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\left(\xi_i^j+\xi_i^{j-1}\right)\cdot\frac12\mathbf{n}_{ik}\times\eta_i^{j-\frac12}+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\frac{\xi_i^j+\xi_i^{j-1}}{2}\cdot\mathbf{n}_{ik}\\
&\times\left(\mathbf{n}_{ik}\times\left[\frac{\xi_i^j+\xi_i^{j-1}}{2}\right]\right)\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\left(\eta_i^{j+\frac12}+\eta_i^{j-\frac12}\right)\cdot\frac12\mathbf{n}_{ik}\times\xi_k^j-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\left(\eta_i^{j+\frac12}+\eta_i^{j-\frac12}\right)\cdot\frac12\mathbf{n}_{ik}\times\xi_i^j\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\right]\right)\\
=&\sum_i\int_{\partial T_i}\frac12\mathbf{n}_i\times\xi_i^{j-1}\cdot\eta_i^{j-\frac12}-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\xi_i^j\cdot\frac12\mathbf{n}_{ik}\times\eta_k^{j-\frac12}\\
&-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\xi_i^{j-1}\cdot\frac12\mathbf{n}_{ik}\times\eta_k^{j-\frac12}+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\eta_i^{j+\frac12}\cdot\frac12\mathbf{n}_{ik}\times\xi_k^j\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\eta_i^{j-\frac12}\cdot\frac12\mathbf{n}_{ik}\times\xi_k^j-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\eta_i^{j+\frac12}\cdot\frac12\mathbf{n}_{ik}\times\xi_i^j\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\frac{\xi_i^j+\xi_i^{j-1}}{2}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\frac{\xi_i^j+\xi_i^{j-1}}{2}\right]\right)\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\right]\right)\\
=&\sum_i\int_{\partial T_i}\frac12\mathbf{n}_i\times\xi_i^{j-1}\cdot\eta_i^{j-\frac12}-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\xi_i^{j-1}\cdot\frac12\mathbf{n}_{ik}\times\eta_k^{j-\frac12}\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\eta_i^{j+\frac12}\cdot\frac12\mathbf{n}_{ik}\times\xi_k^j-\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\eta_i^{j+\frac12}\cdot\frac12\mathbf{n}_{ik}\times\xi_i^j\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\frac{\xi_i^j+\xi_i^{j-1}}{2}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\frac{\xi_i^j+\xi_i^{j-1}}{2}\right]\right)\\
&+\sum_{T_i}\sum_{k\in v_i}\int_{a_{ik}}\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\cdot\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\frac{\eta_i^{j+\frac12}+\eta_i^{j-\frac12}}{2}\right]\right),
\end{aligned}$$

which concludes the proof. □

Theorem 3.1. *Under the assumption that the time step τ satisfies a CFL condition such as:*

$$\tau = \min\left\{\frac{1}{8}, \frac{1}{2\omega_{pe}}, \frac{1}{2\omega_{pm}}, \frac{h}{5C_{inv}C_v}, \frac{h}{5C_{inv}^2 C_v}\right\}, \tag{31}$$

and the solution of (1)–(4) has the following regularity:

$$\boldsymbol{E}_{tt}, \boldsymbol{H}_{tt}, \boldsymbol{J}_{tt}, \boldsymbol{K}_{tt}, \nabla\times\boldsymbol{E}_{tt}, \nabla\times\boldsymbol{H}_{tt} \in L^2(0,T;L^2(\Omega)^3),$$
$$\boldsymbol{E}, \boldsymbol{H} \in L^\infty(0,T;H^{s+1}(\Omega)^3), \quad \forall s \geqslant 0,$$

we have the following error estimate

$$\max_{1\leqslant n}\left(\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_0 + \|\boldsymbol{H}^{n+\frac12} - \boldsymbol{H}_h^{n+\frac12}\|_0 + \|\boldsymbol{J}^{n+\frac12} - \boldsymbol{J}_h^{n+\frac12}\|_0 + \|\boldsymbol{K}^n - \boldsymbol{K}_h^n\|_0\right)$$
$$\leqslant C(\tau^2 + Th^{\min\{s,k\}})$$
$$+ C\left(\|\boldsymbol{E}^0 - \boldsymbol{E}_h^0\|_0 + \|\boldsymbol{H}^{\frac12} - \boldsymbol{H}_h^{\frac12}\|_0 + \|\boldsymbol{J}^{\frac12} - \boldsymbol{J}_h^{\frac12}\|_0 + \|\boldsymbol{K}^0 - \boldsymbol{K}_h^0\|_0\right).$$

where k is the degree of the basis function in the finite element space (7), and the constant

$$C = C(\epsilon_0, \mu_0, \omega_{pe}, \omega_{pm}, \Gamma_e, \Gamma_m, \boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J}, \boldsymbol{K})$$

is independent of both time step τ and mesh size h.

Remark 3.1. When the initial approximations are accurate enough, such as

$$\|\boldsymbol{E}^0 - \boldsymbol{E}_h^0\|_0 + \|\boldsymbol{H}^{\frac12} - \boldsymbol{H}_h^{\frac12}\|_0 + \|\boldsymbol{J}^{\frac12} - \boldsymbol{J}_h^{\frac12}\|_0 + \|\boldsymbol{K}^0 - \boldsymbol{K}_h^0\|_0$$
$$\leqslant C(\tau^2 + h^{\min\{s,k\}}),$$

then the error estimate of Theorem 3.1 becomes

$$\max_{1\leqslant n}\left(\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_0 + \|\boldsymbol{H}^{n+\frac12} - \boldsymbol{H}_h^{n+\frac12}\|_0 + \|\boldsymbol{J}^{n+\frac12} - \boldsymbol{J}_h^{n+\frac12}\|_0 + \|\boldsymbol{K}^n - \boldsymbol{K}_h^n\|_0\right)$$
$$\leqslant C(\tau^2 + Th^{\min\{s,k\}}),$$

which shows that the error grows linearly in time. Note that this L^2 error estimate is sub-optimal in spacial error, since the basis function is k-th order.

3.1. Proof of Theorem 3.1

Integrating the governing Eqs. (1) and (4) from t_{j-1} to t_j, and (2) and (3) from $t_{j-\frac12}$ to $t_{j+\frac12}$, then multiplying the respective resultants by $\frac{\boldsymbol{u}_h}{\tau}, \frac{\boldsymbol{v}_h}{\tau}, \frac{\boldsymbol{\phi}_h}{\tau}, \frac{\psi_h}{\tau}$ and integrating over Ω, we obtain

$$\epsilon_0\left(\frac{\boldsymbol{E}^j - \boldsymbol{E}^{j-1}}{\tau}, \boldsymbol{u}_h\right) - \left(\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\nabla\times\boldsymbol{H}(s)\,ds, \boldsymbol{u}_h\right)$$
$$+ \left(\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{J}(s)\,ds, \boldsymbol{u}_h\right) = 0, \tag{32}$$

$$\mu_0\left(\frac{\boldsymbol{H}^{j+\frac12} - \boldsymbol{H}^{j-\frac12}}{\tau}, \boldsymbol{v}_h\right) + \left(\frac{1}{\tau}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\nabla\times\boldsymbol{E}(s)\,ds, \boldsymbol{v}_h\right)$$
$$+ \left(\frac{1}{\tau}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\boldsymbol{K}(s)\,ds, \boldsymbol{v}_h\right) = 0, \tag{33}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\left(\frac{\boldsymbol{J}^{j+\frac12} - \boldsymbol{J}^{j-\frac12}}{\tau}, \boldsymbol{\phi}_h\right) + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\left(\frac{1}{\tau}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\boldsymbol{J}(s)\,ds, \boldsymbol{\phi}_h\right)$$
$$= \left(\frac{1}{\tau}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\boldsymbol{E}(s)\,ds, \boldsymbol{\phi}_h\right), \tag{34}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\left(\frac{\boldsymbol{K}^j - \boldsymbol{K}^{j-1}}{\tau}, \psi_h\right) + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}\left(\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{K}(s)\,ds, \psi_h\right)$$
$$= \left(\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{H}(s)\,ds, \psi_h\right). \tag{35}$$

