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THE L_p-ALEKSANDROV PROBLEM FOR L_p-INTEGRAL CURVATURE

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Abstract

It is shown that within the L_p-Brunn–Minkowski theory that Aleksandrov's integral curvature has a natural L_p extension, for all real p. This raises the question of finding necessary and sufficient conditions on a given measure in order for it to be the L_p-integral curvature of a convex body. This problem is solved for positive p and is answered for negative p provided the given measure is even.

1. Introduction

Fundamental in convex geometric analysis are both *curvature measures* and *area measures* of convex bodies. They play key roles in the Brunn–Minkowski theory of convex bodies. The most studied of the area measures is the *surface area measure* defined by Aleksandrov [**2**] and Fenchel & Jessen [**13**], while the best known curvature measure is Aleksandrov's *integral curvature* (also called *integral Gauss curvature*) which was defined and studied by Aleksandrov [**3**].

The *support function*, $h_Q : S^{n-1} \to \mathbb{R}$, of a compact convex subset Q of Euclidean n-space, $\mathbb{R}^n$, determines Q uniquely and is defined by $h_Q(u) = \max\{u \cdot x : x \in Q\}$, for $u \in S^{n-1}$, where $u \cdot x$ is the standard inner product of u and x in $\mathbb{R}^n$.

Oliker [**43**] (see also [**44**]) showed that there is a PDE associated with the "Minkowski problem" for Aleksandrov's integral curvature. Specifically, the "Minkowski problem" for Aleksandrov's integral curvature asks: Given a (data) function $g : S^{n-1} \to [0,\infty)$ is there a support function $h : S^{n-1} \to (0,\infty)$ that satisfies the Monge–Ampère type equation

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on S^{n-1}:

$$\frac{h}{(|\nabla h|^2 + h^2)^{n/2}} \det(\nabla^2 h + Ih) = g, \tag{1.1}$$

where ∇h is the gradient of the (unknown) function h, while $\nabla^2 h$ is the Hessian matrix of h, and I is the identity matrix, with respect to an orthonormal frame on S^{n-1}.

The main aim of this work is to demonstrate that for each real p, there is a geometrically natural L_p extension of Aleksandrov's integral curvature. As will be shown, it turns out that the PDE associated with the "Minkowski problem" for L_p-Aleksandrov integral curvature asks: Given a (data) function $g : S^{n-1} \to [0, \infty)$, is there a support function $h : S^{n-1} \to (0, \infty)$ satisfying the Monge–Ampère type equation on S^{n-1}:

$$\frac{h^{1-p}}{(|\nabla h|^2 + h^2)^{n/2}} \det(\nabla^2 h + Ih) = g. \tag{1.2}$$

Unfortunately, for applications, this PDE must be solved for the case where the "data" may well be a measure and not just a function. And the techniques required in this more general situation turn out to be far more delicate.

The "Minkowski problem" for Aleksandrov's integral curvature was originally solved by Aleksandrov himself [**3**]. As will be seen, to demonstrate existence for the "Minkowski problem" for L_p-integral curvature requires an approach to the Aleksandrov problem radically different from that taken by either Aleksandrov [**3**] or Oliker [**45**].

The surface area measure can be viewed as a differential of the volume functional (Lebesgue measure) of convex bodies via Aleksandrov's variational formula for volume. We provide some details.

Let $\mathcal{K}_o^n$ denote the class of convex bodies (compact convex subsets) in Euclidean n-space $\mathbb{R}^n$ that contain the origin in their interiors. For $K, L \in \mathcal{K}_o^n$ and $t \geq 0$, the Minkowski linear combination $K + tL \in \mathcal{K}_o^n$ is defined by

$$h_{K+tL} = h_K + th_L.$$

As will be explained in Section 2, the body $K + tL \in \mathcal{K}_o^n$ can be defined for negative t of sufficiently small absolute value. Aleksandrov's variational formula for volume, V, defines a Borel measure on S^{n-1}, called the *surface area measure* $S(K, \cdot)$ of the convex body $K \in \mathcal{K}_o^n$ via the integral representation

$$\frac{d}{dt} V(K + tQ)\Big|_{t=0} = \int_{S^{n-1}} h_Q(u)\, dS(K, u),$$

which holds for each $Q \in \mathcal{K}_o^n$.

The classical *Minkowski problem* asks: Given a Borel measure μ on S^{n-1} (called the *data*) what are necessary and sufficient conditions on the measure μ to guarantee the existence of a body $K \in \mathcal{K}_o^n$ such that $\mu = S(K, \cdot)$, and if such a body exists to what extent is it unique?

Minkowski [40, 41] himself solved the polytope case using a variational argument. Aleksandrov [1, 2] and Fenchel & Jessen [13], independently gave a complete solution by using a variational method similar to that used by Minkowski. The variational approach is based on the fact, as presented above, that the surface area measure is a differential of volume, and, thus, the solution to the Euler–Lagrange equation of a volume-maximization problem will provide the solution to the Minkowski problem. From the point of view of partial differential equations, the solution of the Minkowski problem amounts to solving a degenerate fully nonlinear partial differential equation. The study of the regularity of the solutions to the Minkowski problem has a long history and strong influence on both the Brunn–Minkowski theory and the theory of fully nonlinear partial differential equations. See, e.g., [9, 7, 42, 46, 52].

The *L_p-Brunn–Minkowski theory* is an extension of the classical Brunn–Minkowski theory. The roots of the L_p-Brunn–Minkowski theory date back to the middle of the twentieth century, but its active development had to await the emergence of the concept of *L_p-surface area measure* in [34] in the early 1990's. For each real $p \geq 1$, Firey (see, e.g., [47]) defined what has become known as the Minkowski–Firey L_p-combination $K +_p t \cdot L \in \mathcal{K}_o^n$ for $K, L \in \mathcal{K}_o^n$ and $t \geq 0$ by letting

$$h^p_{K+_p t\cdot L} = h^p_K + t h^p_L.$$

Note that "$\cdot$" is written without its subscript p. In the early 1990's (as will be explained in Section 2) it was shown that these Minkowski–Firey L_p combinations can be fruitfully defined for negative t of sufficiently small absolute value. This led to the notion of the L_p-surface area measure, $S_p(K, \cdot)$, for each body $K \in \mathcal{K}_o^n$, via the variational formula:

$$\frac{d}{dt} V(K +_p t\cdot Q)\Big|_{t=0} = \frac{1}{p} \int_{S^{n-1}} h_Q(u)^p \, dS_p(K, u),$$

which holds for each $Q \in \mathcal{K}_o^n$. It was also shown in [34] that for each $K \in \mathcal{K}_o^n$,

$$dS_p(K, \cdot) = h_K^{1-p} dS(K, \cdot),$$

which shows that L_p-surface area measure may be extended to all $p \in \mathbb{R}$ in a completely obvious manner.

The associated *L_p-Minkowski problem* in the L_p-Brunn–Minkowski theory (first studied in [34]) asks: For fixed $p \in \mathbb{R}$, given a Borel measure μ on S^{n-1} (the *data*) what are necessary and sufficient conditions on the measure μ to guarantee the existence of a body $K \in \mathcal{K}_o^n$ such that $\mu = S_p(K, \cdot)$, and if such a body K exists to what extent is K unique? The classical Minkowski problem ($p = 1$) becomes a special case of the L_p-Minkowski problem, while the *logarithmic Minkowski problem* ($p = 0$) and the *centro-affine Minkowski problem* ($p = -n$) are two special unsolved cases which are major open problems; see, e.g., [**6**] and Chow & Wang [**10**]. A number of works contributed to solving various cases of the L_p-Minkowski problem; see, e.g., [**8**, **10**, **22**, **23**, **25**, **26**, **27**, **31**, **30**, **35**, **38**, **51**, **49**, **50**, **55**, **56**, **57**, **58**]. In [**11**, **21**, **37**, **39**, **54**] affine Sobolev inequalities were obtained by using the solution of the Minkowski problem and the L_p-Minkowski problem (together with L_p-affine isoperimetric inequalities from [**36**, **20**]). Connections between the L_p-Minkowski problem and curvature flows can be found in, e.g., [**4**, **5**].

If the measure μ has a density function $g : S^{n-1} \to [0, \infty)$, then the partial differential equation that is associated with the L_p-Minkowski problem (with data g) is the Monge–Ampère type equation on S^{n-1}:

$$h^{1-p} \det(\nabla^2 h + Ih) = g, \tag{1.3}$$

where $\nabla^2 h$ is the Hessian matrix of the (unknown) function h and I is the identity matrix with respect to an orthonormal frame on S^{n-1}.

The centro-affine Minkowski problem corresponds to the case $p = -n$. Its partial differential equation is:

$$h^{n+1} \det(\nabla^2 h + Ih) = g. \tag{1.4}$$

Solving this PDE is a longstanding open problem, even when g is assumed to be an even function. For special cases, see the recent paper [**27**].

To state the Aleksandrov problem and its new proposed L_p analogue, in full generality, we shall investigate the *entropy functional*, defined for $Q \in \mathcal{K}_o^n$ by

$$\mathcal{E}(Q) = -\int_{S^{n-1}} \log h_Q(v)\, dv, \tag{1.5}$$

where the integration is with respect to spherical Lebesgue measure. Recall that the polar, Q^*, of the convex body $Q \in \mathcal{K}_o^n$ is the body in $\mathcal{K}_o^n$ defined by

$$Q^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \text{ for all } y \in Q\}.$$

By combining the notion of polarity with that of L_p-combinations we arrive at another important (in both convex geometric and functional analysis) way of combining bodies in $\mathcal{K}_o^n$: *harmonic L_p-combinations.* For each real $p \ge 1$, the harmonic L_p-combination $K \hat{+}_p t \cdot L \in \mathcal{K}_o^n$ of $K, L \in \mathcal{K}_o^n$ and $t \ge 0$, is defined by

$$K \hat{+}_p t{\cdot}L = (K^* +_p t{\cdot}L^*)^*.$$

Note that, by abuse of notation, "$\cdot$" is written on the left without sub or superscripts. As will be shown in Section 2, the body $K \hat{+}_p t{\cdot}L \in \mathcal{K}_o^n$

can be defined for all $p \in \mathbb{R}$ and even for negative t of sufficiently small absolute value.