Summing all elements together for (8)–(11) with $n = j - 1$, we have

$$\left(\epsilon_0\frac{\boldsymbol{E}_h^j - \boldsymbol{E}_h^{j-1}}{\tau}, \boldsymbol{u}_h\right) - \left(\nabla\times\boldsymbol{H}_h^{j-\frac12}, \boldsymbol{u}_h\right) + \left(\boldsymbol{J}_h^{j-\frac12}, \boldsymbol{u}_h\right)$$
$$-\sum_{T_i}\sum_{k\in\nu_i}\int_{a_{ik}}\boldsymbol{u}_i\cdot\frac12\mathbf{n}_{ik}\times\left(\left[\boldsymbol{H}_i^{j-\frac12}\right] - \mathbf{n}_{ik}\times\left[\boldsymbol{E}_i^{\{j-\frac12\}}\right]\right) = 0, \tag{36}$$

$$\left(\mu_0\frac{\boldsymbol{H}_h^{j+\frac12} - \boldsymbol{H}_h^{j-\frac12}}{\tau}, \boldsymbol{v}_h\right) + \left(\nabla\times\boldsymbol{E}_h^j, \boldsymbol{v}_h\right) + \left(\boldsymbol{K}_h^j, \boldsymbol{v}_h\right)$$
$$+\sum_{T_i}\sum_{k\in\nu_i}\int_{a_{ik}}\boldsymbol{v}_i\cdot\frac12\mathbf{n}_{ik}\times\left(\mathbf{n}_{ik}\times\left[\boldsymbol{H}_i^{\{j\}}\right] + \left[\boldsymbol{E}_i^j\right]\right) = 0, \tag{37}$$

$$\frac{1}{\epsilon_0\omega_{pe}^2}\left(\frac{\boldsymbol{J}_h^{j+\frac12} - \boldsymbol{J}_h^{j-\frac12}}{\tau}, \boldsymbol{\phi}_h\right) + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\left(\frac{\boldsymbol{J}_h^{j+\frac12} + \boldsymbol{J}_h^{j-\frac12}}{2}, \boldsymbol{\phi}_h\right) = \left(\boldsymbol{E}_h^j, \boldsymbol{\phi}_h\right), \tag{38}$$

$$\frac{1}{\mu_0\omega_{pm}^2}\left(\frac{\boldsymbol{K}_h^j - \boldsymbol{K}_h^{j-1}}{\tau}, \psi_h\right) + \frac{\Gamma_m}{\mu_0\omega_{pm}^2}\left(\frac{\boldsymbol{K}_h^j + \boldsymbol{K}_h^{j-1}}{2}, \psi_h\right) = \left(\boldsymbol{H}_h^{j-\frac12}, \psi_h\right). \tag{39}$$

Denote $\xi_h^j = P_h\boldsymbol{E}^j - \boldsymbol{E}_h^j$, $\eta_h^j = P_h\boldsymbol{H}^j - \boldsymbol{H}_h^j$, $\tilde{\xi}_h^j = P_h\boldsymbol{J}^j - \boldsymbol{J}_h^j$, $\tilde{\eta}_h^j = P_h\boldsymbol{K}^j - \boldsymbol{K}_h^j$, and the backward operator $\delta_\tau u^j = (u^j - u^{j-1})/\tau$.

Subtracting (36)–(39) from (32)–(35) with the corresponding flux terms added, and using the identity

$$\left(\nabla\times\eta_h^{j-\frac12}, \boldsymbol{u}_h\right) = \left(\eta_h^{j-\frac12}, \nabla\times\boldsymbol{u}_h\right) + \sum_i\int_{\partial T_i}\mathbf{n}_i\times\eta_i^{j-\frac12}\cdot\boldsymbol{u}_i,$$

we can obtain the following error equations:

(i)
$$\epsilon_0\left(\frac{\xi_h^j - \xi_h^{j-1}}{\tau}, \boldsymbol{u}_h\right) - \left(\eta_h^{j-\frac12}, \nabla\times\boldsymbol{u}_h\right) - \sum_i\int_{\partial T_i}\mathbf{n}_i\times\eta_i^{j-\frac12}\cdot\boldsymbol{u}_i$$
$$-\sum_{T_i}\sum_{k\in\nu_i}\int_{a_{ik}}u_i\cdot\frac12\mathbf{n}_{ik}\times\left(\left[\eta_i^{j-\frac12}\right] - \mathbf{n}_{ik}\times\left[\frac{\xi_h^j + \xi_h^{j-1}}{2}\right]\right)$$
$$= \epsilon_0(\delta_t(P_h\boldsymbol{E}^j - \boldsymbol{E}^j), \boldsymbol{u}_h) - \left(\nabla\times\left(P_h\boldsymbol{H}^{j-\frac12} - \frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{H}(s)\,ds\right), \boldsymbol{u}_h\right)$$
$$+\left(-\tilde{\xi}_h^{j-\frac12} + P_h\boldsymbol{J}^{j-\frac12} - \frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{J}(s)\,ds, \boldsymbol{u}_h\right) - \sum_{T_i}\sum_{k\in\nu_i}\int_{a_{ik}}u_i\cdot\frac12\mathbf{n}_{ik}$$
$$\times\left(\left[P_h\boldsymbol{H}^{j-\frac12} - \boldsymbol{H}^{j-\frac12}\right] - \mathbf{n}_{ik}\times\left[\frac{P_h\boldsymbol{E}^j - \boldsymbol{E}^j + P_h\boldsymbol{E}^{j-1} - \boldsymbol{E}^{j-1}}{2}\right]\right), \tag{40}$$

(ii)
$$\mu_0\left(\frac{\eta_h^{j+\frac12} - \eta_h^{j-\frac12}}{\tau}, \boldsymbol{v}_h\right) + \left(\nabla\times\xi_h^j, \boldsymbol{v}_h\right) + \sum_{T_i}\sum_{k\in\nu_i}\int_{a_{ik}}v_i\cdot\frac12\mathbf{n}_{ik}$$
$$\times\left(\mathbf{n}_{ik}\times\left[\frac{\eta_i^{j+\frac12} + \eta_i^{j-\frac12}}{2}\right] + \left[\xi_i^j\right]\right)$$
$$= \mu_0\left(\delta_t\left(P_h\boldsymbol{H}^{j+\frac12} - \boldsymbol{H}^{j+\frac12}\right), \boldsymbol{v}_h\right) + \left(\nabla\times\left(P_h\boldsymbol{E}^j - \frac{1}{\tau}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\boldsymbol{E}(s)\,ds\right), \boldsymbol{v}_h\right)$$
$$+\left(-\tilde{\eta}_h^j + P_h\boldsymbol{K}^j - \frac{1}{\tau}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\boldsymbol{K}(s)\,ds, \boldsymbol{v}_h\right) + \sum_{T_i}\sum_{k\in\nu_i}\int_{a_{ik}}v_i\cdot\frac12\mathbf{n}_{ik}$$
$$\times\left(\mathbf{n}_{ik}\times\left[\frac{P_h\boldsymbol{H}^{j+\frac12} - \boldsymbol{H}^{j+\frac12} + P_h\boldsymbol{H}^{j-\frac12} - \boldsymbol{H}^{j-\frac12}}{2}\right] + \left[P_h\boldsymbol{E}^j - \boldsymbol{E}^j\right]\right), \tag{41}$$

(iii)
$$\frac{1}{\epsilon_0\omega_{pe}^2}\left(\frac{\tilde{\xi}_h^{j+\frac12} - \tilde{\xi}_h^{j-\frac12}}{\tau}, \boldsymbol{\phi}_h\right) + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\left(\frac{\tilde{\xi}_h^{j+\frac12} + \tilde{\xi}_h^{j-\frac12}}{2}, \boldsymbol{\phi}_h\right)$$
$$= \frac{1}{\epsilon_0\omega_{pe}^2}\left(\delta_\tau\left(P_h\boldsymbol{J}^{j+\frac12} - \boldsymbol{J}^{j-\frac12}\right), \boldsymbol{\phi}_h\right) + \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\left(\frac{P_h\boldsymbol{J}^{j+\frac12} + P_h\boldsymbol{J}^{j-\frac12}}{2} - \frac{1}{\tau}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\boldsymbol{J}(s)\,ds, \boldsymbol{\phi}_h\right)$$
$$+\left(\frac{1}{\tau}\int_{t_{j-\frac12}}^{t_{j+\frac12}}\boldsymbol{E}(s)\,ds - P_h\boldsymbol{E}^j + \xi_h^j, \boldsymbol{\phi}_h\right), \tag{42}$$

(iv) $$\frac{1}{\mu_0\omega_{pm}^2}\left(\frac{\tilde{\eta}_h^j-\tilde{\eta}_h^{j-1}}{\tau},\boldsymbol{\psi}_h\right)+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\left(\frac{\tilde{\eta}_h^j+\tilde{\eta}_h^{j-1}}{2},\boldsymbol{\psi}_h\right)$$
$$=\frac{1}{\mu_0\omega_{pm}^2}(\delta_\tau(P_h\boldsymbol{K}^j-\boldsymbol{K}^j),\boldsymbol{\psi}_h)$$
$$+\frac{\Gamma_m}{\mu_0\omega_{pm}^2}\left(\frac{P_h\boldsymbol{K}^j+P_h\boldsymbol{K}^{j+1}}{2}-\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{K}(s)\,\mathrm{d}s,\boldsymbol{\psi}_h\right)$$
$$+\left(\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{H}(s)\,\mathrm{d}s-P_h\boldsymbol{H}^{j-\frac{1}{2}}+\eta_h^{j-\frac{1}{2}},\boldsymbol{\psi}_h\right). \tag{43}$$

Choosing $\boldsymbol{u}_h=\tau\left(\xi_h^j+\xi_h^{j-1}\right)$, $\boldsymbol{v}_h=\tau\left(\eta_h^{j+\frac{1}{2}}+\eta_h^{j-\frac{1}{2}}\right)$, $\boldsymbol{\phi}_h=\tau\left(\tilde{\xi}_h^{j+\frac{1}{2}}+\tilde{\xi}_h^{j-\frac{1}{2}}\right)$, $\boldsymbol{\psi}_h=\tau\left(\tilde{\eta}_h^j+\tilde{\eta}_h^{j-1}\right)$ in (40)–(43), respectively, then summing up the resultants from $j=1$ to $j=n$, dropping those zero terms by the projection property of P_h, and using (30) and the following identities:

$$\left(\nabla\times\xi_h^j,\eta_h^{j+\frac{1}{2}}+\eta_h^{j-\frac{1}{2}}\right)-\left(\eta_h^{j-\frac{1}{2}},\nabla\times(\xi_h^j+\xi_h^{j-1})\right)$$
$$=\left(\nabla\times\xi_h^j,\eta_h^{j+\frac{1}{2}}\right)-\left(\nabla\times\xi_h^{j-1},\eta_h^{j-\frac{1}{2}}\right),$$

and

$$-\left(\tilde{\xi}_h^{j-\frac{1}{2}},\xi_h^j+\xi_h^{j-1}\right)-\left(\tilde{\eta}_h^j,\eta_h^{j+\frac{1}{2}}+\eta_h^{j-\frac{1}{2}}\right)+\left(\xi_h^j,\tilde{\xi}_h^{j+\frac{1}{2}}+\tilde{\xi}_h^{j-\frac{1}{2}}\right)$$
$$+\left(\eta_h^{j-\frac{1}{2}},\tilde{\eta}_h^j+\tilde{\eta}_h^{j-1}\right)$$
$$=\left[\left(\tilde{\xi}_h^{j+\frac{1}{2}},\xi_h^j\right)-\left(\tilde{\xi}_h^{j-\frac{1}{2}},\xi_h^{j-1}\right)\right]-\left[\left(\tilde{\eta}_h^j,\eta_h^{j+\frac{1}{2}}\right)-\left(\tilde{\eta}_h^{j-1},\eta_h^{j-\frac{1}{2}}\right)\right],$$