The *Aleksandrov integral curvature*, $J(K, \cdot)$, of a body $K \in \mathcal{K}_o^n$ is a Borel measure on S^{n-1} that (as will be shown) can be defined by the variational formula

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(K \hat{+}_0 t\cdot Q)\Big|_{t=0} \\ = -\int_{S^{n-1}} \log \rho_Q(u)\, dJ(K, u), \end{aligned} \tag{1.6}$$

for each $Q \in \mathcal{K}_o^n$. It should be emphasized that (1.6) is not Aleksandrov's definition of this classical and fundamental concept.

The Aleksandrov problem is a "Minkowski problem" for Aleksandrov's integral curvature: What are necessary and sufficient conditions on a given Borel measure (the data) on the unit sphere so that the measure is the integral curvature of a convex body? And to what extent is the body (the solution) uniquely determined by the given data measure? Aleksandrov [**3**] gave a complete solution to the problem. He settled the polytope case using his "mapping lemma" and then the general case using an approximation argument.

The problem (posed by Aleksadrov) of finding a direct variational proof demonstrating existence of solutions to the Aleksandrov problem—a proof similar to the variational approach used to demonstrate the existence of solutions to the Minkowski problem—was first studied by Oliker [**45**]. Oliker also considered the polytope case first, but used a variational and mass transport approach to replace the mapping lemma, and then applied Aleksandrov's approximation argument to the general case.

One of the aims of this work is to provide a new direct variational proof demonstrating the existence of a solution of the classical Aleksandrov problem. Unfortunately, the proof presented in this paper (Theorem 7.2) only establishes necessary and sufficient conditions for the existence of solutions in the origin-symmetric case.

Regularity of solutions to the Aleksandrov problem was investigated by Guan–Li [**17**] and Oliker [**43**] (see also [**44**]). For regularity regarding more general problems, see the recent paper Li–Sheng–Wang [**29**] and its references.

General area measures of convex bodies were introduced by Aleksandrov and Fenchel & Jessen. General curvature measures for sets of positive reach were discovered by Federer [**12**], and their restriction to convex bodies were treated directly by Schneider [**48**]. Aleksandrov–Minkowski-type problems for other curvature measures were studied by, e.g., Guan–Lin–Ma [**18**] and Guan–Li–Li [**19**].

In this work, for each $p \in \mathbb{R}$, we define an *L_p-integral curvature*, $J_p(K, \cdot)$, of a convex body $K \in \mathcal{K}_o^n$, as a Borel measure on S^{n-1}, such that:

$$\frac{d}{dt}\mathcal{E}(K \,\hat{+}_p\, t\cdot Q)\Big|_{t=0} = \frac{1}{p}\int_{S^{n-1}} \rho_Q(u)^{-p}\, dJ_p(K,u),$$

for each $Q \in \mathcal{K}_o^n$. The existence of the limit and of L_p-integral curvature will be demonstrated. It turns out (as will be shown) that for each $K \in \mathcal{K}_o^n$,

$$dJ_p(K,\cdot) = \rho_K^p dJ(K,\cdot). \tag{1.7}$$

Note that given the classical definition of Aleksandrov integral curvature, (1.7) could be used to define L_p-integral curvature for all $p \in \mathbb{R}$, although this could rightly be viewed as artificial and unmotivated.

The **L_p-Aleksandrov problem** we pose asks: *For fixed $p \in \mathbb{R}$, given a Borel measure μ on S^{n-1} what are necessary and sufficient conditions on the measure μ to guarantee the existence of a body $K \in \mathcal{K}_o^n$ such that $\mu = J_p(K,\cdot)$? And if such a body exists, to what extent is it unique?* If the measure μ has a density function $g : S^{n-1} \to \mathbb{R}$, then the L_p-Aleksandrov problem (with data g) becomes the PDE (1.2). The aim of this paper is not only to introduce the concept of L_p-integral curvature but to establish existence results for various cases of the associated L_p-Aleksandrov problem (i.e., the "Minkowski problem" for L_p-integral curvature) in the L_p-Brunn–Minkowski theory.

We shall investigate entropy maximization problems for convex bodies. The solutions to the Euler–Lagrange equations for the entropy maximization problems will provide our solutions to the L_p-Aleksandrov problem. A solution to the case where $p > 0$ will be presented (Theorem 7.1). Sufficient conditions will be given in the symmetric case when $p < 0$ (Theorem 7.3). These conditions are also necessary if the given measure has a density. In particular, our results imply that the PDE,

$$\frac{h^{n+1}}{(|\nabla h|^2 + h^2)^{n/2}} \det(\nabla^2 h + Ih) = g \tag{1.8}$$

has a strictly positive solution whenever $g : S^{n-1} \to [0,\infty)$ is an integrable even function whose integral over S^{n-1} is positive. In view of similarities between the PDEs (1.4) and (1.8), this may shed new light on the unsolved centro-affine Minkowski problem. As noted above, for the case of $p = 0$, the classical Aleksandrov problem, we give a new direct variational proof of Aleksandrov's result (Theorem 7.2)—but also here only for even measures where the solutions turn out to be origin symmetric. Unfortunately, uniqueness is not established for the main theorems to be presented.

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2. Preliminaries

In this section, we list some basic facts for quick later reference. Schneider's book [47] is the standard reference regarding convex bodies. The books [14, 15] are also good references.

For $x \in \mathbb{R}^n$, let $|x| = \sqrt{x \cdot x}$ be the Euclidean norm of x. For $x \in \mathbb{R}^n \setminus \{0\}$, define $\bar{x} \in S^{n-1}$ by $\bar{x} = x/|x|$. For a subset E in $\mathbb{R}^n \setminus \{0\}$, let $\bar{E} = \{\bar{x} : x \in E\}$. The origin-centered unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$ is always denoted by B, and its boundary by S^{n-1}. Write ω_n for the volume of B and o_n for the surface area of S^{n-1}. Recall that $o_n = n\omega_n$.

For the set of continuous functions defined on the unit sphere S^{n-1} write $C(S^{n-1})$, and for $f \in C(S^{n-1})$ write $\|f\| = \max_{v \in S^{n-1}} |f(v)|$. We shall view $C(S^{n-1})$ as endowed with the topology induced by this *max-norm*. We write $C^+(S^{n-1})$ for the set of strictly positive functions in $C(S^{n-1})$, and $C_e^+(S^{n-1})$ for the set of functions in $C^+(S^{n-1})$ that are even.

If μ is a fixed non-zero finite Borel measure on S^{n-1}, for $f \in C^+(S^{n-1})$, define

$$\|f : \mu\|_p = \left(\frac{1}{|\mu|} \int_{S^{n-1}} f^p \, d\mu \right)^{1/p}, \quad p \neq 0,$$

and

$$\|f : \mu\|_0 = \|f\|_0 = \exp\left(\frac{1}{|\mu|} \int_{S^{n-1}} \log f \, d\mu \right).$$

If μ is spherical Lebsgue measure, we will write $\|f\|_p$ instead of $\|f : \mu\|_p$.

If $K \subset \mathbb{R}^n$ is compact and convex, the support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of K is defined by $h_K(x) = \max\{x \cdot y : y \in K\}$, for $x \in \mathbb{R}^n$. The support function is convex and homogeneous of degree 1. A compact convex subset of $\mathbb{R}^n$ is uniquely determined by its support function.

Denote by $\mathcal{K}^n$ the space of compact convex sets in $\mathbb{R}^n$ endowed with the *Hausdorff metric*; i.e., the distance between $K, L \in \mathcal{K}^n$ is $\|h_K - h_L\|$. By a *convex body* in $\mathbb{R}^n$ we will always mean a compact convex set with nonempty interior. Denote by $\mathcal{K}_o^n$ the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and denote by $\mathcal{K}_e^n$ the class of origin-symmetric convex bodies in $\mathbb{R}^n$.

Let $K \subset \mathbb{R}^n$ be compact and star-shaped with respect to the origin. The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is defined by

$$\rho_K(x) = \max\{\lambda : \lambda x \in K\},$$

for $x \neq 0$. A compact star-shaped (about the origin) set is uniquely determined by its radial function on S^{n-1}. Denote by $\mathcal{S}^n$ the set of compact star-shaped sets. A star body is a compact star-shaped set with respect to the origin whose radial function is continuous and positive. If K is a star body, then obviously

$$\partial K = \{\rho_K(u)u : u \in S^{n-1}\}.$$

Denote by $\mathcal{S}_o^n$ the space of star bodies in $\mathbb{R}^n$ endowed with the *radial metric*; i.e., the distance between $K, L \in \mathcal{S}_o^n$, is $\|\rho_K - \rho_L\|$. Note that $\mathcal{K}_o^n \subset \mathcal{S}_o^n$ and that on the space $\mathcal{K}_o^n$ the Hausdorff metric and radial metric are equivalent, and, thus, $\mathcal{K}_o^n$ is a subspace of $\mathcal{S}_o^n$.