we obtain

$$\epsilon_0\left(\|\xi_h^n\|_0^2-\|\xi_h^0\|_0^2\right)+\mu_0\left(\|\eta_h^{n+\frac{1}{2}}\|_0^2-\|\eta_h^{\frac{1}{2}}\|_0^2\right)+\frac{1}{\epsilon_0\omega_{pe}^2}\left(\|\tilde{\xi}_h^{n+\frac{1}{2}}\|_0^2-\|\tilde{\xi}_h^{\frac{1}{2}}\|_0^2\right)$$
$$+\frac{1}{\mu_0\omega_{pm}^2}\left(\|\tilde{\eta}_h^n\|_0^2-\|\tilde{\eta}_h^0\|_0^2\right)+\frac{\tau\Gamma_e}{2\epsilon_0\omega_{pe}^2}\sum_{j=1}^n\|\tilde{\xi}_h^{j+\frac{1}{2}}+\tilde{\xi}_h^{j-\frac{1}{2}}\|_0^2+\frac{\tau\Gamma_m}{2\mu_0\omega_{pm}^2}\sum_{j=1}^n\|\tilde{\eta}_h^j$$
$$+\tilde{\eta}_h^{j-1}\|_0^2=-\tau\sum_{j=1}^n\left(\nabla\times\left(P_h\boldsymbol{H}^{j-\frac{1}{2}}-\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{H}(s)\,\mathrm{d}s\right),\xi_h^j+\xi_h^{j-1}\right)$$
$$+\tau\sum_{j=1}^n\left(P_h\boldsymbol{J}^{j-\frac{1}{2}}-\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{J}(s)\,\mathrm{d}s,\xi_h^j+\xi_h^{j-1}\right)$$
$$+\tau\sum_{j=1}^n\left(\nabla\times\left(P_h\boldsymbol{E}^j-\frac{1}{\tau}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\boldsymbol{E}(s)\,\mathrm{d}s\right),\eta_h^{j+\frac{1}{2}}+\eta_h^{j-\frac{1}{2}}\right)$$
$$+\tau\sum_{j=1}^n\left(P_h\boldsymbol{K}^j-\frac{1}{\tau}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\boldsymbol{K}(s)\,\mathrm{d}s,\eta_h^{j+\frac{1}{2}}+\eta_h^{j-\frac{1}{2}}\right)$$
$$+\frac{\tau\Gamma_e}{\epsilon_0\omega_{pe}^2}\sum_{j=1}^n\left(\frac{1}{2}\left(P_h\boldsymbol{J}^{j+\frac{1}{2}}+P_h\boldsymbol{J}^{j-\frac{1}{2}}\right)-\frac{1}{\tau}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\boldsymbol{J}(s)\,\mathrm{d}s,\tilde{\xi}_h^{j+\frac{1}{2}}+\tilde{\xi}_h^{j-\frac{1}{2}}\right)$$
$$+\tau\sum_{j=1}^n\left(\frac{1}{\tau}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\boldsymbol{E}(s)\,\mathrm{d}s-P_h\boldsymbol{E}^j,\tilde{\xi}_h^{j+\frac{1}{2}}+\tilde{\xi}_h^{j-\frac{1}{2}}\right)$$
$$+\frac{\tau\Gamma_m}{\mu_0\omega_{pm}^2}\sum_{j=1}^n\left(\frac{1}{2}\left(P_h\boldsymbol{K}^j+P_h\boldsymbol{K}^{j-1}\right)-\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{K}(s)\,\mathrm{d}s,\tilde{\eta}_h^j+\tilde{\eta}_h^{j-1}\right)$$
$$+\tau\sum_{j=1}^n\left(\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{H}(s)\,\mathrm{d}s-P_h\boldsymbol{H}^{j-\frac{1}{2}},\tilde{\eta}_h^j+\tilde{\eta}_h^{j-1}\right)+\tau\left(\tilde{\xi}_h^{n+\frac{1}{2}},\xi_h^n\right)-\tau\left(\tilde{\xi}_h^{\frac{1}{2}},\xi_h^0\right)$$
$$-\tau\left(\tilde{\eta}_h^n,\eta_h^{n+\frac{1}{2}}\right)+\tau\left(\tilde{\eta}_h^0,\eta_h^{\frac{1}{2}}\right)+\tau\left(\nabla\times\xi_h^0,\eta_h^{\frac{1}{2}}\right)-\tau\left(\nabla\times\xi_h^n,\eta_h^{n+\frac{1}{2}}\right)$$
$$+\sum_i\frac{\tau}{2}\int_{\partial T_i}\eta_i^{n+\frac{1}{2}}\cdot\mathbf{n}_{ik}\times\xi_i^n-\sum_i\frac{\tau}{2}\int_{\partial T_i}\mathbf{n}_i\times\xi_i^0\cdot\eta_i^{\frac{1}{2}}-\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\eta_i^{n+\frac{1}{2}}$$
$$\cdot\mathbf{n}_{ik}\times\xi_k^n+\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\xi_i^0\cdot\mathbf{n}_{ik}\times\eta_k^{\frac{1}{2}}-\sum_{j=1}^n\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\left(\xi_i^j+\xi_i^{j-1}\right)\cdot\mathbf{n}_{ik}$$
$$\times\left(\left[P_h\boldsymbol{H}^{j-\frac{1}{2}}-\boldsymbol{H}^{j-\frac{1}{2}}\right]-\mathbf{n}_{ik}\times\left[\frac{P_h\boldsymbol{E}^j-\boldsymbol{E}^j+P_h\boldsymbol{E}^{j-1}-\boldsymbol{E}^{j-1}}{2}\right]\right)$$
$$+\sum_{j=1}^n\sum_{T_i}\sum_{k\in v_i}\frac{\tau}{2}\int_{a_{ik}}\left(\eta_i^{j+\frac{1}{2}}+\eta_i^{j-\frac{1}{2}}\right)\cdot\mathbf{n}_{ik}$$
$$\times\left(\mathbf{n}_{ik}\times\left[\frac{P_h\boldsymbol{H}^{j+\frac{1}{2}}-\boldsymbol{H}^{j+\frac{1}{2}}+P_h\boldsymbol{H}^{j-\frac{1}{2}}-\boldsymbol{H}^{j-\frac{1}{2}}}{2}\right]+\left[P_h\boldsymbol{E}^j-\boldsymbol{E}^j\right]\right)=\sum_{i=1}^{20}Err_i. \tag{44}$$

Substituting all estimates of Err_i (see Appendix) into (44), and first choosing τ small enough (such as (31)) so that those terms $\|\xi_h^n\|_0^2$, $\left\|\eta_h^{n+\frac{1}{2}}\right\|_0^2$, $\left\|\tilde{\xi}_h^{n+\frac{1}{2}}\right\|_0^2$ and $\|\tilde{\eta}_h^n\|_0^2$ on the right-hand sides of Err_i, $i=9,\ldots,18$, can be controlled by the corresponding terms on the left-hand side of (44), then taking the maximum of the resultant for n and choosing parameters δ_i small enough such as

$$\delta_1=\delta_2=\delta_{19}=\frac{\epsilon_0}{8T},\quad \delta_3=\delta_4=\delta_{20}=\frac{\mu_0}{8T}, \tag{45}$$

$$\delta_5=\delta_7=1,\quad \delta_6=\frac{1}{10T\epsilon_0\omega_{pe}^2},\quad \delta_8=\frac{1}{10T\mu_0\omega_{pm}^2}, \tag{46}$$

we obtain

$$\max_{1\leqslant n}\left(\epsilon_0\|\xi_h^n\|_0^2+\mu_0\|\eta_h^{n+\frac{1}{2}}\|_0^2+\frac{1}{\epsilon_0\omega_{pe}^2}\|\tilde{\xi}_h^{n+\frac{1}{2}}\|_0^2+\frac{1}{\mu_0\omega_{pm}^2}\|\tilde{\eta}_h^n\|_0^2\right)$$
$$\leqslant C(\tau^4+T^2h^{\min\{s,k\}})$$
$$+C\left(\epsilon_0\|\xi_h^0\|_0^2+\mu_0\|\eta_h^{\frac{1}{2}}\|_0^2+\frac{1}{\epsilon_0\omega_{pe}^2}\|\tilde{\xi}_h^{\frac{1}{2}}\|_0^2+\frac{1}{\mu_0\omega_{pm}^2}\|\tilde{\eta}_h^0\|_0^2\right). \tag{47}$$

By using the triangle inequality, (47), and the estimates (29), we get

$$\max_{1\leqslant n}\left(\|\boldsymbol{E}^n-\boldsymbol{E}_h^n\|_0+\|\boldsymbol{H}^{n+\frac{1}{2}}-\boldsymbol{H}_h^{n+\frac{1}{2}}\|_0+\|\boldsymbol{J}^{n+\frac{1}{2}}-\boldsymbol{J}_h^{n+\frac{1}{2}}\|_0+\|\boldsymbol{K}^n-\boldsymbol{K}_h^n\|_0\right)$$
$$\leqslant C(\tau^2+Th^{\min\{s,k\}})$$
$$+C\left(\|\boldsymbol{E}^0-\boldsymbol{E}_h^0\|_0+\|\boldsymbol{H}^{\frac{1}{2}}-\boldsymbol{H}_h^{\frac{1}{2}}\|_0+\|\boldsymbol{J}^{\frac{1}{2}}-\boldsymbol{J}_h^{\frac{1}{2}}\|_0+\|\boldsymbol{K}^0-\boldsymbol{K}_h^0\|_0\right),$$

which completes the proof. □

4. Numerical results

In this section, we present some 2D numerical results supporting our theoretical analysis. Note that all our analysis holds true for 2D problems by noting the scalar and vector curl operators

$$\mathrm{curl}\,\boldsymbol{H}=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y},\quad \nabla\times E=\left(\frac{\partial E}{\partial y},-\frac{\partial E}{\partial x}\right)',$$

where we consider the TM_z mode, which involves a vector magnetic field $\boldsymbol{H}=(H_x,H_y)$ and a scalar electric field E.