If $K \in \mathcal{K}_o^n$, then it is easily seen that the radial function and the support function of K are related by,

$$h_K(v) = \max_{u \in S^{n-1}} (u \cdot v)\, \rho_K(u), \quad v \in S^{n-1}, \tag{2.1}$$

$$1/\rho_K(u) = \max_{v \in S^{n-1}} (u \cdot v)/h_K(v), \quad u \in S^{n-1}. \tag{2.2}$$

From the definition of the polar body, we see that on $\mathbb{R}^n \setminus \{0\}$,

$$\rho_K = 1/h_{K^*} \quad \text{and} \quad h_K = 1/\rho_{K^*}. \tag{2.3}$$

From this, it's trivial to see that,

$$K^{**} = K. \tag{2.4}$$

In this paper, a *convex cone* $\gamma \subset \mathbb{R}^n$ is a convex set such that for all $t \geq 0$ and for each $x \in \gamma$, we have $tx \in \gamma$. The polar cone γ^* is defined by

$$\gamma^* = \{y \in \mathbb{R}^n : x \cdot y \leq 0 \text{ for all } \ x \in \gamma\}.$$

As noted above, the intersection of a convex cone γ with the unit sphere S^{n-1} is denoted by $\bar{\gamma}$; i.e.,

$$\bar{\gamma} = S^{n-1} \cap \gamma.$$

A set $\omega \subset S^{n-1}$ is called convex if there exists a convex cone γ contained in an open half-space of $\mathbb{R}^n$ so that

$$\omega = \bar{\gamma} = S^{n-1} \cap \gamma.$$

The polar ω^* of ω is defined by

$$\omega^* = \overline{\gamma^*} = S^{n-1} \cap \gamma^*,$$

that is,

$$\omega^* = \{v \in S^{n-1} : v \cdot u \leq 0 \text{ for all } \ u \in \omega\}.$$

Denote by $\Omega \subset S^{n-1}$ a closed set that always will be assumed not to be contained in any closed hemisphere of S^{n-1}, and suppose that the function $h : \Omega \to (0, \infty)$ is continuous. The *Wulff shape* $[h] \in \mathcal{K}_o^n$, also known as the *Aleksandrov body*, determined by h is the convex body defined by

$$[h] = \{x \in \mathbb{R}^n : x \cdot v \leq h(v) \text{ for all } v \in \Omega\}.$$

Note that when $h_K : S^{n-1} \to (0, \infty)$ is the support function of a body $K \in \mathcal{K}_o^n$, we have

$$[h_K] = K.$$

Suppose $\rho : \Omega \to (0, \infty)$ is continuous. Since $\Omega \subset S^{n-1}$ is closed, and ρ is continuous, $\{\rho(u)u : u \in \Omega\} \subset \mathbb{R}^n$ is compact. Hence, the convex hull $\langle \rho \rangle$ generated by ρ,

$$\langle \rho \rangle = \operatorname{conv}\{\rho(u)u : u \in \Omega\}$$

is compact as well (see Schneider [47], Theorem 1.1.11). Since Ω is not contained in any closed hemisphere of S^{n-1}, the compact convex set $\langle \rho \rangle \in \mathcal{K}_o^n$. Obviously, if $K \in \mathcal{K}_o^n$, and we consider $\rho_K : S^{n-1} \to (0, \infty)$, we have

$$\langle \rho_K \rangle = K. \tag{2.5}$$

The support function of the convex hull $\langle \rho \rangle$ is given by

$$h_{\langle \rho \rangle}(v) = \max_{u \in \Omega} (v \cdot u)\rho(u), \quad v \in S^{n-1}. \tag{2.6}$$

The Wulff shape $[h]$ of a continuous function $h : \Omega \to (0, \infty)$ and the convex hull $\langle 1/h \rangle$ generated by its reciprocal are polar reciprocals of each other; i.e.,

$$[h]^* = \langle 1/h \rangle.$$

See [24] for the easy proof of this.

The L_p Minkowski combination is the basic concept in the L_p-Brunn–Minkowski theory. Fix a real p. For $K, L \in \mathcal{K}_o^n$, and $a, b \geq 0$, define the *L_p Minkowski combination*, $a \cdot K +_p b \cdot L \in \mathcal{K}_o^n$, via the Wulff shape:

$$a \cdot K +_p b \cdot L = [(a h_K^p(v) + b h_L^p(v))^{1/p}], \tag{2.7}$$

when $p \neq 0$. Note that the notion of Wulff shape allows us to consider an L_p-combination where either a or b may be negative, as long the function $a h_K^p + b h_L^p$ is strictly positive on S^{n-1}. When $p = 0$, define $a \cdot K +_0 b \cdot L$ via the Wulff shape

$$a \cdot K +_0 b \cdot L = [h_K^a h_L^b]. \tag{2.8}$$

Define the *L_p-harmonic combination* $a \cdot K \hat{+}_p b \cdot L$ by

$$a \cdot K \hat{+}_p b \cdot L = (a \cdot K^* +_p b \cdot L^*)^*. \tag{2.9}$$

Note that "$\cdot$" is written without either a sub or superscript. Note, as an aside, that when $a + b = 1$,

$$\lim_{p \to 0} a \cdot K +_p b \cdot L = a \cdot K +_0 b \cdot L.$$

If $a, b \in [0, \infty)$, not both 0, and $p = 1$, then $a \cdot K +_p b \cdot L$ is just written as $aK + bL$. Note that for $a > 0$,

$$h_{aK} = a h_K \text{ and } \rho_{aK} = a \rho_K. \tag{2.10}$$

3. L_p-integral curvature and the L_p-Aleksandrov problem

For a convex body K in $\mathbb{R}^n$, and for $v \in S^{n-1}$, the hyperplane

$$H_K(v) = \{x \in \mathbb{R}^n : x \cdot v = h_K(v)\}$$

is called the *supporting hyperplane to K with unit normal v.*

For $\sigma \subset \partial K$, the *spherical image*, $\boldsymbol{\nu}_K(\sigma)$, of σ is defined by

$$\boldsymbol{\nu}_K(\sigma) = \{v \in S^{n-1} : x \in H_K(v) \text{ for some } x \in \sigma\} \subset S^{n-1}.$$

For $\eta \subset S^{n-1}$, the *reverse spherical image*, $\boldsymbol{x}_K(\eta)$, of η is defined by

$$\boldsymbol{x}_K(\eta) = \{x \in \partial K : x \in H_K(v) \text{ for some } v \in \eta\} \subset \partial K.$$

Let $\sigma_K \subset \partial K$ be the set consisting of all $x \in \partial K$, for which the set $\boldsymbol{\nu}_K(\{x\})$, abbreviated as $\boldsymbol{\nu}_K(x)$, contains more than a single element. It is well known that $\mathcal{H}^{n-1}(\sigma_K)$, the $(n-1)$-dimensional Hausdorff measure of σ_K, is 0 (see p. 84 of Schneider [47]). The function

$$\nu_K : \partial K \setminus \sigma_K \to S^{n-1}, \tag{3.1}$$

defined by letting $\nu_K(x)$ be the unique element in $\boldsymbol{\nu}_K(x)$, for each $x \in \partial K \setminus \sigma_K$, is called the *spherical image map* of K and is well known to be continuous (see Lemma 2.2.12 of Schneider [47]). The set $\eta_K \subset S^{n-1}$ consisting of all $v \in S^{n-1}$, for which the set $\boldsymbol{x}_K(v)$ contains more than a single element, is well known to be of $\mathcal{H}^{n-1}$-measure 0 (see Theorem 2.2.11 of Schneider [47]). The function

$$x_K : S^{n-1} \setminus \eta_K \to \partial K, \tag{3.2}$$

defined, for each $v \in S^{n-1} \setminus \eta_K$, by letting $x_K(v)$ be the unique element in $\boldsymbol{x}_K(v)$, is called the *reverse spherical image map*. The vectors in $S^{n-1} \setminus \eta_K$ are called the *regular normal vectors* of K. Thus, $v \in S^{n-1}$ is a regular normal vector of K if and only if $\partial K \cap H_K(v)$ consists of a single point. The function x_K is well known to be continuous (see Lemma 2.2.12 of Schneider [47]).

For $K \in \mathcal{K}_o^n$, define the *radial map* of K,

$$r_K : S^{n-1} \to \partial K \text{ by } r_K(u) = \rho_K(u)u \in \partial K,$$

for $u \in S^{n-1}$. Note that $r_K^{-1} : \partial K \to S^{n-1}$ is just the restriction of the map $\bar{\cdot} : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ to ∂K. The radial map is a homeomorphism.

For $\omega \subset S^{n-1}$, define the *radial Gauss image of ω* by

$$\boldsymbol{\alpha}_K(\omega) = \boldsymbol{\nu}_K(r_K(\omega)) \subset S^{n-1}.$$

Thus, for $u \in S^{n-1}$,

$$\boldsymbol{\alpha}_K(u) = \{v \in S^{n-1} : r_K(u) \in H_K(v)\}. \tag{3.3}$$

Observe, that from the definition we immediately see that for all $\lambda > 0$, for the homothet λK we have

$$\boldsymbol{\alpha}_{\lambda K} = \boldsymbol{\alpha}_K. \tag{3.4}$$

Define the *radial Gauss map* of the convex body $K \in \mathcal{K}_o^n$

$$\alpha_K : S^{n-1} \setminus \omega_K \to S^{n-1} \text{ by } \alpha_K = \nu_K \circ r_K,$$

where $\omega_K = \overline{\sigma_K} = r_K^{-1}(\sigma_K)$. Since $r_K^{-1} = \bar{\cdot}$ is a bi-Lipschitz map between the spaces ∂K and S^{n-1} it follows that ω_K has spherical Lebesgue measure 0. Observe that if $u \in S^{n-1} \setminus \omega_K$, then $\boldsymbol{\alpha}_K(u)$ contains only the element $\alpha_K(u)$. Note that since both ν_K and r_K are continuous, α_K is continuous.

For $\eta \subset S^{n-1}$, define the *reverse radial Gauss image* of η by

$$\boldsymbol{\alpha}_K^*(\eta) = r_K^{-1}(\boldsymbol{x}_K(\eta)) = \overline{\boldsymbol{x}_K(\eta)}. \tag{3.5}$$

Thus,

$$\boldsymbol{\alpha}_K^*(\eta) = \{\bar{x} : \ x \in \partial K \text{ where } x \in H_K(v) \text{ for some } v \in \eta\}.$$

Observe, that from the definition we immediately see that for all $\lambda > 0$, for the homothet λK we have

$$\boldsymbol{\alpha}_{\lambda K}^* = \boldsymbol{\alpha}_K^*. \tag{3.6}$$

Define the *reverse radial Gauss map* of the convex body $K \in \mathcal{K}_o^n$,

$$\alpha_K^* : S^{n-1} \setminus \eta_K \to S^{n-1}, \text{ by } \alpha_K^* = r_K^{-1} \circ x_K. \tag{3.7}$$

Note that since both r_K^{-1} and x_K are continuous, α_K^* is continuous.

If $\eta \subset S^{n-1}$ is a Borel set, then $\boldsymbol{\alpha}_K^*(\eta) = \overline{\boldsymbol{x}_K(\eta)} \subset S^{n-1}$ is spherical Lebesgue measurable. This fact is Lemma 2.2.14 of Schneider [**47**], an alternate proof of which was given in [**24**].

It was shown in [**24**] that on subsets of S^{n-1} the reverse radial Gauss image, $\boldsymbol{\alpha}_K^*$, of a convex body, K, and the radial Gauss image, $\boldsymbol{\alpha}_{K^*}$, of its polar body, K^*, agree; i.e., for each $K \in \mathcal{K}_o^n$ and each $\eta \subset S^{n-1}$,

$$\boldsymbol{\alpha}_K^*(\eta) = \boldsymbol{\alpha}_{K^*}(\eta). \tag{3.8}$$

It follows that $\boldsymbol{\alpha}_{K^*}(\eta)$ is spherical Lebesgue measurable for each $K \in \mathcal{K}_o^n$ and for each Borel set $\eta \subset S^{n-1}$. Since $^* : \mathcal{K}_o^n \to \mathcal{K}_o^n$ is a bijection, we conclude that $\boldsymbol{\alpha}_K(\omega)$ is spherical Lebesgue measurable for each $K \in \mathcal{K}_o^n$ and for each Borel set $\omega \subset S^{n-1}$.

The *integral curvature*, $J(K,\cdot)$, of $K \in \mathcal{K}_o^n$ is a Borel measure on S^{n-1} defined by

$$J(K,\omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega)), \tag{3.9}$$

for each Borel set $\omega \subset S^{n-1}$; i.e., $J(K,\omega)$ is the spherical Lebesgue measure of $\boldsymbol{\alpha}_K(\omega)$. The total integral curvature, $J(K, S^{n-1})$, of a convex body K is the surface area, o_n, of the unit sphere S^{n-1} in $\mathbb{R}^n$. The concept of integral curvature was introduced by Aleksandrov.

Note that from (3.4) and definition (3.9) it follows immediately that for all $\lambda > 0$, for the homothet λK we have

$$J(\lambda K,\cdot) = J(K,\cdot). \tag{3.10}$$

We note, as an aside that, if the convex body K is C^2 smooth with positive Gauss curvature, then the integral curvature has a continuous density,

$$\frac{h}{(|\nabla h|^2+h^2)^{\frac{n}{2}}}\det(\nabla^2 h+Ih), \tag{3.11}$$

where $h=1/\rho_K$, while ∇h is the gradient of h on S^{n-1}, $\nabla^2 h$ is the Hessian matrix of h on S^{n-1}, and I is the identity matrix, with respect to an orthonormal frame on S^{n-1}.

For each $p\in\mathbb{R}$, define the L_p*-integral curvature* $J_p(K,\cdot)$ of $K\in\mathcal{K}_o^n$ as a Borel measure such that

$$\int_{S^{n-1}} f(u)\,dJ_p(K,u)=\int_{S^{n-1}} f(\alpha_K^*(u))\rho_K^p(\alpha_K^*(u))\,du, \tag{3.12}$$

for each continuous $f: S^{n-1}\to\mathbb{R}$, and the integration is with respect to spherical Lebesgue measure. From (3.12) we see that for each Borel set $\omega\subset S^{n-1}$,

$$J_p(K,\omega)=\int_{S^{n-1}}\mathbb{1}_\omega(\alpha_K^*(u))\rho_K^p(\alpha_K^*(u))\,du=\int_{\boldsymbol{\alpha}_K(\omega)}\rho_K^p(\alpha_K^*(u))\,du, \tag{3.13}$$

where the last identity comes from the fact (see (2.21) in [**24**]) that $\alpha_K^*(u)\in\omega$ if and only if $u\in\boldsymbol{\alpha}_K(\omega)$, for almost all u with respect to spherical Lebesgue measure.

From (3.9), we see that for each Borel set $\omega\subset S^{n-1}$,

$$\int_{S^{n-1}}\mathbb{1}_\omega(u)\,dJ(K,u)=\int_{S^{n-1}}\mathbb{1}_{\boldsymbol{\alpha}_K(\omega)}(u)\,du=\int_{S^{n-1}}\mathbb{1}_\omega(\alpha_K^*(u))\,du, \tag{3.14}$$

where again the last identity comes from the fact that $u\in\boldsymbol{\alpha}_K(\omega)$, if and only if $\alpha_K^*(u)\in\omega$, for almost all u with respect to spherical Lebesgue measure. But from (3.14), it follows that

$$\int_{S^{n-1}} f(u)\,dJ(K,u)=\int_{S^{n-1}} f(\alpha_K^*(u))\,du,$$

or equivalently, that

$$\int_{S^{n-1}} f(u)\rho_K^p(u)\,dJ(K,u)=\int_{S^{n-1}} f(\alpha_K^*(u))\rho_K^p(\alpha_K^*(u))\,du, \tag{3.15}$$

for each continuous $f: S^{n-1}\to\mathbb{R}$. When (3.15) is combined with (3.12) we have

$$dJ_p(K,\cdot)=\rho_K^p\,dJ(K,\cdot). \tag{3.16}$$

Thus, $J_0(K,\cdot)=J(K,\cdot)$ for each $K\in\mathcal{K}_o^n$; i.e., Aleksandrov's integral curvature is the special case $p=0$ of L_p-integral curvature. Observe that from (2.10), (3.16) and (3.10), it follows that for $K \in \mathcal{K}_o^n$ and $\lambda > 0$,

$$J_p(\lambda K, \cdot) = \lambda^p J_p(K, \cdot). \tag{3.17}$$

Note, as an aside, that if the convex body K is C^2 smooth with positive Gauss curvature, then it follows from (3.16) and (3.11) that the L_p-integral curvature has a continuous density, given by

$$\frac{h^{1-p}}{(|\nabla h|^2 + h^2)^{\frac{n}{2}}} \det(\nabla^2 h + Ih), \tag{3.18}$$

where $h = 1/\rho_K$.

It will be seen in the next section (Proposition 4.2) that L_p-integral curvature arises naturally in the L_p-Brunn–Minkowski theory.

It is easy to show that the integral curvature of a convex body is not concentrated in any closed hemisphere, and it was shown that the total measure of the integral curvature of a convex body is the surface area of the unit sphere. It is natural to try to find a complete set of properties that characterize integral curvature.

The Aleksandrov problem. *For a given finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions so that μ is the integral curvature $J(K, \cdot)$ of a convex body $K \in \mathcal{K}_o^n$?*

This problem was solved by Aleksandrov – completely. His solution (of which we will make no use) is:

Theorem 3.1. *If μ is a finite Borel measure on S^{n-1}, then μ is the integral curvature of a convex body in $\mathcal{K}_o^n$ if and only if μ satisfies $|\mu| = o_n$ and*

$$\mu(S^{n-1} \setminus \omega^*) > \mathcal{H}^{n-1}(\omega),$$

for each convex $\omega \subset S^{n-1}$.

We formulate the following problem for the L_p-integral curvature.

The L_p-Aleksandrov problem. *For a fixed $p \in \mathbb{R}$, and a given Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions so that μ is the L_p-integral curvature $J_p(K, \cdot)$ of a convex body $K \in \mathcal{K}_o^n$? And if such a body K exists, to what extent is it unique?*

We note, as an aside, that from (3.18) it follows that a particular case of the L_p-Aleksandrov problem asks: Under what conditions on a given (data) function $g : S^{n-1} \to [0, \infty)$ does there exist a solution $h : S^{n-1} \to (0, \infty)$, that is the support function of a convex body, to the Monge–Ampère equation on S^{n-1}

$$\frac{h^{1-p}}{(|\nabla h|^2 + h^2)^{\frac{n}{2}}} \det(\nabla^2 h + Ih) = g.$$

4. Variational formulas for entropy of convex bodies

Let $\Omega \subset S^{n-1}$ be a closed set that is not contained in any closed hemisphere of S^{n-1}. Let $f : \Omega \to \mathbb{R}$ be continuous, and $\delta > 0$. Let $h_t : \Omega \to (0, \infty)$ be a continuous function defined for each $t \in (-\delta, \delta)$ by

$$\log h_t(v) = \log h(v) + t f(v) + o(t, v), \tag{4.1}$$

where $o : (-\delta, \delta) \times S^{n-1} \to \mathbb{R}$ is such that $o(t, \cdot) : S^{n-1} \to \mathbb{R}$ is continuous, for each t, and $\lim_{t\to 0} o(t, \cdot)/t = 0$, uniformly on Ω. Denote by

$$[h_t] = \{x \in \mathbb{R}^n : x \cdot v \le h_t(v) \text{ for all } v \in \Omega\},$$

the Wulff shape determined h_t. We shall call $[h_t]$ a *logarithmic family of Wulff shapes formed by* (h, f, o). On occasion, we shall write $[h_t]$ as $[h, f, t]$, and if h happens to be the support function of a convex body K perhaps as $[K, f, t]$. We call $[K, f, t]$ a logarithmic family of Wulff shapes formed by (K, f, o).