Example 1. In the first example, we create an exact solution to check the optimal order convergence rate. For simplicity, we assume that $\Omega=[0,1]^2$, $\epsilon_0=\mu_0=1$, $\omega_{pe}=\omega_{pm}=\Gamma_m=\Gamma_e=\pi$, and the exact solution to (1)–(4) with added right-hand side source terms f and g is

$$H_x(x,y,t)=\sin(\pi x)\cdot\cos(\pi y)e^{-\pi t},$$
$$H_y(x,y,t)=-\cos(\pi x)\cdot\sin(\pi y)e^{-\pi t},$$
$$E(x,y,t)=\sin(\pi x)\cdot\sin(\pi y)e^{-\pi t},$$
$$K_x(x,y,t)=\pi^2t\cdot\sin(\pi x)\cdot\cos(\pi y)e^{-\pi t},$$
$$K_y(x,y,t)=-\pi^2t\cdot\cos(\pi x)\cdot\sin(\pi y)e^{-\pi t},$$
$$J(x,y,t)=\pi^2t\sin(\pi x)\cdot\sin(\pi y)e^{-\pi t},$$

where the source terms

$$f(x,y,t) = (-3\pi + \pi^2 t)e^{-\pi t}\sin(\pi x)\cdot\sin(\pi y),$$
$$g_x(x,y,t) = \pi^2 te^{-\pi t}\sin(\pi x)\cdot\cos(\pi y),$$
$$g_y(x,y,t) = -\pi^2 te^{-\pi t}\cos(\pi x)\cdot\sin(\pi y),$$

are added to the right-hand sides of (1) and (2), respectively. In implementation, we have to add a term $\int_{T_i} f^{n+\frac{1}{2}}\cdot u_i$ and a term $\int_{T_i} \boldsymbol{g}^{n+1}\cdot \boldsymbol{v}_i$ to the right-hand sides of (8) and (9), respectively.

We solved the problem using various time step sizes τ and mesh sizes with different orders of basis functions. For a fixed small time step τ and smooth solutions, our numerical results showed that the error estimate is as follows:

$$\max_{1\leqslant n}\left(\|\boldsymbol{E}^n - \boldsymbol{E}_h^n\|_0 + \|\boldsymbol{H}^{n+\frac{1}{2}} - \boldsymbol{H}_h^{n+\frac{1}{2}}\|_0 + \|\boldsymbol{J}^{n+\frac{1}{2}} - \boldsymbol{J}_h^{n+\frac{1}{2}}\|_0 + \|\boldsymbol{K}^n - \boldsymbol{K}_h^n\|_0\right) \leqslant CTh^{k+1},$$

which has one order higher convergence rate than that proved in Theorem 3.1. Recall that k denotes the order of the basis function. This phenomenon happened for the upwind flux has been mentioned in [15, p. 205]. But the rigorous proof of $O(h^{k+1})$ for the upwind flux case is still open.

In Tables 1–3, we present the numerical results obtained using a fixed $\tau = 10^{-6}$ on uniformly refined triangular meshes. The results show the convergence rate $O(h^{k+1})$ clearly.

The linear growth of errors with T is also observed in our numerical tests. In Tables 4 and 5, we presented the numerical results obtained with the same conditions as used for Tables 1 and 2 except the errors are recorded after 10 time steps. Comparing Tables 4 and 5 with Tables 1 and 2, we can see that the errors of K_x, K_y, J in Tables 1 and 2 are almost 10 times large than those in Tables 4 and 5. The errors for the first three solutions do not change that much with the number of time steps, this is due to the fact that the exact solutions (H_x, H_y, E) change very little with respect to time since $e^{-\pi t} \approx 1$. We like to mention that $O(\tau^2)$ could not be observed due to the CFL condition $\tau = O(h)$, since $O(\tau^2 + h^{k+1})$ is always dominated by $O(\tau^2)$.

Example 2. We use this example to demonstrate the effectiveness of our DG method in solving discontinous media problems. For simplicity, we consider a case when the permittivity is discontinuous: $\mu_0 = 1$, $\omega_{pe} = \omega_{pm} = \Gamma_m = \Gamma_e = \pi$, $\epsilon_0 = 1$ in a subdomain $[0.25, 0.75]^2$ and $\epsilon_0 = 100$ elsewhere. For this example, we use the

Table 1
The L^2 errors obtained after 100 steps on uniform triangular meshes with $\tau = 10^{-6}$ and $k = 1$.

Meshes	$h=\frac{1}{4}$	$h=\frac{1}{8}$	Rate	$h=\frac{1}{16}$	Rate	$h=\frac{1}{32}$	Rate	$h=\frac{1}{64}$	Rate
H_x	0.0455	0.0118	1.9471	0.0030	1.9758	7.5453e−4	1.9913	1.8875e−4	1.9991
H_y	0.0517	0.0128	2.0140	0.0032	2.0000	7.9346e−4	2.0118	1.9849e−4	1.9991
E	0.0486	0.0123	1.9823	0.0031	1.9883	7.7451e−4	2.0009	1.9370e−4	1.9995
K_x	4.4914e−5	1.1673e−5	1.9440	2.9630e−6	1.9780	7.4472e−7	1.9923	1.8634e−7	1.9988
K_y	5.1037e−5	1.2628e−5	2.0149	3.1376e−6	2.0089	7.8287e−7	2.0028	1.9582e−7	1.9992
J	4.8251e−5	1.2236e−5	1.9794	3.0708e−6	1.9944	7.6850e−7	1.9985	1.9217e−7	1.9997

Table 2
The L^2 errors obtained after 100 steps on uniform triangular meshes with $\tau = 10^{-6}$ and $k = 2$.

Meshes	$h=\frac{1}{4}$	$h=\frac{1}{8}$	Rate	$h=\frac{1}{16}$	Rate	$h=\frac{1}{32}$	Rate	$h=\frac{1}{64}$	Rate
H_x	0.0047	6.4854e−4	2.8574	8.6694e−5	2.9032	1.1357e−5	2.9324	1.4646e−6	2.9550
H_y	0.0047	6.4894e−4	2.8565	8.7021e−5	2.8986	1.1404e−5	2.9318	1.4699e−6	2.9558
E	0.0047	6.5112e−4	2.8517	8.6896e−5	2.9056	1.1322e−5	2.9402	1.4544e−6	2.9606
K_x	4.6862e−6	6.4001e−7	2.8723	8.5541e−8	2.9034	1.1202e−8	2.9329	1.4434e−9	2.9562
K_y	4.6398e−6	6.4030e−7	2.8572	8.5847e−8	2.8989	1.1245e−8	2.9325	1.4481e−9	2.9571
J	4.6696e−6	6.4594e−7	2.8538	8.6171e−8	2.9061	1.1216e−8	2.9416	1.4370e−9	2.9644

Table 3
The L^2 errors obtained after 100 steps on uniform triangular meshes with $\tau = 10^{-6}$ and $k = 3$.

Meshes	$h=\frac{1}{4}$	$h=\frac{1}{8}$	Rate	$h=\frac{1}{16}$	Rate	$h=\frac{1}{32}$	Rate	$h=\frac{1}{64}$	Rate
H_x	7.3281e−4	5.3071e−5	3.7874	3.6439e−6	3.8644	2.4100e−7	3.9184	1.5618e−8	3.9478
H_y	7.5179e−4	5.5864e−5	3.7503	3.8611e−6	3.8548	2.5572e−7	3.9164	1.6659e−8	3.9402
E	6.4847e−4	4.9042e−5	3.7250	3.4892e−6	3.8130	2.3667e−7	3.8819	1.5848e−8	3.9005
K_x	7.2325e−7	5.2389e−8	3.7872	3.5986e−9	3.8638	2.3817e−10	3.9174	1.5410e−11	3.9501
K_y	7.4200e−7	5.5150e−8	3.7500	3.8135e−9	3.8542	2.5272e−10	3.9155	1.6391e−11	3.9466
J	6.4357e−7	4.8697e−8	3.7242	3.4676e−9	3.8118	2.3542e−10	3.8806	1.5601e−11	3.9155

Table 4
The L^2 errors obtained after 10 steps on uniform triangular meshes with $\tau = 10^{-6}$ and $k = 1$.