Let $g : \Omega \to \mathbb{R}$ be continuous and $\delta > 0$. Let $\rho_t : \Omega \to (0, \infty)$ be a continuous function defined for each $t \in (-\delta, \delta)$ by

$$\log \rho_t(u) = \log \rho(u) + t g(u) + o(t, u), \tag{4.2}$$

where again the function $o : (-\delta, \delta) \times S^{n-1} \to \mathbb{R}$ is such that $o(t, \cdot) : S^{n-1} \to \mathbb{R}$ is continuous, for each t, and $\lim_{t\to 0} o(t, \cdot)/t = 0$, uniformly on Ω. Denote by

$$\langle \rho_t \rangle = \mathrm{conv}\{\rho_t(u) u : u \in \Omega\},$$

the convex hull generated by ρ_t. We will call $\langle \rho_t \rangle$ a *logarithmic family of convex hulls generated by* (ρ, g, o). On occasion, we shall write $\langle \rho_t \rangle$ as $\langle \rho, g, t \rangle$, and if ρ happens to be the radial function of a convex body K as $\langle K, g, t \rangle$. We call $\langle K, g, t \rangle$ a logarithmic family of convex hulls generated by (K, g, o).

The dual entropy $E(K)$ of a convex body $K \in \mathcal{K}_o^n$ is defined by

$$E(K) = \int_{S^{n-1}} \log \rho_K(u)\, du. \tag{4.3}$$

From (2.3) we see that the dual entropy of $K \in \mathcal{K}_o^n$ and the entropy of the polar of K are the same; i.e.,

$$E(K) = \mathcal{E}(K^*). \tag{4.4}$$

The following variational formulas for the entropy and dual entropy of convex bodies were established in Lemmas 4.6 and 4.7 of [**24**].

Lemma 4.1. *Suppose $\Omega \subset S^{n-1}$ is a closed set that is not contained in any closed hemisphere of S^{n-1}. Suppose also that $o : (-\delta, \delta) \times S^{n-1} \to \mathbb{R}$ is such that $o(t, \cdot) : S^{n-1} \to \mathbb{R}$ is continuous, for each t, and $\lim_{t\to 0} o(t, \cdot)/t = 0$, uniformly on Ω. Further, suppose also that $K \in \mathcal{K}_o^n$*

and $f, g : \Omega \to \mathbb{R}$ *are continuous. Then if* $\langle K, g, t\rangle$ *is a logarithmic family of convex hulls generated by* (K, g, o), *then*

$$\frac{d}{dt}\mathcal{E}(\langle K, g, t\rangle)\Big|_{t=0} = -\int_\Omega g(u)\, dJ(K, u), \tag{4.5}$$

and if $[K, f, t]$ *is a logarithmic family of Wulff shapes formed by* (K, f, o), *then*

$$\frac{d}{dt}E([K, f, t])\Big|_{t=0} = \int_\Omega f(v)\, dJ(K^*, v). \tag{4.6}$$

The variational formulas above can be employed to obtain the variational formulas for the entropy of harmonic L_p-Minkowski combinations:

Proposition 4.2. *Let* $K, L \in \mathcal{K}_o^n$. *Then, for* $p \neq 0$,

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_p\, t{\cdot}L)\Big|_{t=0} = \frac{1}{p}\int_{S^{n-1}} \rho_L(v)^{-p}\, dJ_p(K, v), \tag{4.7}$$

while for $p = 0$,

$$\frac{d}{dt}\mathcal{E}(K \hat{+}_0\, t{\cdot}L)\Big|_{t=0} = -\int_{S^{n-1}} \log \rho_L(v)\, dJ(K, v). \tag{4.8}$$

Proof. For $p \neq 0$, let

$$h_t = (h_K^p + t h_L^p)^{1/p},$$

and choose a positive δ such that

$$\delta < \begin{cases} [\min_{v\in S^{n-1}} h_K(v)/\max_{v\in S^{n-1}} h_L(v)]^p & \text{for } p > 0,\\ [\min_{v\in S^{n-1}} h_L(v)/\max_{v\in S^{n-1}} h_K(v)]^{-p} & \text{for } p < 0.\end{cases}$$

Then

$$\log h_t = \log h_K + \frac{1}{p}\Big(\frac{h_L}{h_K}\Big)^p t + o_p(t, \cdot),$$

where $o_p : (-\delta, \delta) \times S^{n-1} \to \mathbb{R}$ is such that each $o_p(t, \cdot) : S^{n-1} \to \mathbb{R}$ is continuous and $\lim_{t\to 0} o_p(t, \cdot)/t = 0$, uniformly on S^{n-1}. Choose

$$f = \frac{1}{p}\Big(\frac{h_L}{h_K}\Big)^p,$$

and note that

$$K +_p t{\cdot}L = [h_t] = [K, f, t].$$

This, (2.3), (3.16), and (4.6) give

$$\frac{d}{dt}E(K +_p t{\cdot}L)\Big|_{t=0} = \frac{1}{p}\int_{S^{n-1}} h_L^p(v)\, dJ_p(K^*, v),$$

when $p \neq 0$. In the above display replace K, L by K^*, L^*; use (4.4), (2.3) and (2.4), and we obtain the desired result (4.7).

For the case of $p = 0$, let

$$h_t = h_K h_L^t.$$

Then

$$\log h_t = \log h_K + t \log h_L.$$

Let

$$f = \log h_L.$$

Then

$$K +_0 t\cdot L = [K, f, t].$$

This and (4.6) give

$$\frac{d}{dt} E(K +_0 t\cdot L)\Big|_{t=0} = \int_{S^{n-1}} \log h_L(v)\, dJ(K^*, v).$$

In the above display replace K, L by K^*, L^*; use (2.3), (2.4), (4.4) and we obtain the desired result (4.8). q.e.d.

The variational formulas in Proposition 4.2 show how L_p-integral curvature arises naturally in the L_p-Brunn–Minkowski theory.

5. Maximizing the entropy of convex bodies

Fix $p \in \mathbb{R}$. We show that the L_p-Aleksandrov problem can be reduced to the Euler–Lagrange equation of a maximization problem. For a given Borel measure μ, the maximization problem is:

$$\sup\{\mathcal{E}(\langle f\rangle)/o_n + \log \|f : \mu\|_{-p} : f \in C^+(S^{n-1})\}. \tag{5.1}$$

On $C^+(S^{n-1})$, the class of strictly positive continuous functions on S^{n-1}, define the functional $\Phi : C^+(S^{n-1}) \to \mathbb{R}$, by letting

$$\Phi(f) = \mathcal{E}(\langle f\rangle)/o_n + \log \|f : \mu\|_{-p}, \tag{5.2}$$

for each $f \in C^+(S^{n-1})$. The convex hull,

$$\langle f\rangle = \operatorname{conv}\{f(u)u : u \in S^{n-1}\}$$

is in $\mathcal{K}_o^n$ since f is strictly positive. We first observe that Φ is homogeneous of degree 0, in that for all $\lambda > 0$, and all $f \in C^+(S^{n-1})$,

$$\Phi(\lambda f) = \Phi(f).$$

To see this, note that $f \mapsto \|f : \mu\|_{-p}$ is obviously homogeneous of degree 1. Clearly $\langle \lambda f\rangle = \lambda\langle f\rangle$ and, thus, $h_{\langle \lambda f\rangle} = \lambda h_{\langle f\rangle}$. The homogeneity of degree 0 of Φ now follows immediately from definitions (1.5) and (5.2).

We observe that $\Phi : C^+(S^{n-1}) \to \mathbb{R}$ is continuous. To see this, recall that if $f_0, f_1, \ldots \in C^+(S^{n-1})$, are such that $\lim_{k\to\infty} f_k = f_0$, uniformly on S^{n-1}, then $\langle f_k\rangle \to \langle f_0\rangle$, in $\mathcal{K}_o^n$. The continuity of Φ now follows immediately from the continuity of $\mathcal{E} : \mathcal{K}_o^n \to \mathbb{R}$ and that of $\|\cdot : \mu\|_{-p} : C^+(S^{n-1}) \to (0, \infty)$.

Lemma 5.1. *Suppose $p \in \mathbb{R}$. A convex body $K \in \mathcal{K}_o^n$ is a solution of the maximization problem,*

$$\sup\{\mathcal{E}(Q)/o_n + \log \|\rho_Q : \mu\|_{-p} : \ Q \in \mathcal{K}_o^n\},$$

if and only if ρ_K is a solution of the maximization problem,

$$\sup\{\mathcal{E}(\langle f \rangle)/o_n + \log \|f : \mu\|_{-p} : f \in C^+(S^{n-1})\}.$$

Proof. Consider the maximization problem

$$\sup\{\Phi(f) : f \in C^+(S^{n-1})\}. \tag{5.3}$$

For the convex hull $\langle f \rangle = \operatorname{conv}\{f(u)u : u \in S^{n-1}\}$, of $f \in C^+(S^{n-1})$, we clearly have $\rho_{\langle f \rangle} \geq f$ and, thus, $\|\rho_{\langle f \rangle} : \mu\|_{-p} \geq \|f : \mu\|_{-p}$. Also, since from (2.5), we have $\langle \rho_{\langle f \rangle} \rangle = \langle f \rangle$, we see that $\mathcal{E}(\langle \rho_{\langle f \rangle} \rangle) = \mathcal{E}(\langle f \rangle)$. Thus, directly from (5.2), we have

$$\Phi(f) \leq \Phi(\rho_{\langle f \rangle}).$$

This tells us that in searching for the supremum in (5.3) we can restrict our attention to the radial functions of bodies in $\mathcal{K}_o^n$; i.e.,

$$\sup\{\Phi(f) : f \in C^+(S^{n-1})\} = \sup\{\Phi(\rho_Q) : Q \in \mathcal{K}_o^n\}.$$

Therefore, a convex body $K \in \mathcal{K}_o^n$ is a solution of the maximization problem,

$$\sup\{\mathcal{E}(Q)/o_n + \log \|\rho_Q : \mu\|_{-p} : \ Q \in \mathcal{K}_o^n\},$$

if and only if

$$\Phi(\rho_K) = \sup\{\Phi(f) : f \in C^+(S^{n-1})\}.$$ q.e.d.

Lemma 5.2. *Suppose $p \in \mathbb{R}$. Let μ be a finite Borel measure on S^{n-1} and $K \in \mathcal{K}_o^n$ satisfying*

$$\int_{S^{n-1}} \rho_K^{-p} \, d\mu = o_n. \tag{5.4}$$

If K is a solution of the maximization problem

$$\sup\{\mathcal{E}(Q)/o_n + \log \|\rho_Q : \mu\|_{-p} : \ Q \in \mathcal{K}_o^n\}, \tag{5.5}$$

then

$$\mu = J_p(K, \cdot).$$

Proof. Since $\langle \rho_Q \rangle = Q$, for each $Q \in \mathcal{K}_o^n$, the fact that K is a solution of the maximization problem (5.5) can be rewritten, in light of (5.2), as:

$$\Phi(\rho_K) = \sup\{\Phi(\rho_Q) : Q \in \mathcal{K}_o^n\}. \tag{5.6}$$

Lemma 5.1, and the fact that K is a solution of the maximization problem (5.6), tells us that

$$\Phi(\rho_K) = \sup\{\Phi(f) : f \in C^+(S^{n-1})\}.$$

Suppose $g \in C^+(S^{n-1})$ is fixed. Define

$$\rho_t = \rho(t, \cdot) = \rho_K e^{tg},$$

that is,

$$\log \rho_t = \log \rho_K + tg. \tag{5.7}$$

From Lemma 4.1 we have

$$\frac{d}{dt}\mathcal{E}(\langle \rho_t \rangle)\Big|_{t=0} = -\int_{S^{n-1}} g(u)\, dJ(K,u). \tag{5.8}$$

We now show that for each $p \in \mathbb{R}$,

$$\frac{d}{dt}\log \|\rho_t : \mu\|_{-p}\Big|_{t=0} = \frac{1}{o_n}\int_{S^{n-1}} \rho_K^{-p}(u) g(u)\, d\mu(u). \tag{5.9}$$

To see this, first suppose $p \neq 0$. Since $|e^s - 1 - s| \leq es^2$, for all $s \in (-1,1)$, we see that

$$\Big| \frac{e^{-tpg(u)} - 1}{t} + pg(u) \Big| \ \leq ep^2 g(u)^2 |t|,$$

for all $u \in S^{n-1}$, and all t such that $|t| < 1/(|p| \max_{u \in S^{n-1}} g(u))$. Since g is continuous on S^{n-1} we conclude that, as $t \to 0$

$$\frac{\rho_t^{-p} - \rho_0^{-p}}{t} = \frac{e^{-tpg} - 1}{t}\rho_K^{-p} \longrightarrow -pg\, \rho_K^{-p}, \text{ uniformly on } S^{n-1}.$$

From this, by recalling (5.4), we immediately get the desired (5.9). The case $p = 0$ is simpler: From (5.7) we see that

$$\log \|\rho_t : \mu\|_0 = \frac{1}{|\mu|}\int_{S^{n-1}} \log \rho_t \, d\mu = \frac{1}{|\mu|}\int_{S^{n-1}} (tg + \log \rho_K)\, d\mu,$$

which quickly gives (5.9) for the case $p = 0$, by recalling that here (5.4) is $|\mu| = o_n$.

The Euler–Lagrange equation,

$$\frac{d}{dt}\Phi(\rho_t)\Big|_{t=0} = \frac{d}{dt}\left(\mathcal{E}(\langle \rho_t \rangle)/o_n + \log \|\rho_t : \mu\|_{-p}\right)\Big|_{t=0} = 0,$$

together with (5.8) and (5.9) gives

$$-\int_{S^{n-1}} g(u)\, dJ(K,u) + \int_{S^{n-1}} \rho_K^{-p}(u) g(u)\, d\mu(u) = 0.$$

Since g was arbitrary, we conclude that $\rho_K^{-p} d\mu = dJ(K,\cdot)$. When this is combined with (3.16) we obtain the desired conclusion that $\mu = J_p(K,\cdot)$. q.e.d.

Recall that a Borel measure on S^{n-1} is called *even* if it assumes the same values on antipodal Borel subsets of S^{n-1}. For origin-symmetric convex bodies in $\mathbb{R}^n$ and even measures on S^{n-1}, we have the following similar lemma.

Lemma 5.3. *Suppose $p \in \mathbb{R}$. Let μ be a finite, even, Borel measure on S^{n-1} and $K \in \mathcal{K}_e^n$ a body satisfying*

$$\int_{S^{n-1}} \rho_K^{-p}\, d\mu = o_n.$$

If K is a solution of the maximization problem

$$\sup\{\mathcal{E}(Q)/o_n + \log \|\rho_Q : \mu\|_{-p} : \ Q \in \mathcal{K}_e^n\},$$

then

$$\mu = J_p(K, \cdot).$$

The proof is the same (mutatis mutandis) as that of Lemma 5.2.

6. Existence of solutions to maximization problems for entropy

In this section, we shall establish results regarding the existence of solutions to maximization problems for entropy of convex bodies. These results will yield existence of solutions to L_p-Aleksandrov problems.

For $v_0 \in S^{n-1}$ and $0 < r < 1$, define $\omega_r(v_0)$ and $\omega_r'(v_0)$ by:

$$\omega_r(v_0) = \{u \in S^{n-1} : u \cdot v_0 \geq r\}, \tag{6.1}$$

$$\omega_r'(v_0) = \{u \in S^{n-1} : |u \cdot v_0| \geq r\}. \tag{6.2}$$

We shall make use of the fact that if μ is a positive (non-zero) Borel measure on S^{n-1} that's not concentrated on any closed hemisphere of S^{n-1}, then for each $v_0 \in S^{n-1}$, we must have $\mu(\omega_\delta(v_0)) > 0$, for all sufficiently small δ. Otherwise, we would have

$$\mu(\{u \in S^{n-1} : u \cdot v_0 > 0\}) = \lim_{n \to \infty} \mu(\omega_{\frac{1}{n}}(v_0)) = 0,$$

which would imply that the measure μ is concentrated in the closed hemisphere $\{u \in S^{n-1} : u \cdot v_0 \leq 0\}$. Observe that

$$\omega_r'(v_0) = \omega_r(v_0) \cup \omega_r(-v_0). \tag{6.3}$$

Lemma 6.1. *Suppose $0 < r < 1$ and $v_0 \in S^{n-1}$. If K_i is a sequence of convex bodies in $\mathcal{K}_o^n$, then*

$$\lim_{i \to \infty} h_{K_i}(v_0) = 0 \textit{ implies } \rho_{K_i} \to 0, \textit{ uniformly on } \omega_r(v_0).$$

Proof. Note that (2.1) tells us that $(u \cdot v_0)\rho_{K_i}(u) \leq h_{K_i}(v_0)$, for all $u \in S^{n-1}$. But by definition, $u \in \omega_r(v_0)$ means that $u \cdot v_0 \geq r$. Hence, $\rho_{K_i}(u) \leq h_{K_i}(v_0)/r$, for all $u \in \omega_r(v_0)$, which allows us to conclude the desired result that $\lim_{i \to \infty} h_{K_i}(v_0) = 0$, implies $\rho_{K_i} \to 0$, uniformly on $\omega_r(v_0)$. q.e.d.

From (6.3) and Lemma 6.1 we immediately have:

Lemma 6.2. *Suppose $0 < r < 1$ and $v_0 \in S^{n-1}$. If K_i is a sequence of convex bodies in $\mathcal{K}^n_e$, then*

$$\lim_{i\to\infty} h_{K_i}(v_0) = 0, \text{ implies } \rho_{K_i} \to 0, \text{ uniformly on } \omega'_r(v_0).$$

For $p \in \mathbb{R}$, define $\mathcal{F}_p : \mathcal{K}^n_o \to \mathbb{R}$, by letting

$$\mathcal{F}_p(Q) = \mathcal{E}(Q)/o_n + \log \|\rho_Q : \mu\|_{-p}, \tag{6.4}$$

for each $Q \in \mathcal{K}^n_o$. Obviously, $\mathcal{F}_p$ is homogeneous of degree 0; i.e., for each $Q \in \mathcal{K}^n_o$, we have $\mathcal{F}_p(\lambda Q) = \mathcal{F}_p(Q)$, for each $\lambda > 0$.