Meshes	$h=\frac{1}{4}$	$h=\frac{1}{8}$	Rate	$h=\frac{1}{16}$	Rate	$h=\frac{1}{32}$	Rate	$h=\frac{1}{64}$	Rate
H_x	0.0455	0.0118	1.9471	0.0030	1.9758	7.5479e−4	1.9908	1.8890e−4	1.9985
H_y	0.0517	0.0128	2.0140	0.0032	2.0000	7.9325e−4	2.0122	1.9840e−4	1.9994
E_z	0.0487	0.0123	1.9853	0.0031	1.9883	7.7529e−4	1.9995	1.9388e−4	1.9996
K_x	4.4922e−6	1.1675e−6	1.9440	2.9636e−7	1.9780	7.4496e−8	1.9921	1.8644e−8	1.9985
K_y	5.1046e−6	1.2630e−6	2.0149	3.1378e−7	2.0090	7.8288e−8	2.0029	1.9580e−8	1.9994
J_z	5.0436e−6	1.2790e−6	1.9794	3.2102e−7	1.9943	8.0347e−8	1.9983	2.0093e−8	1.9996

Table 5
The L^2 errors obtained after 10 steps on uniform triangular meshes with $\tau = 10^{-6}$ and $k = 2$.

Meshes	$h = \frac{1}{4}$	$h = \frac{1}{8}$	Rate	$h = \frac{1}{16}$	Rate	$h = \frac{1}{32}$	Rate	$h = \frac{1}{64}$	Rate
H_x	0.0047	6.4858e−4	2.8573	8.6680e−5	2.9035	1.1349e−5	2.9331	1.4616e−6	2.9569
H_y	0.0047	6.4881e−4	2.8568	8.6976e−5	2.8991	1.1390e−5	2.9328	1.4659e−6	2.9579
E_z	0.0047	6.5149e−4	2.8508	8.6889e−5	2.9065	1.1302e−5	2.9426	1.4459e−6	2.9665
K_x	4.6873e−7	6.4012e−8	2.8723	8.5547e−9	2.9036	1.1200e−9	2.9332	1.4424e−10	2.9570
K_y	4.6404e−7	6.4033e−8	2.8574	8.5838e−9	2.8991	1.1241e−9	2.9328	1.4466e−10	2.9580
J_z	4.8815e−7	6.7516e−8	2.8540	9.0042e−9	2.9066	1.1712e−9	2.9426	1.4981e−10	2.9668

same f and g as Example 1, but with the following initial conditions:

$$H_x(x,y,\tau/2) = \sin\left(\frac{\pi x}{4}\right)\cdot\cos\left(\frac{\pi y}{4}\right)e^{-\pi\tau/2},$$

$$H_y(x,y,\tau/2) = -\cos\left(\frac{\pi x}{4}\right)\cdot\sin\left(\frac{\pi y}{4}\right)e^{-\pi\tau/2},$$

$$E(x,y,0) = \sin(\pi x)\cdot\sin(\pi y),$$

$$K_x(x,y,0) = K_y(x,y,0) = 0,$$

$$J(x,y,\tau/2) = \pi^2\frac{\tau}{2}\sin(\pi x)\cdot\sin(\pi y)e^{-\pi\tau/2}.$$

Since the exact solution to this problem is unknown, we just plot the numerical solutions obtained on different meshes. To verify the long term stability, we solved this example to 10000 time steps with $\tau = 10^{-6}$. The obtained numerical magnetic field $H = (H_x, H_y)$ and electric field E on various uniform triangular meshes are presented in Figs. 1–4, which show that the solutions are convergent.

5. Conclusions

We developed a leap-frog DG method for the Maxwell's equations in metamaterials. Rigorous stability and error estimates are

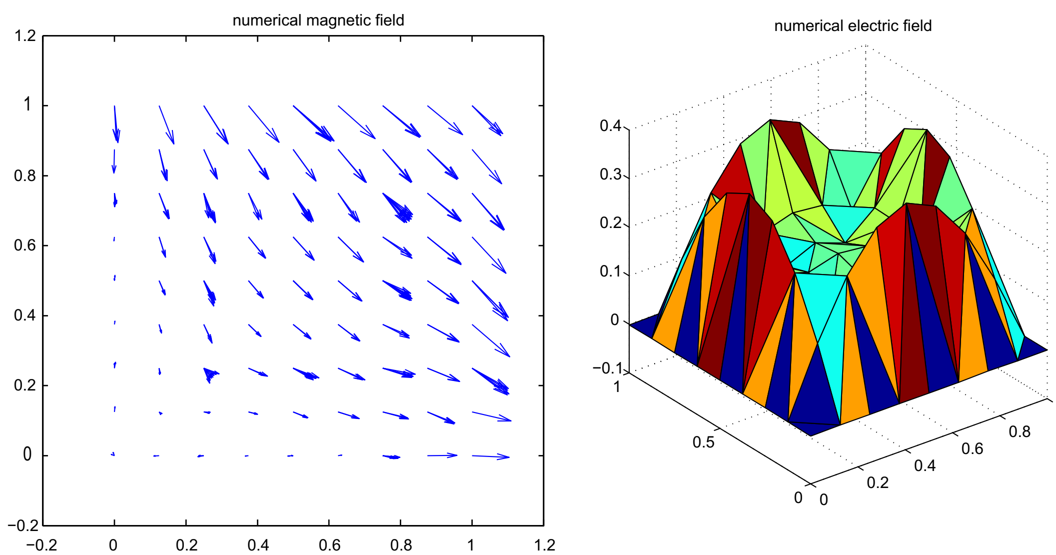

Fig. 1. Numerical solutions after 10000 time steps with $h = 1/4$: (Left) The magnetic field $\boldsymbol{H} = (Hx, Hy)$; (Right) The electric field E.

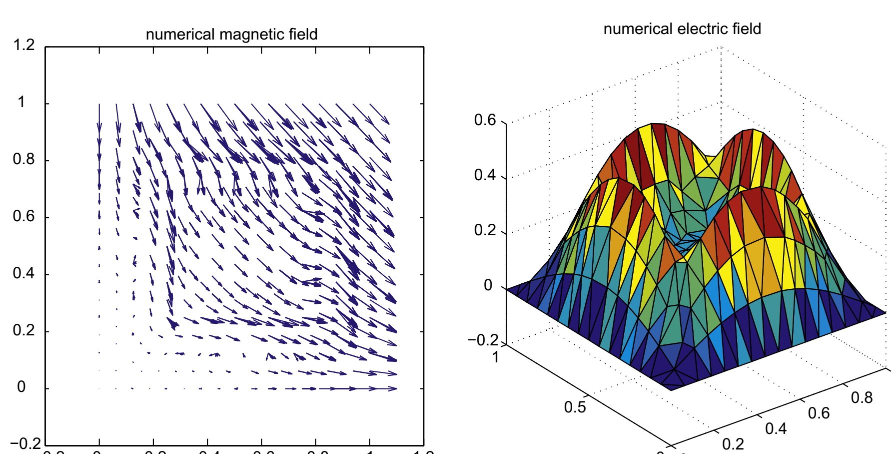

Fig. 2. Numerical solutions after 10000 time steps with $h = 1/8$: (Left) The magnetic field $\boldsymbol{H} = (Hx, Hy)$; (Right) The electric field E.

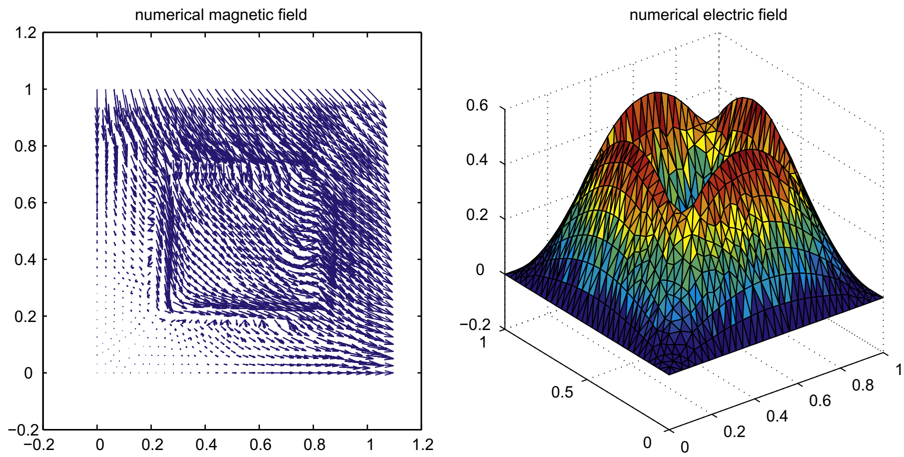

Fig. 3. Numerical solutions after 10000 time steps with $h = 1/16$: (Left) The magnetic field $\boldsymbol{H} = (Hx, Hy)$; (Right) The electric field E.

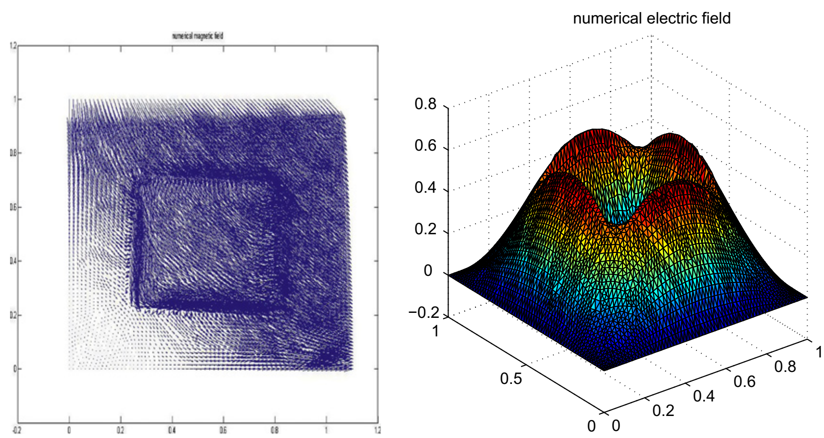

Fig. 4. Numerical solutions after 10000 time steps with $h = 1/32$: (Left) The magnetic field $\boldsymbol{H} = (Hx, Hy)$; (Right) The electric field E.

proved for the scheme. Numerical results supporting the analysis are presented. How to prove the optimal L^2 error estimate observed with upwind flux (even for Maxwell's equations in free space) is still open. More interesting simulations such as invisibility cloak with metamaterials will be investigated in the future.

Acknowledgments

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Appendix. Estimates of Err_i, $i = 1, \ldots, 20$

Here we will estimate each Err_i. Below we frequently use the arithmetic-geometric mean inequality

$$(a,b) \leqslant \frac{\delta}{2}\|a\|_0^2 + \frac{1}{2\delta}\|b\|_0^2, \tag{48}$$

where δ is an arbitrary positive constant.

Using integration by parts, the projection property, and Lemma 3.1(ii), we obtain

$$\begin{aligned} Err_1 &\leqslant \tau\delta_1\sum_{j=1}^{n}\left(\left\|\xi_h^j\right\|_0^2 + \|\xi_h^{j-1}\|_0^2\right) + \frac{\tau}{2\delta_1}\sum_{j=1}^{n}\|\nabla\times\boldsymbol{H}^{j-\frac{1}{2}} \\ &\quad - \frac{1}{\tau}\int_{t_{j-1}}^{t_j}\nabla\times\boldsymbol{H}(s)\,ds\|_0^2 \leqslant \tau\delta_1\cdot 2\sum_{j=1}^{n}\|\xi_h\|_{l^\infty(L^2)}^2 + \tau\delta_1\left\|\xi_h^0\right\|_0^2 \\ &\quad + \frac{\tau}{2\delta_1}\sum_{j=1}^{n}\frac{\tau^3}{4}\int_{t_{j-1}}^{t_j}\|\nabla\times\boldsymbol{H}_{tt}(s)\|_0^2\,ds \leqslant 2T\delta_1\|\xi_h\|_{l^\infty(L^2)}^2 + \tau\delta_1\left\|\xi_h^0\right\|_0^2 \\ &\quad + \frac{\tau^4}{8\delta_1}\|\nabla\times\boldsymbol{H}_{tt}\|_{L^2(0,T;L^2(\Omega)^3)}, \end{aligned}$$

where we introduced the notation $\|\xi_h\|_{l^\infty(L^2)} = \max_{j\geqslant 1}\left\|\xi_h^j\right\|_0$.

Similarly, we can obtain

$$\begin{aligned} Err_2 &= \tau\sum_{j=1}^{n}\left(\boldsymbol{J}^{j-\frac{1}{2}} - \frac{1}{\tau}\int_{t_{j-1}}^{t_j}\boldsymbol{J}(s)\,ds, \xi_h^j + \xi_h^{j-1}\right) \\ &\leqslant 2T\delta_2\|\xi_h\|_{l^\infty(L^2)}^2 + \tau\delta_2\left\|\xi_h^0\right\|_0^2 + \frac{\tau^4}{8\delta_2}\|\boldsymbol{J}_{tt}\|_{L^2(0,T;L^2(\Omega)^3)}. \end{aligned}$$

Using integration by parts, the projection property, and Lemma 3.1(i), we have

$$\begin{aligned} Err_3 &= \tau\sum_{j=1}^{n}\left(\nabla\times\boldsymbol{E}^j - \frac{1}{\tau}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\nabla\times\boldsymbol{E}(s)\,ds, \eta_h^{j+\frac{1}{2}} + \eta_h^{j-\frac{1}{2}}\right) \\ &\leqslant \tau\sum_{j=1}^{n}\delta_3\left(\left\|\eta_h^{j+\frac{1}{2}}\right\|_0^2 + \left\|\eta_h^{j-\frac{1}{2}}\right\|_0^2\right) + \frac{\tau}{2\delta_3}\sum_{j=1}^{n}\frac{\tau^3}{4}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}}\|\nabla\times\boldsymbol{E}_{tt}(s)\|_0^2\,ds \\ &\leqslant 2T\delta_3\|\eta_h\|_{l^\infty(L^2)}^2 + \tau\delta_3\left\|\eta_h^{\frac{1}{2}}\right\|_0^2 + \frac{\tau^4}{8\delta_3}\int_{t_{\frac{1}{2}}}^{t_{n+\frac{1}{2}}}\|\nabla\times\boldsymbol{E}_{tt}(s)\|_0^2\,ds. \end{aligned}$$

By similar arguments, we have

$$Err_4 \leqslant 2T\delta_4\|\eta_h\|^2_{l^\infty(L^2)} + \tau\delta_4\left\|\eta_h^{\frac{1}{2}}\right\|_0^2 + \frac{\tau^4}{8\delta_4}\int_{t_{\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \|\boldsymbol{K}_{tt}(s)\|_0^2\,ds.$$

Similarly, we have

$$Err_5 \leqslant \frac{\tau\Gamma_e}{\epsilon_0\omega_{pe}^2}\left[\sum_{j=1}^n \frac{\delta_5}{2}\left\|\tilde{\xi}_h^{j+\frac{1}{2}} + \tilde{\xi}_h^{j-\frac{1}{2}}\right\|_0^2 + \sum_{j=1}^n \frac{1}{2\delta_5}\left\|\frac{1}{2}\left(\boldsymbol{J}^{j+\frac{1}{2}} + \boldsymbol{J}^{j-\frac{1}{2}}\right) - \frac{1}{\tau}\int_{t_{j-\frac{1}{2}}}^{t_{j+\frac{1}{2}}} \boldsymbol{J}(s)\,ds\right\|_0^2\right]$$

$$\leqslant \frac{\Gamma_e}{\epsilon_0\omega_{pe}^2}\left[\frac{\tau\delta_5}{2}\sum_{j=1}^n\left\|\tilde{\xi}_h^{j+\frac{1}{2}} + \tilde{\xi}_h^{j-\frac{1}{2}}\right\|_0^2 + \frac{\tau^4}{8\delta_5}\int_{t_{\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \|\boldsymbol{J}_{tt}(s)\|_0^2\,ds\right].$$