Lemma 6.3. *Suppose $p > 0$ and μ is a finite Borel measure that is not concentrated on any closed hemisphere of S^{n-1}. Then there exists a body $K_0 \in \mathcal{K}^n_o$ such that*

$$\sup\{\mathcal{E}(Q)/o_n + \log \|\rho_Q : \mu\|_{-p} : \ Q \in \mathcal{K}^n_o\} = \mathcal{F}_p(K_0). \tag{6.5}$$

Proof. Let,

$$\mathcal{K} = \{Q \in \mathcal{K}^n_o : \int_{S^{n-1}} h^p_Q \, d\mu = o_n\}. \tag{6.6}$$

Note that each $K \in \mathcal{K}^n_o$ has a dilate that belongs to $\mathcal{K}$. In particular, the ball $r_{np\mu}B$ of radius $r_{np\mu} = (o_n/|\mu|)^{1/p}$ belongs to $\mathcal{K}$.

Consider the continuous function $v \mapsto \int_{S^{n-1}} (v\cdot u)^p_+ \, d\mu(u)$. The function is strictly positive since μ is not concentrated on a closed hemisphere of S^{n-1}. Let $v_\mu \in S^{n-1}$ be such that,

$$\int_{S^{n-1}} (v\cdot u)^p_+ \, d\mu(u) \geq \int_{S^{n-1}} (v_\mu \cdot u)^p_+ \, d\mu(u) > 0, \tag{6.7}$$

for all $v \in S^{n-1}$.

We show that $\mathcal{K}$ is bounded. Suppose $Q \in \mathcal{K}$ and let

$$\max_{v\in S^{n-1}} \rho_Q(v) = \rho_Q(v_Q), \tag{6.8}$$

for some $v_Q \in S^{n-1}$. Since $\rho_Q(v_Q)v_Q \in Q$,

$$\rho_Q(v_Q)(v_Q \cdot u)_+ \leq h_Q(u), \quad \text{for all } u \in S^{n-1}.$$

Since $Q \in \mathcal{K}$, we have

$$\rho_Q(v_Q)^p \int_{S^{n-1}} (v_Q \cdot u)^p_+ \, d\mu(u) \leq \int_{S^{n-1}} h^p_Q(u) \, d\mu(u) = o_n. \tag{6.9}$$

Combining (6.8), (6.9) with (6.7) yields

$$\max_{v\in S^{n-1}} \rho_Q(v) = \rho_Q(v_Q) \leq o_n^{1/p} \Big(\int_{S^{n-1}} (v_\mu \cdot u)^p_+ \, d\mu(u) \Big)^{-1/p} = c_{np\mu}.$$

This shows that all bodies in $\mathcal{K}$ are contained in the ball $c_{np\mu}B$.

From definitions (6.4) and (6.6), by using (2.3), we see that

$$\mathcal{F}_p(Q) = \mathcal{E}(Q)/o_n - \frac{1}{p} \log \frac{o_n}{|\mu|}, \text{ whenever } Q^* \in \mathcal{K}. \tag{6.10}$$

Since $\mathcal{F}_p$ is homogeneous of degree 0, we may choose a maximizing sequence K_i for $\mathcal{F}_p$ with each K_i having been dilated so that $K_i^* \in \mathcal{K}$. Let $L_i = K_i^*$. Since $r_{np\mu} B \in \mathcal{K}$, where $r_{np\mu} = (o_n/|\mu|)^{1/p}$, it follows that for all sufficiently large i,

$$\mathcal{F}_p(K_i) > \mathcal{F}_p(r_{np\mu} B) = \mathcal{F}_p(B) = 0, \tag{6.11}$$

unless a ball is our desired solution.

Since $\mathcal{K}$ is bounded and the $L_i \in \mathcal{K}$, the sequence L_i has a convergent subsequence, denoted again by L_i, such that

$$L_i \longrightarrow L_0,$$

for some compact convex L_0. To see that L_0 has non-empty interior, and that the origin is not a point on its boundary, we shall argue by contradiction. If the origin $o \in \partial L_0$, then L_0 is contained in a closed half-space $\{x \in \mathbb{R}^n : x \cdot u_0 \leq 0\}$, for some $u_0 \in S^{n-1}$. Then $h_{L_0}(u_0) = 0$. Thus, $h_{L_i}(u_0) \to 0$. Consider $\omega_\delta(u_0)$, defined in (6.1), for some fixed small $\delta > 0$. It follows from Lemma 6.1, that $\rho_{L_i} \to 0$ uniformly on $\omega_\delta(u_0)$. Thus, since the $K_i^* = L_i \in \mathcal{K}$, and all bodies in $\mathcal{K}$ are contained in $c_{np\mu} B$, we have from definition (1.5), and (2.3),

$$\begin{aligned}
\mathcal{E}(K_i) &= \mathcal{E}(L_i^*) \\
&= \int_{S^{n-1}} \log \rho_{L_i}(u)\, du \\
&\leq \int_{\omega_\delta(u_0)} \log \rho_{L_i}(u)\, du \;+\; \int_{S^{n-1} \setminus \omega_\delta(u_0)} \log c_{np\mu}\, du \\
&\leq \int_{\omega_\delta(u_0)} \log \rho_{L_i}(u)\, du \;+\; \mathcal{H}^{n-1}(S^{n-1} \setminus \omega_\delta(u_0)) \log c_{np\mu}.
\end{aligned}$$

Since $\rho_{L_i} \to 0$, uniformly on $\omega_\delta(u_0)$, we conclude that

$$\int_{\omega_\delta(u_0)} \log \rho_{L_i}\, du \to -\infty,$$

forcing $\mathcal{E}(K_i) \to -\infty$, and, thus, since $K_i^* \in \mathcal{K}$, from (6.10) we conclude that $\mathcal{F}_p(K_i) \to -\infty$, in contradiction to (6.11), and, thus, L_0 must be a convex body that contains the origin in its interior.

Since L_0 contains the origin in its interior, from $L_i \to L_0$ we conclude $L_i^* \to L_0^*$ or equivalently $K_i \to L_0^* \in \mathcal{K}_o^n$, which shows that L_0^* is the desired limit of the maximizing sequence for $\mathcal{F}_p$. q.e.d.

Define $\mathcal{G} : \mathcal{K}_o^n \to \mathbb{R}$, by

$$\mathcal{G}(Q) = \mathcal{E}(Q) + \int_{S^{n-1}} \log \rho_Q(u)\, d\mu(u), \tag{6.12}$$

for $Q \in \mathcal{K}_o^n$. Note that $\mathcal{G}$ is homogeneous of degree 0 when $|\mu| = o_n$; i.e., $\mathcal{G}(\lambda Q) = \mathcal{G}(Q)$, for each $\lambda > 0$.

Lemma 6.4. *Suppose μ is a finite even Borel measure that is not concentrated on any great sub-sphere of S^{n-1} and that has total mass $|\mu| = o_n$. Then there exists an $L \in \mathcal{K}_e^n$, such that*

$$\sup\{\mathcal{E}(Q) + \int_{S^{n-1}} \log \rho_Q \, d\mu : Q \in \mathcal{K}_e^n\} = \mathcal{G}(L). \tag{6.13}$$

Proof. Let

$$\mathcal{K} = \{Q \in \mathcal{K}_e^n : \mathcal{E}(Q) = 0\}.$$

Since $\mathcal{G}$ is homogeneous of degree 0, the maximization problem for $\mathcal{G}$ is equivalent to

$$\sup\{\int_{S^{n-1}} \log \rho_Q \, d\mu : Q \in \mathcal{K}\},$$

and, thus, in searching for a maximum for $\mathcal{G}$ we shall restrict our attention to bodies from $\mathcal{K}$ exclusively. We observe that the unit ball B belongs to $\mathcal{K}$.

For $K \in \mathcal{K}_e^n$, let

$$R_K = \max_{u \in S^{n-1}} \rho_K(u) = \rho_K(u_K), \tag{6.14}$$

where u_K is one of the unit vectors at which the maximum occurs. Then, since K is origin-symmetric and $\{\lambda u_K : -R_K \leq \lambda \leq R_K\} \subset K$, we have

$$R_K |u_K \cdot v| \leq h_K(v), \text{ for all } v \in S^{n-1}.$$

Integrating this (over S^{n-1}) we see that if $K \in \mathcal{K}$, by using definition (1.5) and the definition of $\mathcal{K}$,

$$o_n \log R_K + \int_{S^{n-1}} \log |u_K \cdot v| \, dv \leq \int_{S^{n-1}} \log h_K(v) \, dv = -\mathcal{E}(K) = 0. \tag{6.15}$$

Since the integral on the left is independent of u_K, (6.15) implies that R_K is bounded and, thus, there exists an $m \in (0, \infty)$ such that every set in $\mathcal{K}$ is contained in the ball mB.

Let $K_i \in \mathcal{K}$ be a maximizing sequence for $\mathcal{G}$. Since obviously $B \in \mathcal{K}$, for all sufficiently large i,

$$\mathcal{G}(K_i) > \mathcal{G}(B) = 0, \tag{6.16}$$

unless a ball is our desired solution. Since $\mathcal{K}$ is bounded, K_i has a convergent subsequence, denoted again by K_i, which converges to an origin-symmetric compact convex set L.

We show that L has non-empty interior arguing by contradiction; specifically by assuming that L is contained in a co-dimension 1 subspace, say $v_0^{\perp}$. For small $\delta > 0$, let $\omega'_\delta(v_0)$ be defined by (6.2). Since $h_L(v_0) = 0$ and $K_i \to L$, it follows that $\lim_{i\to\infty} h_{K_i}(v_0) = 0$. Lemma 6.2, now tells us that $\rho_{K_i} \to 0$, uniformly on $\omega'_\delta(v_0)$. Since μ is not concentrated on a great sub-sphere of S^{n-1}, we conclude that $\mu(\omega'_{r_0}(v_0)) > 0$, for a sufficiently small $r_0 > 0$. Thus, using the fact that all $K_i \subset mB$

$$\begin{aligned}\int_{S^{n-1}} \log \rho_{K_i} d\mu &\le \int_{\omega'_{r_0}(v_0)} \log \rho_{K_i} d\mu + \int_{S^{n-1}\setminus\omega'_{r_0}(v_0)} \log m \, d\mu \\ &\le \int_{\omega'_{r_0}(v_0)} \log \rho_{K_i} d\mu + \mu(S^{n-1} \setminus \omega'_{r_0}(v_0)) \log m.\end{aligned} \tag{6.17}$$

Since $\rho_{K_i} \to 0$, uniformly on $\omega'_{r_0}(v_0)$, we conclude that

$$\int_{\omega'_{r_0}(v_0)} \log \rho_{K_i} \, d\mu \to -\infty,$$

which forces $\mathcal{G}(K_i) \to -\infty$, producing the desired contradiction with (6.16). Therefore, L is a solution of the maximization problem (6.13) in $\mathcal{K}_e^n$. q.e.d.

For negative p, we have:

Lemma 6.5. *Suppose $p \in (-\infty, 0)$. If μ is a finite, non-zero, even Borel measure on S^{n-1} that vanishes on all great sub-spheres of S^{n-1}, then there exists an $L \in \mathcal{K}_e^n$, such that*

$$\sup\{\mathcal{E}(K)/o_n + \log \|\rho_K : \mu\|_{-p} : \ K \in \mathcal{K}_e^n\} = \mathcal{F}_p(L). \tag{6.18}$$

Proof. Let

$$\mathcal{K} = \{K \in \mathcal{K}_e^n : \mathcal{E}(K) = 0\}.$$

Using the same argument used in the proof of Lemma 6.4, we conclude the existence of an $m \in (0, \infty)$ such that all bodies in $\mathcal{K}$ are contained in the ball mB.

Since $\mathcal{F}_p$ is homogeneous of degree 0, we may choose a maximizing sequence $K_i \in \mathcal{K}_e^n$ for $\mathcal{F}_p$ each K_i having been dilated precisely so that $K_i \in \mathcal{K}$. Thus,

$$\lim_{i\to\infty} \mathcal{F}_p(K_i) = \sup\{\log \|\rho_Q : \mu\|_{-p} : Q \in \mathcal{K}\}.$$

Since $B \in \mathcal{K}$, and $\mathcal{F}_p(B) = 0$, it follows that for all sufficiently large i,

$$\mathcal{F}_p(K_i) > \mathcal{F}_p(B) = 0, \tag{6.19}$$

unless a ball is our desired solution.

Since $\mathcal{K}$ is bounded, the maximizing sequence $K_i \in \mathcal{K}_e^n$ has a convergent subsequence, denoted again by K_i, which converges to an origin symmetric compact convex set L. We shall show that L has non-empty interior by contradiction. Assume not; i.e., L is contained in the $(n-1)$-dimensional subspace $v_0^\perp$, for some $v_0 \in S^{n-1}$.

Then since $h_L(v_0) = 0$, from $K_i \to L$, we have $h_{K_i}(v_0) \to 0$. It follows from Lemma 6.2, that

$$\rho_{K_i} \to 0 \quad \text{uniformly on } \omega'_\delta(v_0), \tag{6.20}$$

whenever $0 < \delta < 1$.

Since $K_i \subset mB$, and $p < 0$, we have

(6.21)
$$\begin{aligned}\int_{S^{n-1}} \rho_{K_i}^{-p}(u)\, d\mu(u) &= \int_{\omega'_\delta(v_0)} \rho_{K_i}^{-p}(u)\, d\mu(u) + \int_{S^{n-1}\setminus\omega'_\delta(v_0)} \rho_{K_i}^{-p}(u)\, d\mu(u) \\ &\le \int_{\omega'_\delta(v_0)} \rho_{K_i}^{-p}(u)\, d\mu(u) + m^{-p}\mu(S^{n-1}\setminus\omega'_\delta(v_0)).\end{aligned}$$

Since by hypothesis μ vanishes on all great sub-spheres of S^{n-1}, we know that $\mu(S^{n-1} \cap v_0^\perp) = 0$. Choose a sequence $1 > \delta_1 > \delta_2 > \cdots > \delta_j \to 0$. Observe, that

$$S^{n-1}\setminus\omega'_{\delta_1}(v_0) \;\supset\; S^{n-1}\setminus\omega'_{\delta_2}(v_0) \;\supset\; \cdots,$$

with

$$\bigcap_{j=1}^{\infty}(S^{n-1}\setminus\omega'_{\delta_j}(v_0)) \;=\; S^{n-1}\cap v_0^\perp.$$

Thus, since μ is a finite measure,

$$\lim_{j\to\infty}\mu(S^{n-1}\setminus\omega'_{\delta_j}(v_0)) = \mu(S^{n-1}\cap v_0^\perp) = 0.$$

For $\varepsilon > 0$, choose j_0 so that

$$m^{-p}\mu(S^{n-1}\setminus\omega'_{\delta_{j_0}}(v_0)) < \varepsilon/2. \tag{6.22}$$

From (6.20), we know that $\rho_{K_i}^{-p} \to 0$ uniformly on $\omega'_{\delta_{j_0}}(v_0)$. Therefore, we can find an i_0, so that for all $i \ge i_0$

$$\int_{\omega'_{\delta_{j_0}}(v_0)} \rho_{K_i}^{-p}(u)\, d\mu(u) < \varepsilon/2. \tag{6.23}$$

Combining (6.21) with (6.22) and (6.23) shows that

$$\int_{S^{n-1}} \rho_{K_i}^{-p}(u)\, d\mu(u) \longrightarrow 0,$$

as $i \to \infty$. We conclude that $\|\rho_{K_i} : \mu\|_{-p} \to 0$, and, thus,

$$\mathcal{F}_p(K_i) = \log\|\rho_{K_i} : \mu\|_{-p} \to -\infty,$$

as $i \to \infty$, in contradiction to (6.19). q.e.d.

7. Existence of solutions to the L_p-Aleksandrov problem

The following theorem gives a complete solution to the existence part of the L_p-Aleksandrov problem for the case where $p > 0$.

Theorem 7.1. *Suppose $p \in (0, \infty)$. If μ is a finite Borel measure on S^{n-1}, then there exists a convex body $K \in \mathcal{K}_o^n$ such that μ is the L_p-integral curvature of K if and only if μ is not concentrated in any closed hemisphere of S^{n-1}.*

Proof. The necessity is obvious, and the sufficiency follows from combining Lemmas 5.2 and 6.3. q.e.d.

The following theorem gives the complete solution to the existence part of the classical Aleksandrov problem for even measures. We give a direct variational proof. Finding a similar proof of the classical Aleksandrov problem is an open and interesting problem. Our proof here gives an answer for the symmetric case. The authors believe that the ideas developed here might well be helpful for the general case, but technical obstacles will need to be overcome.

Theorem 7.2. *If μ is a finite even Borel measure on S^{n-1}, then there exists an origin symmetric convex body K in $\mathbb{R}^n$ so that μ is the integral curvature of K if and only if μ is not concentrated on a great sub-sphere of S^{n-1} and $|\mu| = o_n$.*

Proof. The necessity is obvious, and the sufficiency follows from combining Lemmas 5.3 and 6.4. q.e.d.

Note that the conditions for the existence of a solution to the classical Aleksandrov problem: $\mu(S^{n-1} \setminus \omega^*) > \mathcal{H}^{n-1}(\omega)$ for each convex set ω in S^{n-1}, holds trivially whenever μ is an even measure. Note also that the necessary and sufficient conditions for the Aleksandrov problem in the symmetric case are greatly simplified.

The following theorem provides a sufficient condition for the existence of solutions to the L_p-Aleksandrov problem for the case where $p < 0$ and where the measure is even.

Theorem 7.3. *Suppose $p \in (-\infty, 0)$. If μ is a finite, even, non-zero Borel measure that vanishes on great sub-spheres of S^{n-1}, then there exists a convex body K in $\mathbb{R}^n$ so that μ is the L_p-integral curvature of K.*

Proof. Combine Lemmas 5.3 and 6.5. q.e.d.

Finally, we state implications of our results above for the existence of strictly positive weak solutions to the PDE (1.2) on S^{n-1},

$$\frac{h^{1-p}}{(|\nabla h|^2 + h^2)^{n/2}} \det(\nabla^2 h + Ih) = g,$$

where $p \in (-\infty, \infty)$ and $g : S^{n-1} \to [0, \infty)$ is integrable.

1) When $p > 0$, the PDE (1.2) has a strictly positive solution h if and only if

$$\int_{\Theta} g(u)\, du > 0,$$

for each hemisphere $\Theta \subset S^{n-1}$.

2) When $p < 0$ and g is even, the PDE (1.2) has a strictly positive even solution h if and only if

$$\int_{S^{n-1}} g(u)\,du > 0.$$

3) When $p = 0$, necessary and sufficient conditions on g for the existence of solutions to the PDE (1.2) can be derived from Aleksandrov's solution to the Aleksandrov problem.

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