By similar arguments, we have

$$Err_6 \leqslant 2T\delta_6\|\tilde{\xi}_h\|^2_{l^\infty(L^2)} + \tau\delta_6\left\|\tilde{\xi}_h^{\frac{1}{2}}\right\|_0^2 + \frac{\tau^4}{8\delta_6}\int_{t_{\frac{1}{2}}}^{t_{n+\frac{1}{2}}} \|\boldsymbol{E}_{tt}(s)\|_0^2\,ds.$$

Similarly, we obtain

$$Err_7 \leqslant \frac{\Gamma_m}{\mu_0\omega_{pm}^2}\left[\frac{\tau\delta_7}{2}\sum_{j=1}^n \|\tilde{\eta}_h^j + \tilde{\eta}_h^{j-1}\|_0^2 + \frac{\tau^4}{8\delta_7}\int_{t_0}^{t_n} \|\boldsymbol{K}_{tt}(s)\|_0^2\,ds\right]$$

and

$$Err_8 \leqslant 2T\delta_8\|\tilde{\eta}_h\|^2_{l^\infty(L^2)} + \tau\delta_8\|\tilde{\eta}_h^0\|_0^2 + \frac{\tau^4}{8\delta_8}\int_{t_0}^{t_n} \|\boldsymbol{H}_{tt}(s)\|_0^2\,ds.$$

Using the Cauchy–Schwarz inequality, we obtain

$$Err_9 = \tau\left(\tilde{\xi}_h^{n+\frac{1}{2}}, \xi_h^n\right) \leqslant \frac{\epsilon_0\omega_{pe}^2\tau^2}{2}\|\xi_h^n\|_0^2 + \frac{1}{2\epsilon_0\omega_{pe}^2}\left\|\tilde{\xi}_h^{n+\frac{1}{2}}\right\|_0^2, \quad (49)$$

$$Err_{10} = \tau\left(\tilde{\xi}_h^{\frac{1}{2}}, \xi_h^0\right) \leqslant \frac{\epsilon_0\omega_{pe}^2\tau^2}{2}\|\xi_h^0\|_0^2 + \frac{1}{2\epsilon_0\omega_{pe}^2}\left\|\tilde{\xi}_h^{\frac{1}{2}}\right\|_0^2, \quad (50)$$

$$Err_{11} = \tau\left(\tilde{\eta}_h^n, \eta_h^{n+\frac{1}{2}}\right) \leqslant \frac{\mu_0\omega_{pm}^2\tau^2}{2}\left\|\eta_h^{n+\frac{1}{2}}\right\|_0^2 + \frac{1}{2\mu_0\omega_{pm}^2}\|\tilde{\eta}_h^n\|_0^2, \quad (51)$$

$$Err_{12} = \tau\left(\tilde{\eta}_h^0, \eta_h^{\frac{1}{2}}\right) \leqslant \frac{\mu_0\omega_{pm}^2\tau^2}{2}\left\|\eta_h^{\frac{1}{2}}\right\|_0^2 + \frac{1}{2\mu_0\omega_{pm}^2}\|\tilde{\eta}_h^0\|_0^2. \quad (52)$$

By the Cauchy–Schwarz inequality and the inverse estimate (15), and recall the notation $C_v = 1/\sqrt{\mu_0\epsilon_0}$, we have

$$Err_{13} \leqslant \tau \cdot C_{inv}h^{-1}\|\xi_h^0\|_0\left\|\eta_h^{\frac{1}{2}}\right\|_0 = \tau C_{inv}h^{-1}C_v\sqrt{\epsilon_0}\|\xi_h^0\|_0 \cdot \sqrt{\mu_0}\left\|\eta_h^{\frac{1}{2}}\right\|_0$$

$$\leqslant \frac{C_{inv}C_v\tau}{2h}\left[\epsilon_0\|\xi_h^0\|_0^2 + \mu_0\left\|\eta_h^{\frac{1}{2}}\right\|_0^2\right],$$

$$Err_{14} \leqslant \tau \cdot C_{inv}h^{-1}\|\xi_h^n\|_0\left\|\eta_h^{n+\frac{1}{2}}\right\|_0$$

$$= \tau C_{inv}h^{-1}C_v\sqrt{\epsilon_0}\|\xi_h^n\|_0 \cdot \sqrt{\mu_0}\left\|\eta_h^{n+\frac{1}{2}}\right\|_0$$

$$\leqslant \frac{C_{inv}C_v\tau}{2h}\left[\epsilon_0\|\xi_h^n\|_0^2 + \mu_0\left\|\eta_h^{n+\frac{1}{2}}\right\|_0^2\right],$$

$$Err_{15} = \sum_i \frac{\tau}{2}\int_{\partial T_i} \eta_i^{n+\frac{1}{2}} \cdot \mathbf{n}_{ik} \times \xi_i^n$$

$$\leqslant \frac{\tau}{2}C_{inv}^2 h^{-1}c_v\sum_i\sqrt{\epsilon_0}\|\xi_i^n\|_{0,T_i}\sqrt{\mu_0}\left\|\eta_i^{n+\frac{1}{2}}\right\|_{0,T_i}$$

$$\leqslant \frac{\tau C_{inv}^2 C_v}{4h}\left[\epsilon_0\|\xi_h^n\|_0^2 + \mu_0\left\|\eta_h^{n+\frac{1}{2}}\right\|_0^2\right].$$

Similar to Err_{15}, we have

$$Err_{16} \leqslant \frac{\tau C_{inv}^2 C_v}{4h}\left(\epsilon_0\|\xi_h^0\|_0^2 + \mu_0\left\|\eta_h^{\frac{1}{2}}\right\|_0^2\right),$$

$$Err_{17} \leqslant \frac{\tau C_{inv}^2 C_v}{4h}\left(\epsilon_0\|\xi_h^n\|_0^2 + \mu_0\left\|\eta_h^{n+\frac{1}{2}}\right\|_0^2\right),$$

$$Err_{18} \leqslant \frac{\tau C_{inv}^2 C_v}{4h}\left(\|\xi_h^0\|_0^2 + \left\|\eta_h^{\frac{1}{2}}\right\|_0^2\right).$$

By the Cauchy–Schwarz inequality, the inverse estimate (15) and the projection error estimate (29), we have

$$Err_{19} \leqslant \sum_{j=1}^n \frac{\tau}{2}C_{inv}^2 h^{-1}\sum_{T_i}\left\|\xi_i^j + \xi_i^{j-1}\right\|_{0,T_i} \cdot Ch^{min(s,k)+1}\left(\|\boldsymbol{H}\|_{L^\infty(0,T;H^{s+1}(\Omega))} + \|\boldsymbol{E}\|_{L^\infty(0,T;H^{s+1}(\Omega))}\right)$$

$$\leqslant \tau\sum_{j=1}^n\left[\delta_{19}\sum_{T_i}\left(\left\|\xi_h^j\right\|_0^2 + \left\|\xi_h^{j-1}\right\|_0^2\right) + \frac{C_{inv}^4}{2\delta_{19}} \cdot Ch^{2min(s,k)}\right]$$

$$\leqslant 2T\delta_{19}\|\xi_h\|^2_{l^\infty(L^2)} + \tau\delta_{19}\|\xi_h^0\|_0^2 + \frac{TC_{inv}^4}{2\delta_{19}} \cdot Ch^{2min(s,k)}.$$

By the same arguments, we have

$$Err_{20} \leqslant 2T\delta_{20}\|\eta_h\|^2_{l^\infty(L^2)} + \tau\delta_{20}\|\eta_h^0\|_0^2 + \frac{TC_{inv}^2}{2\delta_{20}} \cdot Ch^{2min(s,k)}.$$

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